Texas A&M University
Department of Electrical and Computer Engineering
ELEN 444 - Digital Signal Processing
Fall 2024

Problem Set 6
Assigned: Thursday, Sept. 26, 2024
Due: Thursday, Oct. 10, 2024

Transform analysis of LTI systems

5.1: Proakis and Manolakis 3.36

5.2: Proakis and Manolakis 3.40

5.3: Proakis and Manolakis 3.47

5.4: Proakis and Manolakis 3.51

5.4: Proakis and Manolakis 3.55

Computer Assignment 5: Transform analysis of LTI systems

L5.1: Suppose the system function of a causal system is given by

$$H(z) = \frac{z^{-1} - a^*}{1 - az^{-1}}.$$

Assume $a = re^{j\theta}$ (with $r < 1$), a) find the magnitude, phase, and group delay of $H(e^{j\omega})$ and confirm that $|H(e^{j\omega})| = 1$, i.e., $H(e^{j\omega})$ is an allpass filter; b) Use Matlab to plot the log magnitude, phase, and group delay of the system for $a = 0.4$ and $a = -0.4$.

L5.2: A lowpass FIR filter is given as: $h[0] = \frac{(1+\sqrt{3})}{4\sqrt{2}}, h[1] = \frac{(3+\sqrt{3})}{4\sqrt{2}}, h[2] = \frac{(3-\sqrt{3})}{4\sqrt{2}}, h[3] = \frac{(1-\sqrt{3})}{4\sqrt{2}}$, and $h(n) = 0$ for other n. Now define a highpass FIR filter $g[n]$ based on $h[n]$ as $g[n] = (-1)^n h[n]$. We know that the DC gain of $h[n]$ is $H(e^{j\omega})|_{\omega=0} = \sum_{n=-\infty}^{\infty} h[n] = \sqrt{2}$ and $G(e^{j\omega}) = H(e^{j(\omega+\pi)})$. It also can be shown that $|H(e^{j\omega})|^2 + |G(e^{j\omega})|^2 = 2$. In other words, $|H(e^{j\omega})|^2 + |H(e^{j(\omega+\pi)})|^2 = 2$. This is one of the conditions that $h[n]$ has to satisfy in order to be called an orthogonal wavelet filter.

(a) Use "freqz" in Matlab to plot the log magnitude and phase of $h[n]$ and $g[n]$

(b) Plot $|H(e^{j\omega})|^2 + |H(e^{j(\omega+\pi)})|^2$ and check if it is indeed a constant that equals to 2.

5.1 $H(z) = \dfrac{1-2z^{-1}+2z^{-2}-z^{-3}}{(1-z^{-1})(1-\frac{1}{2}z^{-1})(1-\frac{1}{5}z^{-1})}, \quad \frac{1}{2} < |z| < 1$

↳ Simplify Numerator

$$\frac{\cancel{(1-z^{-1})}(1-z^{-1}+z^{-2})}{\cancel{(1-z^{-1})}(1-\frac{1}{2}z^{-1})(1-\frac{1}{5}z^{-1})} = \frac{(1-z^{-1}+z^{-2})}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{5}z^{-1})}$$

QUADRATIC FORMULA:

$z^2 - z^{-1} + 1 = a^2 + b + c \qquad x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

$x = \dfrac{1 \pm \sqrt{1-4}}{2(1)} \rightarrow z_1 = z_2 = \dfrac{1 \pm \sqrt{-3}}{2}$ AND poles AT $p = \frac{1}{2}$ and $p = \frac{1}{5}$

$$\frac{(1-z^{-1}+z^{-2})}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{5}z^{-1})} = A + \frac{B}{1-\frac{1}{2}z^{-1}} + \frac{C}{1-\frac{1}{5}z^{-1}}$$

$\rightarrow 1 - z^{-1} + z^{-2} = A(1-\frac{1}{2}z^{-1})(1-\frac{1}{5}z^{-1}) + B(1-\frac{1}{5}z^{-1}) + C(1-\frac{1}{2}z^{-1})$

$= A(1 - \frac{7}{10}z^{-1} + \frac{1}{10}z^{-2}) + B - \frac{B}{5}z^{-1} + C - \frac{C}{2}z^{-1}$

$\rightarrow 1 - z^{-1} + z^{-2} = A + B + C + (-\frac{7A}{10} - \frac{B}{5} - \frac{C}{2})z^{-1} + \frac{A}{10}z^{-2}$

$A + B + C = 1$

$-\frac{7A}{10} - \frac{B}{5} - \frac{C}{2} = -1$

$\frac{A}{10} = 1 \rightarrow A = 10$

$B + C = -9$

$-\frac{7(10)}{10} - \frac{B}{5} - \frac{C}{2} = -1$

$-\frac{B}{5} - \frac{C}{2} = 6$

$2(B + C = -9)$

$-2B - 5C = 60$

$C = -14$

$B = 5$

$$H(z) = 1 + \frac{5}{1-\frac{1}{2}z^{-1}} - \frac{14}{1-\frac{1}{5}z^{-1}}$$

INVERSE Z-TRANSFORM

$$h(n) = \delta[n] + \left[5\left(\tfrac{1}{2}\right)^n - 14\left(\tfrac{1}{5}\right)^n\right]u(n)$$

5.2 INPUT

$$x(n) = \left(\frac{1}{2}\right)^n u(n) - \frac{1}{4}\left(\frac{1}{2}\right)^{n-1} u(n-1)$$

OUTPUT

$$y(n) = \left(\frac{1}{3}\right)^n u(n)$$

(a) Z - TRANSFORMS

$$x(n) \rightarrow X(z) = \frac{1}{1-\frac{1}{2}z^{-1}} - \frac{1/4\, z^{-1}}{1-\frac{1}{2}z^{-1}}$$

$$= \frac{1-1/4\, z^{-1}}{1-1/2\, z^{-1}} \qquad y(n) \rightarrow Y(z) = \frac{1}{1-1/3\, z^{-1}}$$

$$H(z) = Y(z)|X(z) = \frac{\left(\frac{1}{1-1/3\, z^{-1}}\right)}{\left(\frac{1-1/4\, z^{-1}}{1-1/2\, z^{-1}}\right)}$$

$$H(z) = \frac{A}{1-1/4\, z^{-1}} + \frac{B}{1-1/3\, z^{-1}} = 1-\frac{1}{2}z^{-1} \rightarrow 1-\frac{1}{2}z^{-1} = A\left(1-\frac{1}{3}z^{-1}\right) + B\left(1-\frac{1}{4}z^{-1}\right)$$

$$1-\frac{1}{2}z^{-1} = (A+B) + \left(-\frac{A}{3}-\frac{B}{4}\right)z^{-1}$$

$$A+B=1, \quad -\frac{A}{3}-\frac{B}{4} = -\frac{1}{2}$$

$$B = 1-A \quad \rightarrow -\frac{A}{3} - \frac{1-A}{4} = -\frac{1}{2}$$

$$\rightarrow -4A - 3(1-A) = -6 \qquad 3+B=1$$

$$-4A - 3 + 3A = -6 \qquad B = -2$$

$$\underline{A = 3}$$

$$H(z) = \frac{3}{1-\frac{1}{4}z^{-1}} - \frac{2}{1-\frac{1}{3}z^{-1}} \rightarrow h(n) = 3\left(\frac{1}{4}\right)^n u(n) - 2\left(\frac{1}{3}\right)^n u(n)$$

(b)

$$\frac{Y(z)}{X(z)} = \frac{1-1/2\, z^{-1}}{\left(1-\frac{1}{4}z^{-1}\right)\left(1-1/3\, z^{-1}\right)} \rightarrow Y(z)\left(1-\frac{1}{4}z^{-1}\right)\left(1-1/3\, z^{-1}\right) = X(z)\left(1-1/2\, z^{-1}\right)$$

$$= \left(1-\frac{1}{4}z^{-1}-\frac{1}{3}z^{-1}+\frac{1}{12}z^{-2}\right)Y(z) = \left(1-\frac{1}{2}z^{-1}\right)X(z)$$

$$Y(z) - \frac{1}{4}z^{-1}Y(z) - \frac{1}{3}z^{-1}Y(z) + \frac{1}{12}z^{-2}Y(z) = X(z) - \frac{1}{2}z^{-1}X(z)$$

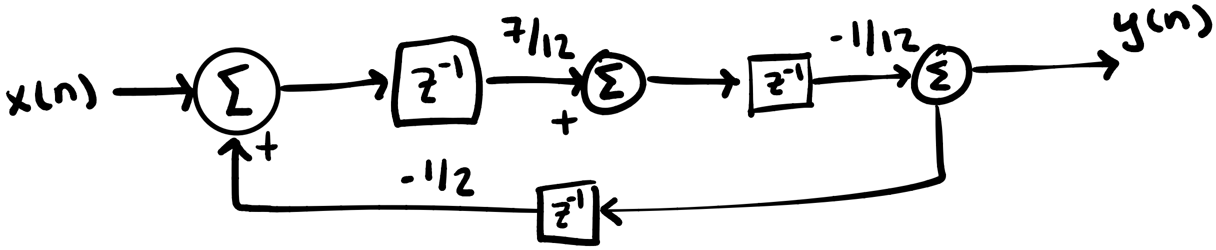
$$y(n) - \frac{1}{4}y(n-1) - \frac{1}{3}y(n-1) + \frac{1}{12}y(n-2) = x(n) - \frac{1}{2}x(n-1)$$

$$\rightarrow y(n) - \frac{7}{12}y(n-1) + \frac{1}{12}y(n-2) = x(n) - \frac{1}{2}x(n-1)$$

c. 2 delays

$\rightarrow y(n) - \frac{7}{12}y(n-1) + \frac{1}{12}y(n-2) = x(n) - \frac{1}{2}x(n-1)$

$$\frac{y(n)}{x(n)} = \frac{-\,{}^1/_2\,x(n-1)}{\frac{7}{12}y(n-1) - \frac{1}{12}y(n-2)}$$



(d). Poles are $\frac{1}{3}$ AND $\frac{1}{4}$.

5.3 3.47

(a). $y(0) = x_1(0)x_2(0) = 1$

$y(1) = x_1(0)x_2(1) + x_1(1)x_2(0) = 2$

$y(2) = x_1(0)x_2(2) + x_1(2)x_2(1) + x_1(2)x_2(0) = 3$

$y(3) = 3$

$y(4) = 3$

$y(5) = 2$

$y(6) = 1$

$y(n) = \{1,2,3,3,3,2,1\}$

$X_1(z) = 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4}$

$X_2(z) = 1 + z^{-1} + z^{-2}$

$Y(z) = X_1(z)X_2(z) = 1 + 2z^{-1} + 3z^{-2} + 3z^{-3} + 3z^{-4} + 2z^{-5} + z^{-6}$

$y(n) = \{1,2,3,3,3,2,1\}$

(b). $y(n) = \sum_{k=0}^{n} x_1(k)x_2(n-k)$

$\rightarrow y(n) = \sum_{k=0}^{n} \left(\frac{1}{2}\right)^k \left(\frac{1}{3}\right)^{n-k}$

$\rightarrow Y(z) = \frac{z^2}{(z-\frac{1}{2})(z-\frac{1}{3})}$

$y(n) = \frac{(3/2)^{n+1} - 1}{3/2 - 1} = \frac{z}{z-\frac{1}{3}} \cdot \frac{z}{z-\frac{1}{2}}$

$\frac{z^2}{(z-\frac{1}{2})(z-\frac{1}{3})} = \frac{A}{z-\frac{1}{2}} + \frac{B}{z-\frac{1}{3}}$

$z^2 = A(z-\frac{1}{3}) + B(z-\frac{1}{2})$

$z = 1/3$ AND $z = 1/2$ $B = -2, A = 3$

$Y(z) = \frac{3}{z-\frac{1}{2}} + \frac{-2}{z-\frac{1}{3}}$

$y(n) = 3\left(\frac{1}{2}\right)^n - 2\left(\frac{1}{3}\right)^n u(n)$

(c). $x_1(n) = \{1,\underset{\uparrow}{2},3,4\}$, $x_2(n) = \{4,3,\underset{\uparrow}{2},1\}$

$4,3,\underset{\uparrow}{2},1$ $y(n) = \{1,3,\underset{\uparrow}{10},20,25,24,16\}$

$y(-2) = 1$, $y(-1) = 3$, $y(0) = 3+4+3$

$y(1) = 4+6+6+4$

$y(2) = 8+9+8$

$y(3) = 12+12$

$y(4) = 16$

$y(n) = z^{+2} + 3z^{1} + 10 + 20z^{-1} + 25z^{-2} + 24z^{-3} + 16z^{-4}$

(d) $x_1(n) = \{\underset{\uparrow}{1}, 1, 1, 1, 1\}$ $x_2(n) = \{1, \underset{\uparrow}{1}, 1\}$

$y(-3) = 0$ $y(-2) = 1$ $y(-1) = 2$ $y(0) = 3$ $y(1) = 3$ $y(2) = 3$ $y(3) = 2$ $y(4) = 1$

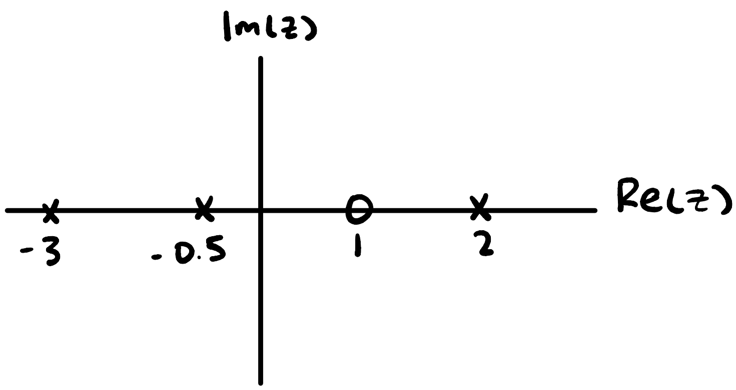
$y(n) = \{1, 2, 3, 3, 3, 2, 1\}$ ✓

Multiply such as part a

MATCHES ✓

5.4

(a)



For system to be stable, ROC in unit circle $|z| = 1$

For stability, the ROC should be outside outer pole

Outermost pole is $z = 2$, for stability $|z| > 2$

(b) Casual - ROC extends past $|z| = 2$

To be casual and stable, $|z| > 2$

(c) 3 poles, $p_1 = -3$, $p_2 = -0.5$, $p_3 = 2$.

1. $z = 2$, $|z| > 2$, casual and stable
2. $z = -0.5$ and $z = 2$, $0.5 < |z| < 2$, anti-casual, unstable
3. $z = -3$, $|z| < 0.5$, anti-casual, unstable.

5.5 $s(n) = \left(\frac{1}{3}\right)^{n-2} \mu(n+2)$

(a) $S(z) = \frac{H(z)}{1-z^{-1}}$, Shifted $s'(n) = \left(\frac{1}{3}\right)^{n} u(n)$

$$S'(z) = \frac{z}{z - 1/3} = \left(\frac{1}{3}\right)^n u(n)$$

$$S(z) = z^{-2} \cdot \frac{z}{z-\frac{1}{3}} = \frac{z^{-1}}{z - 1/3}$$

FIND H(z)

$$H(z) = S(z) \cdot (1 - z^{-1}) = \frac{z^{-1}}{z - 1/3} \cdot (1 - z^{-1}) = \frac{z - 1}{z(z - \frac{1}{3})}$$

Zeros: 1

Poles: $P_1 = 0$, $P_2 = 1/3$

(b) Impulse Response

$$H(z) = \frac{z-1}{z(z-1/3)} = \underbrace{\frac{1}{z}}_{\delta(n)} - \underbrace{\frac{1}{z - 1/3}}_{\left(\frac{1}{3}\right)^n \mu(n)}$$

$\hookrightarrow h(n) = \delta(n) - \left(\frac{1}{3}\right)^n \mu(n)$

(c) BOTH $\delta(n)$ and $\mu(n)$ are zero when $n<0$

System is Casual

Poles are in unit circle $|z| = 1$

System is stable