MATH 311, HW 8

Due 11:00pm on Thursday April 6, 2023

Remember to scan/upload exactly the 6 pages of this template.

Problem 1. [10pts]

1. (3pts) Find the point on the line $y = 2x$ that is closest to the point $(7, 2)$.

Ans:	$\left(\frac{11}{5}, \frac{22}{5}\right)$

Slope to $y = 2x$ is $-\frac{1}{2}$

$y - 2 = \left(-\frac{1}{2}\right)(x-7)$

$\rightarrow y - 2 = -\frac{1}{2}x + \frac{7}{2}$

$y = -\frac{1}{2}x + \frac{7}{2} + 2$

$y = -\frac{1}{2}x + \frac{11}{2}$

Find Intersection

$2x = -\frac{1}{2}x + \frac{11}{2}$

$2x + \frac{1}{2}x = \frac{11}{2}$

$\frac{5}{\cancel{2}}x = \frac{11}{\cancel{2}}$

$\rightarrow 5x = 11 \rightarrow x = \frac{11}{5}$

So, $y = 2x$, $y = 2\left(\frac{11}{5}\right)$

$y = \frac{22}{5}$

2. (3pts) Find the distance from the point $(7, 2)$ to the line $y = 2x$. $y - 2x = 0$

x y

Ans:	$D = \frac{12}{\sqrt{5}} = \frac{12\sqrt{5}}{5}$

$D = \frac{|(2) - 2(7)|}{\sqrt{(-2)^2 + (1)^2}} \rightarrow \sqrt{4+1}$

$= \frac{-12}{\sqrt{5}} = \frac{12\sqrt{5}}{5}$

3. (4pts) Find the point on the line $y = 2x + 1$ that is closest to the point $(7, 2)$.

Ans:	$\left(\frac{9}{5}, \frac{23}{5}\right)$

1. $y - 2 = -\frac{1}{2}(x - 7)$

$\rightarrow y - 2 = -\frac{1}{2}x + \frac{7}{2}$

$y = -\frac{1}{2}x + \frac{11}{2}$

$2x + 1 = -\frac{1}{2}x + \frac{11}{2}$

$2x + \frac{1}{2}x = \frac{11}{2} - 1$

$\frac{5}{2}x = \frac{9}{2} \rightarrow x = \frac{9}{5}$

$y = 2\left(\frac{9}{5}\right) + 1$

$y = \frac{23}{5}$

Problem 2. [10pts] Show that if S is a subspace of a vector space V, then the orthogonal complement $S^\perp$ is also a vector space.

① $S^\perp$ is non-empty

It is a vector space since it is contained in $S^\perp$

② $(v_1+v_2)\cdot w = e(v_1\cdot w)+(v_2\cdot w)=0$

Here, v_1+v_2 is contained in $S^\perp$

③ It is closed under scalar multiplication

· If v is in W, then αv is in W.

Problem 3. [10pts] Consider the weights vector $r_1=\frac{1}{2}$, $r_2=\frac{1}{3}$, $r_3=\frac{1}{6}$ to define $\langle\cdot,\cdot\rangle_{\frac{1}{2},\frac{1}{3},\frac{1}{6}}$ the inner product with weight in $\mathbb{R}^3$ (as seen in class). Let

$$x=\begin{pmatrix}-6\\3\\2\end{pmatrix},\qquad y=\begin{pmatrix}2\\4\\6\end{pmatrix}$$

1. (2pts) Show that x and y are orthogonal with respect to this inner product.

$\langle -6,3,2\rangle \cdot \langle 2,4,6\rangle$

$-12+12+12$

$-12\cdot\frac{1}{2}=-6 \quad 12\times\frac{1}{3}=4 \quad 12\times\frac{1}{6}=2$

$-6+4+2=0$

Since the answer is 0, x and y are orthogonal with respect to the inner product.

2. (8pts) Compute the norms $\|x\|$, $\|y\|$ with respect to this inner product.

$\|x\|=$	$\sqrt{\frac{65}{3}}$
$\|y\|=$	$\sqrt{\frac{40}{3}}$

$\sqrt{\frac{(-6)^2}{2}+\frac{(3)^2}{3}+\frac{(2)^2}{6}}=\sqrt{18+3+\frac{2}{3}}=\sqrt{\frac{65}{3}}$

$\sqrt{\frac{2^2}{2}+\frac{4^2}{3}+\frac{6^2}{6}}=\sqrt{\frac{40}{3}}$

Problem 4. [20pts] Given

$$A = \begin{pmatrix} 4 & 1 & 2 \\ -2 & 0 & -1 \\ -2 & 1 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 2 & -2 \\ 1 & -1 & -3 \end{pmatrix}$$

Consider the inner product in the vector space of 3×3 matrices $\mathbb{R}^{3\times 3}$, as seen in class. We can define a norm $\|\cdot\|_F$ for matrices using this inner product (this is known as the Frobenius norm). Determine the value of each of the following:

1. (4pts) $\langle A, B\rangle$ $\quad \cancel{4}+\cancel{4}-4+\cancel{2}-2-\cancel{1}-3=0$
2. (2pts) $\|A\|_F$ $\quad \sqrt{16+1+4+4+1+4+2} = \sqrt{32}$
3. (2pts) $\|B\|_F$ $\quad \sqrt{1+4+4+4+4+1+1+9} = \sqrt{28}$
4. (4pts) $\|A+B\|_F$ $\begin{pmatrix} 4 & 1 & 2 \\ -2 & 0 & 1 \\ -2 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 2 \\ 2 & 2 & -2 \\ 1 & -1 & -3 \end{pmatrix} = \begin{pmatrix} 5 & 1 & 4 \\ 0 & 2 & -3 \\ -1 & 0 & -2 \end{pmatrix} = \sqrt{25+1+16+4+9+1+4}$
5. (4pts) Find the 'angle' θ between A and B.
6. (4pts) Find the projection p of A onto B.

$\langle A, B\rangle =$	0
$\|A\| =$	$\sqrt{32}$
$\|B\| =$	$\sqrt{28}$
$\|A+B\| =$	$\sqrt{60}$
$\theta =$	$90°$

$\dfrac{A\cdot B}{\|A\|\|B\|} = 0 \qquad \cos(\theta)=0 \qquad \cos(0)^{-1} = 90$

$p =$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Problem 5. [20pts] Consider the vector space of continuous functions on $\underset{a,b}{[0,1]}$ with the inner product explained in class, and consider the functions

$$f(x) = e^x, \qquad g(x) = e^{-x}$$

Using the inner product we can define a norm $\|\cdot\|$ for functions. Determine the value of each of the following:

$\langle f,g\rangle = \int_a^b f(x)g(x)\,dx \to \int_0^1 e^x \cdot e^{-x}\,dx = [x]_0^1 = 1-0 = 1$

$\langle f,f\rangle = \int_a^b (f(x))^2\,dx \to \int_0^1 (e^x)^2\,dx \to \int_0^1 e^{2x}\,dx \to \left[\frac{1}{2}e^{2x}\right]_0^1 \to \frac{e^2}{2}-\frac{1}{2}$

$\langle g,g\rangle = \int_a^b (g(x))^2\,dx \to \int_0^1 (e^{-x})^2\,dx \to \int_0^1 e^{-2x}\,dx \to -\frac{1}{2}e^{-2x} \to \left[-\frac{1}{2e^{2x}}\right]_0^2 \quad -\frac{1}{2e^2}+\frac{1}{2} \quad \underline{e^{-2}}$

$\langle f+g\rangle = \int_0^1 (e^x+e^{-x})^2\,dx \to \int_0^1 e^{2x}+e^{-2x}+2\,dx$

$\to \frac{1}{2}(e^2-1)+2+\frac{1}{2}(1-e^{-2})$

$= \sqrt{2+\frac{1}{2}(e^2-e^{-2}}$

1. (4pts) $\langle f, g\rangle$
2. (2pts) $\|f\|$
3. (2pts) $\|g\|$
4. (4pts) $\|f+g\|$
5. (4pts) Find the 'angle' θ between f and g.
6. (4pts) Find the projection p of f onto g.

$\langle f, g\rangle =$	1
$\|f\| =$	$\sqrt{\frac{e^2-1}{2}}$
$\|g\| =$	$\sqrt{\frac{-e^2+1}{2}}$
$\|f+g\| =$	$\sqrt{\frac{5}{2}(1-e^{-2})}$
$\theta =$	about $31.7°$

$\frac{\langle f(x), g(x)\rangle}{\|f(x)\|\,\|g(x)\|}$

$\cos^{-1}\left(\frac{2e}{e^2-1}\right)$

$p =$	$\frac{2}{(1-e^{-2})e^x}$

6. $\frac{\langle f,g\rangle}{\|g\|^2}\cdot g = \frac{1}{\frac{1}{2}(1-e^{-2})}\cdot e^{-x} \to \frac{2}{e^x(1-e^{-2})}$

Problem 6. [10pts]

1. (7pts) Find the best least squares fit by a linear function to the data

x	4	2	1	0
y	-1	0	1	2

$y =$ $y = -\frac{26}{35}x + \frac{9}{5}$

$$1\cdot\alpha + 4\beta = -1$$
$$1\cdot\alpha + 2\beta = 0$$
$$1\cdot\alpha + 1\beta = 1$$
$$1\cdot\alpha + 0\beta = 2$$
$$\rightarrow \begin{pmatrix} 1 & 4 \\ 1 & 2 \\ 1 & 1 \\ 1 & 0 \end{pmatrix}$$

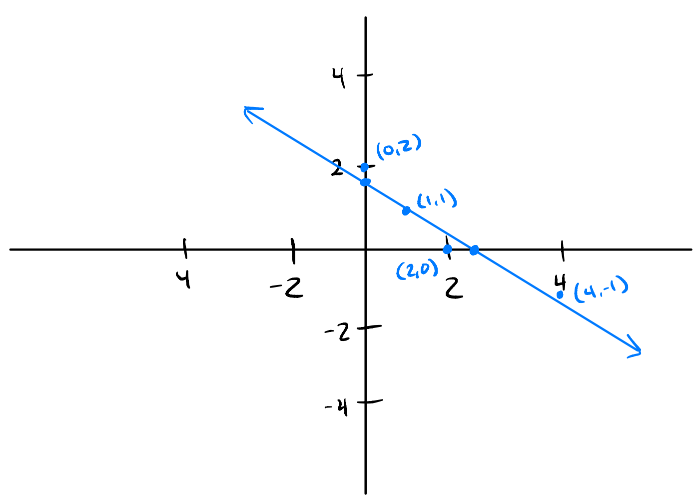
$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 4 & 2 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 4 \\ 1 & 2 \\ 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{bmatrix} 4 & 7 \\ 7 & 21 \end{bmatrix}^{-1}$$

$$= \frac{1}{35}\begin{pmatrix} 21 & -7 \\ -7 & 4 \end{pmatrix} =$$

$$\frac{1}{35}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 4 & 2 & 1 & 0 \end{pmatrix}\begin{pmatrix} -1 \\ 0 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 9/5 \\ -26/35 \end{pmatrix}$$

$$y = -\frac{26}{35}x + \frac{9}{5}$$

2. (3pts) Plot your linear function, along with the data on a coordinate system.



Problem 7. [20pts] Given the basis $\left\{ \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right\}$ for $\mathbb{R}^3$, use the Gram–Schmidt process to obtain an orthonormal basis $\{u_1, u_2, u_3\}$.

	(4pts)	(8pts)	(8pts)
Ans:	$u_1 = \frac{1}{3}\begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}$	$u_2 = \frac{1}{3}\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$	$u_3 = \frac{1}{3}\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$

① $\mu_1 = \frac{1}{\|v_1\|}(v_1)$ find $p_1 = \langle v_2, \mu_1 \rangle \langle \mu_1 \rangle$

$\|v_1\| = 3$

$\frac{1}{3}\begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}$ $\mu_1 = \frac{1}{3}\begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}$

② $\mu_2 = v_2 - \frac{\langle v_3, \mu_1 \rangle}{(v_1)^2} \cdot \mu_1 = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \cdot \frac{-4+6+4}{9} \left\langle \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} \right\rangle$

$= \begin{pmatrix} 6/3 \\ 9/3 \\ 12/3 \end{pmatrix} - \begin{pmatrix} -4/3 \\ 4/3 \\ 2/3 \end{pmatrix} = \begin{pmatrix} 10/3 \\ 5/3 \\ 10/3 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$

③ $\mu_3 = v_3 - \frac{\langle v_3, \mu_1 \rangle}{(v_1)^2} - v_1 \frac{\langle v_3, \mu_2 \rangle}{\langle v_2 \rangle^2} v_2$

$\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} - \frac{1}{3}\begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} - 2\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$

$\frac{1}{3}\begin{pmatrix} 3 \\ 6 \\ 3 \end{pmatrix} - \frac{1}{3}\begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} - \begin{pmatrix} 4 \\ 6 \\ 8 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$