MATH 311, HW 3

Due 11:00pm on Thursday February 9, 2023

Remember that you must provide an explanation of how you computed the answers.

$\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

Problem 1. [10 pts] Find the determinant and inverse of the following matrix: $A = \begin{pmatrix} -1 & 2 \\ -4 & 3 \end{pmatrix}$

$\det(A) =$	5	$A^{-1} =$	$\begin{pmatrix} 3/5 & -2/5 \\ 4/5 & -1/5 \end{pmatrix}$

$\frac{1}{(-1)(3)-(2)(-4)}$

$-3+8=5$

$= \frac{1}{5}\begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix} \to \begin{pmatrix} 3/5 & -2/5 \\ 4/5 & -1/5 \end{pmatrix}$

Problem 2. [25 pts] Consider the following matrix: $A = \begin{pmatrix} -1 & 3 & 0 \\ 2 & -1 & 2 \\ 5 & -4 & 3 \end{pmatrix}$

1. (10pts) Find its determinant.

$\det(A) =$	7

$\det(A) = -1\begin{vmatrix} -1 & 2 \\ -4 & 3 \end{vmatrix} - 3\begin{vmatrix} 2 & 2 \\ 5 & 3 \end{vmatrix} + \cancel{0}$

$= -1(-3-(-8)) - 3(6-10)$

$= -5 + 12 = 7 \to \det(a) = \frac{1}{7}$

2. (10pts) Find the inverse of A
   (or explain why it is non-invertible).

$A^{-1} =$	$\begin{pmatrix} 5/7 & -9/7 & 6/7 \\ 4/7 & -3/7 & 2/7 \\ -3/7 & 11/7 & -5/7 \end{pmatrix}$

$\begin{pmatrix} +\begin{vmatrix} -1 & 2 \\ -4 & 3 \end{vmatrix} & -\begin{vmatrix} 2 & 2 \\ 5 & 3 \end{vmatrix} & +\begin{vmatrix} 2 & -1 \\ 5 & -4 \end{vmatrix} \\ -\begin{vmatrix} 3 & 0 \\ -4 & 3 \end{vmatrix} & +\begin{vmatrix} -1 & 0 \\ 5 & 3 \end{vmatrix} & -\begin{vmatrix} -1 & 3 \\ 5 & -4 \end{vmatrix} \\ +\begin{vmatrix} 3 & 0 \\ -1 & 2 \end{vmatrix} & -\begin{vmatrix} -1 & 0 \\ 2 & 2 \end{vmatrix} & +\begin{vmatrix} -1 & 3 \\ 2 & -1 \end{vmatrix} \end{pmatrix}$

$\frac{1}{7}\begin{pmatrix} 5 & 4 & -3 \\ -9 & -3 & 11 \\ 6 & 2 & -5 \end{pmatrix}^T \to \frac{1}{7}\begin{pmatrix} 5 & -9 & 6 \\ 4 & -3 & 2 \\ -3 & 11 & -5 \end{pmatrix} \to \begin{pmatrix} 5/7 & -9/7 & 6/7 \\ 4/7 & -3/7 & 2/7 \\ -3/7 & 11/7 & -5/7 \end{pmatrix}$

3. (5pts) Use A^{-1} to find the solutions to the following system of equations:

$$\begin{cases} -x_1 + 3x_2 &= 1 \\ 2x_1 - x_2 + 2x_3 &= 2 \\ 5x_1 - 4x_2 + 3x_3 &= 1 \end{cases}$$

$A = \begin{pmatrix} -1 & 3 & 0 \\ 2 & -1 & 2 \\ 5 & -4 & 3 \end{pmatrix}$, $A^{-1} = \frac{1}{7}\begin{pmatrix} 5 & -9 & 6 \\ 4 & -3 & 2 \\ -3 & 11 & -5 \end{pmatrix}$

Answer:	$\begin{cases} x_1 = 1 \\ x_2 = 0 \\ x_3 = 2 \end{cases}$

$\underbrace{\frac{1}{7}\begin{pmatrix} 5 & -9 & 6 \\ 4 & -3 & 2 \\ -3 & 11 & -5 \end{pmatrix}}_{A^{-1}}\underbrace{\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}}_{b} = \frac{1}{7}\begin{pmatrix} 7 \\ 0 \\ 14 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$

$\begin{cases} x_1 = 1 \\ x_2 = 0 \\ x_3 = 2 \end{cases}$

$a_{11} = 5 - 18 + 6 = 7$

$a_{21} = 4 - 6 + 2 = 0$

$a_{31} = -3 + 22 - 5 = 14$

Problem 3. [10pts] If A, B are 3×3 matrices and $\det(A) = 2$ and $\det(B) = 3$, find:

$\det\left(2(A^{-1}B)^2 B^T\right) =$	54

$\det\left(2(A^{-1}B)^2 B^T\right) =$

$\rightarrow 2^3 \det\left((A^{-1}B)^2 B^T\right) \rightarrow$

$\rightarrow 2^3(\det(B))\left(\left(\frac{\det(B)}{\det(A)}\right)^2\right) \rightarrow 2^3 \cdot 3 \cdot \left(\frac{3}{2}\right)^2 = 54$

Problem 4. [40 pts] Consider the following matrix: $A = \begin{pmatrix} 2 & 3 & 0 & 1 \\ 0 & -1 & 3 & 0 \\ 0 & 2 & -1 & 2 \\ 1 & 5 & -4 & 3 \end{pmatrix} \xrightarrow{R_3 \to R_3 + (2R_2)} \begin{pmatrix} 2 & 3 & 0 & 1 \\ 0 & -1 & 3 & 0 \\ 0 & 0 & 5 & 2 \\ 1 & 5 & -4 & 3 \end{pmatrix}$

1. (15pts) Find its determinant.

$\det(A) =$	1

$$\begin{pmatrix} 2 & 3 & 0 & 1 \\ 0 & -1 & 3 & 0 \\ 0 & 0 & 5 & 2 \\ 1 & 5 & -4 & 3 \end{pmatrix} \xrightarrow{R_4 \to R_4 + -\frac{1}{2}(R_1)} \begin{pmatrix} 2 & 3 & 0 & 1 \\ 0 & -1 & 3 & 0 \\ 0 & 0 & 5 & 2 \\ 0 & 7/2 & -4 & 5/2 \end{pmatrix} \xrightarrow{R_4 \to R_4 + \frac{7}{2}(R_2)} \begin{pmatrix} 2 & 3 & 0 & 1 \\ 0 & -1 & 3 & 0 \\ 0 & 0 & 5 & 2 \\ 0 & 0 & 13/2 & \frac{5}{2} \end{pmatrix} \xrightarrow{R_4 \to R_4 + R_3\left(-\frac{13}{10}\right)} \begin{pmatrix} 2 & 3 & 0 & 1 \\ 0 & -1 & 3 & 0 \\ 0 & 0 & 5 & 2 \\ 0 & 0 & 0 & -\frac{1}{10} \end{pmatrix} \rightarrow 2 \times -1 \times 5 \times -\frac{1}{10} = 1$$

$\det(A) = 1$

2. (20pts) Find the inverse of A (or explain why it is non-invertible).

$$A^{-1} = \begin{pmatrix} 7 & -12 & 16 & -13 \\ -6 & 11 & -15 & 12 \\ -2 & 4 & -5 & 4 \\ 5 & -9 & 13 & -10 \end{pmatrix}$$

$$\left(\begin{array}{cccc|cccc} 2 & 3 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 3 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & -1 & 2 & 0 & 0 & 1 & 0 \\ 1 & 5 & -4 & 3 & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_1 \to R_4} \left(\begin{array}{cccc|cccc} 1 & 5 & -4 & 3 & 0 & 0 & 0 & 1 \\ 0 & -1 & 3 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & -1 & 2 & 0 & 0 & 1 & 0 \\ 2 & 3 & 0 & 1 & 1 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_4 \to R_4 + (-2)R_1} \left(\begin{array}{cccc|cccc} 1 & 5 & -4 & 3 & 0 & 0 & 0 & 1 \\ 0 & -1 & 3 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & -1 & 2 & 0 & 0 & 1 & 0 \\ 0 & -7 & 8 & -5 & 1 & 0 & 0 & -2 \end{array}\right)$$

$$\xrightarrow{\text{Switch sign}} \left(\begin{array}{cccc|cccc} 1 & 5 & -4 & 3 & 0 & 0 & 0 & 1 \\ 0 & 1 & -3 & 0 & 0 & -1 & 0 & 0 \\ 0 & 2 & -1 & 2 & 0 & 0 & 1 & 0 \\ 0 & -7 & 8 & -5 & 1 & 0 & 0 & -2 \end{array}\right) \xrightarrow{R_1 \to R_1 + (-5 \times R_2)}$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 11 & 3 & 0 & 5 & 0 & 1 \\ 0 & 1 & -3 & 0 & 0 & -1 & 0 & 0 \\ 0 & 2 & -1 & 2 & 0 & 0 & 1 & 0 \\ 0 & -7 & 8 & -5 & 1 & 0 & 0 & -2 \end{array}\right) \xrightarrow[\substack{-1+6=5 \\ 2+-4=-2}]{R_3 \to R_3 + (-2R_2)} \left(\begin{array}{cccc|cccc} 1 & 0 & 11 & 3 & 0 & 5 & 0 & 1 \\ 0 & 1 & -3 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 5 & 2 & 0 & 2 & 1 & 0 \\ 0 & -7 & 8 & -5 & 1 & 0 & 0 & -2 \end{array}\right) \xrightarrow{R_4 + R_4 + (7R_2)}$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 11 & 3 & 0 & 5 & 0 & 1 \\ 0 & 1 & -3 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 5 & 2 & 0 & 2 & 1 & 0 \\ 0 & 0 & -13 & -5 & 1 & -7 & 0 & -2 \end{array}\right) \xrightarrow{R_3 \to \frac{1}{5}R_3} \left(\begin{array}{cccc|cccc} 1 & 0 & 11 & 3 & 0 & 5 & 0 & 1 \\ 0 & 1 & -3 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 2/5 & 0 & 2/5 & 1/5 & 0 \\ 0 & 0 & -13 & -5 & 1 & -7 & 0 & -2 \end{array}\right) \xrightarrow[\substack{-7+(13 \times 2/5) = -9/5 \\ 0+(13 \times 1/5) = 13/5}]{R_4 \to R_4 + (13 \times R_3)}$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 11 & 3 & 0 & 5 & 0 & 1 \\ 0 & 1 & -3 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 2/5 & 0 & 2/5 & 1/5 & 0 \\ 0 & 0 & 0 & 1/5 & 1 & -9/5 & \frac{13}{5} & -2 \end{array}\right) \xrightarrow[\substack{-1+(3 \times 2/5) = 1/5 \\ 0+3 \times 1/5 = 3/5}]{R_2 \to R_2 + (3R_3)} \left(\begin{array}{cccc|cccc} 1 & 0 & 11 & 3 & 0 & 5 & 0 & 1 \\ 0 & 1 & 0 & 6/5 & 0 & 1/5 & 3/5 & 0 \\ 0 & 0 & 1 & 2/5 & 0 & 2/5 & 1/5 & 0 \\ 0 & 0 & 0 & 1/5 & 1 & -9/5 & \frac{13}{5} & -2 \end{array}\right) \xrightarrow[\substack{3+(-11(2/5)) \\ 5+(-11 \times 2/5) = 3/5 \\ 0+(-11 \times 1/5) = -11/5}]{R_1 \to R_1 + (-11R_3)}$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & -7/5 & 0 & 3/5 & -\frac{11}{5} & 1 \\ 0 & 1 & 0 & 6/5 & 0 & 1/5 & 3/5 & 0 \\ 0 & 0 & 1 & 2/5 & 0 & 2/5 & 1/5 & 0 \\ 0 & 0 & 0 & 1/5 & 1 & -9/5 & \frac{13}{5} & -2 \end{array}\right) \xrightarrow{R_4 \to R_4 \times 5} \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & -7/5 & 0 & 3/5 & -\frac{11}{5} & 1 \\ 0 & 1 & 0 & 6/5 & 0 & 1/5 & 3/5 & 0 \\ 0 & 0 & 1 & 2/5 & 0 & 2/5 & 1/5 & 0 \\ 0 & 0 & 0 & 1 & 5 & -9 & 13 & -10 \end{array}\right) \xrightarrow[R_1 \to R_1 + (\frac{7}{5}R_4)]{\substack{R_3 \to R_3 + (-\frac{2}{5}R_4) \\ R_2 \to R_2 + (-\frac{6}{5}R_4)}}$$

$$\xrightarrow[R_1 \to R_1 + (\frac{7}{5}R_4)]{\substack{R_3 \to R_3 + (-\frac{2}{5}R_4) \\ R_2 \to R_2 + (-\frac{6}{5}R_4)}} \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 7 & -12 & 16 & -13 \\ 0 & 1 & 0 & 0 & -6 & 11 & -15 & 12 \\ 0 & 0 & 1 & 0 & -2 & 4 & -5 & 4 \\ 0 & 0 & 0 & 1 & 5 & -9 & 13 & -10 \end{array}\right) \quad \xrightarrow{\text{Answer}} \begin{pmatrix} 7 & -12 & 16 & -13 \\ -6 & 11 & -15 & 12 \\ -2 & 4 & -5 & 4 \\ 5 & -9 & 13 & -10 \end{pmatrix}$$

$R_3 \to R_3 + \left(-\frac{2}{5}R_4\right)$

$0 + \left(-\frac{2}{5}(5)\right) = -2$

$\frac{2}{5} + \left(-\frac{2}{5}(-9)\right) = 4$

$\frac{1}{5} + \left(-\frac{2}{5}(13)\right) = -5$

$0 + \left(-\frac{2}{5}(-10)\right) = 4$

$R_2 \to R_2 + \left(-\frac{6}{5}R_4\right)$

$0 + \left(-\frac{6}{5}(5)\right) = -6$

$\frac{1}{5} + \left(-\frac{6}{5}(-9)\right) = 11$

$\frac{3}{5} + \left(-\frac{6}{5}(13)\right) = -15$

$0 + \left(-\frac{6}{5}(-10)\right) = 12$

$R_1 \to R_1 + \left(\frac{7}{5}R_4\right)$

$0 + \left(7/5(5)\right) = 7$

$\frac{3}{5} + \left(7/5(-9)\right) = -12$

$-\frac{11}{5} + \left(\frac{7}{5}(13)\right) = 16$

$1 + \left(\frac{7}{5}(-10)\right) = -13$

3. (5pts) Use A^{-1} to find the solutions to the following system of equations:

$$\begin{cases} 2x_1 + 3x_2 + x_4 &= 3 \\ -x_2 + 3x_3 &= 1 \\ 2x_2 - x_3 + 2x_4 &= 0 \\ x_1 + 5x_2 - 4x_3 + 3x_4 &= -1 \end{cases}$$

$x = A^{-1}b$

$$b = \begin{pmatrix} 3 \\ 1 \\ 0 \\ -1 \end{pmatrix} \rightarrow \begin{array}{c} A^{-1} \\ 1\begin{pmatrix} 7 & -12 & 16 & -13 \\ -6 & 11 & -15 & 12 \\ -2 & 4 & -5 & 4 \\ 5 & -9 & 13 & 10 \end{pmatrix} \end{array} \begin{array}{c} b \\ \begin{pmatrix} 3 \\ 1 \\ 0 \\ -1 \end{pmatrix} \end{array} = \begin{pmatrix} 22 \\ -19 \\ -6 \\ -4 \end{pmatrix}$$

$21 + -12 + 0 + 13 = 22$
$-18 + 11 + 0 + -12 = -19$
$-6 + 4 + 0 - 4 = -6$
$15 + -9 + 0 - 10 = -4$

Answer:	$x_1 = 22$ $x_2 = -19$ $x_3 = -6$ $x_4 = -4$

Problem 5. [15 pts] Let $\mathbb{R}_+$ be the set of positive real numbers. Define scalar multiplication ($\circ$) and addition ($\boxplus$) by

$$\alpha \circ x = x^\alpha \qquad \text{for all } x \in \mathbb{R}_+, \alpha \in \mathbb{R}$$

$$x \boxplus y = xy, \qquad \text{for all } x, y \in \mathbb{R}_+$$

For example, the scalar product of -4 times $\frac{1}{2}$ is $-4 \circ \frac{1}{2} = \left(\frac{1}{2}\right)^{-4} = 16$, and the addition of 3 and 7 is $3 \boxplus 7 = 3 \cdot 7 = 21$.

1. (5pts) Considering the operations $\circ$ and $\boxplus$, is $\mathbb{R}_+$ a vector space? Yes ☒ No ☐

2. (10pts) Justify your answer (briefly explain why the axioms A1 to A8 hold or fail)

C1) $2 \circ 3 = 2^3 = 8$ ✓ Positive Real #

C2) $2 \boxplus 3 = 2 \cdot 3 = 6$ ✓

A1) $x + y = y + x$
$x \boxplus y = x \cdot y = y \cdot x = y \boxplus x$ ✓

A2) $(x+y)+z = x+(y+z)$
$(x \boxplus y) \boxplus z = (x \cdot y) \cdot z = x \cdot (y \cdot z) = x \boxplus (y \boxplus z)$ ✓

A3) $x + 0 = x$
$x \boxplus 1 = x \cdot 1 = x$ for $x \in \mathbb{R}^+$ ✓

A4) $x \boxplus (-x) = \vec{0} = -x \boxplus x$
$x \boxplus \left(\frac{1}{x}\right) = x \cdot \left(\frac{1}{x}\right) = \underset{\text{equals zero vector}}{1} = \vec{0}$ ✓

A5) $\alpha(x+y) = \alpha x + \alpha y$,
$\alpha \circ (x \boxplus y) = (x \cdot y)^\alpha = x^\alpha \cdot y^\alpha = (\alpha \circ x) \boxplus (\alpha \circ y)$

A6) $(\alpha \boxplus \beta) \circ x = \alpha x \boxplus \beta x$ ✓
$(\alpha \circ x) \boxplus (\alpha \circ x) = x^\alpha \boxplus x^\beta = x^{\alpha+\beta} = \underline{(\alpha+\beta) \circ x}$

A7) $\alpha \circ (\beta \circ x) = (x^\beta)^\alpha = x^{\beta\alpha} = x^{\alpha\beta} = (\alpha\beta) \circ x$

A8) $1 \circ x = x^1 = x$ ✓

I verified that all Axioms are satisfied.
$\mathbb{R}^+$ is a vector space with the operations.