Exam 2 - Diff. Equations Joaquin Salas

(1) Let $T(t)$ = temp inside at any time t.
$H(t)$ = heat produced by people, light, machines
$U(t)$ = heating & cooling by AC or furnace
$m(t)$ = outside temp

Givens: Δ in external temp w/ respect to time → $\frac{dm}{dt} = a$, $m(15) = 98°F$, $m(24) = 75°F$
& $T(15) = 72°F$

Newton's Law of cooling

$$Q(t) = km(t) + \underbrace{H(t) + U(t)}_{0}$$

$$\int dm = \int a\,dt$$

$$\rightarrow m = at + b$$

$$T(t) = e^{-kt}\left(\int e^{kt} Q\,dt\right)$$

$$T(t) = e^{-kt}\left(\int e^{kt} km\,dt\right) \rightarrow T(t) = ke^{-kt}\left(\int e^{kt}(at+b)\,dt\right)$$

$u = at+b \quad dv = e^{kt}dt$

$du = a\,dt \quad v = \frac{e^{kt}}{k}$

$$T(t) = ke^{-kt}\left[\frac{e^{kt}}{k}(at+b) - \frac{a}{k}\int e^{kt}\,dt\right]\frac{1}{k}$$

$uv - \int v\,du$

$$\rightarrow T(t) = ke^{-kt}\left[\frac{e^{kt}}{k}(at+b) - \frac{a}{k^2}e^{kt} + c\right] \text{ distribute}$$

$$\rightarrow T(t) = \cancel{ke^{-kt}}\left[\frac{a}{k}e^{kt} - \frac{b}{k}e^{kt} - \frac{a}{k^2}e^{kt} + c\right]$$

$$\rightarrow T(t) = at + b - \frac{a}{k} + ce^{-kt},$$

$m(15) = 98°F$
$15a + b = 98$
$b = 98 - 15a$
$b = 98 - 15(-2.56)$
$\boxed{b = 136}$

$m(24) = 75°F$
$24a + b = 75$
$24a + (98 - 15a) = 75$
$9a = -23$
$\boxed{a = -2.56}$

$$T = -2.56t + 136 + \frac{+2.56}{1/3} + ce^{-t/3}$$

$$T = -2.56 + 144 + ce^{-t/3}$$

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$T(15) = 72°F \qquad T(t) = -2.56t + 144 + Ce^{-t/3}$

So, $\quad 72 = -2.56(15) + 144 + Ce^{-5}$

$$\frac{72 + 2.56(15) - 144}{e^{-5}} = C$$

$$-4968 = C$$

$$T(t) = -2.56t + 144 - 4986e^{-t/3}$$

max when $T'(x) = 0$,

$$T'(t) \rightarrow \quad 0 = -2.56 + \frac{4968}{3} e^{-t/3}$$

$$\frac{2.56}{1662} = \frac{1662e^{-t/3}}{1662}$$

$$\ln 0.00154 = \ln e^{-t/3}$$

$$-3(\ln(0.00154)) = -\frac{t}{3} \cdot -3$$

$$\boxed{t = 19.428 \text{ or } 7{:}26 \text{ P.m}}$$

The temperature inside the building will reach a maximum at 7:26 P.m.

Q.2

a) $\frac{dQ}{dt} = (\text{Toxin Rate})_{in} - (\text{Toxin Rate})_{out}$

Pond A,

$(\text{Toxin Rate})_{in} = 1000 \times 0$
$= 0$

$(\text{Toxin Rate})_{out} = \frac{Q_A}{500{,}000} \times 1000$
$= \frac{Q_A}{500}$

$\frac{dQ_A}{dt} = 0 - \frac{Q_A}{500} \Rightarrow \boxed{\frac{dQ_1}{dt} = -\frac{Q_A}{500}, \quad Q_1(0) = 1000}$

For Pond B,

$(\text{Toxin Rate})_{in} = \frac{Q_A}{500}$

$(\text{Toxin Rate})_{out} = \frac{Q_A}{200{,}000} \times 1000 = \frac{Q_B}{200}$

$\boxed{\frac{dQ_B}{dt} = \frac{Q_A}{500} - \frac{Q_B}{200}, \quad Q_B(0) = 0}$

(b) $\frac{dQ_A}{dt} = -\frac{Q_A}{500}, \quad Q_A(0) = 1000$ and $\frac{dQ_B}{dt} = \frac{Q_A}{500} - \frac{Q_B}{200}, \quad Q_B(0) = 0$

$\int \frac{1}{Q_A}\, dQ_A = \int -\frac{1}{500}\, dt$

$\ln(Q_A) = -\frac{t}{500} + C_1$

$= Q_A = Ce^{-\frac{t}{500}}$

$= \ln(1000) = C$

$\boxed{Q_A(t) = 1000e^{-\frac{t}{500}}}$

$\frac{dQ_B}{dt} = \frac{\cancel{1000}e^{-\frac{t}{500}}}{\cancel{500}} - \frac{Q_B}{200}$

$\frac{dQ_B}{dt} = 2e^{-\frac{t}{500}} - \frac{Q_B}{200}$

$\frac{dQ_B}{dt} + \frac{Q_B}{200} = 2e^{-\frac{t}{500}} \qquad I.F. = e^{\int \frac{dt}{200}}$
$= e^{\frac{t}{200}}$

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Integrating factor $= e^{\int \frac{dt}{200}}$

$= e^{\frac{t}{200}}$

$$Q_B e^{\frac{t}{200}} = \int e^{\frac{t}{200}} \left(2e^{-\frac{t}{500}}\right) dt$$

$$\frac{Q_B e^{\frac{t}{200}}}{e^{\frac{t}{200}}} = 2\int e^{\frac{t}{200}-\frac{t}{500}} dt$$

$$Q_B e^{\frac{t}{200}} = \frac{2000}{3} e^{-\frac{t}{500}} + C_2 e^{-\frac{t}{200}} \qquad \text{Sub } Q_B(0) = 0$$

$$= \frac{2000}{3} e^{-\frac{0}{200}} + C_2 e^{-\frac{0}{200}} = 0 \qquad C_2 = -\frac{2000}{3}$$

$$= \frac{2000}{3} e^{-\frac{t}{500}} - \frac{2000}{3} e^{-\frac{t}{200}} \Rightarrow \boxed{Q_B(t) = \frac{2000}{3}\left(e^{-\frac{t}{500}} - e^{-\frac{t}{200}}\right)}$$

(c) $$\frac{dQ_B}{dt} = \frac{Q_A}{500} - \frac{Q_B}{200} \qquad \text{when } \frac{dQ_B}{dt} = 0$$

$$\frac{Q_A}{500} - \frac{Q_B}{200} = 0$$

$$Q(t) = \frac{1000e^{-\frac{t}{500}}}{500} - \frac{\frac{2000}{3}\left(e^{-\frac{t}{500}} - e^{-\frac{t}{200}}\right)}{200}$$

$$\Rightarrow 2e^{-\frac{t}{500}} - \frac{10}{3}\left(e^{-\frac{t}{500}} - e^{-\frac{t}{200}}\right) = 0$$

$$\rightarrow \frac{3}{10} \cdot \frac{2e^{-\frac{t}{500}}}{e^{-\frac{t}{500}}} = \frac{10}{3}\frac{\left(e^{-\frac{t}{500}} - e^{-\frac{t}{200}}\right)}{e^{-t/500}} \cdot \frac{3}{10}$$

$$\rightarrow \frac{6}{20}$$

$$\rightarrow \frac{3t}{1000} = \ln\left(\frac{5}{2}\right) \rightarrow t = \frac{1000}{3}\ln\left(\frac{5}{2}\right) \rightarrow \boxed{t \approx 305 \text{ Hours}}$$

$$\text{Toxin} = \frac{2000}{3}\left(e^{-\frac{305.4}{500}} - e^{-\frac{305.4}{500}}\right)$$

$$\boxed{= 217 \text{ lbs of toxin}}$$

(d) $Q_A(t) = 1/2$ lbs

$Q_A(t) = 1000e^{-\frac{t}{500}}$

$1000e^{-\frac{t}{500}} = \frac{1}{2}$ $\rightarrow \ln e^{-\frac{t}{500}} = \ln \frac{1}{2000} \rightarrow -\frac{t}{500} = \ln(2000)$

$t = 500 \ln(2000)$ $t \approx 3800.45$ hours for pond A.

$Q_B(t) = 1/5$ lbs

$Q_B(t) = \frac{2000}{3}\left(e^{-\frac{t}{500}} - e^{-\frac{t}{200}}\right)$

$\frac{2000}{3}\left(e^{-\frac{t}{500}} - e^{-\frac{t}{200}}\right) = \frac{1}{5}$

$= e^{-\frac{t}{500}} - e^{-\frac{t}{200}} = \frac{3}{10000}$

graphically, t is 0.25 or 3598,
Since it is going up at 0.25,

$\boxed{t = 3598 \text{ Hours}}$

(3) $\frac{dP}{dt} = 0.25P - \frac{25}{10^6}$

(a) find $P(t)$

$$\frac{1}{\left(0.25P - \frac{25}{10^6}P\right)} dP = dt$$

Integrate by parts: $\frac{A}{P} + \frac{B}{.25 - \frac{25}{10^6}P} = \frac{.25 - \frac{25}{10^6}PA + BP}{P(.25 - \frac{25}{10^6}P)} = \frac{1}{P(.25 - \frac{25}{10^6}P)}$

$-\frac{25}{10^6}PA + BP = 0$ $\quad .25A = 1 \to \boxed{A = \frac{1}{.25}}$

$P\left(-\frac{25}{10^6}A + B\right) = 0$

$P = \left(-\frac{25}{10^6}\left(\frac{1}{.25}\right) + B\right) = 0$ $\quad \boxed{B = 10^4}$

$P(10^{-4} + B) = 0 \to \frac{B\cancel{P}}{\cancel{P}} = 10^4$

$$\to .25 \times \frac{1}{.25}\int\left[\frac{1}{P} + \frac{25\times10^{-6}}{.25 - 25\times10^{-6}P}\right] dP = .25\,dt$$

$$= \ln|P| + \frac{\cancel{25\times10^{-6}}^{\,1}}{-\cancel{25\times10^{-6}}} \ln\left[.25 - 25\times10^{-6}P\right] = .25t + c$$

$= \ln_e\left|\frac{P}{.25 - 25\times10^{-6}P}\right| = e^{\frac{t}{4} + c}$ Solve for "P"

$= \frac{P}{.25 - 25\times10^{-6}P} = Ce^{t/4} \to P = Ce^{\frac{t}{4}}(25) - Ce^{\frac{t}{4}}(25\times10^{-6}P)$

$\to P + Ce^{t/4}(25\times10^{-6})P = Ce^{t/4}(.25)$

$\to P(1 + Ce^{t/4}(25\times10^{-6})) = Ce^{t/4}(.25)$

$\to \boxed{P = \frac{.25Ce^{t/4}}{1 + Ce^{t/4}(25\times10^{-6})}}$ $\quad P(0) = 4$

$P(0) = 4$

$\to P(t) = \frac{C}{4 + Ce^{t/4}\times10^{-6}} \to 4 = \frac{C}{4 + C\times10^{-6}}$

$\to 16 + 4C\times10^{-6} = C$

$\to 16 = C - 4C\times10^{-6}$

$\to 16 = C(1 - 4\times10^{-6})$

$1 - .0004$

$16 = (.99996)(C)$

$\boxed{\frac{16}{.99996} = C}$ $\to$ Next page

$$\rightarrow \quad P(t) = \frac{\overset{4}{\cancel{.25}}\left(\frac{\cancel{16}}{.99996}\right)e^{t/4}}{1+\overset{4}{\left(\frac{\cancel{16}}{.99996}\right)}e^{t/4}\left(\cancel{25}\times 10^{-6}\right)^{-4}}$$

$$= \frac{10^4\left(\frac{4}{.99996}\right)e^{t/4}}{1+4e^{t/4}} \longrightarrow \boxed{P(t) = \frac{\left(\frac{4}{.99996}\right)e^{t/4}}{1+\left(\frac{4}{.99996}\right)e^{t/4}\times 10^{-4}}}$$

(b) How many people were infected by the end of 4 weeks

t = days

$4 \times 7 = \underline{28 \text{ days}}$

$$P(28) = \frac{\left(\frac{4}{.99996}\right)\cancel{\times 10^4}\, e^{28/4}}{1+\left(\frac{4}{.99996}\right)e^{28/4}\times 10^{-4}} = 3049$$

$\boxed{\text{After 4 weeks, or 28 days, 3049 people will be infected}}$

(c) limiting size of the population that becomes infected?

$$\lim_{t\to\infty} P(t) = \lim_{t\to\infty} \frac{\left(\frac{4}{.99996}\right)^{4} e^{t/4}}{1+\left(\frac{4}{.99996}\right)e^{t/4}\times 10^{-4}}$$

$$= \lim_{t\to\infty} \frac{\cancel{4}\times 10^4}{1+\cancel{4}e^{-\infty t/4}\to 0}$$

$$= \frac{10^4}{1} = \boxed{10{,}000}$$

(d) $P = \frac{10000}{2} = 5000 \quad t = ? \qquad 5000 = \frac{\left(\frac{4}{.99996}\right)e^{t/4}}{1+\left(\frac{4}{.99996}\right)e^{t/4}\times 10^{-4}}$

$\rightarrow 5000\left(1+4e^{t/4}\times 10^{-4}\right) = 4e^{t/4}$

$\rightarrow 5000 + \frac{\cancel{2e^{t/4}}}{\cancel{2e^{t/4}}} = \frac{\cancel{4e^{t/4}}}{\cancel{2e^{t/4}}}$

$\rightarrow \frac{5000}{2} = \frac{2e^{t/4}}{2} \rightarrow \ln 2500 = \ln e^{t/4}$

$4 \cdot \ln 2500 = \frac{t}{4}\cdot 4 \qquad t = 31.3$

$\boxed{\text{In 31 days, half of limiting size will be infected}}$

(4) Rate in = 2.5 gal/min Rate out = 4 gal/min

(a) Rate of water lost per min $(4-2.5) = 1.5$ gal

Initial water → $\frac{400 \text{ gallons}}{1.5 \text{ min}} = \boxed{266.67 \text{ minutes}}$

(b) $\frac{dy}{dt} = (\text{rate in}) - (\text{rate out})$ rate in $= (2.5 \times 4)$ lb/min $= 10$ lb/min of salt

rate out $= \left[\frac{y(t)}{v(t)} \times 4 \text{ gal/min}\right]$

y = concentration in

v = volume

$400 + (2.5-4)t$
$= 400 - 1.5t$

Therefore $\frac{y}{400-1.5} \times 4 = \frac{4y}{400-1.5t}$ (lb/min)

linear

$\frac{dy}{dt} + P_y = Q$

$\frac{dy}{dt} = 10 - \frac{4y}{400-1.5t} \rightarrow \frac{dy}{dt} + \frac{4y}{400-1.5t} = 10$

I.F. $e^{\int \frac{4}{400-1.5t} dt}$

$e^{\frac{4 \ln(400-1.5t)}{-1.5}} \rightarrow e^{-\frac{8}{3} \ln(400-1.5)} \rightarrow$

$e^{\ln(400-1.5t)^{-8/3}}$
$= (400-1.5t)^{-8/3}$ $y \text{ I.F.} = \int Q(\text{IF})\, dt + c$

$\rightarrow y(400-1.5t)^{-8/3} = \int 10 \cdot (400-1.5t)^{-8/3}$

$\rightarrow y(400-1.5t)^{-8/3} = \frac{10(400-1.5t)^{-8/3+1}}{(-8/3)+1\,(1.5)} + c$

$\rightarrow y(400-1.5t)^{-8/3} = \frac{10}{-5/3+1.5}(400-1.5t)^{-5/3} + c$

$\rightarrow y(400-1.5t)^{-8/3} = 4(400-1.5t)^{-5/3} + c$

$\rightarrow y = \frac{4(400-1.5t)^{-5/3} + c}{(400-1.5t)^{-8/3}} \rightarrow y = \frac{4(400-1.5t)^{-8/3}}{(400-1.5t)^{-8/3}} + \frac{c}{(400-1.5t)^{-8/3}}$

$y = 4(400-1.5t) + c(400-1.5t)^{8/3}$ — (2)

At $t=0$, $y=0$

$0 = 4(400-1.5(0)) + c(400-1.5(0))^{8/3} \rightarrow 0 = 1600 + c(400)^{8/3}$

$c = \frac{-1600}{(400)^{8/3}} = -4 \times \frac{400}{(400)^{8/3}} = \frac{-4}{(400)^{8/7}}$

From eq (2)

$$\boxed{x(t) = 4(400-1.5t) - \frac{4(400-1.5t)^{8/3}}{(400)^{5/3}}}$$

(C) 100 gal left so,

$$\frac{= 300}{4-(2.5)} = \underline{\text{After } 200 \text{ minutes}}$$

Salt present?

$$x(t) = 4(400-1.5t) - \frac{4(400-1.5t)^{8/3}}{(400)^{5/3}}$$

$$x(200) = 4(400-1.5(200)) - \frac{4(400-1.5(200))^{8/3}}{(400)^{5/3}}$$

$= 360 \ \underline{\text{lbs}}$

When 100 gallons of brine are left, there will be 360 lbs of salt present

(d) To find max let $x=0$ of $\frac{d}{dt}$

$$\frac{d}{dt}\left(4(400-1.5t) - \frac{4(400-1.5t)^{8/3}}{(400)^{5/3}}\right)$$

$$= \quad -6 - \frac{4 \times (400-1.5t)^{8/3}}{20^{10/3}}$$

$$0 \quad -6 - \frac{(800-3t)(8000-30t)^{2/3}}{2000}$$

= 555 is the max amount of salt present and it happens in 119 minutes from $t=0$

(5) $m(t) = m_0 + A\sin\left(\omega t - \frac{\pi}{2}\right)$

let m_0 = Average outside temp

A = Positive constant

$\omega = \pi/12$

(a) $m(t) = m_0 + A\sin\left(\omega t - \pi/2\right)$

$= m_0 + A\sin\left(-\left(-\omega t + \frac{\pi}{2}\right)\right)$

$\boxed{= m_0 - A\cos \omega t}$

(b) $m_0 = m_0 - A\cos(0) = m_0 - A$

$\rightarrow m_0 - A$

$\boxed{\text{Outside temperature at midnight, Minimum temperature}}$

$m(12) = m_0 - A\cos(\pi) = m_0 + A$

$\rightarrow m_0 + A$

$\boxed{\text{Outside temperature at noon, Maximum temperature}}$

(16) Consider a projectile of constant mass being fired Vertically from Earth.

(a) $F = \frac{GMm}{r^2}$ $\cancel{m}a = \frac{GM\cancel{m}}{r^2}$

$a = \frac{GM}{r^2}$ $a = \frac{dv}{dt} = -\frac{dv}{dx} \cdot \frac{dx}{dt} - V$

$V \cdot \frac{dv}{dx} = -\frac{GM}{r^2}$ and $\frac{GM}{R^2} = g$

$$\boxed{V\frac{dv}{dx} = -\frac{gR^2}{r^2}}$$

(b) $\frac{v}{v}\frac{dv}{dx} = \frac{-gR^2}{r^2} \longrightarrow V \cdot dV = \frac{-gR^2}{r^2}dx$

$\rightarrow \int_{v}^{v} v\,dv = \int_{R}^{r} -\frac{gR^2}{r^2}dx \rightarrow \frac{V^2 - V_0^2}{2} = \frac{gR^2}{R} = gR$

where V_0 is initial velocity

So, $\boxed{V^2 = V_0^2 + 2gR}$

(c)

$\lim_{r \to \infty} \frac{2gR^2}{r} + V_0^2 - 2gR$

$V^2 = V_0^2 - 2gR$ $\boxed{V_0 = V^2 + 2gR}$

(d) if $g = -9.81\ m/sec^2$ & $R = 6370\ km$

Earths escape velocity

$V_0 = \sqrt{V^2 + 2gR}$, $V = 0$

$V_0 = \sqrt{2gR} \rightarrow V_0 = \sqrt{2(9.81\ m/sec^2)(6370\ km)\times 10^3}$

$$\boxed{V_0 = 11179.42\ m/s}$$

(e) $g_m = \frac{g}{6}$ the radius of the moon $R_m = 1738\ km$

or $\underline{1738000\ m}$

Escape velocity of the moon?

$$V_{om} = \sqrt{2gR} \qquad g = \frac{g}{6} = \frac{9.81}{6} = 1.635$$

$$= \sqrt{2(1.635)(1738000\ m)}$$

$$\boxed{V_{om} = 2383.96\ m/s}$$

#7

(a) $$\frac{dP}{dt} = 0.2357P - \frac{4.7140}{10^5}P^2$$

$$P^{-2}\frac{dP}{dt} - 0.2357P^{-1} = -\frac{4.7140}{10^5}$$

Let $v = P^{-1}$ and $\frac{dv}{dt} = -P^{-2}\frac{dP}{dt}$:

$$\rightarrow \frac{dv}{dt} + 0.2357v = \frac{4.714}{10^5} \leftarrow \text{linear}$$

Integrating factor $\mu = e^{\int .2357\,dt}$ $\boxed{= e^{0.2357t}}$

$$\rightarrow \int \left(e^{0.2357t}\right)\frac{dv}{dt} + \left(e^{0.2357t}\right)\frac{0.2357v}{0.2357} = \int \frac{4.714}{23570}\left(e^{0.2357t}\right)$$

$$\rightarrow \frac{\cancel{e^{0.2357t}}}{P} = \left(\frac{4.7140}{23570}\right)\left(\cancel{e^{0.2357t}}\right) + c$$

$$\rightarrow \frac{1}{P} = \frac{4.7140}{23570} + Ce^{-0.2357t}$$

$$\rightarrow \frac{1}{P} = 0.0002 + Ce^{-0.2357t}$$

$P(0) = 50$ solve for "C"

$$\rightarrow \frac{1}{50} - 0.0002 = C\cancel{e^{-0.2357(0)}}^{1}$$

$C = 0.0198$

$$\rightarrow \boxed{P(t) = \frac{1}{0.0002 + 0.0198e^{-0.2357t}}}$$

(b) $$\lim_{t\to\infty} \frac{1}{0.0002 + 0.0198\cancel{e^{-0.2357(\infty)}}^{0}} \Rightarrow P(t) = \frac{1}{0.0002}$$

$\boxed{P = 5000}$

$\boxed{P = 5000 \text{ is the carrying capacity.}}$

(c) $$P(10) = \frac{1}{0.0002 + 0.0198e^{-0.2357(10)}}$$

$= 481.9$

After 10 years, there will be about 482 bison.

(d)

$$2500 = \frac{1}{0.0002 + 0.0198e^{-0.2357t}}$$

$$\rightarrow \frac{1}{2500} = 0.0002 + 0.0198e^{-0.2357t}$$

$$\rightarrow \frac{1}{2500} - 0.0002 = \frac{0.0198e^{-0.2357t}}{0.0198}$$

$$\rightarrow \frac{.0002}{0.0198} = e^{-0.2357t}$$

ln ln

$$\rightarrow \frac{\ln\left(\frac{.0002}{0.0198}\right)}{-0.2357} = t \implies t = 19.5 \text{ years}$$

After 19 years, the population of bison will reach one-half of the carrying capacity

#8

$$m\frac{dv}{dt} = -\frac{GMm}{r^2} + kv^2$$

$$\frac{dy}{dx} + P(x)y = Q(x) \leftarrow \text{linear eq.}$$

$$\rightarrow \frac{dv}{dt} - \underbrace{km^{-1}}_{P(x)} v^2 = \underbrace{-GMmr^{-2}}_{Q(x)}$$

I.F. $I = e^{\int -km^{-1}dt}$

$$= e^{-\frac{kt}{m}}$$

$$\rightarrow e^{-\frac{kt}{m}}\left(\frac{dv}{dt} - km^{-1}v^2\right) = e^{-\frac{kt}{m}}\left(-GMmr^{-2}\right)$$

$$\rightarrow \left(e^{-\frac{kt}{m}} \cdot \frac{dv}{dt} + \frac{k}{m}e^{-\frac{kt}{m}}v^2\right) = \int e^{-\frac{kt}{m}}\left(-GMmr^{-2}\right)dt$$

$$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\rightarrow 2\int\left(e^{-\frac{kt}{m}} \cdot v^2\right)dv = -\frac{GMm}{r^2}\int e^{-\frac{kt}{m}}dt \quad \text{let } u = -\frac{kt}{m}$$

$$\rightarrow \frac{2}{3}v^3e^{-\frac{kt}{m}}$$

$$\rightarrow -\frac{GMm}{r^2}\int e^{-\frac{kt}{m}}dt$$

$$\rightarrow -\frac{GMm}{r^2}\int\left(-\frac{me^u}{k}du\right)$$

$$\rightarrow \frac{GMm}{r^2}\cdot\frac{m}{k}\int e^u du$$

$$\rightarrow \frac{GMm^2}{r^2k} \times e^{-kt}$$

So,

$$\frac{\frac{2}{3}v^3\cancel{e^{-\frac{kt}{m}}}}{\cancel{e^{-\frac{kt}{m}}}} = \frac{\frac{GMm^2}{r^2k}\cancel{e^{-\frac{kt}{m}}} + C}{\cancel{e^{-\frac{kt}{m}}}}$$

$$v = \frac{3}{2}\left[\frac{GMm^2}{r^2k} + Ce^{-\frac{kt}{m}}\right]^{3/2}$$

when the body reaches height "h" the velocity is 0.

$V = 0$ when $r = R-e+h$

(b) find impact velocity

when $h = 0$ So,

$$\boxed{V = \frac{3}{2}\left[\frac{GMm^2}{(R-e+h)^2 k} + Ce^{-\frac{kt}{m}}\right]^{3/2}}$$

when $h=0$ will equal impact velocity.