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Math 2415 Exam 3

Be sure to state the test you use to determine convergence or divergence. SHOW ALL WORK. Put your thoughts (as discussed in class) with each problem, whether you agree and why, or whether you don't agree, and what you think it should be.

Here, the bounds are given in the "R =" and after that, simple integration gets me the answer of $5 + \frac{25\pi}{8} u^2$ since this is area

CHAPTER 14 EXAM

Evaluate the following integrals.

Problem 1. $\iint_R (\sec^2 x + y)\, dA;\ R = \left\{(x,y)\colon 0 \le x \le \frac{\pi}{4}, 0 \le y \le 5\right\}$ $\quad \int_0^{\frac{\pi}{4}}\int_0^5 dy\,dx$

$$\int_0^{\frac{\pi}{4}}\int_0^5 (\sec^2 x + y)\,dy\,dx$$

1) $\int_0^5 \sec^2 x\,dy + \int_0^5 y\,dy = y\sec^2 x + \frac{1}{2}y^2\Big|_0^5 \rightarrow \underline{5\sec^2 x + \frac{25}{2}} - \left(0\cancel{\sec^2 x} + \frac{1}{2}0^2\right)$

2) $\int_0^{\frac{\pi}{4}} 5\sec^2 x + \frac{25}{2}\,dx$ or $5\int_0^{\frac{\pi}{4}}\left(\sec^2 x + \frac{5}{2}\right)dx = 5\left[\int_0^{\frac{\pi}{4}}\sec^2 x\,dx + \int_0^{\frac{\pi}{4}}\left(\frac{5}{2}\right)\right]dx$

$= 5\left[\tan(x) + \frac{5}{2}(x)\right]_0^{\pi/4} = \tan\left(\frac{\pi}{4}\right) + \frac{5}{2}\left(\frac{\pi}{4}\right) - \left(\tan(0) + \frac{5}{2}(0)\right)$

$= 5\left(1 + \frac{5\pi}{8}\right) = \boxed{5 + \frac{25\pi}{8}u^2}$ Answer

Problem 2. $\int_0^{\pi}\int_0^{\pi}\int_0^{\sin x} \sin y\,dz\,dx\,dy$

$\int_0^{\sin(x)} \sin(y)\,dz \rightarrow = z\sin(y)\Big|_0^{\sin(x)} \rightarrow \underline{\sin(x)\sin(y)}$

$\rightarrow \int_0^{\pi} \sin(x)\sin(y)\,dx \rightarrow = \cos(x)\sin(y)\Big|_0^{\pi} \rightarrow (-1)\sin(y) - ((1)\sin(y))$

$= -2\sin(y)$

$\rightarrow -2\int_0^{\pi}\sin(y)\,dy \rightarrow \overset{-2}{=}\cos(y)\Big|_0^{\pi} \overset{-2}{=} (-1-1) = -2$

$(-2)(-2) = \boxed{4}$

This is a triple integration, no extra work is needed in this problem, my answer is 4.

Problem 3. $\iint_R x\,dA$ where R is bounded by $y = 2x + 2$ and $y = x^2 - 1$. $\quad A = \int_c^d \int_{g(y)}^{h(y)} x\,dy\,dx$

I.P.: $2x + 2 = x^2 - 1$
$x^2 - 2x - 3 = 0$
$(x+1)(x-3) = 0$
$x = -1$
$x = 3$

Bounds are $\int_{-1}^{3}\int_{x^2-1}^{2x+2}$

Answer: $\frac{32}{3}u^2$

So, $\int_{-1}^{3}\int_{x^2-1}^{2x+2} x\,dy\,dx$

$\rightarrow \int_{x^2-1}^{2x+2} x\,dy = yx\Big|_{x^2-1}^{2x+2} = $

$= (2x+2)x - (x^2-1)x$
$= 2x^2 + 2x - x^3 + x$
$= \underline{-x^3 + 2x^2 + 3x}$

$\int_{-1}^{3} -x^3 + 2x^2 + 3x\,dx$

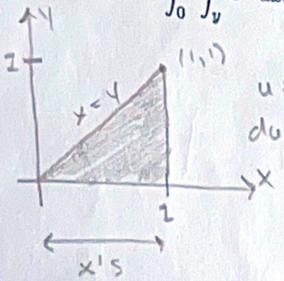
$= -\frac{x^4}{4} + \frac{2x^3}{3} + \frac{3x^2}{2}\Big|_{-1}^{3}$

$= -\frac{3^4}{4} + \frac{2(3)^3}{3} + \frac{3(3)^2}{2} - \left(-\frac{-1^4}{4} + \frac{-2^3}{3} + \frac{-3^2}{2}\right)$

$= -\frac{81}{4} + 18 + \frac{27}{2} - \frac{7}{12} = \underline{\frac{32}{3}}$

First find the I.P. like in the notes and $x^2 - 1$ to $2x+2$ for 2^{nd} Intergal bound and integrate

Problem 4. $\int_0^1 \int_y^1 e^{x^2}\, dx\, dy$



$u = x^2$
$du = 2x$

$$\int_0^1 \int_0^x e^{x^2}\, dy\, dx$$

$$\to e^{x^2} \int_0^x 1\, dy = \left[y e^{x^2} \right]_0^x = e^{x^2}(x-0) = \underline{x e^{x^2}}$$

$$\to \int_0^1 x e^{x^2} dx = \int_0^1 \overset{u}{e^{x^2}} \cdot \underbrace{2x\, dx}_{du}\, du \qquad \frac{1}{2}e - \frac{1}{2}$$

$$\to \frac{1}{2} \int_0^1 e^u du = \frac{1}{2} e^u \Big|_0^1 = \frac{1}{2}(e^1 - e^0) \qquad \boxed{= \frac{1}{2}(e-1)}$$

$$= \underline{\frac{1}{2}e^1 - \frac{1}{2}}$$

Problem 5. Determine the volume of the solid bounded by $z = x^2 + y^2$ and $z = 25$.

$$\int_0^{2\pi} \int_0^5 (25 - r^2)\, r\, dr\, d\theta$$

$$\to \int_0^5 (25 - r^2)\, r\, dr$$

let $u = 25 - r^2$
$du = -2r\,dr$
$dr = \frac{du}{-2r}$

$$\to \int_0^5 -\frac{1}{2} u\, du$$

$$\to -\frac{1}{2}\left[\frac{u^2}{2} \right]_0^5 = -\frac{1}{2}\left(\frac{(25-r^2)^2}{2} \right) \Bigg|_0^5$$

$$\to -\frac{1}{2}\left(\frac{\cancel{(25-(5)^2)^2}}{\cancel{2}} - \frac{(25-0^2)^2}{2} \right) = -\frac{1}{2}\left(-\frac{625}{2}\right)$$

$$= \underline{\frac{625}{4}}$$

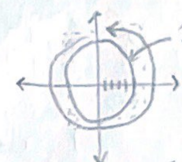
$$\int_0^{2\pi} \frac{625}{4} d\theta$$

$$\to \frac{625}{4} \theta \Big|_0^{2\pi}$$

$$\to \frac{625}{4}(2\pi) - \cancel{\frac{625}{4}(0)}$$

$$\boxed{V = \frac{625\pi}{2} u^3}$$

I.P. : $x^2 + y^2 = 25$

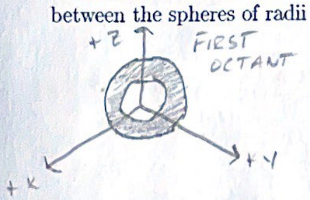


So, $\int_0^{2\pi} \int_0^5 (\text{Top} - \text{Bottom})\, dy\, dx$

Rectangular to Polar
let $x^2 + y^2 = r^2$
$0 \le \theta \le 2\pi$
$0 \le r \le 5$

Formula
$V = \int_R \int f(r, \theta)\, r\, dr\, d\theta$

Problem 6. Evaluate the following integral over D which is the region in the first octant (where x, y, and z are positive) between the spheres of radii 1 and 2 centered at the origin.



$$\iiint_D \sqrt{x^2 + y^2 + z^2}\, dV =$$

$$\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_1^2 (\rho^2)^{1/2} \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta = \int_0^{\pi/2} \int_0^{\pi/2} \int_1^2 \rho^3 \sin\phi\, d\rho\, d\phi\, d\theta$$

$$\to \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \sin\phi \left[\frac{\rho^4}{4} \right]_1^2 d\phi\, d\theta = \left(\frac{2^4}{4} - \frac{1^4}{4} \right) = \left(4 - \frac{1}{4}\right) = \frac{15}{4}$$

$$\to \int_0^{\frac{\pi}{2}} \frac{15}{4} \left[-\cos\phi \right]_0^{\frac{\pi}{2}} d\theta = -\cos\frac{\pi}{2} - (-\cos(0)) = 0 - (-1) = +1$$

$$\to \int_0^{\frac{\pi}{2}} \frac{15}{4} d\theta = \frac{15}{4}\left[\theta \right]_0^{\frac{\pi}{2}} = \frac{15}{4}\left[\frac{\pi}{2} - 0 \right]$$

$$\boxed{= \frac{15\pi}{8} \text{ Answers}}$$

Converting from Rectangular to Spherical, $\rho^2 = x^2 + y^2 + z^2$ and add "$\rho^2 \sin\phi$" like found in the notes, integrate and answer!

$$\bar{x} = \frac{M_y}{m}\iint_R x\rho(x,y)\,dA \qquad \bar{y} = \frac{M_x}{m}\iint_R y\rho(x,y)\,dA$$

Problem 7. Determine the center of mass of the plate with density $\rho = y$ that is bounded by the semicircle $x^2 + y^2 = 9$ with $y \geq 0$.

let $x^2 + y^2 = r^2$

$x = r\cos\theta$

$y = r\sin\theta$

(Polar)

$r = 3$

$$m = \iint_R \rho(x,y)\,dA$$

$$m = \int_0^{\pi}\int_0^{3} r\sin\theta \, r\,dr\,d\theta$$

$$= \int_0^{\pi}\frac{r^3}{3}\sin\theta \Big|_0^3 = \frac{3^3}{3} - \frac{0^3}{3} = [9]$$

$$= \int_0^{\pi}\sin\theta\,d\theta\,[9]$$

$$= 9\left[-\cos\theta\Big|_0^{\pi}\right] = [-(-1-1)]9$$

$$= 2[9]$$

$m = 18$

Center of mass:

$$M_x = \frac{1}{18}\int_0^{\pi}\int_0^{3}\cos\theta\sin\theta\, r^3\,dr\,d\theta$$

$$= \frac{1}{18}\int_0^{\pi}\cos\theta\sin\theta\,d\theta \cdot \int_0^3 r^3\,dr = \left[\frac{r^4}{4}\right]_0^3 = \frac{3^4}{4} - 0 = \frac{81}{4}$$

$M_x = 0$

$$= \frac{1}{18}\left[-\sin\theta\cos\theta\right]_0^{\pi} = -\cancel{\sin\pi}\cos\pi - (-\cancel{\sin 0}\cos 0) = \frac{1}{18}\left[\frac{81}{4}(0)\right]$$

$$M_y = \frac{1}{18}\int_0^{\pi}\int_0^{3}\sin^2\theta\, r^3\,dr\,d\theta = \frac{1}{18}\left[\int_0^{\pi}\sin^2\theta\,d\theta \cdot \int_0^3 r^3\,dr\right] = \frac{1}{18}\left[\frac{1-\cos(2x)}{2}\Big|_0^{\pi} \cdot \frac{r^4}{4}\Big|_0^3\right]$$

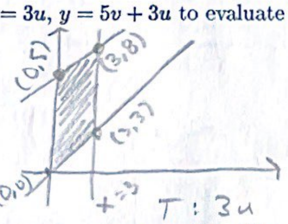
$$\frac{1-\cos(2\pi)}{2} - \frac{1-\cos 2(0)}{2}$$

$$\frac{1}{18}\left[\frac{81}{4}\cdot\emptyset\right]$$

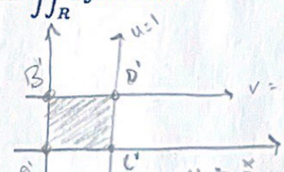
$$= \frac{1}{18}\left[\frac{81}{4}\cdot\frac{\pi}{2}\right] = \frac{9\pi}{16} = M_y$$

So $\boxed{\text{Center point } \left(0, \frac{9\pi}{16}\right)}$

Problem 8. Use the change of variables $x = 3u$, $y = 5v + 3u$ to evaluate the integral $\iint_R xy\,dA$ where $R = \{(x,y)\colon 0 \leq x \leq 3, x \leq y \leq x+5\}$.



$\rightarrow$



$T: 3u$

$y = 5v + 3u$

$u = \frac{x}{3}$

$v = \frac{y-3u}{5}$

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v}\end{vmatrix} = \begin{vmatrix}3 & 0 \\ 3 & 5\end{vmatrix}$$

$= 15$

So, $dy\,dx = 15\,du\,dv$

$$\iint_R xy\,dA = \iint_{R'} 3u(5v+3u)|15|\,du\,dv \rightarrow \int_0^1\int_0^1 3u(5v+3u)15\,du\,dv$$

$= 45u(5v+3u)$

$= 225uv + 135u^2$

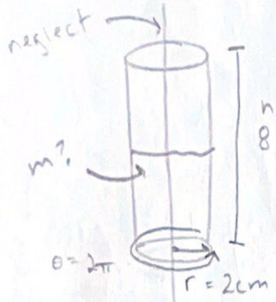
$$= \int_0^1 \frac{225vu^2}{2} + \frac{135u^3}{3}\Big|_0^1 dv = \frac{225v(1)^2}{2} + \frac{135(1)^3}{3} - \left(\cancel{\frac{225v(0)}{2}} + \cancel{\frac{135(0)^3}{3}}\right)$$

$$= \int_0^1 \frac{225v}{2} + 45\,dv = \frac{225v^2}{4} + 45v\Big|_0^1 = \frac{225(1)^2}{4} + 45(1) - \left(\cancel{\frac{225(0)^2}{4}} + 45(0)\right)$$

$$= \boxed{\frac{405}{4}}$$

* Used Jacobian to find scale factor and solved the intergal

Problem 9. (BONUS) A right circular cylinder with a height of 8 cm and radius 2 cm is filled with water. A heated filament running through the center of the cylinder produces a variable density in the water given by $\rho = \left(1 - 0.05e^{-0.01r^2}\right)$ g/cm^3 where r is the distance away from the center of the cylinder. Assuming that the filament is small enough to neglect its volume, determine the mass of the water in the cylinder.



$$D = \{(r,\theta,z) \mid \alpha \le \theta \le \beta,\ g(\theta) \le r \le h(\theta),\ G(r,\theta) \le z \le H(r,\theta)\}$$

$$\int_\alpha^\beta \int_{g(\theta)}^{h(\theta)} \int_{G(r,\theta)}^{H(r,\theta)} f(r,\theta,z)\, r\, dz\, dr\, d\theta.$$

$$D = \{(r,\theta,z) \mid 0 \le r \le 2,\ 0 \le \theta \le 2\pi,\ 0 \le z \le 8\}$$

$$\int_0^{2\pi}\int_0^{2}\int_0^{8} \left(1 - 0.05e^{-0.01r^2}\right) r\, dz\, dr\, d\theta$$

Integrate

$$\int_0^{2\pi}\int_0^{2}\int_0^{8} \left(1 - 0.05e^{-0.01r^2}\right) r\, dz\, dr\, d\theta$$

1) $\left(1 - 0.05e^{-0.01r^2}\right) r \int dz = [z]_0^8 = (8-0) = 8$

2) $8\int_0^2 r - 0.05e^{-0.01r^2}\, dr$

$$= 8\left[\frac{r^2}{2} - \frac{0.05}{0.02} e^{-0.01r^2}\right]_0^2$$

$$= \left(\frac{2^2}{2} - 2.5e^{-0.01(2)^2}\right) - \left(\cancel{\frac{0^2}{2}}^{\,0} - 2.5 e^{-0.01(0)^2}\right) \quad +2.5 \quad 1$$

$$= \underline{4.5 - 2.5e^{-0.04}}$$

3) $8\int_0^{2\pi}\left(4.2 - 2.5e^{-0.04}\right)\int_0^{2\pi} d\theta$

$$= 4.2 - 2.5e^{-0.04}\,[\theta]_0^{2\pi} = \left(4.2 - 2.5e^{-0.04}\right)(2\pi - 0)$$

$$8\left[9\pi - 5\pi e^{-0.04}\right] = 72\pi - 40\pi e^{-0.04}$$

or

$$72\pi - \frac{40\pi}{e^{0.04}} = 105.45$$