MATH 311, HW 11

Due 11:00pm on Thursday April 27, 2023

Remember to scan/upload exactly the 8 pages of this template, in order.

Problem 1. [30 pts] Calculate the scalar line integral $\int_x f ds$ for the scalar field f and the path x given below:

1. (5pts) $f\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1 + 2x_2$, and

   $x(t) = \begin{pmatrix} 2-3t \\ 4t-1 \end{pmatrix}$ for $t \in [0,2]$.

$\int_x f ds =$	50

$$\int_a^b f(x(t)) \, ||x'(t)|| \, dt$$

$$f(x(t)) = 2 - 3t + 2(4t-1)$$
$$= 5t$$

$$x'(t) = 3, 4 = \sqrt{3^2+4^2} = 5$$

Plug in

$$\int_0^2 (5)(5t)\,dt = \int_0^2 25t\,dt = \left[\frac{25t^2}{2}\right]_0^2 = 50$$

2. (5pts) $f\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1 + 2x_2$, and

   $x(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$ for $t \in [0,2\pi]$.

$\int_x f ds =$	0

$$f(x(t)) = \cos t + 2\sin t$$

$$x'(t) = -\sin t, \cos t$$

$$\sqrt{-\sin^2 t + \cos^2 t}$$
$$= 1$$

$$\int_0^{2\pi} \cos t + 2\sin t \, dt = \left[\sin t - 2\cos t\right]_0^{2\pi}$$

$$= 0 - 2 - [0-2] = 0$$

3. (10pts) $f\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = x_1x_2x_3$, and

$f(x(t)) = (t)(2t)(3t)$

$= \underline{6t^3}$

$x(t) = \begin{pmatrix}t\\2t\\3t\end{pmatrix}$ for $t \in [0,2]$.

$||x'(t)|| = 1, 2, 3 = \sqrt{1^2+2^2+3^2}$

$= \sqrt{14}$

$\int_x f ds =$	$24\sqrt{14}$

Integrate

$\sqrt{14}\int_0^2 6t^3 dt = \left[\frac{6}{4}t^4\right]_0^2$

$\frac{6}{4}(2)^4 = \underline{24}$

$\sqrt{14}\cdot 24$

4. (10pts) $f\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = 3x_1 + x_1x_2 + x_3^2$, and

$\underline{f(x(t)) = 3\cos(4t) + \sin(4t)\cos(4t) + 9t^2}$

$x(t) = \begin{pmatrix}\cos(4t)\\\sin(4t)\\3t\end{pmatrix}$ for $t \in [0, 2\pi]$.

$||x'(t)|| = \sqrt{16\sin^2(4t) + 16\cos^2(4t) + 9}$

$16(1) + 9 = \sqrt{25} = 5$

$\int_x f ds =$	$120\pi^3$

Integrate

$= 5\int_0^{2\pi} 3\cos(4t) + \sin(4t) + 9t^2 dt$

$\rightarrow 5\left[\frac{3}{4}\sin(4t) + 3t^3 - \frac{1}{8}\cos^3(4t)\right.$

$\rightarrow 5\left[0 + 3(2\pi)^3 \cancel{- \frac{1}{8} + \frac{1}{8}}\right]$

$\rightarrow 5\left[24\pi^3\right] = \underline{120\pi^3}$

Problem 2. [30 pts] Calculate the vector line integral $\int_x F \cdot ds$ for the vector field F and the path x given below:

1. (5pts) $F\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$,

   and $x(t) = \begin{pmatrix} 2t+1 \\ t+2 \end{pmatrix}$ for $t \in [0, 1]$.

$\int_x F \cdot ds =$	$\frac{13}{2}$

$F(x(t)) = (2t+1, t+2)$

$||x(t)|| = 2, 1$

Plug in

$F(x(t)) \cdot x'(t) = 4t + 2 + t + 2$

$= 5t + 4$

Integrate

$\int_0^1 5t + 4 \, dt$

$\rightarrow \left[\frac{5t^2}{2} + 4t\right]_0^1 = \frac{5}{2} + 4 = \frac{13}{2}$

2. (5pts) $F\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 - x_1 \\ x_1^4 x_2^3 \end{pmatrix}$,

   and $x(t) = \begin{pmatrix} t^2 \\ t^3 \end{pmatrix}$ for $t \in [-1, 1]$.

$\int_x F \cdot ds =$	$\frac{4}{5}$

$F(x(t)) = (t^3 - t^2, t^8 t^9)$

$x'(t) = (2t, 3t^2)$

Integrate

$\int_{-1}^1 2t^4 - 2t^3 + 3t^{19} dt$

$\rightarrow \left[\frac{2t^5}{5} - \frac{2t^4}{4} + \frac{3t^{20}}{20}\right]_{-1}^1$

$\rightarrow \frac{2}{5} - \cancel{\frac{2}{4}} + \cancel{\frac{3}{20}} - \left(-\frac{2}{5} - \cancel{\frac{2}{4}} + \cancel{\frac{3}{20}}\right)$

$\frac{2}{5} + \frac{2}{5} = \frac{4}{5}$

3. (10pts) $F\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_1x_2 \\ x_1x_2x_3 \end{pmatrix}$,

and $x(t) = \begin{pmatrix} 3\cos t \\ 2\sin t \\ 5t \end{pmatrix}$ for $t \in [0, 2\pi]$.

$\int_x F \cdot ds =$	-75π

$F(x) = 3\cos t,\ 3\sin(2t),\ 15t\sin 2t$

$x'(t) = \langle -3\cos t,\ 2\cos t,\ 5\rangle$

DOT PROD

$F(x'(t)) = -9\cos t\sin t + 6\sin(2t)\cos t + 75t\sin(2t))$

$$\int_0^{2\pi} -9\cos t\sin t + 6\sin(2t)\cos t + 75t\sin(2t))$$

$$= \left[\cancel{\tfrac{9}{2}\sin^2 t} - \cancel{4\cos^3 t} + 75\left(\cancel{-\tfrac{1}{2}}t\cancel{\cos(2t)} + \cancel{\tfrac{1}{4}\sin(2t)}\right)\right|_{\cancel{0}}^{2\pi}$$

-75π

4. (10pts) $F\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -3x_2 \\ x_1 \\ 3x_3^2 \end{pmatrix}$,

and $x(t) = \begin{pmatrix} 2t+1 \\ t^2+t \\ e^t \end{pmatrix}$ for $t \in [0, 1]$.

$\int_x F \cdot ds =$	$-\frac{5}{3} + e^3$

$F(x(t)) = -3t^2 - 3t,\ 2t+1,\ 3e^{2t}$

$x'(t) = 2,\ 2t+1,\ e^t - 1$

DOT PROD

$F(x'(t)) = (-6t^2 - 6t,\ 4t^2 + 4t + 1,\ 3e^{3t})$

Integrate

$$\int_0^1 (-6t^2 - 6t,\ 4t^2 + 4t + 1,\ 3e^{3t})$$

$$= \left[-\frac{2t^3}{3} - \frac{2t^2}{2} + t + e^{3t}\right]_0^1 = -\frac{2}{3}\ \cancel{-1+1} + e^3 - [1]$$

$$= -\frac{2}{3}e^3 - \frac{3}{3} = -\frac{5}{3} + e^t$$

Problem 3. [40 pts] Verify Green's Theorem for the following vector field and region D. Namely, compute both, the line integral and the double integral.

1. (10pts) $F\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_1^2 x_2 \\ x_1 x_2^2 \end{pmatrix}$, and D is the disk such that $x_1^2 + x_2^2 \leq 4$.

$\oint_C (F_1 dx_1 + F_2 dx_2) =$	8π

① $F\langle x, y\rangle = \langle -x^2 y, xy^2\rangle$

$x^2 + y^2 \leq 4$ Ex. from Class

$$\int_0^{2\pi} (-4\cos^2 t)(2\sin t)(-2\sin t)\,dt + (2\cos t)(4\sin^2 t)(2\cos t)\,dt$$

$$= \int_0^{2\pi} 32 \sin^2 t \cos^2 t \, dt = 32\int_0^{2\pi} \frac{1-\cos(4t)}{8}\,dt = \frac{32}{8}\int_0^{2\pi} 1 - \cos(4t)\,dt$$

$$= \frac{32}{8}\cdot\left[t - \frac{1}{4}\sin(4t)\right]_0^{2\pi} = \frac{32}{8}[2\pi] = 8\pi$$

$\iint_D \left(\frac{\partial F_2}{x_1} - \frac{\partial F_1}{x_2}\right) dx_1 dx_2 =$	8π

$$\int_0^{2\pi}\int_0^2 (y^2+x^2)\,dA \rightarrow \int_0^{2\pi}\int_0^2 r^2 \cdot r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2 r^3\,dr\,d\theta = \int_0^{2\pi} d\theta \times \int_0^2 r^3\,dr$$

$$= [\theta]_0^{2\pi} \times \left.\frac{r^4}{4}\right|_0^2$$

$$2\pi \times \frac{8}{4} = 4\pi$$

2. (10pts) $F\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_1^2 \end{pmatrix}$, and D is the square with vertices $(1,1), (-1,1), (-1,-1)$, and $(1,-1)$.

$\oint_C (F_1 dx_1 + F_2 dx_2) =$	-4

$$\int_{-1}^{1} y\,dx + x^2\,dy = \int_{-1}^{1} x\,dx + \int_{-1}^{1} 1\,dy + \int_{-1}^{1} x\,dx + \int_{-1}^{1} 1\,dy$$

$$\left[\frac{x^2}{2} + \frac{x^2}{2}\right]_{1}^{-1} \qquad [y + y]_{1}^{-1}$$

$$= \frac{1}{2} \cancel{- \frac{1}{2}} \quad 2(-1-1) = -4$$

$\iint_D \left(\frac{\partial F_2}{x_1} - \frac{\partial F_1}{x_2}\right) dx_1 dx_2 =$	-4

$$\int_{-1}^{1}\int_{-1}^{1} 2x - 1\,dx\,dy$$

$$\int_{-1}^{1} \frac{2x^2}{2} - x\,dy$$

$$= x^2 - x \Big|_{-1}^{1} = 1 - 1 - [1+1]$$

$$\underline{= -2}$$

$$\int_{-1}^{1} -2\,dy = -2y\Big|_{-1}^{1}$$

$$-2 - [2] = -4$$

3. (10pts) $F\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}2x_2\\x_1\end{pmatrix}$, and D is the semicircular region $x_1^2+x_2^2\le 9$, $y\ge 0$.

$x = 3\cos t \qquad y = 3\sin t$

$dx = -3\sin t \qquad dy = 3\cos t$

$\oint_C(F_1dx_1+F_2dx_2)=$	$-\frac{9}{2}\pi$

$$\int_0^\pi (6\sin t)(-3\sin t)dt + (3\cos t)(3\cos t)dt$$

$$= \int_0^\pi -18\sin^2 t + 9\cos^2 t\, dt = \Big[\left(-18\sin^2(t)+9\cos^2(t)\right)t + \left(-\tfrac{1}{2}t\cos(2t)+\tfrac{1}{4}\sin(2t)\right)27\Big]_0^\pi$$

$$= -\frac{9}{2}\pi$$

$\iint_D\left(\frac{\partial F_2}{x_1}-\frac{\partial F_1}{x_2}\right)dx_1dx_2=$	$-\frac{9}{2}\pi$

$$\int_0^\pi\int_0^3 (1-2\ r)\ dr\,d\theta$$

$$= \int_0^\pi\int_0^3 -r\,dr\,d\theta = \int_0^\pi -\frac{9}{2}\,d\theta = -\frac{9}{2}\pi$$

$$\left[-\frac{r^2}{2}\right]_0^3 = \frac{-9}{2}$$

4. (10pts) $F\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}3x_2\\-4x_1\end{pmatrix}$, and D is the elliptical region $x_1^2 + 2x_2^2 \leq 4$.

$x = 2\cos t \qquad y = 2\sin t$

$dx = -2\sin t \qquad dy = 2\cos t$

$\oint_C (F_1 dx_1 + F_2 dx_2) =$	-28π

$$\int_0^{2\pi} (6\sin t)(-2\sin t)\,d\theta + (-8\cos t)(2\cos t)\,dt$$

$$= \int_0^{2\pi} -12\sin^2 t\,dt - 16\cos^2 t\,dt$$

$$= 14t - \sin(2t)\Big|_0^{2\pi}$$

$$= -28\pi$$

$$= \underline{-28\pi}$$

$\iint_D \left(\frac{\partial F_2}{x_1} - \frac{\partial F_1}{x_2}\right) dx_1 dx_2 =$	-28π

$$\int_0^{2\pi}\int_0^{2} (-4-3r)\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{2} -7r\,dr\,d\theta = \left[\frac{7r^2}{2}\right]_0^2 = \frac{-28}{2} = -14$$

$$\int_0^{2\pi} -14\,d\theta = -14\theta\Big|_0^{2\pi} = \underline{-28\pi}$$