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Texas A&M University
Department of Electrical and Computer Engineering
ELEN 444 - Digital Signal Processing
Fall 2024

Problem Set 2
Assigned: Thursday, Aug. 29, 2024
Due: Thursday, Sept 12, 2024

Fourier Transforms

2.1: The impulse response $h[n]$ of an LTI system is defined as $h[n] = u[n] - u[n-6]$. (a) find the response $y[n]$ of this system to the input $x[n] = h[n]$ using the convolution sum $y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$. (b) Check the results of $y[n]$ by first finding $Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$, and then taking the inverse Fourier transform of $Y(e^{j\omega})$.

2.2: Redo the above problem with $x[n] = u[n] - u[n-5]$ and $h[n] = u[n] - u[n-7]$.

2.3: (a) Show that the Fourier transform of the function $x[n]$ defined by

$$x[n] = \begin{cases} 1 - \frac{|n|}{2L}, & \text{if } \ |n| \leq 2L, \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

is

$$X(e^{j\omega}) = \frac{1}{2L}[\frac{sin(\omega L)}{sin(\omega/2)}]^2. \tag{2}$$

Hint: Try to leverage what you learned from Problem 2.1.

(b) Use Matlab to plot $X(e^{j\omega})$.

2.4: Proakis and Manolakis 4.9. (a,b,c,d,f,g)

2.5: Proakis and Manolakis 4.11.

2.6: Proakis and Manolakis 4.13 (use Matlab to plot $X(e^{j\omega})$).

2.7: Proakis and Manolakis 4.17.

Computer Assignment 2: Discrete-Time Signals and Systems

L2.1: The impulse response of the moving average system is

$$h[n] = \begin{cases} \frac{1}{M_1+M_2+1}, & \text{if } \ -M_1 \leq n \leq M_2, \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

In the class, we have shown that its frequency response is

$$H(e^{j\omega}) = \frac{1}{M_1 + M_2 + 1} e^{-j\omega(M_2 - M_1)/2} \frac{sin[\omega(M_1 + M_2 + 1)/2]}{sin(\omega/2)}.$$

Plot two periods ($-2\pi < \omega \leq 2\pi$) of $H(e^{j\omega})$ with $M_1 = 0$ and $M_2 = 5$ (both the magnitude and the phase).

L2.2: Write a Matlab script to simulate Gibbs phenomenon shown in Fig. 4.2.5 in Proakis and Manolakis. Try $\omega_c = \frac{\pi}{25}$ and $\omega_c = \frac{\pi}{15}$.

2.1: The impulse response $h[n]$ of an LTI system is defined as $h[n] = u[n] - u[n-6]$. (a) find the response $y[n]$ of this system to the input $x[n] = h[n]$ using the convolution sum $y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$. (b) Check the results of $y[n]$ by first finding $Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$, and then taking the inverse Fourier transform of $Y(e^{j\omega})$.

(a) FIND $y[n]$ INPUT $x[n] = h[n]$

USING $y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$.

$\rightarrow y[n] = h[n] * h[n]$

$n=0 \rightarrow 1$

$y[1] = h[0]h[1] + h[1]h[0] = 2$

$y[2] = 3$

$y[3] = 4$

$y[4] = 5$

$y[5] = 6$

$y[6] = 5$

$y[7] = 4$

$y[8] = 3$

$y[9] = 2$

$y[10] = 1$

ANSWER: $y[n] = \{\underset{\uparrow 0}{1}, 2, 3, 4, 5, 6, 5, 4, 3, 2, 1\}$

$$y[n] = \begin{cases} n+1, & 0 \le n \le 5 \\ 10-n, & 6 \le n \le 10 \\ 0, & \text{else} \end{cases}$$

(b)

$$H(\omega) = \sum_{n=-\infty}^{\infty} h(n) e^{-j\omega n}$$

$$\rightarrow = \sum_{n=-\infty}^{\infty} (u[n] - u[n-6]) e^{-j\omega n}$$

$$\rightarrow = \sum_{n=0}^{5} e^{-j\omega n} = 1 + e^{-j\omega} + e^{-2j\omega} + e^{-3j\omega} + e^{-4j\omega} + e^{-5j\omega}$$

CALCULATE $Y(\omega) = H(\omega) \cdot H(\omega)$

$$\rightarrow Y(\omega) = \left(1 + e^{-j\omega} + e^{-2j\omega} + e^{-3j\omega} + e^{-4j\omega} + e^{-5j\omega}\right)^2$$

$$\rightarrow Y(\omega) = 1 + 2e^{-j\omega} + 3e^{-2j\omega} + 4e^{-3j\omega} + 5e^{-4j\omega} + 6e^{-5j\omega} + 5e^{-6j\omega} + 4e^{-7j\omega} + 3e^{-8j\omega} + 2e^{-9j\omega} + e^{-10j\omega}$$

INVERSE FOURIER TRANSFORM

$$y[n] = \delta(n) + 2\delta(n-1) + 3\delta(n-2) + 4\delta(n-3) + 5\delta(n-4) + 6\delta(n-5) + 5\delta(n-6) + 4\delta(n-7) + 3\delta(n-8) + 2\delta(n-9) + \delta(n-10)$$

ANSWER: $y[n] = \{\underset{\uparrow}{1}, 2, 3, 4, 5, 6, 5, 4, 3, 2, 1\}$

2.2: Redo the above problem with $x[n] = u[n] - u[n-5]$ and $h[n] = u[n] - u[n-7]$.

(a) $y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h(n-k]$

$x[n] = \begin{cases} 1, & 0 \le n \le 4 \\ 0, & \text{else} \end{cases}$

$h(n] = \begin{cases} 1, & 0 \le n < 6 \\ 0, & \text{else} \end{cases}$

n=0:
$y[0] = 1$
$y[1] = 2$
$y[2] = 3$
$y[3] = 4$
$y[4] = 5$
$y[5] = 4$

$y[6] = 3$
$y[7] = 2$
$y[8] = 1$

ANSWER

$y[n] = \{\underset{\uparrow}{1}, 2, 3, 4, 5, 4, 3, 2, 1\}$

(b) $X(j\omega) = \sum_{n=0}^{4} e^{-j\omega n} \longrightarrow X(\omega) = 1 + e^{-j\omega} + e^{-2j\omega} + e^{-3j\omega} + e^{-4j\omega}$

FOURIER $h[n]$:

$H(\omega) = \sum_{n=0}^{6} e^{-j\omega n} \rightarrow H(\omega) = 1 + e^{-j\omega} + e^{-2j\omega} + e^{-3j\omega} + e^{-4j\omega} + e^{-5j\omega} + e^{-6j\omega}$

$Y(\omega) = X(\omega) \cdot H(\omega)$:

$\rightarrow = (1 + e^{-j\omega} + e^{-2j\omega} + e^{-3j\omega} + e^{-4j\omega}) \cdot (1 + e^{-j\omega} + e^{-2j\omega} + e^{-3j\omega} + e^{-4j\omega} + e^{-5j\omega} + e^{-6j\omega})$

$= Y(\omega) = 1 + 2e^{-j\omega} + 3e^{-2j\omega} + 4e^{-3j\omega} + 5e^{-4j\omega} + 4e^{-5j\omega} + 3e^{-6j\omega} + 2e^{-7j\omega} + e^{-8j\omega}$

$\rightarrow y[n] = \delta[n] + 2\delta[n-1] + 3\delta[n-2] + 4\delta(n-3] + 5\delta[n-4] + 4\delta(n-5) + 3\delta[n-6] + 2\delta[n-7] + \delta[n-8]$

ANSWER

$y[n] = \{\underset{\uparrow}{1}, 2, 3, 4, 5, 4, 3, 2, 1\}$

2.3: (a) Show that the Fourier transform of the function $x[n]$ defined by

$$x[n] = \begin{cases} 1 - \frac{|n|}{2L}, & \text{if } |n| \leq 2L, \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

is

$$X(e^{j\omega}) = \frac{1}{2L}[\frac{sin(\omega L)}{sin(\omega/2)}]^2. \tag{2}$$

Hint: Try to leverage what you learned from Problem 2.1.

(b) Use Matlab to plot $X(e^{j\omega})$.

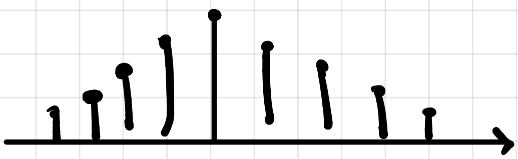
FT

$$X(e^{jw}) = \sum_{n=-\infty}^{\infty} x[n] e^{-jwn}$$

$x[n] = 0$ when $|n| > 2L$ $\quad X(e^{jw}) = \sum_{n=-2L}^{2L} \left(1 - \frac{|n|}{2L}\right) e^{-jwn}$

Plot

$x[n]$



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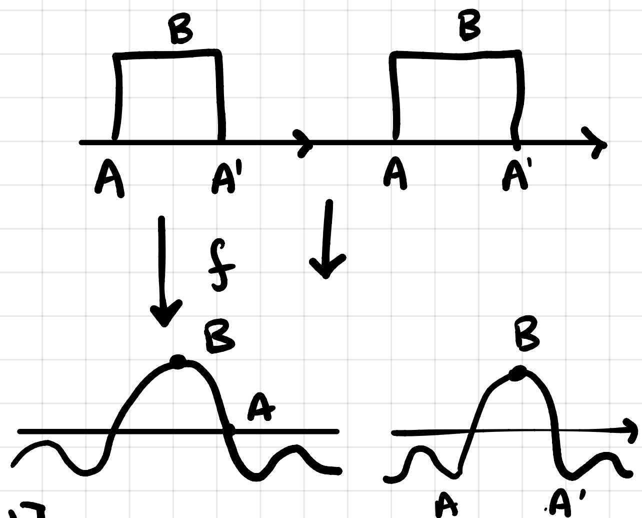

$x[n]$ is triangular Represented by:

$$x[n] = rect[n] * rect[n]$$

Fourier Transform

$$Rect[n] = u[n+L] - u[n-L-1]$$

$$x[n] = Rect[n] * Rect[n] \xrightarrow{FT}$$

$$Rect(e^{jw}) = \sum_{-\infty}^{\infty} x\langle n] e^{-jwL} = \frac{1}{2L} \sum_{k=-2L}^{2L} e^{-jkw} =$$

$$\rightarrow = \frac{1}{2L} \cdot e^{-jwL} \frac{1 - e^{jw(2L+1)}}{1 - e^{jw}} = \frac{1}{2L} \left(\frac{\sin(wL)}{\sin(w/2)} \right)^2$$

Squared because

$*$ in time domain

$=$ $\times$ in Fourier Domain.

4.9 Compute the Fourier transform of the following signals.

(a) $x(n) = u(n) - u(n-6)$

(b) $x(n) = 2^n u(-n)$

(c) $x(n) = (\frac{1}{4})^n u(n+4)$

(d) $x(n) = (\alpha^n \sin \omega_0 n) u(n), \qquad |\alpha| < 1$

(e) $x(n) = |\alpha|^n \sin \omega_0 n, \qquad |\alpha| < 1$

(f) $x(n) = \begin{cases} 2 - (\frac{1}{2})n, & |n| \le 4 \\ 0, & \text{elsewhere} \end{cases}$

(g) $x(n) = \{-2, -1, \underset{\uparrow}{0}, 1, 2\}$

(a) $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$

x(n) is non-zero from n = 0 to n = 5

$\rightarrow X(e^{j\omega}) = \sum_{n=0}^{5} e^{-j\omega n}$

GEOMETRIC SEQUENCE

$\sum_{n=0}^{N} r^n = \frac{1 - r^{N+1}}{1-r} \rightarrow \sum_{n=0}^{5} e^{-j\omega n} = X(e^{j\omega}) = \frac{1 - e^{-6j\omega}}{1 - e^{-j\omega}}$

(b) $x(n) = 2^n, \quad n \le 0$
$x(n) = 0, \quad n > 0$

Let $m = -n$ AND $n = -m$

$\rightarrow X(e^{j\omega}) = \sum_{n=-\infty}^{0} 2^n e^{-j\omega n}$

$\rightarrow \sum_{m=0}^{\infty} 2^{-m} e^{j\omega n}$ $\quad * \; r = 2^{-1} e^{j\omega} = \frac{e^{j\omega}}{2}$

$\rightarrow X(e^{j\omega}) = \sum_{m=0}^{\infty} \left(\frac{e^{j\omega}}{2}\right)^m \rightarrow X(e^{j\omega}) = \frac{1}{1 - \frac{e^{j\omega}}{2}}$

$\rightarrow = \frac{1}{1 - \frac{\cos(\omega) + j\sin(\omega)}{2}} \cdot \frac{1 - \frac{\cos(\omega) - j\sin(\omega)}{2}}{1 - \frac{\cos(\omega) - j\sin(\omega)}{2}} \rightarrow \frac{1}{1 - \frac{1}{2}\cos(\omega) - \frac{1}{2}j\sin(\omega)} \cdot \frac{1 - \frac{1}{2}\cos(\omega) + \frac{1}{2}j\sin(\omega)}{1 - \frac{1}{2}\cos(\omega) + \frac{1}{2}j\sin(\omega)}$

$\rightarrow X(e^{j\omega}) = \frac{2}{2 - (\cos(-\omega) + j\sin(-\omega))} = \frac{2}{2 - e^{-j\omega}} = X(e^{j\omega})$

(c) $x(n) = \frac{1}{4}^{n+4}, \quad n \geq -4$

$$X(e^{j\omega}) = \sum_{n=-4}^{\infty} \left(\frac{1}{4}\right)^{n+4} e^{-j\omega n} = \left(\frac{1}{4}\right)^4 \sum_{n=-4}^{\infty} \left(\frac{1}{4}\right)^n e^{-j\omega n} = \frac{1}{4^4} \sum_{n=-4}^{\infty} \left(\frac{1}{4}\right)^n e^{-j\omega n}$$

Let $m = n+4$

$$\rightarrow X(e^{j\omega}) = \frac{1}{4^4} \sum_{m=0}^{\infty} \left(\frac{1}{4}\right)^{m-4} e^{-j\omega(m-4)} = \frac{e^{4j\omega}}{4^4} \sum_{m=0}^{\infty} \left(\frac{1}{4}\right)^m e^{-j\omega m}$$

$$\rightarrow X(e^{j\omega}) = \frac{e^{4j\omega}}{4^4} \sum_{m=0}^{\infty} \left(\frac{e^{-j\omega}}{4}\right)^m = \frac{e^{4j\omega}}{4^4} \cdot \frac{1}{1 - \frac{e^{-j\omega}}{4}}$$

$$\rightarrow X(e^{j\omega}) = \frac{4^4 e^{4j\omega}}{4 - e^{-j\omega}}$$

(d)

$$\sin(\omega_0 n) = \frac{e^{j\omega_0 n} - e^{-j\omega_0 n}}{2j}$$

Plug in:

$$x(n) = \frac{a^n}{2j}\left(e^{j\omega_0 n} - e^{j\omega_0 n}\right) u(n)$$

$$\rightarrow X(e^{j\omega}) = \sum_{n=0}^{\infty} \left[\frac{a^n e^{j\omega_0 n}}{2j}\right] - \sum_{n=0}^{\infty} \left[\frac{a^n e^{-j\omega_0 n}}{2j}\right]$$

$$\rightarrow X(e^{j\omega}) = \frac{1}{2j}\left[\frac{1}{1-\alpha e^{-j(\omega-\omega_0)}} - \frac{1}{1-\alpha e^{-j(\omega+\omega_0)}}\right]$$

(f)

$$X(e^{j\omega}) = \sum_{n=-4}^{4} \left[2 - \left(\frac{1}{2}\right)n\right] e^{-j\omega n}$$

$$= 2 \cdot \frac{e^{-4j\omega}}{1 - e^{-j\omega}} - \frac{1}{2}\left[-4e^{4j\omega} - 3e^{3j\omega} - 2e^{2j\omega} - e^{j\omega} + e^{-j\omega} + 2e^{-2j\omega} + 3e^{-3j\omega} + 4e^{-4j\omega}\right]$$

$$X(e^{j\omega}) = 2 \cdot \frac{e^{-4j\omega}}{1 - e^{-j\omega}} + 4j\sin 4\omega + 3j\sin 3\omega + 2j\sin 2\omega + \sin\omega$$

(g) $x(n) = \{-2, -1, 0, 1, 2\} = -2e^{2j\omega} - e^{j\omega} + e^{-j\omega} + 2e^{-2j\omega}$

$$= -2j\sin 2\omega - j\cancel{\sin\omega + j}\sin\omega + 2j\sin 2\omega$$

$$X(e^{j\omega}) = -2j(\sin\omega + 2\sin 2\omega)$$

4.11 Consider the signal

$$x(n) = \{1, 0, -1, \underset{\uparrow}{2}, 3\}$$

with Fourier transform $X(\omega) = X_R(\omega) + j(X_I(\omega))$. Determine and sketch the signal $y(n)$ with Fourier transform

$$Y(\omega) = X_I(\omega) + X_R(\omega)e^{j2\omega}$$

$$X_{even}(n) = \frac{x(n)+x(-n)}{2} \qquad X_{odd}(n) = \frac{x(n)-x(-n)}{2}$$

$x(-3) = \frac{1}{2}$ $\quad x(-1) = \frac{-1+3}{2} = -2$ $\quad x(-3) = 1/2$ $\quad x(0) = 0$

$x(1) = 2$

$x(-2) = 0$ $\quad x(0) = 2$ $\quad x(1) = 1/2$ $\quad x(-2) = 0$ $\quad x(2) = 0$

$x(-1) = -2$ $\quad x(3) = 1/2$

$$= \{\tfrac{1}{2}, 0, 1, \underset{\uparrow}{2}, 1, 0, 1/2\} \qquad = \{\tfrac{1}{2}, 0, -2, \underset{\uparrow}{0}, 2, 0, \tfrac{1}{2}\}$$

$$F^{-1}\{Y(\omega)\} = F^{-1}\{X_I(\omega)\} + F^{-1}\{X_R(\omega)e^{2j\omega}\}$$

$$y(n) = -jx_o(n) + x_e(n+2)$$

$$y(n) = \{\tfrac{1}{2}, 0, 1-\tfrac{1}{2}j, 2, 1+\tfrac{1}{2}j, 0, \tfrac{1}{2}-2j, 0, \tfrac{j}{2}\}$$

4.13 In Example 4.4.2, the Fourier transform of the signal

$$x(n) = \begin{cases} 1, & -M \le n \le M \\ 0, & \text{otherwise} \end{cases}$$

was shown to be

$$X(\omega) = 1 + 2\sum_{n=1}^{M} \cos \omega n$$

Show that the Fourier transform of

$$x_1(n) = \begin{cases} 1, & 0 \le n \le M \\ 0, & \text{otherwise} \end{cases}$$

and

$$x_2(n) = \begin{cases} 1, & -M \le n \le -1 \\ 0, & \text{otherwise} \end{cases}$$

are, respectively,

$$X_1(\omega) = \frac{1 - e^{-j\omega(M+1)}}{1 - e^{-j\omega}}$$

$$X_2(\omega) = \frac{e^{j\omega} - e^{j\omega(M+1)}}{1 - e^{j\omega}}$$

Thus prove that

$$X(\omega) = X_1(\omega) + X_2(\omega) = \frac{\sin(M + \frac{1}{2})\omega}{\sin(\omega/2)}$$

and therefore,

$$1 + 2\sum_{n=1}^{M} \cos \omega n = \frac{\sin(M + \frac{1}{2})\omega}{\sin(\omega/2)}$$

GEOMETRIC SEQUENCE

$$\sum_{n=0}^{N} r^n = \frac{1 - r^{N+1}}{1 - r}$$

$$X_1(e^{jw}) = \sum_{n=0}^{m} e^{-jwn} \rightarrow \frac{1 - e^{-jw(m+1)}}{1 - e^{-jw}} \checkmark$$

$$X_2(e^{jw}) = \sum_{n=-m}^{-1} e^{jwn} \xrightarrow{m=-n} \sum_{n=1}^{m} e^{jwn}$$

$$\rightarrow \left(\frac{1 - e^{jwm}}{1 - e^{jw}} e^{jw}\right) \checkmark$$

$$X_1(\omega) = \frac{1 - e^{-j\omega(M+1)}}{1 - e^{-j\omega}} \cdot \frac{1 - e^{j\omega}}{1 - e^{j\omega}} + X_2(\omega) = \frac{e^{j\omega} - e^{j\omega(M+1)}}{1 - e^{j\omega}} \cdot \frac{1 - e^{-j\omega}}{1 - e^{-j\omega}}$$

$$= \frac{1 - e^{-jw(m+1)} - e^{jw} + e^{jw} - 1 - e^{jw(m+1)} + e^{jwm} + e^{-jwm}}{2 - e^{-jw} - e^{jw}}$$

$$= \frac{1}{m + 1/2} e^{-jw(m+1)/2} \cdot \frac{\sin(w(m+1)/2)}{\sin(w/2)} = \frac{\sin(m + 1/2)w}{\sin(w/2)}$$

4.17 Let $x(n)$ be an arbitrary signal, not necessarily real valued, with Fourier transform $X(\omega)$. Express the Fourier transforms of the following signals in terms of $X(\omega)$.

(a) $x^*(n)$

(b) $x^*(-n)$

(c) $y(n) = x(n) - x(n-1)$

(d) $y(n) = \sum_{k=-\infty}^{n} x(k)$

(e) $y(n) = x(2n)$

(f) $y(n) = \begin{cases} x(n/2), & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$

(a) $\sum_{n=-\infty}^{\infty} x(n) e^{j\omega n}$ Complex Conjugate of FT

$\rightarrow$ Sub $-\omega$ for ω

$X(-\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j(-\omega)n} \rightarrow X(-\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{j\omega n}$

$\rightarrow X(\omega) = X^*(-\omega)$

(b) $x^*(n) = \sum_{n=-\infty}^{\infty} x^*(-n) e^{-j\omega n} = \sum_{n=-\infty}^{\infty} x^*(n) e^{j\omega n}$

$\rightarrow X^*(\omega) = x^*(-n)$

(c) $\rightarrow \sum x(n) e^{-j\omega n} - \sum x(n-1) e^{-j\omega n}$

$\rightarrow Y(\omega) = X(\omega) + X(\omega) e^{-1j\omega}$

(d) $y(n) = x(n) * u(n)$

$u(n) = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$

$U(\omega) = \frac{1}{1-e^{-j\omega}} + \pi\delta(\omega)$

$Y(\omega) = X(\omega) \cdot U(\omega)$

$U(\omega) = \frac{1}{1-e^{-j\omega}} \rightarrow Y(\omega) = X(\omega) \cdot \frac{1}{1-e^{-j\omega}}$

(e) $Y(\omega) = \sum x(2n) e^{-j\omega n}$ Let $m = 2n$

$Y(\omega) = \sum x(m) e^{-j\omega \frac{1}{2} m} \rightarrow Y(\omega) = X\left(\frac{\omega}{2}\right)$

For even, so replicate for 2π $Y(\omega) = \frac{1}{2}\left[X\left(\frac{\omega}{2}\right) + X\left(\frac{\omega}{2} + \pi\right)\right]$

(f) $Y(\omega) = \sum_{n=0,2,4\ldots} x\left(\frac{n}{2}\right) e^{-j\omega n}$ Let $m = n/2$

$\rightarrow \sum x(m) e^{-2j\omega n} = X(2\omega)$