MATH 311, HW 10

Due 11:00pm on Thursday April 20, 2023

Remember to scan/upload exactly the 6 pages of this template, in order.

Problem 1. [10 pts] Consider the scalar field $f : \mathbb{R}^3 \to \mathbb{R}$ such that $f(x) = \|x\|^7$. Compute the gradient of f.

$\nabla f(x) =$	$\begin{pmatrix} 7x_1 \|x\|^5 \\ 7x_2 \|x\|^5 \\ 7x_3 \|x\|^5 \end{pmatrix}$

$\nabla f(x) = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3} \right)$

$f(x) = \left(\sqrt{x_1^2 + x_2^2 + x_3^2}\right)^7$

① Compute Partial derivatives

$- \frac{\partial f}{\partial x_1}\left(\sqrt{x_1^2 + x_2^2 + x_3^2}\right)^7 = \frac{\partial f}{\partial x_1}\left(x_1^2 + x_2^2 + x_3^2\right)^{7/2}$

Chain Rule - $\frac{7}{\cancel{2}}\left(x_1^2 + x_2^2 + x_3^2\right)^{5/2} \cdot (\cancel{2}x)$

$\frac{\partial f}{\partial x_1} = 7x_1\left(x_1^2 + x_2^2 + x_3^2\right)^{5/2}$

$\frac{\partial f}{\partial x_2}\left(x_1^2 + x_2^2 + x_3^2\right)^{7/2} =$

$\frac{7}{\cancel{2}}\left(x_1^2 + x_2^2 + x_3^2\right)^{5/2} \cdot \cancel{2}x_2$

$= 7x_2\left(x_1^2 + x_2^2 + x_3^2\right)^{5/2}$

$\frac{\partial f}{\partial x_3} = 7x_3\left(x_1^2 + x_2^2 + x_3^2\right)^{5/2}$

Answer,

$7x_1\left(x_1^2 + x_2^2 + x_3^2\right)^{5/2}, 7x_2\left(x_1^2 + x_2^2 + x_3^2\right)^{5/2},$

$7x_3\left(x_1^2 + x_2^2 + x_3^2\right)^{5/2}$

Simplified

$\nabla f(x) = \begin{pmatrix} 7\|x\|^5 \cdot x_1 \\ 7\|x\|^5 \cdot x_2 \\ 7\|x\|^5 \cdot x_3 \end{pmatrix}$

Problem 2. [10 pts] Consider the scalar field $f : \mathbb{R}^3 \setminus \{\vec{0}\} \to \mathbb{R}$ such that $f(x) = \ln \|x\|$. Compute the gradient of f.

$\nabla f(x) =$	$\begin{pmatrix} \frac{x_1}{x_1^2 + x_2^2 + x_3^2} \\ \frac{x_2}{x_1^2 + x_2^2 + x_3^2} \\ \frac{x_3}{x_1^2 + x_2^2 + x_3^2} \end{pmatrix}$

$f\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \ln\left(\sqrt{x_1^2 + x_2^2 + x_3^2}\right)$

$\frac{\partial f}{\partial x_1}\left(\ln\left(\sqrt{x_1^2 + x_2^2 + x_3^2}\right)\right) = \frac{1}{2}\ln\left(x_1^2 + x_2^2 + x_3^2\right)$

$\ln(x)' = \frac{1}{x}$ so,

$\frac{\partial f}{\partial x_1} = \frac{1}{\cancel{2}} \cdot \frac{1}{x_1^2 + x_2^2 + x_3^2} \cancel{2}x_1 = \frac{x_1}{x_1^2 + x_2^2 + x_3^2}$

$\frac{\partial f}{\partial x_2} = \frac{1}{2} \cdot \frac{1}{x_1^2 + x_2^2 + x_3^2} 2x_2 = \frac{x_2}{x_1^2 + x_2^2 + x_3^2}$

$\frac{\partial f}{\partial x_3} = \frac{1}{2} \cdot \frac{1}{x_1^2 + x_2^2 + x_3^2} 2x_3 = \frac{x_3}{x_1^2 + x_2^2 + x_3^2}$

Problem 3. [20 pts] Consider the vector field $F : \mathbb{R}^3 \to \mathbb{R}^3$ such that $F(x) = \|x\|^7 x$. Compute the divergence of F and the curl of F.

$\nabla \cdot F(x) =$	$\|x\|^7 + 7\|x\|^5 x^2$
$\nabla \times F(x) =$	$\{\vec{0}\}$

DOT PRODUCT

$$\nabla \cdot f = \frac{\partial f}{\partial x_1} + \frac{\partial f}{\partial x_2} + \frac{\partial f}{\partial x_3}$$

$$\frac{\partial F}{\partial x_1} = (x_1^2+x_2^2+x_3^2)^{7/2} + x_1(x_1^2+x_2^2+x_3^2)^{5/2}(7/2)\cdot 2x_1$$

$$\frac{\partial F}{\partial x_2} = (x_1^2+x_2^2+x_3^2)^{7/2} + x_2(x_1^2+x_2^2+x_3^2)^{5/2}(7/2)\cdot 2x_2$$

$$\frac{\partial F}{\partial x_3} = (x_1^2+x_2^2+x_3^2)^{7/2} + x_3(x_1^2+x_2^2+x_3^2)^{5/2}(7/2)\cdot 2x_3$$

$$= \underline{\|x\|^7 + 7x_1^2\|x\|^5}$$

$$F(x) = \begin{pmatrix} x_1(x_1^2+x_2^2+x_3^2)^{7/2} \\ x_2(x_1^2+x_2^2+x_3^2)^{7/2} \\ x_3(x_1^2+x_2^2+x_3^2)^{7/2} \end{pmatrix}$$

$$\text{Curl } F = \nabla \times F(x) = \begin{pmatrix} i & j & k \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ F_1(x) & F_2(x) & F_3(x) \end{pmatrix} = \begin{pmatrix} \frac{\partial F_3}{\partial x_2} - \frac{\partial F_2}{\partial x_3} \\ \frac{\partial F_3}{\partial x_1} - \frac{\partial F_1}{\partial x_3} \\ \frac{\partial F_2}{\partial x_3} - \frac{\partial F_1}{\partial x_2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{7}{2}x_3\|x\|^5(2x_2) - \frac{7}{2}x_3\|x\|^5(2x_3) \\ \frac{7}{2}x_1\|x\|^5(2x_3) - \frac{7}{2}x_1\|x\|^5(2x_1) \\ \frac{7}{2}x_2\|x\|^5(2x_2) - \frac{7}{2}x_2\|x\|^5(2x_2) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \{\vec{0}\}$$

Problem 4. [30 pts] For each of the following vector fields F find the divergence.

1. (5pts) $F : \mathbb{R}^2 \to \mathbb{R}^2$ such that $F\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1^2 \\ x_2^2 \end{pmatrix}$.

$\nabla \cdot F(x) =$	$2(x_1 + x_2)$

DOT PRODUCT

$$\nabla \cdot f = \begin{pmatrix} \frac{\partial F}{\partial x_1} \\ \frac{\partial F}{\partial x_2} \end{pmatrix} = \frac{\partial F_1(x)}{\partial x_1} + \frac{\partial F_2(x)}{\partial x_2}$$

$$= \frac{\partial}{\partial x_1}(x_1^2) + \frac{\partial}{\partial x_2}(x_2^2) \to \underline{2x_1 + 2x_2}$$

$$\nabla \cdot F(x) = 2(x_1 + x_2)$$

2. (5pts) $F : \mathbb{R}^2 \to \mathbb{R}^2$ such that $F\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2^2 \\ x_1^2 \end{pmatrix}$.

$\nabla \cdot F(x) =$	0

$$= \frac{\partial}{\partial x_1}(F_1) + \frac{\partial}{\partial x_2}(F_2)$$

$$= \frac{\partial}{\partial x_1}(x_2^2) + \frac{\partial}{\partial x_2}(x_1^2)$$

$$\nabla \cdot F(x) = 0$$

3. (5pts) $F : \mathbb{R}^3 \to \mathbb{R}^3$ such that $F\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_2 + x_3 \\ x_1 + x_3 \end{pmatrix}$. $= \frac{\partial}{\partial x_1}(F_1) + \frac{\partial}{\partial x_2}(F_2) + \frac{\partial}{\partial x_3}(F_3)$

$\nabla \cdot F(x) =$	3

$$= \frac{\partial}{\partial x_1}(x_1 + x_2) + \frac{\partial}{\partial x_2}(x_2 + x_3) + \frac{\partial}{\partial x_3}(x_1 + x_3)$$

$$= (1+0) + (1+0) + (0+1)$$

$$\nabla \cdot F(x) = \underline{3}$$

4. (5pts) $F : \mathbb{R}^3 \to \mathbb{R}^3$ such that $F\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_3\cos(e^{x_2^2}) \\ x_1\sqrt{x_3^2+1} \\ e^{2x_2}\sin(3x_1) \end{pmatrix}$.

$\nabla \cdot F(x) =$	0

DOT PRODUCT

$$\frac{\partial}{\partial x_1} F_1 + \frac{\partial}{\partial x_2} F_2 + \frac{\partial}{\partial x_3} F_1$$

$$= \frac{\partial}{\partial x_1}\left(x_3 \cos(e^{x_2^2})\right) + \frac{\partial}{\partial x_2}\left(x_1 \sqrt{x_3^2+1}\right) + \frac{\partial}{\partial x_3}\left(e^{2x_2} \sin(3x_1)\right)$$

$$= 0 + 0 + 0 = 0 \,?$$

5. (5pts) $F : \mathbb{R}^n \to \mathbb{R}^n$ such that $F\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x_1^2 \\ 2x_2^2 \\ \vdots \\ nx_n^2 \end{pmatrix}$.

$\nabla \cdot F(x) =$	$2x_1 + 4x_2 + 6x_3 + \dots + 2 \cdot n x_n$

$$\nabla \cdot F(x) = \frac{\partial}{\partial x_1}(x_1^2) + \frac{\partial}{\partial x_2}(2x_2^2) + \frac{\partial}{\partial x_3}(3x_3^2) + \dots + \frac{\partial}{\partial x_n}(nx_n^2)$$

or $\sum_{k=1}^{n} \frac{\partial}{\partial x_k}\left(k x_k^2\right)$ AND SOLUTION IS...

$$\underline{\nabla \cdot F(x) = 2x_1 + 4x_2 + 6x_3 + \dots + 2 \cdot n x_n}$$

6. (5pts) $F : \mathbb{R}^n \to \mathbb{R}^n$ such that $F\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x_1 \\ 2x_1 \\ \vdots \\ nx_1 \end{pmatrix}$.

$\nabla \cdot F(x) =$	1

$$\nabla \cdot F(x) = \frac{\partial}{\partial x_1}(x_1) + \frac{\partial}{\partial x_2}(2x_1) + \frac{\partial}{\partial x_3}(3x_1) + \dots$$

or $\sum_{n=1}^{K} \frac{\partial}{\partial x_n}(n x_1) = 1 + 0 + 0 + 0 + \dots + 0$

Formula For $\nabla \times F(x)$

$$\nabla \times F(x) = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \vdots \\ \frac{\partial}{\partial x_3} \end{pmatrix} \times \begin{pmatrix} F_1(x) \\ \vdots \\ F_3(x) \end{pmatrix} = \begin{pmatrix} \frac{\partial F_3}{\partial x_2} - \frac{\partial F_2}{\partial x_3} \\ \frac{\partial F_1}{\partial x_3} - \frac{\partial F_3}{\partial x_1} \\ \frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2} \end{pmatrix}$$

Problem 5. [30pts] For each of the following vector fields F find the curl.

1. (5pts) $F : \mathbb{R}^3 \to \mathbb{R}^3$ such that $F\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1^2 \\ x_1 e^{x_2} \\ 2x_1x_2x_3 \end{pmatrix}$.

$\nabla \times F(x) =$	$\begin{pmatrix} 2x_1x_3 \\ -2x_2x_3 \\ e^{x_2} \end{pmatrix}$

Since Equation is written above, I will only plug in the values

$$= \begin{pmatrix} \frac{\partial}{\partial x_2}(2x_1x_2x_3) - \frac{\partial}{\partial x_3}(x_1e^{x_2}) \\ \frac{\partial}{\partial x_3}(x_1^2) - \frac{\partial}{\partial x_1}(2x_1x_2x_3) \\ \frac{\partial}{\partial x_1}(x_1e^{x_2}) - \frac{\partial}{\partial x_2}(x_1^2) \end{pmatrix} \xrightarrow{\text{Next line}}$$

$$\to \begin{pmatrix} 2x_1x_3 - 0 \\ 0 - 2x_2x_3 \\ e^{x_2} - 0 \end{pmatrix} = \begin{pmatrix} 2x_1x_3 \\ -2x_2x_3 \\ e^{x_2} \end{pmatrix}$$

2. (5pts) $F : \mathbb{R}^3 \to \mathbb{R}^3$ such that $F\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_1 \end{pmatrix}$.

$\nabla \times F(x) =$	$\begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} \frac{\partial}{\partial x_2}(x_1) - \frac{\partial}{\partial x_3}(x_2) \\ \frac{\partial}{\partial x_3}(x_1) - \frac{\partial}{\partial x_1}(x_1) \\ \frac{\partial}{\partial x_1}(x_2) - \frac{\partial}{\partial x_2}(x_1) \end{pmatrix} = \begin{pmatrix} 0 - 0 \\ 0 - 1 \\ 0 - 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$$

3. (5pts) $F : \mathbb{R}^3 \to \mathbb{R}^3$ such that $F\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 + x_2x_3 \\ x_2 + x_3x_1 \\ x_3 + x_1x_2 \end{pmatrix}$.

$\nabla \times F(x) =$	$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ for $\{\vec{0}\}$

$$\begin{pmatrix} \frac{\partial}{\partial x_2}(\cancel{x_3} + x_1x_2) - \frac{\partial}{\partial x_3}(\cancel{x_2} + x_3x_1) \\ \frac{\partial}{\partial x_3}(\cancel{x_1} + x_2x_3) - \frac{\partial}{\partial x_1}(\cancel{x_3} + x_1x_2) \\ \frac{\partial}{\partial x_1}(\cancel{x_2} + x_3x_1) - \frac{\partial}{\partial x_2}(\cancel{x_1} + x_2x_3) \end{pmatrix}$$

$$= \begin{pmatrix} x_1 - x_1 \\ x_2 - x_2 \\ x_3 - x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

4. (5pts) $F : \mathbb{R}^3 \to \mathbb{R}^3$ such that $F\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \cos(x_2x_3) - x_1 \\ \cos(x_3x_1) - x_2 \\ \cos(x_1x_2) - x_3 \end{pmatrix}$.

$\nabla \times F(x) =$	Look below

$$\begin{pmatrix} \frac{\partial}{\partial x_2}\left(\cos(x_1x_2) \cancel{- x_3}\right) - \frac{\partial}{\partial x_3}\left(\cos(x_3x_1) \cancel{- x_2}\right) \\ \frac{\partial}{\partial x_3}\left(\cos(x_2x_3) \cancel{- x_1}\right) - \frac{\partial}{\partial x_1}\left(\cos(x_1x_2) \cancel{- x_3}\right) \\ \frac{\partial}{\partial x_1}\left(\cos(x_3x_1) \cancel{- x_2}\right) - \frac{\partial}{\partial x_2}\left(\cos(x_2x_3) \cancel{- x_1}\right) \end{pmatrix}$$

$$= \begin{pmatrix} -x_1\sin(x_1x_2) + x_1\sin(x_1x_3) \\ -x_2\sin(x_2x_3) + x_2\sin(x_1x_2) \\ -x_3\sin(x_3x_1) + x_3\sin(x_2x_3) \end{pmatrix} =$$

$$\begin{pmatrix} -x_1\sin(x_1x_2) + x_1\sin(x_1x_3) \\ -x_2\sin(x_2x_3) + x_2\sin(x_1x_2) \\ -x_3\sin(x_3x_1) + x_3\sin(x_2x_3) \end{pmatrix}$$

5. (5pts) $F : \mathbb{R}^3 \to \mathbb{R}^3$ such that $F\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_2^2x_3 \\ e^{x_1x_2x_3} \\ x_1^2x_2 \end{pmatrix}$.

$\nabla \times F(x) =$	$\begin{pmatrix} x_1^2 - x_1x_2e^{x_1x_2x_3} \\ x_2^2 - 2x_1x_2 \\ x_2x_3e^{x_1x_2x_3} - 2x_2x_3 \end{pmatrix}$

$$\begin{pmatrix} \frac{\partial}{\partial x_2}\left(x_1^2x_2\right) - \frac{\partial}{\partial x_3}\left(e^{x_1x_2x_3}\right) \\ \frac{\partial}{\partial x_3}\left(x_2^2x_3\right) - \frac{\partial}{\partial x_1}\left(x_1^2x_2\right) \\ \frac{\partial}{\partial x_1}\left(e^{x_1x_2x_3}\right) - \frac{\partial}{\partial x_2}\left(x_2^2x_3\right) \end{pmatrix} \longrightarrow$$

$$\nabla \times F(x) = \begin{pmatrix} x_1^2 - x_1x_2e^{x_1x_2x_3} \\ x_2^2 - 2x_1x_2 \\ x_2x_3e^{x_1x_2x_3} - 2x_2x_3 \end{pmatrix}$$

6. (5pts) $F : \mathbb{R}^3 \to \mathbb{R}^3$ such that $F\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3x_1^2x_3 \\ 2x_1x_2 \\ x_1^3 - 2x_3 \end{pmatrix}$.

$\nabla \times F(x) =$	$\begin{pmatrix} 0 \\ 0 \\ 2x_2 \end{pmatrix}$

$$\begin{pmatrix} \frac{\partial}{\partial x_2}\left(x_1^3 - 2x_3\right) - \frac{\partial}{\partial x_3}\left(2x_1x_2\right) \\ \frac{\partial}{\partial x_3}\left(3x_1^2x_3\right) - \frac{\partial}{\partial x_1}\left(x_1^3 - 2x_3\right) \\ \frac{\partial}{\partial x_1}\left(2x_1x_2\right) - \frac{\partial}{\partial x_2}\left(3x_1^2x_3\right) \end{pmatrix} \longrightarrow$$

$$= \begin{pmatrix} 0 - 0 \\ 3x_1^2 - 3x_1^2 \\ 2x_2 - 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2x_2 \end{pmatrix}$$