MATH 311, HW 9

Due 11:00pm on Thursday April 13, 2023

Remember to scan/upload exactly the 8 pages of this template, in order.

Since each problem use the answer from the previous problems. **Please verify your answer before proceeding to the next problem.** Otherwise, you may be deducted several marks.

For Problems 1 to 5 we consider the matrix $A = \begin{pmatrix} 2 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{pmatrix}$.

Problem 1. [7pts] Find the characteristic polynomial of A.

$p_A(\lambda) =$	$\lambda^3 + 2\lambda^2 + \lambda - 2$

① $A = \begin{pmatrix} 2-\lambda & 2 & 1 \\ 0 & 1-\lambda & 2 \\ 0 & 0 & -1-\lambda \end{pmatrix}$

② $2-\lambda((1-\lambda)(-1-\lambda)-0) - 2(0) + 1(0)$

$\rightarrow 2-\lambda(-1-\lambda+\lambda+\lambda^2)$

$\rightarrow 2-\lambda(\lambda^2-1)$

$\rightarrow \underline{2\lambda^2 - 2 + \lambda^3 + \lambda}$

$\rightarrow (\lambda^3 + 2\lambda^2) + (\lambda - 2)$

$= -\lambda^2(\lambda - 2) + 1(\lambda - 2)$

λ

Problem 2. [9pts] Find the eigenvalues $\lambda_1 \geq \lambda_2 \geq \lambda_3$ of A. (When providing your answer write the eigenvalues in decreasing order).

Ans:	$\lambda_1 =$ 2	$\lambda_2 =$ 1	$\lambda_3 =$ -1

$(\lambda^3 + 2\lambda^2) + 1\lambda - 2)$

$\rightarrow -\lambda^2(\lambda - 2) + 1(\lambda - 2)$

$(\lambda - 2)(\lambda - 1)(\lambda + 1)$

$\lambda_1 = 2$

$\lambda_2 = 1$

$\lambda_3 = -1$

Problem 3. [12pts] Find the **unit** (with norm 1) eigenvectors u_1, u_2, u_3 corresponding to each of the eigenvalues $\lambda_1 \geq \lambda_2 \geq \lambda_3$ respectively.

Ans:	$u_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$	$u_2 = \frac{1}{\sqrt{5}} \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$	$u_3 = \frac{1}{\sqrt{19}} \begin{pmatrix} 1 \\ -3 \\ 3 \end{pmatrix}$

FIND NULL SPACE

For $\lambda_1 = 2$

$$A = \begin{pmatrix} 2-\lambda & 2 & 1 \\ 0 & 1-\lambda & 2 \\ 0 & 0 & -1-\lambda \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 2 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & -3 \end{pmatrix} \xrightarrow{REF} \begin{array}{c} \begin{matrix} \alpha & y & z \end{matrix} \\ \begin{bmatrix} \boxed{0} & 1 & 1/2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\rightarrow \quad \begin{matrix} y + \frac{1}{2}z = 0 \\ z = 0 \\ x = \alpha \end{matrix} \quad \rightarrow \quad \begin{matrix} x = \alpha \\ y = 0 \\ z = 0 \end{matrix} \quad \text{So, } \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\} \quad x = \frac{1}{\|x\|} x \quad x = \frac{1}{1}$$

For $\lambda_2 = 1$

$$A = \begin{pmatrix} 2-1 & 2 & 1 \\ 0 & 1-1 & 2 \\ 0 & 0 & -1-1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & -2 \end{pmatrix} \xrightarrow[N]{\substack{REF \\ R_3 = R_3 + R_2}} \begin{array}{c} \begin{matrix} x & y & z \end{matrix} \\ \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\rightarrow \quad \begin{matrix} x + 2y + z = 0 \\ 2z = 0 \\ y = \text{FREE VAR.} \end{matrix} \quad \rightarrow \quad \begin{matrix} x = -2\alpha \\ y = \alpha \\ z = 0 \end{matrix} \quad u_1 = \frac{1}{\|x\|} x \quad \operatorname{Span}\left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \right\} \quad \sqrt{2^2 + 1^2} = \sqrt{5} \quad \text{So, } \frac{1}{\sqrt{5}} \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$$

For $\lambda_3 = -1$

$$A = \begin{pmatrix} 2-(-1) & 2 & 1 \\ 0 & 1-(-1) & 2 \\ 0 & 0 & -1-(-1) \end{pmatrix} \xrightarrow{N} \begin{array}{c} \begin{matrix} x & y & \overset{\alpha}{z} \end{matrix} \\ \begin{pmatrix} 3 & 2 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{pmatrix} \end{array} \left\{ \begin{matrix} 3x + 2y + z = 0 \\ 2y + 2z = 0 \\ z = \alpha \end{matrix} \right. \rightarrow \frac{2y}{2} = -\frac{2\alpha}{2} \rightarrow y = -\alpha$$

$$3x = -2(-\alpha) - (\alpha) \quad x = \frac{\alpha}{3}$$

$$\operatorname{Span}\left\{ \begin{pmatrix} \alpha/3 \\ -\alpha \\ \alpha \end{pmatrix} \right\} = \operatorname{Span}\left\{ \begin{pmatrix} 1/3 \\ -1 \\ 1 \end{pmatrix} \right\} \quad \sqrt{\left(\tfrac{1}{3}\right)^2 - (1)^2 + (1)^2} = \frac{1}{9} + 2 = \sqrt{\frac{19}{9}} \rightarrow \frac{1}{\frac{\sqrt{19}}{3}} \cdot \frac{3}{1} = \frac{3}{\sqrt{19}}$$

$$\frac{3}{\sqrt{19}} \begin{pmatrix} 1/3 \\ -1 \\ 1 \end{pmatrix} \rightarrow \frac{1}{\sqrt{19}} \begin{pmatrix} 1 \\ -3 \\ 3 \end{pmatrix} \qquad \underset{\wedge}{\sqrt{19}} \qquad \rightarrow \begin{pmatrix} 1/3 \\ -1 \\ 1 \end{pmatrix}$$

Problem 4. [7pts] Factor the matrix A into a product XDX^{-1}, where D is diagonal.

Ans:	$X = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 3 \end{pmatrix}$	$D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	$X^{-1} = \begin{pmatrix} 1 & 2 & 5/3 \\ 0 & 1 & 1 \\ 0 & 0 & 1/3 \end{pmatrix}$

D = EIGENVALUES

$$D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$X = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 3 \end{pmatrix}$$

$$X^{-1} = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 3 \end{pmatrix}^{-1}$$

① $\det \left(1(3-0) - ((-2)(0)) + 1(0)\right)$
$= 3 - 0 - 0 = \frac{1}{3}$

② $3 \quad -0 \quad +0$
$+6 \quad +3 \quad -0$
$+|+6-1| \quad -(-3) \quad +1$
$= \begin{pmatrix} 3 & 0 & 0 \\ +6 & 3 & 0 \\ 5 & 3 & 1 \end{pmatrix}$

③ TRANSPOSE $\frac{1}{3}\begin{pmatrix} 3 & +6 & 5 \\ 0 & 3 & 3 \\ 0 & 0 & 1 \end{pmatrix}$

$$X^{-1} = \begin{pmatrix} 1 & +2 & 5/3 \\ 0 & 1 & 1 \\ 0 & 0 & 1/3 \end{pmatrix}$$

Problem 5. [15pts] Compute the following matrices:

1.	(5pts) $A^5 =$	$\begin{pmatrix} 32 & 62 & 51 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{pmatrix}$

$A^5 = (XDX^{-1})^5$
$\rightarrow (XDX^{-1})(XDX^{-1})(XDX^{-1})(XDX^{-1})(XDX^{-1})$
$\rightarrow (XD^5X^{-1}) = A^5$

$$X \begin{pmatrix} \lambda_1^5 & 0 & 0 \\ 0 & \lambda_2^5 & 0 \\ 0 & 0 & \lambda_3^5 \end{pmatrix} X^{-1} \rightarrow X \begin{pmatrix} 32 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} X^{-1}$$

Compute

①
$$\underset{X}{\begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 3 \end{pmatrix}} \underset{D^5}{\begin{pmatrix} 32 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}} = \underset{XD}{\begin{pmatrix} 32 & -2 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & -3 \end{pmatrix}} \cdot \underset{X^{-1}}{\begin{pmatrix} 1 & 2 & 5/3 \\ 0 & 1 & 1 \\ 0 & 0 & 1/3 \end{pmatrix}} =$$

$$\begin{pmatrix} 32 & 62 & 51 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{pmatrix} \Leftarrow \begin{matrix} 32 & 64-2 & 32(\frac{5}{3}) - 2 - \frac{1}{3} \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{matrix}$$

$$A = (XDX^{-1})^2 + (XDX^{-1}) + I$$
$$= X(D^2 + D + I)X^{-1}$$

(5pts)	
2. $A^2 + A + 1 =$	$\begin{pmatrix} 7 & 8 & 20/3 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

PLUG IN $D^2 = \begin{pmatrix} \lambda_1^2 + \lambda + 1 & 0 & 0 \\ 0 & \lambda_2^2 + \lambda_2 + 1 & 0 \\ 0 & 0 & \lambda_3^2 + \lambda_3 + 1 \end{pmatrix} \rightarrow \begin{pmatrix} 7 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

$$\begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 7 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 7 & -6 & 3 \\ 0 & 3 & -9 \\ 0 & 0 & 9 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 5/3 \\ 0 & 1 & 1 \\ 0 & 0 & 1/3 \end{pmatrix} = \begin{pmatrix} 7 & 8 & 20/3 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

(5pts)	
3. $e^A =$	

$$\begin{pmatrix} 1/3 & -2 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} e^{-1} & 0 & 0 \\ 0 & e^{1} & 0 \\ 0 & 0 & e^{-2} \end{pmatrix}$$

$$= \begin{pmatrix} 1/3e & -2e & e^2 \\ -1/e & e & 0 \\ 1/e & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & 5/3 \end{pmatrix}$$

$$= \begin{pmatrix} e^2 & 2e^2 - 2e & \frac{5e^2 - 5e}{3} \\ 0 & e & \frac{e^2 - 1}{e} \\ 0 & 0 & 1/e \end{pmatrix}$$

For Problems 6 to 10 we consider the matrix $A = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 3 \\ 1 & 1 & -1 \end{pmatrix}$.

Problem 6. [10pts] Find the characteristic polynomial of A.

$p_A(\lambda) =$	$-\lambda^3 + \lambda^2 + 4\lambda + 4$

$$\begin{pmatrix} 1-\lambda & 0 & 0 \\ -2 & 1-\lambda & 3 \\ 1 & 1 & -1-\lambda \end{pmatrix}$$

det $= 1-\lambda((1-\lambda)(-1-\lambda) - 3)$

$\rightarrow 1-\lambda(-1-\cancel{\lambda}+\cancel{\lambda}+\lambda^2-3)$

$\rightarrow 1-\lambda(\lambda^2-4) =$

$\rightarrow \lambda^2-4-\lambda^3+4\lambda$

$\rightarrow (-\lambda^3+\lambda^2)+(4\lambda-4)$

Problem 7. [9pts] Find the eigenvalues $\lambda_1 \geq \lambda_2 \geq \lambda_3$ of A. (When providing your answer write the eigenvalues in decreasing order).

Ans:	$\lambda_1 = 2$	$\lambda_2 = 1$	$\lambda_3 = -2$

$(-\lambda^3+\lambda^2)+(4\lambda-4)$

$-\lambda^2(\lambda-1)+4(\lambda-1)$

$(-\lambda^2+4)(\lambda-1)$

$-\lambda^2+4=0$ $\qquad \lambda = 1$

$\sqrt{4} = \sqrt{\lambda^2}$

$\lambda_2 = -2$

$\lambda_3 = 2$

Problem 8. [9pts] Find the **unit** (with norm 1) eigenvectors u_1, u_2, u_3 corresponding to each of the eigenvalues $\lambda_1 \geq \lambda_2 \geq \lambda_3$ respectively.

Ans:	$u_1 = \frac{1}{\sqrt{10}}\begin{pmatrix}0\\3\\1\end{pmatrix}$	$u_2 = \frac{1}{\sqrt{14}}\begin{pmatrix}3\\1\\2\end{pmatrix}$	$u_3 = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\-1\\1\end{pmatrix}$

$\underline{\lambda_1 = 2}$

$$A = \begin{pmatrix}1-\lambda & 0 & 0\\ -2 & 1-\lambda & 3\\ 1 & 1 & -1-\lambda\end{pmatrix} \to \begin{pmatrix}-1 & 0 & 0\\ -2 & -1 & 3\\ 1 & 1 & -3\end{pmatrix} \xrightarrow{REF} \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & -3\\ 0 & 0 & 0\end{pmatrix}\end{array}$$

$$\to \left.\begin{array}{r} x = 0\\ y - 3z = 0\\ z = \alpha \end{array}\right\} \to \begin{array}{l} x = 0\\ y = 3\alpha\\ z = \alpha \end{array} \qquad \cancel{x = -3\alpha} \qquad \operatorname{Span}\left\{\begin{pmatrix}0\\-3\\1\end{pmatrix}\right\} \qquad \sqrt{3^2+3^2+1} \quad 9+1 \quad \sqrt{10}$$

$\lambda_2 = 1$

$$A = \begin{pmatrix}0 & 0 & 0\\ -2 & 0 & 3\\ 1 & 1 & -2\end{pmatrix} \xrightarrow{REF} \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{pmatrix}1 & 0 & -3/2\\ 0 & 1 & -1/2\\ 0 & 0 & 0\end{pmatrix}\end{array} \to \begin{array}{l} x = 3/2\, z\\ y = 1/2\, z\\ z = \alpha \end{array} \qquad \begin{array}{l} x = 3/2\,\alpha\\ y = 1/2\,\alpha\\ z = \alpha \end{array}$$

$$\begin{pmatrix}3/2\\1/2\\1\end{pmatrix} = \begin{pmatrix}3\\1\\2\end{pmatrix} \qquad 9+1+4 = \sqrt{14}$$

$\underline{\lambda_3 = -2}$

$$A = \begin{pmatrix}3 & 0 & 0\\ -2 & 3 & 3\\ 1 & 1 & 1\end{pmatrix} \to \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 1\\ 0 & 0 & 0\end{pmatrix}\end{array} \quad \left.\begin{array}{r} x = 0\\ y + z = 0\\ z = \alpha \end{array}\right\} \quad \begin{array}{l} x = 0\\ y = \alpha\\ z = \alpha \end{array} \quad \operatorname{Span}\left\{\begin{pmatrix}0\\-1\\1\end{pmatrix}\right\}$$

$$\frac{1}{\sqrt{2}}\begin{pmatrix}0\\-1\\1\end{pmatrix}$$

Problem 9. [7pts] Factor the matrix A into a product XDX^{-1}, where D is diagonal.

Ans:	$X = \begin{pmatrix} 0 & 3/2 & 0 \\ -3 & 1/2 & -1 \\ 1 & 1 & 1 \end{pmatrix}$	$D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$	$X^{-1} = \begin{pmatrix} -1/4 & 1/4 & 1/4 \\ 2/3 & 0 & 0 \\ -5/12 & -1/4 & 3/4 \end{pmatrix}$

$$X^{-1} = \begin{pmatrix} -3 & 3/4 & 0 \\ 3 & 1/2 & -1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} -1/4 & 1/4 & 1/4 \\ 2/3 & 0 & 0 \\ -5/12 & -1/4 & 3/4 \end{pmatrix}$$

Problem 10. [15pts] Compute the following matrices:

	(5pts)	
1.	$A^5 =$	$\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 3 \\ 1 & 1 & -1 \end{pmatrix}$

$$X \begin{pmatrix} \lambda_1^5 & & \\ & \lambda_2^5 & \\ & & \lambda_3^5 \end{pmatrix} X^{-1}$$

<u>Plug in</u>

$$X \begin{pmatrix} 32 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -32 \end{pmatrix} X^{-1}$$

$$\begin{pmatrix} 0 & 3/2 & 0 \\ 3 & 1/2 & -1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 32 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -32 \end{pmatrix} \begin{pmatrix} -1/4 & 1/4 & 1/4 \\ 2/3 & 0 & 0 \\ -5/12 & -1/4 & 3/4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 3 \\ 1 & 1 & -1 \end{pmatrix}$$

$2, 1, -2$

(5pts)	
2. $A^2 + A + 1 =$	$\begin{pmatrix} 1 & 0 & 0 \\ -\frac{37}{6} & 1 & 3 \\ \frac{31}{6} & 1 & -1 \end{pmatrix}$

$= X(D^2 + D + I)X^{-1}$

$$\to X \begin{pmatrix} \lambda_1^2 + \lambda_1 + 1 & 0 & 0 \\ 0 & \lambda_2^2 + \lambda_2 + 1 & 0 \\ 0 & 0 & \lambda_3^2 + \lambda_3 + 1 \end{pmatrix} X^{-1}$$

$$\to X \begin{pmatrix} 4+2+1 & 0 & 0 \\ 0 & 1+1+1 & 0 \\ 0 & 0 & 4+1+1 \end{pmatrix} X^{-1}$$

$$\to \begin{pmatrix} 0 & 3/2 & 0 \\ 3 & 1/2 & -1 \\ 1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 7 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} \cdot \begin{pmatrix} -1/4 & 1/4 & 1/4 \\ 2/3 & 0 & 0 \\ -5/12 & -1/4 & 3/4 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ -37/6 & 1 & 3 \\ 31/6 & 1 & -1 \end{pmatrix}$$

(5pts)	
3. $e^A =$	$e \quad 0 \quad 0$ $-4e^4+4e^3+5$

$$\begin{pmatrix} 0 & 3/2 & 0 \\ 3 & 1/2 & -1 \\ 1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} e^2 & 0 & 0 \\ 0 & e^1 & 0 \\ 0 & 0 & e^{-2} \end{pmatrix}$$

$$\begin{pmatrix} e & 0 & 0 \\ -4e^4 + 4e^3 + 5 & \frac{3e^4+1}{4e^2} & \frac{3e^4-3}{4e^2} \\ \frac{-3e^4+8e^3-3}{12e^2} & \frac{e^4-1}{4e^2} & \frac{e^4+3}{4e^2} \end{pmatrix} = \begin{pmatrix} 0 & 3/2\,e & 0 \\ 3e^2 & e/2 & -1/e^2 \\ e^2 & e & 1/e^2 \end{pmatrix} \begin{pmatrix} -1/4 & 1/4 & 1/4 \\ 2/3 & 0 & 0 \\ -5/12 & -1/4 & 3/4 \end{pmatrix}$$

$$\begin{pmatrix} e & 0 & 0 \\ -4e^4 + 4e^3 + 5 & \frac{3e^4+1}{4e^2} & \frac{3e^4-3}{4e^2} \\ \frac{-3e^4+8e^3-3}{12e^2} & \frac{e^4-1}{4e^2} & \frac{e^4+3}{4e^2} \end{pmatrix}$$