JOAQUIN SALAS 731000141 9|19|24

Texas A&M University
Department of Electrical and Computer Engineering
ELEN 444 - Digital Signal Processing
Fall 2024

Problem Set 3
Assigned: Thursday, Sept. 12, 2024
Due: Thursday, Sept. 19, 2024

Sampling & Fourier Transform

3.1: Proakis and Manolakis 1.5

3.2: Proakis and Manolakis 1.6

3.3: Proakis and Manolakis 1.7

3.4: Proakis and Manolakis 1.9

3.5: Proakis and Manolakis 1.10

3.6: Proakis and Manolakis 1.13

3.7: Proakis and Manolakis 4.10

3.8: Proakis and Manolakis 4.14

3.9: Proakis and Manolakis 4.16

3.10: Proakis and Manolakis 4.18

Computer Assignment 3: Sampling

L3.1: The goal of this computer assignment is to illustrate the Fourier domain relationships of the serveral sampling rate conversions defined in Proakis and Manolakis 4.23 (page 299). Let $x[n]$ and $X(e^{j\omega})$ represent a sequence and its Fourier transform, respectively. Suppose $X(e^{j\omega})$ is given as follows:

$$X(e^{j\omega}) = \begin{cases} 1 - \frac{|\omega|}{\frac{\pi}{8}}, & \text{if } |\omega| \leq \frac{\pi}{8}, \\ 0, & \text{otherwise,} \end{cases} \tag{1}$$

with the 2π periodicity of $X(e^{j\omega})$ understood. We first use the following Matlab script to generate a length-256 sequence $X[n]$, which we assume is the sampled version of $X(e^{j\omega})$, with $X[n] = X(e^{j\frac{2\pi(n-1)}{256}})$, for $1 \leq n \leq 256$.

```
for i=1:16
X(i)=1-(i-1)/16;
end
for i=17:240
X(i)=0;
end
for i=241:256
X(i)=(i-241)/16;
end
```

We then take the inverse Fourier transform of $X[n]$ to get the time-domain sequence $x[n]$ by using `x=ifft(X,256)` in Matlab. You are required to use Matlab to plot the Fourier transform of (a) the *sampler* sequence $y_1[n]$, (b) the *compressor* sequence $y_2[n]$, and (c) the *expander* sequence $y_3[n]$.

Check your experimental results with the following formula:

(a) $Y_1(e^{j\omega}) = \frac{1}{2}X(e^{j\omega}) + \frac{1}{2}X(e^{j(\omega+\pi)})$,

(b) $Y_2(e^{j\omega}) = \frac{1}{2}X(e^{j\frac{\omega}{2}}) + \frac{1}{2}X(e^{j(\frac{\omega}{2}+\pi)})$,

(c) $Y_3(e^{j\omega}) = X(e^{j2\omega})$.

1.5 Consider the following analog sinusoidal signal:

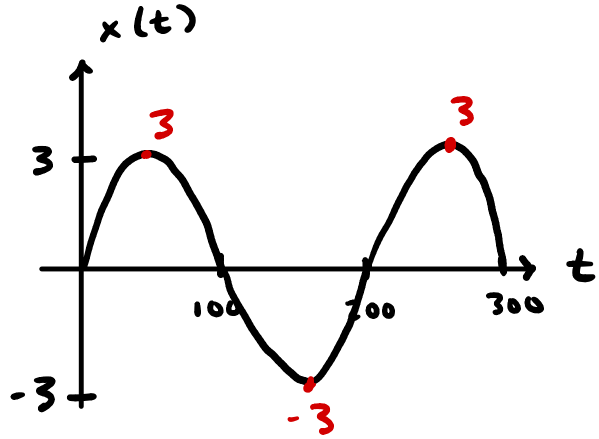
$$x_a(t) = 3\sin(100\pi t)$$

(a) Sketch the signal $x_a(t)$ for $0 \le t \le 30$ ms.

(b) The signal $x_a(t)$ is sampled with a sampling rate $F_s = 300$ samples/s. Determine the frequency of the discrete-time signal $x(n) = x_a(nT)$, $T = 1/F_s$, and show that it is periodic.

(c) Compute the sample values in one period of $x(n)$. Sketch $x(n)$ on the same diagram with $x_a(t)$. What is the period of the discrete-time signal in milliseconds?

(d) Can you find a sampling rate F_s such that the signal $x(n)$ reaches its peak value of 3? What is the minimum F_s suitable for this task?



(a) SKETCH $x_a(t) = 3\sin(100\pi t)$ — PEAK; STARTS AT 0; 0 - 100

(b) $x(n) = x_a(nT)$, $T = 1/F$

$\hookrightarrow x(n) = x_a(n/F)$

$= 3\sin\left(\frac{100\pi}{300}\right)$ ← ω

$f = \frac{\omega}{2\pi} \rightarrow \frac{\left(\frac{100\pi}{300}\right)}{2\pi} = \frac{1}{6}$

$N = \frac{1}{f} = \frac{1}{1/6} = 6$

(c)

$x(0) = 3\sin(0) = 0$

$x(1) = 3 \times \frac{\sqrt{3}}{2} = \frac{3}{\sqrt{2}}$

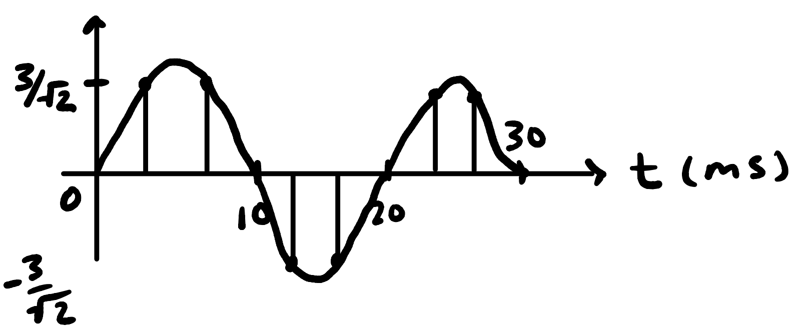
$x(2) = 3/\sqrt{2}$

$x(3) = 3\sin(0) = 0$

$x(4) = 3\sin(4\pi/3) = -3/\sqrt{2}$

$x(5) = -3/\sqrt{2}$

$x(6) = 0$



(d) SET $x(1) = 3$

$\rightarrow x(1) = 3\sin\left(\frac{100\pi(1)}{F}\right) = 1 = \sin\left(\frac{100\pi}{F}\right)$

$\rightarrow 100T_s = \frac{\pi}{2} + 2\pi \rightarrow 100T_s = \frac{1}{2}$

$T_s = \frac{1}{200}$, $F_s = \frac{1}{T} = 200 \frac{\text{SAMPLES}}{\text{S}}$

1.6 A continuous-time sinusoid $x_a(t)$ with fundamental period $T_p = 1/F_0$ is sampled at a rate $F_s = 1/T$ to produce a discrete-time sinusoid $x(n) = x_a(nT)$.

(a) Show that $x(n)$ is periodic if $T/T_p = k/N$ (i.e., T/T_p is a rational number).

(b) If $x(n)$ is periodic, what is its fundamental period T_p in seconds?

(c) Explain the statement: $x(n)$ is periodic if its fundamental period T_p, in seconds, is equal to an integer number of periods of $x_a(t)$.

(a) CONTINUOUS - TIME SIGNAL

$$X_a(t) = A\sin(2\pi f_0 t + \phi) \longleftrightarrow$$

DISCRETE TIME SIGNAL.

$$T = \frac{1}{F},$$

$$X_a(nT) = A\sin(2\pi f(nT) + \phi)$$

IN DT SIGNAL:

Let $f_0 = 1/T_p$

$$\rightarrow x(n) = \cancel{A\sin}\left(2\pi \frac{\cancel{n}T}{T_p} + \cancel{\phi}\right) = \cancel{A\sin}\left(2\pi \frac{(\cancel{n}+N)T}{T_p} + \cancel{\phi}\right)$$

$$\rightarrow 2\pi \frac{NT}{T_p} = 2\pi k \Rightarrow k = \frac{NT}{T_p}$$

$$\rightarrow x(n) = A\sin\left(2\pi \frac{T}{T_p} N + \phi\right)$$

(b) $N = \frac{k}{f} \Rightarrow f = \frac{k}{N}$

Plug into $k = \frac{NT}{T_p} \rightarrow \frac{k}{N} = \frac{T}{T_p} = f = \frac{T_p}{T} \rightarrow T_d = T_p \cdot k$

$$\frac{N}{k} = \frac{T_p}{\cancel{T}} \cdot \frac{1}{\cancel{f}}$$

(c) $X_a(t + T_p) = X_a(t)$

SIGNAL REPEATS AFTER T_p.

$\frac{N}{k} = \frac{T_p}{T}$ ← FROM PART C.

1.7 An analog signal contains frequencies up to 10 kHz.

(a) What range of sampling frequencies allows exact reconstruction of this signal from its samples?

(b) Suppose that we sample this signal with a sampling frequency $F_s = 8$ kHz. Examine what happens to the frequency $F_1 = 5$ kHz.

(c) Repeat part (b) for a frequency $F_2 = 9$ kHz.

(a) $F_{MAX} = 10\,Hz$ SAMPLING FREQ, MUST BE TWICE THE MAX FREQ.

$2 \times F_{max} = F_s$

$F_s = 20\,kHz$

(b)

$F = |F_1 - kF_s|$

$F_s = 8\,kHz$, $F_s/2 = 4\,kHz$

$F_1 = 5\,kHz > F_s/2$ ✓

$F_{ALIAS} = |5\,kHz - 8\,kHz| = 3\,kHz$

(C) $F_s = 8\,kHz \rightarrow$ Divide By 2 $= 4.5\,kHz$

$F_2 = 5\,kH > \frac{9\,kHz}{2}$ ✓

$F_{ALIAS} = |9\,kHz - 8\,kHz| = 1\,kHz$

1.9 An analog signal $x_a(t) = \sin(480\pi t) + 3\sin(720\pi t)$ is sampled 600 times per second.

(a) Determine the Nyquist sampling rate for $x_a(t)$.

(b) Determine the folding frequency.

(c) What are the frequencies, in radians, in the resulting discrete time signal $x(n)$?

(d) If $x(n)$ is passed through an ideal D/A converter, what is the reconstructed signal $y_a(t)$?

(a) Nyquist RATE $= 2 \times F_{max} = 2 \times 720\,Hz = 1440\,Hz$

(b) Folding freq. $= F_s/2 = 600/2 = 300\,Hz$

(C) $\sin(480t) \rightarrow \omega_1 = \frac{480}{600} \times 2\pi = \frac{8\pi}{5}$ (RADIANS)

for $3\sin(720t)$

$\rightarrow \omega_2 = \frac{120}{600} \times 2\pi = \frac{1\pi}{5}$ (RADIANS)

(d)

Aliasing from 720Hz to 120Hz

$Y_a(t) = -2\sin(480\pi t)$

1.10 A digital communication link carries binary-coded words representing samples of an input signal

$$x_a(t) = 3\cos 600\pi t + 2\cos 1800\pi t$$

The link is operated at 10,000 bits/s and each input sample is quantized into 1024 different voltage levels.

(a) What are the sampling frequency and the folding frequency?

(b) What is the Nyquist rate for the signal $x_a(t)$?

(c) What are the frequencies in the resulting discrete-time signal $x(n)$?

(d) What is the resolution Δ?

(a) BITS / SAMPLE

$$\log_2(1024) = 10 \text{ bits/sample}$$

SAMPLING FREQ.

$$F_s = \frac{10{,}000 \text{ bits/s}}{10 \text{ bit/SAMPLE}} = 1{,}000 \text{ SAMPLES/S}$$

FOLDING FREQ

$$F_{Fold} = \frac{F_s}{2} = \frac{1{,}000}{2} = 500 \text{ Hz}$$

(b) Nyquist = 2x highest freq.

$$\frac{1800}{2} = 900 \times 2 \rightarrow F_{Nyq.} = 1{,}800 \text{ Hz}$$

(c) For $600\pi t$. $\omega_1 = 2\pi \frac{300}{1{,}000} = \frac{3}{5}\pi$

For $1800\pi t$. $\omega_2 = 2\pi \frac{900}{1000} = \frac{9}{5}\pi$

(d) $$\Delta = \frac{\text{Voltage RANGE}}{1024} = \frac{5-(-5)}{1024} = \frac{10}{1024}$$

1.13 The discrete-time signal $x(n) = 6.35\cos(\pi/10)n$ is quantized with a resolution (a) $\Delta = 0.1$ or (b) $\Delta = 0.02$. How many bits are required in the A/D converter in each case?

(a) $$L = \frac{\text{Peak to Peak}}{\Delta} = \frac{2 \times 6.35}{0.1} = 127 \text{ Levels}$$

$$\rightarrow B = \log_2(127) \rightarrow 2^7 - 1 = 7 \text{ bits}$$

(b) $$L = \frac{12.7}{0.2} = 63.5 \rightarrow 2^6 - 1 = 63 = 6 \text{ bits}$$

4.10 Determine the signals having the following Fourier transforms.

(a) $X(\omega) = \begin{cases} 0, & 0 \le |\omega| \le \omega_0 \\ 1, & \omega_0 < |\omega| \le \pi \end{cases}$

(b) $X(\omega) = \cos^2 \omega$

(c) $X(\omega) = \begin{cases} 1, & \omega_0 - \delta\omega/2 \le |\omega| \le \omega_0 + \delta\omega/2 \\ 0, & \text{elsewhere} \end{cases}$

(d) The signal shown in Fig. P4.10.

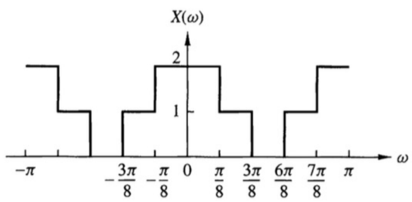


Figure P4.10

(a)

$$x(n) = \frac{1}{2\pi} \int_{-\omega_0}^{\omega_0} e^{j\omega n}\, d\omega$$

RECTANGULAR FREQ. RANGE

$$\rightarrow x(n) = \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn} - \frac{e^{j\omega n}}{jn}\right]$$

$e^{j\theta} - e^{-j\theta} = 2j\sin(\theta)$

$$\rightarrow x(n) = \frac{1}{\cancel{2}\pi}\left[\frac{\cancel{2j}\sin(\omega_0 n)}{\cancel{j}n}\right] \rightarrow x(n) = \frac{\sin(\omega_0 n)}{n\pi}$$

(b)

$$\cos^2(\omega) = \frac{1+\cos(2\omega)}{2} \qquad \mathcal{F}^{-1}\left\{\frac{1}{2}\right\} = \frac{1}{2}\delta(n)$$

$$\mathcal{F}^{-1}\left\{\frac{1}{2}\cos(2\omega)\right\} = \frac{1}{2}\left[\delta(n-k) + \delta(n+k)\right] \quad \hookrightarrow = \frac{1}{4}\left[\delta(n+2) + \delta(n-2)\right]$$

$$x(n) = \frac{1}{2}\delta(n) + \frac{1}{4}\delta(n+2) + \frac{1}{4}\delta(n-2)$$

(c)

$$x(n) = \frac{1}{2\pi}\int_{\omega_0 - \frac{\delta\omega}{2}}^{\omega_0 + \frac{\delta\omega}{2}} e^{j\omega n}\, d\omega \rightarrow x(n) = \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{\omega_0 - \frac{\delta\omega}{2}}^{\omega_0 + \frac{\delta\omega}{2}}$$

$$\rightarrow x(n) = \frac{1}{2\pi}\left(\frac{e^{j(\omega_0 + \frac{\delta\omega}{2})n} - e^{j(\omega_0 - \frac{\delta\omega}{2})n}}{jn}\right)$$

$e^{j\theta} - e^{-j\theta} = 2j\sin(\theta)$

$$\rightarrow x(n) = \frac{1}{\pi n}\sin\left(\frac{\delta\omega n}{2}\right)e^{j\omega_0 n} \quad \rightarrow x(n) = \frac{2}{\delta\omega}\sin\left(\frac{\delta\omega n}{2}\right)e^{j\omega_0 n}$$

(d) Segment 1

$$x(n) = \frac{1}{n\pi}\sin\left(\frac{7\pi n}{8}\right)$$

Segment 2

$$x(n) = \frac{1}{n\pi}\sin\left(\frac{6\pi n}{8}\right)$$

Segment 3

$$x(n) = -\frac{1}{n\pi}\sin\left(\frac{3\pi n}{8}\right)$$

4:

$$x(n) = -\frac{1}{\pi n}\sin\left(\frac{\pi n}{8}\right)$$

COMBINE: $x(n) = \frac{1}{n\pi}\left[\sin\left(\frac{7\pi n}{8}\right) + \sin\left(\frac{6\pi n}{8}\right) - \sin\left(\frac{3\pi n}{8}\right) - \sin\left(\frac{\pi n}{8}\right)\right]$

4.14 Consider the signal

$$x(n) = \{-1, 2, \underset{\uparrow}{-3}, 2, -1\}$$

with Fourier transform $X(\omega)$. Compute the following quantities, without explicitly computing $X(\omega)$:

(a) $X(0)$

(b) $\angle X(\omega)$

(c) $\int_{-\pi}^{\pi} X(\omega)\, d\omega$

(d) $X(\pi)$

(e) $\int_{-\pi}^{\pi} |X(\omega)|^2\, d\omega$

(a) $X(0) = (-1) + 2 + (-3) + 2 + (-1) = -1$

(b) $\angle X(w) = \pi$, $\{\pi, \pi, \pi, \pi, \pi\}$

(c) $x(0) = \int_{-\pi}^{\pi} X(w)\, dw = 2\pi\, x(0) = 2\pi \times -3 = -6\pi$

(d) $x(n) = \sum_n x(n)(-1)^n$

$\hookrightarrow X(\pi) = (-1)(-1)^0 + 2(-1)^1 + (-3)(-1)^2 + 2(-1)^3 + (-1)(-1)^4$

$= -1 - 2 + (-3) + (-2) + 1 = -7$

(e) $\int_{-\pi}^{\pi} |X(w)^2|\, dw$

$\hookrightarrow 2\pi \sum_n |x(n)^2| = (-1)^2 + 2^2 + (-3)^2 + 2^2 + 1^2 = 19$

$\hookrightarrow 19(2\pi) = 38\pi$

4.16 Consider the Fourier transform pair

$$a^n u(n) \overset{F}{\longleftrightarrow} \frac{1}{1 - ae^{-j\omega}}, \qquad |a| < 1$$

Use the differentiation in frequency theorem and induction to show that

$$x(n) = \frac{(n + l - 1)!}{n!(l - 1)!} a^n u(n) \overset{F}{\longleftrightarrow} X(\omega) = \frac{1}{(1 - ae^{-j\omega})^l}$$

$l = k + 1$

$\rightarrow x(n) = \frac{(n+k)!}{n!\,k!} a^n \mu(n)$

$\hookrightarrow X(w) = \frac{1}{(1 - ae^{-jw})} + \frac{1}{(1 - ae^{-jw})^l}$

4.18 Determine and sketch the Fourier transforms $X_1(\omega)$, $X_2(\omega)$, and $X_3(\omega)$ of the following signals.

(a) $x_1(n) = \{1, 1, \underset{\uparrow}{1}, 1, 1\}$

(b) $x_2(n) = \{1, 0, 1, 0, \underset{\uparrow}{1}, 0, 1, 0, 1\}$

(c) $x_3(n) = \{1, 0, 0, 1, 0, 0, \underset{\uparrow}{1}, 0, 0, 1, 0, 0, 1\}$

(d) Is there any relation between $X_1(\omega)$, $X_2(\omega)$, and $X_3(\omega)$? What is its physical meaning?

(e) Show that if

$$x_k(n) = \begin{cases} x\left(\frac{n}{k}\right), & \text{if } n/k \text{ integer} \\ 0, & \text{otherwise} \end{cases}$$

then

$$X_k(\omega) = X(k\omega)$$

(a) $X_1(\omega) = \sum 1 \cdot e^{-j\omega n} = e^{-2j\omega} + e^{-j\omega} + 1 + e^{j\omega} + e^{2j\omega}$

$\hookrightarrow = 1 + 2\cos(\omega) + 2\cos(2\omega)$

(b) $X_2(\omega) = \sum x_2(n) \cdot e^{-j\omega n}$

$\hookrightarrow = e^{-4j\omega} + e^{-2j\omega} + 1 + e^{2j\omega} + e^{4j\omega}$

$\hookrightarrow = 1 + 2\cos 2\omega + 2\cos 4\omega$

(c) $X_3(\omega) = \sum x_3(n) \cdot e^{-j\omega n}$

$= e^{-6j\omega} + e^{-3j\omega} + 1 + e^{3j\omega} + e^{6j\omega}$

$= 1 + 2\cos(3\omega) + 2\cos(6\omega)$

(d) X_2 is equal to every other of X_1

AND X_3 is equal to every 3 of X_1.

(e) $X_k(\omega) = \sum_{n,\, n/k} x_k(n) e^{-j\omega n}$

$= \sum x\left(\frac{n}{k}\right) \cdot e^{-j\omega n}$, k is int.

$\rightarrow = X\left(\frac{n}{k}\omega\right)$