MATH 311, HW 4

Due 11:00pm on Thursday February 16, 2023

Remember that you must provide an explanation of how you computed the answers. Remember to scan/upload exactly the 4 pages of this template.

Problem 1. [55 pts] For each of the following set of vectors in $\mathbb{R}^3$, determine if

1) S is a spanning set for $\mathbb{R}^3$ (5pts),
2) the vectors are linearly independent in $\mathbb{R}^3$ (5pts),
3) the vectors form a basis for $\mathbb{R}^3$ (1pt).

1. (11pts) $S = \left\{ \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} \right\}$

$R_3 - \frac{1}{3}R_1 \quad 3 \cdot \frac{4}{4} - \frac{1}{3} \cdot 4 \quad \frac{12}{12} - \frac{4}{12} = \frac{8}{12} \to \frac{2}{3}$

$$\text{Span}(S) = \begin{bmatrix} 3 & 1 \\ 0 & 2 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \to \begin{pmatrix} 3 & 1 \\ 0 & 2 \\ 1 & 4 \end{pmatrix} \to \begin{pmatrix} 3 & 1 \\ 0 & 2 \\ 0 & 2/3 \end{pmatrix} \xrightarrow[R_3 - 2/3 R_1]{} \begin{pmatrix} 3 & 1 \\ 0 & 2 \\ 0 & 0 \end{pmatrix} \xrightarrow{R_2 \to R_2/2} \begin{pmatrix} 3 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \xrightarrow{R_1 \to R_1/3} \begin{pmatrix} \boxed{1} & 1/3 \\ 0 & \boxed{1} \\ 0 & 0 \end{pmatrix}.$$

$Span(S) = \mathbb{R}^3$?	lin. independent?	basis?
Yes ☐ No ☒	Yes ☒ No ☐	Yes ☐ No ☒

- Row of zeros, not Spanning in $\mathbb{R}^3$
- Pivot in all columns, it is linearly independent

- Does NOT Span, but it is linearly independent. So, it is NOT a Basis.

2. (11pts) $S = \left\{ \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$

$Span(S) = \mathbb{R}^3$?	lin. independent?	basis?
Yes ☒ No ☐	Yes ☒ No ☐	Yes ☒ No ☐

$R_2 - 2R_1 \begin{cases} 2 - 2(1) \\ 1 - 2(2) \\ 1 - 0 \end{cases} \qquad R_2/-3 \begin{cases} 0/-3 = 0 \\ -3/-3 = 1 \\ 1/-3 = -1/3 \end{cases}$

$$\text{Span}(S) = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 1 \\ 4 & 4 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 \\ 0 & -3 & 1 \\ 0 & 2 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1/3 \\ 0 & 2 & 0 \end{bmatrix}$$

$R_3 - 4R_1 \begin{cases} 4 - 4(1) \\ 4 - 2(1) \\ 0 - 2(4) \end{cases}$

$x_1 + 2x_2 = 0$

$x_2 + -1/3\, x_3 = 0$

x_3 is free variable, Non-zero solution

- Yes, no Zero columns, so it does span.
- All columns have a pivot, so linearly independent.
- S is linearly independent, so S is a basis for $\mathbb{R}^3$

$R_3 - 2R_2 \begin{cases} 0 - 2(0) = 0 \\ 2 - 2(1) = 0 \\ 0 - 2(-\frac{1}{3}) = 2/3 \end{cases}$

$$\longrightarrow \begin{bmatrix} \boxed{1} & 2 & 0 \\ 0 & \boxed{1} & -1/3 \\ 0 & 0 & \boxed{1} \end{bmatrix}$$

$R_3 \to R_3 \cdot \frac{3}{2}$

3. (11pts) $S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} \right\}$

$R_2 \to R_2 - R_1 \begin{cases} 1 - 1 = 0 \\ 1 - 0 = 1 \\ 2 + 1 = 3 \end{cases}$

$$\text{Span}(S) = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 1 & 1 & -1 \end{pmatrix} \xleftrightarrow{R_3 \leftrightarrow R_2} \begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{\text{REF}} \begin{pmatrix} \boxed{1} & 0 & 2 \\ 0 & \boxed{1} & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

$Span(S) = \mathbb{R}^3$?	lin. independent?	basis?
Yes ☐ No ☒	Yes ☐ No ☒	Yes ☐ No ☒

- Row of 0's, S does not Span in $\mathbb{R}^3$
- Not all columns have pivots, not linearly independent.

- Not spanning, Not lin. ind. Not basis

4. (11pts) $S = \left\{ \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} \right\}$

$Span(S) = \mathbb{R}^3$?	lin. independent?	basis?
Yes ☒ No ☐	Yes ☒ No ☐	Yes ☒ No ☐

$R_2 \to R_2 + \frac{1}{2}R_1 \begin{cases} -1 + \frac{1}{2}(2) = 0 \\ -2 + 1/2(1) = -3/2 \\ 1 + 0 = 1 \end{cases}$ $R_3 \to R_3 - \frac{4}{3}R_2 \begin{cases} 0 - \frac{4}{3}(0) = 0 \\ -2 + \frac{4}{3} \cdot \frac{3}{2} = 0 \\ -3 - \frac{4}{3} = -\frac{13}{3} \end{cases}$

$$Span(S) = \begin{pmatrix} 2 & 1 & 0 \\ -1 & -2 & 1 \\ 0 & -2 & -3 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & 1 & 0 \\ 0 & -3/2 & 1 \\ 0 & -2 & -3 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & 1 & 0 \\ 0 & -3/2 & 1 \\ 0 & 0 & -13/3 \end{pmatrix} \xrightarrow[\substack{R_2 \to R_2 \cdot -\frac{2}{3} \\ R_3 \to R_3 \cdot 3/5}]{R_1 \to R_1/2} \begin{pmatrix} \boxed{1} & 1/2 & 0 \\ 0 & \boxed{1} & -2/3 \\ 0 & 0 & \boxed{1} \end{pmatrix}$$

- No zero rows, S spans $\mathbb{R}^3$
- Pivots in all columns. Linearly independent
- IT is A Basis for $\mathbb{R}^3$.

5. (11pts) $S = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \right\}$

$Span(S) = \mathbb{R}^3$?	lin. independent?	basis?
Yes ☒ No ☐	Yes ☐ No ☒	Yes ☐ No ☒

$R_2 \to R_2 - R_1 \begin{cases} 1 - 1 = 0 \\ 1 - 0 = 1 \\ 0 - 0 = 0 \\ 3 - 2 = 1 \end{cases}$ $R_3 \to R_3 - R_2 \begin{cases} 0 - 0 = 0 \\ 1 - 1 = 0 \\ 1 - 0 = 1 \\ 1 - 1 = 0 \end{cases}$

$$Span(S) = \begin{pmatrix} 1 & 1 & 0 & 3 \\ 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & 1 \end{pmatrix} \xrightarrow{R_2 \longleftrightarrow R_1} \begin{pmatrix} 1 & 0 & 0 & 2 \\ 1 & 1 & 0 & 3 \\ 0 & 1 & 1 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} \boxed{1} & 0 & 0 & 2 \\ 0 & \boxed{1} & 0 & 1 \\ 0 & 0 & \boxed{1} & 0 \end{pmatrix}$$

- No Zero Rows, It does Span $\mathbb{R}^3$
- Not a pivot in every column. Not lin ind.
- Not a basis because of 2nd point.

Problem 2. [20pts] Determine the null space of each of the following matrices:

Step 1) REF form $\to A\vec{x} = \vec{0}$

Step 2) Equation form

- (5pts) $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$

$N(A) =$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$R_2 \to R_2 - 2R_1 \begin{cases} 2 - 2(1) = 0 \\ 1 - 2(2) = -3 \end{cases}$

$$\left(\begin{array}{cc|c} 1 & 2 & 0 \\ 2 & 1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & -3 & 0 \end{array}\right) \xrightarrow{R_2 \to R_2/-3} \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 1 & 0 \end{array}\right)$$

$x_1 + 2x_2 = 0$ $\quad x_1 = -2x_2$ $\quad \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$x_2 = 0$

Step 1) RREF form $\to$ $A\vec{x}=\vec{0}$

Step 2) Equation form

$R_2 \to R_2 - \frac{1}{2}R_1 \begin{cases} 1-\frac{1}{2}(2)=0 \\ 2-\frac{1}{2}(1)=3/2 \\ -6-\frac{1}{2}(3) = -15/2 \\ 6-\frac{1}{2}(-3) = 15/2 \end{cases}$

- (5pts) $A = \begin{pmatrix} 2 & 1 & 3 & -3 \\ 1 & 2 & -6 & 6 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & 1 & 3 & -3 \\ 0 & 3/2 & -\frac{15}{2} & \frac{15}{2} \end{pmatrix} \xrightarrow{R_2 \to \frac{1}{2}R_1} \left(\begin{array}{cccc|c} 1 & 1/2 & 3/2 & -3/2 & 0 \\ 0 & 1 & -5 & 5 & 0 \end{array}\right)$

$R_2 \to \frac{2}{3}R_2 \begin{cases} \frac{2}{3}(0)=0 \\ 2/3(3/2)=1 \\ -\frac{15}{2}\cdot\frac{2}{3}=-5 \\ \frac{15}{2}\cdot\frac{2}{3}=5 \end{cases}$

$x_3=\alpha$

$x_4=\beta$

$x_1+\frac{1}{2}x_2+\frac{3}{2}\alpha-\frac{3}{2}\beta=0 \longrightarrow x_1=-\frac{1}{2}x_2-\frac{3}{2}\alpha+\frac{3}{2}\beta$

$x_2=5x_3-5x_4$

$x_2=5\alpha-5\beta \to \underline{x_2=5(\alpha-\beta)}$

$x_1=-\frac{1}{2}(5\alpha-5\beta)-\frac{3}{2}\alpha+\frac{3}{2}\beta$

$x_1=-\frac{5}{2}\alpha+\frac{5}{2}\beta-\frac{3}{2}\alpha+\frac{3}{2}\beta$

$x_1=-4\alpha+4\beta$

$\underline{x_1=4(-\alpha+\beta)}$

$N(A) =$	$\begin{pmatrix} 4(-\alpha+\beta) \\ 5(\alpha-\beta) \\ \alpha \\ \beta \end{pmatrix}$ $\beta\ \alpha \in \mathbb{R}$

$*$ Free variable Only when more columns than rows

$R_3 \to R_3+R_1 \begin{cases} -2+2=0 \\ -3+3=0 \\ -6+6=0 \end{cases}$

$R_2 \to R_2-\frac{1}{2}R_1 \begin{cases} 1-\frac{1}{2}(2)=0 \\ 2-\frac{1}{2}(3)=1/2 \\ -1-3=-4 \end{cases}$

- (5pts) $A = \begin{pmatrix} 2 & 3 & 6 \\ 1 & 2 & -1 \\ -2 & -3 & -6 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & 3 & 6 \\ 0 & 1/2 & -4 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow[R_2 \to R_2\cdot 2]{R_1 \to R_1/2} \left(\begin{array}{ccc|c} 1 & 3/2 & 3 & 0 \\ 0 & 1 & -8 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

$x_3=\alpha$

$x_1+\frac{3}{2}x_2+3x_3=0$

$x_1=-\frac{3}{2}x_2-3\alpha$

$x_1=-\frac{3}{2}(8\alpha)-3\alpha$

$x_1=-12\alpha-3\alpha$

$x_2-8x_3=0$

$\underline{x_2=8\alpha}$

$x_1=-15\alpha$

$N(A) =$	$\begin{pmatrix} -15 \\ 8 \\ 1 \end{pmatrix}$

$\begin{pmatrix} -15 \\ 8 \\ 1 \end{pmatrix}$

$R_3=R_3+\frac{1}{3}R_1 \begin{cases} -1+\frac{1}{3}(3)=0 \\ -\frac{3}{3}+\frac{1}{3}=-2/3 \\ 0-\frac{1}{3}=-1/3 \\ 2-4/3=2/3 \end{cases}$

$\frac{6}{3}+-\frac{8}{3}$

$R_2=R_2+\frac{2}{3}R_1 \begin{cases} -2+2/3(3)=0 \\ -1+2/3(1)=-1/3 \\ 1-2/3=1/3 \\ 2+2/3(-4)=-2/3 \end{cases}$

$R_3=R_3-2R_2 \begin{cases} -2/3-2(-1/3)=0 \\ -1/3-2(1/3)=-1 \\ 2/3-2(-2/3)=2 \end{cases}$

- (5pts) $A = \begin{pmatrix} 3 & 1 & -1 & -4 \\ -2 & -1 & 1 & 2 \\ -1 & -1 & 0 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 3 & 1 & -1 & -4 \\ 0 & -1/3 & 1/3 & -2/3 \\ 0 & -2/3 & -1/3 & 2/3 \end{pmatrix} \longrightarrow \begin{pmatrix} 3 & 1 & -1 & -4 \\ 0 & -1/3 & 1/3 & -2/3 \\ 0 & 0 & -1 & 2 \end{pmatrix}$

$R_1=R_1/3$

$R_2=R_2\cdot -3$

$R_3=R_3\cdot -1$

$\to \left(\begin{array}{cccc|c} \boxed{1} & 1/3 & -1/3 & -4/3 & 0 \\ 0 & \boxed{1} & -1 & -2 & 0 \\ 0 & 0 & \boxed{1} & -2 & 0 \end{array}\right)$ $x_4=\alpha$

$x_1+\frac{1}{3}x_2-\frac{1}{3}x_3-\frac{4}{3}\alpha=0 \to x_1=-\frac{1}{3}x_2+\frac{1}{3}x_3+\frac{4}{3}\alpha$

$x_2-x_3-2\alpha=0 \to x_2=x_3+2\alpha$

$x_3-2\alpha=0$ $\quad \underline{x_3=2\alpha}$

$x_2=2\alpha+2\alpha$

$\underline{x_2=4\alpha}$

$x_4=\alpha$

$x_1=-\frac{1}{3}(4\alpha)+\frac{1}{3}(2\alpha)+\frac{4}{3}\alpha$

$x_1=\cancel{-\frac{4}{3}\alpha}+\frac{2}{3}\alpha+\cancel{\frac{4}{3}\alpha}$

$\underline{x_1=\frac{2}{3}\alpha}$

$N(A) =$	$\begin{pmatrix} \frac{2}{3}\alpha \\ 4\alpha \\ 2\alpha \\ \alpha \end{pmatrix}$ $\alpha \in \mathbb{R}$

$\begin{pmatrix} \frac{2}{3}\alpha \\ 4\alpha \\ 2\alpha \\ \alpha \end{pmatrix}$

Problem 3. [10pts] The vectors $v_1 = \begin{pmatrix}1\\2\\2\end{pmatrix}$, $v_2 = \begin{pmatrix}3\\0\\1\end{pmatrix}$, $v_3 = \begin{pmatrix}-2\\2\\1\end{pmatrix}$, $v_4 = \begin{pmatrix}5\\4\\0\end{pmatrix}$, $v_5 = \begin{pmatrix}6\\0\\2\end{pmatrix}$ are a spanning set for $\mathbb{R}^3$. Pare down the set $\{v_1, v_2, v_3, v_4, v_5\}$ (destroy some vectors) to form a basis for $\mathbb{R}^3$.

Basis:	$\left\{ \begin{pmatrix}1\\2\\2\end{pmatrix}, \begin{pmatrix}3\\0\\1\end{pmatrix}, \begin{pmatrix}5\\4\\0\end{pmatrix} \right\}$

$$R_3 = R_3 - 2R_1 \begin{cases} 2-2(1)=0 \\ 1-2(3)=-5 \\ 1+4=5 \\ 0-2(5)=-10 \\ 2-2(6)=-10 \end{cases}$$

$$R_2 = R_2 \cdot -\tfrac{1}{6} \qquad R_3 = R_3 \cdot -\tfrac{1}{5}$$

$$\begin{pmatrix} 1 & 3 & -2 & 5 & 6 \\ 2 & 0 & 2 & 4 & 0 \\ 2 & 1 & 1 & 0 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 3 & -2 & 5 & 6 \\ 0 & -6 & 6 & 14 & 12 \\ 0 & -5 & 5 & -10 & -10 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 3 & -2 & 5 & 6 \\ 0 & 1 & -1 & 1 & -2 \\ 0 & 1 & -1 & 2 & 2 \end{pmatrix}$$

$$R_2 = R_2 - 2R_1 \begin{cases} 2-2(1)=0 \\ 0-2(3)=-6 \\ 2+4=6 \\ 4-5(2)=-6 \\ 0+2(6)=12 \end{cases}$$

$$\xrightarrow{R_3 = R_3 - R_2} \begin{array}{c} \begin{pmatrix} \boxed{1} & 3 & -2 & 5 & 6 \\ 0 & \boxed{1} & -1 & 1 & -2 \\ 0 & 0 & 0 & \boxed{1} & 4 \end{pmatrix} \\ \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \end{array}$$

x_1, x_2, x_4 form basis

Problem 4. [15pts] Given $v_1 = \begin{pmatrix}1\\1\\1\end{pmatrix}$, and $v_2 = \begin{pmatrix}3\\-1\\4\end{pmatrix}$:

$$R_2 \to R_2 - R_1 \begin{cases} 1-1=0 \\ -1-3=-4 \end{cases} \qquad R_3 \to R_3 - R_1 \begin{cases} 1-1=0 \\ 4-3=1 \end{cases} \qquad R_3 \to R_3 + \tfrac{1}{4}R_2 \begin{cases} 1+\tfrac{1}{4}(-4)=0 \\ 0+\tfrac{1}{4}(0)=0 \end{cases}$$

$$R_2 \to R_2/-4$$

$$\begin{pmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 4 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 3 \\ 0 & -4 \\ 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 3 \\ 0 & -4 \\ 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} \boxed{1} & 3 \\ 0 & \boxed{1} \\ 0 & 0 \end{pmatrix}$$

1. (5pts) Do v_1 and v_2 span $\mathbb{R}^3$? Explain.

   Yes ☐ No ☒

   No, A row of zeros when put into REF.

2. (5pts) Do v_1 and v_2 are linearly independent? Explain.

   Yes ☒ No ☐ Yes, Pivot in every column.

3. (5pts) Find a third vector v_3 such that the set $\{v_1, v_2, v_3\}$ is now a basis for $\mathbb{R}^3$.

$$V_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$$