MATH 311, HW 7

Due 11:00pm on Thursday March 23, 2023

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Problem 1. [18 pts] $L : \mathbb{R}^3 \to \mathbb{R}^3$ is the map $L\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) := \begin{pmatrix} x_1 - x_2 \\ x_2 - x_3 \\ x_3 - x_1 \end{pmatrix}$ for all $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3$. Find the following:

1. (8 pts) The matrix M_L representing L with respect to the standard basis ($E = \{e_1, e_2, e_3\}$)

$M_L =$	$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix}$

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 \\ x_2 - x_3 \\ x_3 - x_1 \end{pmatrix}$$

Compute $L\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1-0 \\ 0-0 \\ 0-1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

$L\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

$L\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$

2.

(5 pts) $ker(L) =$ Null(L)	$x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ -1 & 0 & 1 & 0 \end{array}\right] \xrightarrow{R_1+R_3} \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right] \xrightarrow{R_1+R_2} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right]$$

x_3 = free variable

$x_1 - x_3 = 0$ $\quad x_1 = x_3$

$x_2 - x_3 = 0$ $\quad x_2 = x_3$

$x_3 = x_3$

$$\xrightarrow{R_3+R_2} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$x = \begin{bmatrix} x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

3.

(5 pts) $range(L) =$ Col(L)	Span $\left\{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}\right\}$

$$\left[\begin{array}{ccc|c} \boxed{1} & 0 & -1 & 0 \\ 0 & \boxed{1} & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Problem 2. [10pts] Find the orthogonal complements of the following subspaces:

1. (5pts) $S = span\left\{\begin{pmatrix}3\\1\\2\end{pmatrix}, \begin{pmatrix}-1\\2\\1\end{pmatrix}\right\} \subset \mathbb{R}^3$.

$S^\perp =$	$\left\{\begin{pmatrix}-3/7\\-5/7\\1\end{pmatrix}\right\}$

$S^\perp = \left\{ \vec{x} \text{ such that } \vec{x}\cdot\begin{pmatrix}3\\1\\2\end{pmatrix} = 0 \text{ AND } \vec{x}\cdot\begin{pmatrix}-1\\2\\1\end{pmatrix} = 0 \right\}$

$$\left[\begin{array}{ccc|c}3&1&2&0\\-1&2&1&0\end{array}\right] \xrightarrow{R_2 \to R_2 + \frac{1}{3}R_1} \left[\begin{array}{ccc|c}3&1&2&0\\0&7/3&5/3&0\end{array}\right] \xrightarrow[R_2 = R_2 \cdot \frac{3}{7}]{R_1 = \frac{R_1}{3}} \left[\begin{array}{ccc|c}1&\boxed{1/3}&2/3&0\\0&1&5/7&0\end{array}\right]$$

Destroy

$-1 + \frac{1}{3}(3) = 0$
$2 + \frac{1}{3}(1) = 7/3$
$1 + \frac{1}{3}(2) = 5/3$

$$\xrightarrow{R_1 = R_1 - \frac{1}{3}R_2} \left[\begin{array}{ccc|c}1&0&3/7&0\\0&1&5/7&0\end{array}\right]$$

$\frac{2}{3} - \frac{1}{3}\left(\frac{5}{7}\right) = 3/7$

So, $x_1 = -3/7\, x_3$
and $x_2 = -5/7\, x_3$
and $x_3 = x_3$

Basis for $S^\perp = \begin{pmatrix}-3/7\\-5/7\\1\end{pmatrix} x_3$

2. (5pts) $S = span\left\{\begin{pmatrix}3\\1\\0\\2\end{pmatrix}, \begin{pmatrix}-1\\0\\1\\4\end{pmatrix}\right\} \subset \mathbb{R}^4$.

$S^\perp =$	$\left\{\begin{pmatrix}1\\-3\\1\\0\end{pmatrix}, \begin{pmatrix}4\\14\\0\\1\end{pmatrix}\right\}$

$S^\perp = \left\{ \vec{x} \text{ such that } \vec{x}\cdot\begin{pmatrix}3\\1\\0\\2\end{pmatrix} = 0 \right\}$

and $S^\perp = \left\{ \vec{x} \text{ such that } \vec{x}\cdot\begin{pmatrix}-1\\0\\1\\4\end{pmatrix} = 0 \right\}$

$$\left[\begin{array}{cccc|c}3&1&0&2&0\\-1&0&1&4&0\end{array}\right] \xrightarrow{R_2 = R_2 + \frac{1}{3}R_1} \left[\begin{array}{cccc|c}3&1&0&2&0\\0&1/3&1&14/3&0\end{array}\right]$$

$0 + \frac{1}{3}(1) = \frac{1}{3}$
$1 + \frac{1}{3}(0) = 1$
$4 + \frac{1}{3}(2) = \frac{14}{3}$

$$\xrightarrow[R_2 = 3R_2]{R_1 = \frac{R_1}{3}} \left[\begin{array}{cccc|c}1&1/3&0&2/3&0\\0&1&3&14&0\end{array}\right] \xrightarrow{R_1 \to R_1 - \frac{1}{3}R_2} \left[\begin{array}{cccc|c}1&0&-1&-4&0\\0&1&3&14&0\end{array}\right]$$

$1 - \frac{1}{3}(0) = 1$
$\frac{1}{3} - \frac{1}{3}(1) = 0$
$0 - \frac{1}{3}(3) = -1$
$2/3 - 1/3(14) = -4$

$$\xrightarrow{R_1 = R_1 + \frac{1}{3}R_2} \left[\begin{array}{cccc|c}1&0&-1&-4&0\\0&1&3&14&0\end{array}\right]$$

$x_1 - x_3 - 4x_4 = 0$
$x_2 + 3x_3 + 14x_4 = 0$

So, $\begin{pmatrix}1\\-3\\1\\0\end{pmatrix}\alpha, \begin{pmatrix}4\\14\\0\\1\end{pmatrix}\beta$

$x_1 = x_3 + 4x_4$
$x_2 = -3x_3 + 14x_4$
$x_3 = \alpha$
$x_4 = \beta$

Problems 3 to 6 For the rest of the problems, you will be given a pair of vectors v and w in $\mathbb{R}^n$, and you must find the following information:

a) (3pt) The length $\|v\|$

b) (3pt) The length $\|w\|$

c) (3pt) The distance between the endpoints of v and w.

d) (3pt) The angle θ between the vectors v and w (in degrees up to 2 decimal points, you are allowed to use calculators).

e) (3pt) The vector projection p of v onto w.

f) (3pt) The vector projection q of w onto v.

Problem 3. [18pts] $v = \begin{pmatrix} -2 \\ 3 \end{pmatrix}, \quad w = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$ $\langle -2, 3 \rangle \cdot \langle 6, 4 \rangle$

$\|v\| =$	$\sqrt{13}$
$\|w\| =$	$\sqrt{52}$
$\|v - w\| =$	$\sqrt{65}$
$\theta =$	90°

$\sqrt{(-2)^2 + (3)^2} = \sqrt{13}$

$\sqrt{6^2 + 4^2} = \sqrt{52}$

$\begin{pmatrix} -2 \\ 3 \end{pmatrix} - \begin{pmatrix} 6 \\ 4 \end{pmatrix} = \begin{pmatrix} -8 \\ -1 \end{pmatrix} = \sqrt{-8^2 + -1^2} = \sqrt{65}$

$\cos\theta = \frac{u \cdot v}{|u||v|} = \frac{-12+12}{\sqrt{13}\sqrt{52}} = 0 \quad \cos\theta = 0 = 90^\circ$

$= \frac{-12+12}{\sqrt{52}\sqrt{52}} \cdot \langle 6, 4 \rangle$

$= \langle 0, 0 \rangle$

$p =$	$\langle 0, 0 \rangle$
$q =$	$\langle 0, 0 \rangle$

$P_w(v) = \frac{\langle v, w \rangle}{\langle w, w \rangle} \cdot w$

$= \frac{-12+12}{(\sqrt{52})^2} \cdot \langle 6, 4 \rangle$

$= \frac{0}{52} \cdot \langle 6, 4 \rangle = \langle 0, 0 \rangle$

$Q_v(w) = \frac{\langle v, w \rangle}{\langle v, v \rangle} \cdot v$

$= \frac{-12+12}{(\sqrt{13})^2} \cdot \langle -2, 3 \rangle$

$= 0 \cdot \langle -2, 3 \rangle$

$= \langle 0, 0 \rangle$

Problem 4. [18pts] $v = \begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix}, \quad w = \begin{pmatrix} 8 \\ 2 \\ 6 \end{pmatrix}$ $\langle 4,1,3\rangle, \langle 8,2,6\rangle$

$\|v\| =$	$\sqrt{26}$
$\|w\| =$	$\sqrt{104}$
$\|v-w\| =$	$\sqrt{26}$
$\theta =$	0°

$\sqrt{4^2 + 1^2 + 3^2} = \sqrt{26}$

$\sqrt{8^2 + 2^2 + 6^2} = \sqrt{104}$

$\begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix} - \begin{pmatrix} 8 \\ 2 \\ 6 \end{pmatrix} = \begin{pmatrix} -4 \\ -1 \\ -3 \end{pmatrix} \to \sqrt{-4^2 + -1^2 + -3^2} = \sqrt{26}$

$\cos\theta = \frac{u \cdot v}{|u||v|} \longrightarrow \frac{32+2+18}{\sqrt{26}\cdot\sqrt{104}} = \frac{52}{52} = \underline{1}$

$p =$	$\langle 4, 1, 3\rangle$
$q =$	$\langle 8, 2, 6\rangle$

$P_w(v) = \frac{\langle v, w\rangle}{|w|^2} w = \frac{32+2+18}{(\sqrt{104})^2} \cdot \langle 8, 2, 6\rangle$

$= \frac{52}{104} \cdot \langle 8, 2, 6\rangle = \frac{1}{2} \cdot \langle 8, 2, 6\rangle$

$= \underline{\langle 4, 1, 3\rangle}$

$Q_v(w) = \frac{\langle v, w\rangle}{|v|^2} v = \frac{32+2+18}{(\sqrt{26})^2} \cdot \langle 4, 1, 3\rangle$

$= \frac{52}{26} \cdot \langle 4, 1, 3\rangle$

$= 2 \cdot \langle 4, 1, 3\rangle$

$= \langle 8, 2, 6\rangle$

Problem 5. [18pts] $v = \begin{pmatrix} 2 \\ -5 \\ 4 \end{pmatrix}$, $\quad w = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$

$\|v\| =$	$\sqrt{45}$
$\|w\| =$	$\sqrt{6}$
$\|v-w\| =$	$\sqrt{-23}$
$\theta =$	137°

$v = \langle 2, -5, 4 \rangle$

$\|v\| = \sqrt{(2)^2 + (-5)^2 + (4)^2} = \sqrt{45}$

$w = \langle 1, 2, -1 \rangle$

$\|w\| = \sqrt{(1)^2 + (2)^2 + (-1)^2} = \sqrt{6}$

$\begin{pmatrix} 2 \\ -5 \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -7 \\ 5 \end{pmatrix}$ $\quad \|v - w\| = \sqrt{(1)^2 + (-7)^2 + (5)^2} = \sqrt{-23}$

$\cos\theta = \dfrac{v \cdot w}{|v|\,|w|} = \dfrac{2 + (-10) + (-4)}{\sqrt{45} \cdot \sqrt{6}} = \dfrac{-12}{3\sqrt{5} \cdot \sqrt{6}} = 136.9^\circ$ or 137°

$p =$	$\langle -2, -4, 2 \rangle$
$q =$	$\langle -\frac{8}{15}, \frac{4}{3}, -\frac{16}{15} \rangle$

$P_w(v) = \dfrac{\langle v, w \rangle}{|w|^2} w \longrightarrow \dfrac{2 - 10 - 4}{(\sqrt{6})^2} \cdot \langle 1, 2, -1 \rangle$

$= -\frac{12}{6} \cdot \langle 1, 2, -1 \rangle = -2 \cdot \langle 1, 2, -1 \rangle$

$= \langle -2, -4, 2 \rangle$

$Q_v(w) = \dfrac{\langle v, w \rangle}{|v|^2} v$

$= \dfrac{-12}{45} \cdot \langle 2, -5, 4 \rangle$

$= -\frac{4}{15} \langle 2, -5, 4 \rangle$

$= \langle -\frac{8}{15}, \frac{4}{3}, -\frac{16}{15} \rangle$

Problem 6. [18pts] $v = \begin{pmatrix} 0 \\ 2 \\ 8 \\ 2 \end{pmatrix}, \quad w = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}$

$\|v\| =$	$\sqrt{72}$
$\|w\| =$	2
$\|v - w\| =$	92
$\theta =$	118°

$v = \langle 0, 2, 8, 2 \rangle$

$= \sqrt{4 + 64 + 4} = \underline{\sqrt{72}}$

$w = \langle 1, -1, -1, 1 \rangle$

$= \sqrt{4} = \underline{2}$

$\begin{pmatrix} 0 \\ 2 \\ 8 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ 9 \\ 1 \end{pmatrix}, \quad \|v - w\| = \sqrt{1 + 9 + 81 + 1} = \sqrt{92}$

$\cos\theta = \frac{v \cdot w}{|v|\,|w|} = \frac{0 - 2 - 8 + 2}{2\sqrt{72}} = \frac{-8}{2\sqrt{72}} \qquad \underline{\cos^{-1}\left(\frac{-8}{2\sqrt{72}}\right)}$

$p =$	$\langle -2, 2, 2, -2 \rangle$
$q =$	$\langle 0, -\frac{2}{9}, -\frac{8}{9}, -\frac{2}{9} \rangle$

$P_w(v) = \frac{\langle v, w \rangle}{|w|^2} w$

$= \frac{0 - 2 - 8 + 2}{4} \cdot \langle 1, -1, -1, 1 \rangle$

$= -\frac{8}{4} \cdot \langle 1, -1, -1, 1 \rangle$

$= -2 \cdot \langle 1, -1, -1, 1 \rangle$

$= \langle -2, 2, 2, -2 \rangle$

$Q_v(w) = \frac{\langle v, w \rangle}{|v|^2} v$

$= \frac{-8}{(\sqrt{72})^2} \cdot \langle 0, 2, 8, 2 \rangle$

$= -\frac{1}{9} \cdot \langle 0, 2, 8, 2 \rangle$

$= \langle 0, -\frac{2}{9}, -\frac{8}{9}, -\frac{2}{9} \rangle$