Texas A&M University
Department of Electrical and Computer Engineering
ELEN 444 - Digital Signal Processing
Fall 2024

Problem Set 4
Assigned: Thursday, Sept. 19, 2024
Due: Thursday, Sept. 26, 2024

The z-transform

4.1: Proakis and Manolakis 3.1

4.2: Proakis and Manolakis 3.4

4.3: Proakis and Manolakis 3.7

4.4: Proakis and Manolakis 3.8

4.5: Proakis and Manolakis 3.11

4.6: Proakis and Manolakis 3.14 (a,c,f,i,j)

4.7: Proakis and Manolakis 3.15

4.8: Proakis and Manolakis 3.16 (a,c)

4.9: Proakis and Manolakis 3.26

4.10: Proakis and Manolakis 3.38 (b,d)

Computer Assignment 4: The z-transform

L4.1: Ingle and Proakis P4.12

L4.1: Ingle and Proakis P4.13

L4.1: Ingle and Proakis P4.14

L4.1: Ingle and Proakis P4.21

L4.1: Ingle and Proakis P4.22

3.1 Determine the z-transform of the following signals.

(a) $x(n) = \{3, 0, 0, 0, 0, \underset{\uparrow}{6}, 1, -4\}$

(b) $x(n) = \begin{cases} (\frac{1}{2})^n, & n \geq 5 \\ 0, & n \leq 4 \end{cases}$

(a) $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$ (3.1.1)

$x(-5) = 3z^5 + 6\cancel{z^0} + z^{-1} - 4z^{-2}$

$= 6 + z^{-1} - 4z^{-2} + 3z^5$ ROC: ENTRE Z-PLANE EXCEPT $z=0$

(b) $x(n) = (\frac{1}{2})^n$, $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

$\hookrightarrow X(z) = \sum_{n=5}^{\infty} (\frac{1}{2})^n z^{-n} = (\frac{1}{2} z^{-1})^n \leftarrow$ INFINITE GEOMETRIC SERIES.

$1 + A + A^2 + A^3 = \frac{1}{1-A}$

$= (\frac{1}{2} z^{-1})^5 = \frac{z^{-5}}{32}$, $X(z) = \frac{1}{1 - \frac{1}{2} z^{-1}}$ $= \frac{z^{-5}}{32} \cdot \frac{1}{1 - \frac{1}{2} z^{-1}}$ ROC: $|z| > \frac{1}{2}$

3.4 Determine the z-transform of the following signals.

(a) $x(n) = n(-1)^n u(n)$

(b) $x(n) = n^2 u(n)$

(c) $x(n) = -na^n u(-n-1)$

(d) $x(n) = (-1)^n \left(\cos \frac{\pi}{3} n\right) u(n)$

(e) $x(n) = (-1)^n u(n)$

(f) $x(n) = \{\underset{\uparrow}{1}, 0, -1, 0, 1, -1, \ldots\}$

(a) $X(z) = \sum_{n=-\infty}^{-1} (-1)^n z^{-n} = \frac{(-1) z^{-1}}{(1-(-1)z^{-1})^2} = \frac{-z^{-1}}{(1+z^{-1})^2}$ ROC: $|z| > 1$

(b) $X(z) = \sum_{n=0}^{\infty} n^2 z^{-n} = \frac{z(\alpha+1)}{(z-\alpha)^3}$, ROC: $|z| > \alpha \rightarrow \frac{z(2)}{(z-1)^3}$, ROC: $|z| > 1$

(c) $X(z) = \sum_{n=0}^{\infty} -na^n z^{-n} = \sum_{n=-1}^{-\infty} -na^n z^{-n} = \sum_{m=1}^{\infty} ma^{-m} z^m \rightarrow \sum_{m=1}^{\infty} m(\frac{z}{\alpha})^m$ $|z| < |a|$

$= \frac{\alpha z}{(\alpha - z)^2}$

$\rightarrow X(z) = \frac{az}{(a-z)^2}$, ROC: $|z| < |a|$

(d) $X(z) = \sum_{n=0}^{\infty} -1^n \cos(\frac{\pi}{3} n) z^{-n} \rightarrow X(z) = \frac{1 + z\cos \pi/3}{1 + 2z\cos \pi/3 + z^{-2}}$, ROC: $|z| > 1$

(e) $X(z) = \sum_{n=0}^{\infty} -1^n z^{-n} = \frac{1}{1 - \alpha z^{-1}}$, $|z| > |a|$ $= X(z) = \frac{1}{1+z^{-1}}$, ROC: $|z| > 1$

(f) $X(z) = 1 + \cancel{z^0} - z^{-2} + \cancel{z^0} + z^{-4} - z^{-5} = 1 - z^{-2} + z^{-4} - z^{-5}$

when $z \neq 0$

3.7 Compute the convolution of the following signals by means of the z-transform.

$$x_1(n) = \begin{cases} (\frac{1}{3})^n, & n \geq 0 \\ (\frac{1}{2})^{-n}, & n < 0 \end{cases}$$

$$x_2(n) = (\frac{1}{2})^n u(n)$$

$$X(z) = \frac{1}{1-\alpha z^{-1}}, |z| > |\alpha|$$

$$X_1(z) = \sum_{n=0}^{\infty} (\tfrac{1}{3}n) z^{-n} + \sum_{n=-\infty}^{-1} (\tfrac{1}{2}n) z^{-n} = \frac{1}{1-\frac{1}{3}z^{-1}} + \frac{1}{1-\frac{1}{2}z^{-1}} - 1$$

$$X_2(z) = \sum_{n=0}^{\infty} (\tfrac{1}{2})^n z^{-n} = \frac{1}{1-\frac{1}{2}z^{-1}}$$

$$X(z) = X_1(z) \cdot X_2(z) = \left(\frac{z}{z-1/3} + \frac{1}{1-\frac{1}{2}z}\right) \cdot \frac{z}{z+1/2}$$

3.8 Use the convolution property to:

(a) Express the z-transform of

$$y(n) = \sum_{k=-\infty}^{n} x(k) = \sum_{k=-\infty}^{n} x(k) u(n-k) = x(n) * u(n)$$

in terms of $X(z)$.

(a) Z - TRANSFORM

$$Y(z) = X(z) * U(z) : X(z) = Y(z) / U(z) \rightarrow X(z) = Y(z)[1-z^{-1}]$$

(b) SHOW $x(n) = u(n) * u(n) \rightarrow u(n) * u(n) = \sum_{k-\infty}^{\infty} u(k) u(n-k)$

$$\rightarrow \sum_{k=-\infty}^{\infty} (n+1) u(n) z^{-n} = \frac{(1)\cancel{z^{-n+1}}}{(1-(1)z^{-1})^2} \Rightarrow X(z) = \frac{1}{(1-z^{-1})^2}, \text{ROC}: |z| > 1$$

3.11 Using long division, determine the inverse z-transform of

$$X(z) = \frac{1+2z^{-1}}{1-2z^{-1}+z^{-2}}$$

if **(a)** $x(n)$ is causal and **(b)** $x(n)$ is anticausal.

(a) 1^{st} TERM

$$\frac{1}{1} = 1, \quad 1(1-2z^{-1}+z^{-2}) = 1-2z^{-1}+z^{-2}$$

$$\rightarrow (1+2z^{-1}) - (1-2z^{-1}+z^{-2}) = 4z^{-1} - z^{-2}$$

$\underline{2^{nd}}$

$$\frac{4z^{-1}}{1} = 4z^{-1} \rightarrow 4z^{-1}(1-2z^{-1}+z^{-2}) = 4z^{-1} - 8z^{-2} + 4z^{-3}$$

$$\rightarrow (4z^{-1} - z^{-2}) - (4z^{-1} - 8z^{-2} + 4z^{-3}) = 7z^{-2} - 4z^{-3}$$

$\underline{3^{rd}}$

$$7z^{-2} \rightarrow 7z^{-2}(1-2z^{-1}+z^{-2}) = 7z^{-2} - 14z^{-3} + 7z^{-4}$$

$$\rightarrow (7z^{-2} - 4z^{-3}) - (7z^{-2} - 14z^{-3} + 7z^{-4}) = 10z^{3} - 7z^{-4}$$

$$X(z) = 1 + 4z^{-1} + 7z^{-2} + 10z^{-3} \rightarrow x(n) = \{1, 4, 7, 10, \ldots\}$$

(b) 1^{st} TERM $\quad \frac{z^2+2z}{z^2-2z+1}$

$\frac{z^2}{z^2} = 1 \rightarrow z^2-2z+1 \rightarrow (z^2+2z)-(z^2-2z+1) = 4z-1$

2^{nd} TERM

$\frac{4z}{z^2} = 4z^{-1} \rightarrow 4z^{-1}(z^2-2z+1) = 4z-8+4z^{-1} \rightarrow (4z-1)-(4z-8+4z^{-1})$

$= 7-4z^{-1}$

$X(z) = 1+4z+7z^2+10z^3$

$\rightarrow x(n) = \{10, 7, 4, 1\}$

3.14 Determine the causal signal $x(n)$ if its z-transform $X(z)$ is given by:

(a) $X(z) = \frac{1+3z^{-1}}{1+3z^{-1}+2z^{-2}}$

(b) $X(z) = \frac{1}{1-z^{-1}+\frac{1}{2}z^{-2}}$

(c) $X(z) = \frac{z^{-6}+z^{-7}}{1-z^{-1}}$

(d) $X(z) = \frac{1+2z^{-2}}{1+z^{-2}}$

(e) $X(z) = \frac{1}{4}\frac{1+6z^{-1}+z^{-2}}{(1-2z^{-1}+2z^{-2})(1-0.5z^{-1})}$

(f) $X(z) = \frac{2-1.5z^{-1}}{1-1.5z^{-1}+0.5z^{-2}}$

(i) $X(z) = \frac{1-\frac{1}{2}z^{-1}}{1+\frac{1}{2}z^{-1}}$

(j) $X(z) = \frac{1-az^{-1}}{z^{-1}-a}$

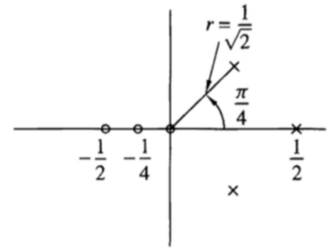

Figure P3.14

(a) $\frac{1+3z^{-1}}{(1+2z^{-1})(1+z^{-1})} = \frac{A}{1+2z^{-1}} + \frac{B}{1+z^{-1}} \rightarrow 1+3z^{-1} = A(1+z^{-1})+B(1+2z^{-1})$

$z^{-1} = -1$

$-2 = -B \rightarrow \underline{B=2}$

$z^{-1} = -1/2 \quad 1+3(-1/2) = A(1-1/2)$

$1-3/2 = A - A/2$

$2-3 = 2A-A \rightarrow \underline{-1=A}$

$x(n) = (-2)^n \mu(n) - (-2)^n \mu(n)$

(c) $X(z) = \frac{z^{-6}}{1-z^{-1}} + \frac{z^{-7}}{1-z^{-1}} \rightarrow x(n) = u(n-6)+u(n-7)$

(f) $1-1.5x+0.5x^2 \rightarrow (1-x)(1-\frac{1}{2}x)$

$\frac{2-1.5z^{-1}}{(1-z^{-1})(1-1/2z^{-1})} = \frac{A}{1-z^{-1}} + \frac{B}{1-1/2z^{-1}} \rightarrow 2-1.5z^{-1} = A(1-\frac{1}{2}z^{-1})+B(1-z^{-1})$

$z^{-1}=1$

$.5 = A(.5) \rightarrow A=1$

$z^{-1}=2 \quad -1 = -B \rightarrow B=1$

$= \frac{1}{1-z^{-1}} + \frac{1}{1-\frac{1}{2}z^{-1}}, \quad x(n) = (\frac{1}{2})^n u(n) + \mu(n)$

(i) $$X(z) = \frac{1 - 1/4\, z^{-1}}{1 + 1/2\, z^{-1}} = \frac{1}{1 + 1/2\, z^{-1}} - \frac{1/4\, z^{-1}}{1 + 1/2\, z^{-1}}$$

$$x(n) = \left(-\frac{1}{2}\right)^n u(n) + \frac{1}{4}\left(-\frac{1}{2}\right)^n u(n)$$

(j) $$X(z) = \frac{1 - a z^{-1}}{z^{-1} - a} \cdot \left(\frac{z}{z}\right)$$

$$\rightarrow X(z) = \frac{z-a}{1-az} = \frac{A}{z-a} + \frac{B}{z} \rightarrow z - a = A(1-az) + B(z)$$

$$\rightarrow X(z) = \frac{1-a^2}{1-az} - \frac{a}{z}$$

$z = 1$

$\cancel{1 - a = A(1-a)} + B$

$B = -a$

$z = a$

$A = 1 - a^2$

$$x(n) = (1-a^2)a^n u(n) - a u(n-1)$$

3.15 Determine all possible signals $x(n)$ associated with the z-transform

$$X(z) = \frac{5z^{-1}}{(1 - 2z^{-1})(3 - z^{-1})} \cdot z^2$$

$$\rightarrow X(z) = \frac{5}{(z-2)(3z-1)} = \frac{A}{z-2} + \frac{B}{3z-1} \rightarrow 5 = A(3z-1) + B(z-2)$$

$$\rightarrow 5 = A3z - A + Bz - 2B$$

$$\rightarrow 5 = (3A+B)z + (-A-2B)$$

$B = -3A$ $-A - 2(-3A) = 5$

$B = -3$ $A = 1$

$$\rightarrow X(z) = \frac{1}{z-2} - \frac{3}{3z-1}$$

$x_1(n) = 2^n u(n)$, $x_2(n) = \left(\frac{1}{3}\right)^n u(n)$

$$x(n) = 2^n u(n) - 3\left(\frac{1}{3}\right)^n u(n)$$

3.16 Determine the convolution of the following pairs of signals by means of the z-transform.

(a) $x_1(n) = (\frac{1}{4})^n u(n-1)$, $\quad x_2(n) = [1 + (\frac{1}{2})^n]u(n)$

(b) $x_1(n) = u(n)$, $\quad x_2(n) = \delta(n) + (\frac{1}{2})^n u(n)$

(c) $x_1(n) = (\frac{1}{2})^n u(n)$, $\quad x_2(n) = \cos \pi n u(n)$

(d) $x_1(n) = n u(n)$, $\quad x_2(n) = 2^n u(n-1)$

(a) $$X_1(z) = z^{-1} \cdot \frac{1}{1 - \frac{1}{4} z^{-1}}, \; |z| > 1/4 = \frac{1}{z - 1/4}$$

$$X_2 = \frac{1}{1 - z^{-1}} + \frac{1}{1 - \frac{1}{2} z^{-1}} = \left(\frac{z}{z-1} + \frac{z}{z - 1/2}\right) \cdot \frac{1}{z - 1/4} = \frac{z}{(z - \frac{1}{4})(z-1)} + \frac{z}{(z-\frac{1}{4})(z-\frac{1}{2})}$$

$$\frac{z}{(z-\frac{1}{4})(z-1)} = \frac{A}{z - \frac{1}{4}} + \frac{B}{z-1} = A(z-1) + B(z - \tfrac{1}{4}) = z$$

$z = 1$ $B(1 - 1/4) = 1 \rightarrow B = 4/3$

$z = 1/4$ $A(1/4 - 1) = 1/4 \rightarrow A = -1/3$

$$= \frac{-1/3}{z - 1/4} + \frac{4/3}{z - 1}$$

(a)

$$\frac{z}{(z-\frac{1}{4})(z-\frac{1}{2})} = \frac{A}{z-\frac{1}{4}} + \frac{B}{z-\frac{1}{2}} = A(z-\tfrac{1}{2})+B(z-\tfrac{1}{4})=z$$

$z = 1/2$

$B(1/2 - 1/4) = 1/2 \rightarrow B = 2$

$z = 1/4$

$A(1/4 - 1/2) = 1/4 \rightarrow A = -1$

First Term

$$= \frac{-1/3}{z-1/4} + \frac{4/3}{z-1}$$

2nd

$$\frac{-1}{z-1/4} + \frac{2}{z-1/2}$$

$$\rightarrow x(n) = \left[-\frac{1}{3}\left(\frac{1}{4}\right)^n + \frac{4}{3}\cdot 1^n\right]u(n) + \left[-\left(\frac{1}{4}\right)^n + 2\left(\frac{1}{2}\right)^n\right]u(n)$$

(c)

$$X_1(z) = \frac{1}{1-\frac{1}{2}z^{-1}}, \quad X_2(z) = \frac{1+z^{-1}}{1+2z^{-1}+z^{-2}}$$

$$X(z) = X_1(z)\cdot X_2(z) = \frac{1}{1-\frac{1}{2}z^{-1}} \cdot \frac{1+z^{-1}}{1+2z^{-1}+z^{-2}} = \frac{1+z^{-1}}{(1-\frac{1}{2}z^{-1})(1+2z^{-1}+z^{-2})}$$

$$\frac{1+z^{-1}}{(1-\frac{1}{2}z^{-1})(1+2z^{-1}+z^{-2})} = \frac{A}{(1-\frac{1}{2}z^{-1})} + \frac{B}{(1+z^{-1})^2} = A(1+z^{-1})^2 + B(1-\tfrac{1}{2}z^{-1}) = 1+z^{-1}$$

$$\rightarrow A + 2Az^{-1} + Az^{-2} + B - \tfrac{1}{2}Bz^{-1} = 1+z^{-1}$$

$$\rightarrow Az^{-2} + (2Az^{-1} - \tfrac{1}{2}Bz^{-1}) + (A+B) = 1+z^{-1}$$

$A=0, \quad 2A - \frac{1}{2}B = 1 \rightarrow B = 1/3$

$A + 1/3 = 1 \rightarrow A = 2/3$

$$\rightarrow \frac{2/3(1+z^{-1})}{1-\frac{1}{2}z^{-1}} + \frac{1/3}{(1+z^{-1})^2}$$

$$x(n) = \frac{2}{3}\cos\pi n\, u(n) + \frac{1}{3}\left(\frac{1}{2}\right)^n u(n)$$

3.26 Determine the signal $x(n)$ with z-transform

$$X(z) = \frac{3}{1 - \frac{10}{3}z^{-1} + z^{-2}}$$

if $X(z)$ converges on the unit circle.

$$z = \frac{-(-\frac{10}{3}) \pm \sqrt{(\frac{10}{3})^2 - 4 \cdot 1 \cdot 1}}{2}, \quad z_1 = \frac{10/3 + 8/3}{2} = 3, \quad z_2 = \frac{\frac{10}{3} - \frac{8}{3}}{2} = \frac{1}{3}$$

$$X(z) = \frac{3z^2}{(z-3)(z-1/3)} = \frac{A}{(z-3)} + \frac{B}{(z-1/3)} = A(z-1/3) + B(z-3) = 3z^2$$

$z = 1/3$

$B(1/3 - 3) = 3 \cdot \frac{1}{3}^2$

$\rightarrow B = 3/8$

$z = 3$

$A(3 - 1/3) = 3(1/3)^2$

$A = -27/8$

$$\rightarrow X(z) = \frac{-27/8}{z-3} + \frac{3/8}{z - 1/3}$$

$$x(n) = -\frac{27}{8} \cdot 3^n u(n) + \frac{3}{8}\left(\frac{1}{3}\right)^n u(n)$$

3.38 Determine the impulse response and the step response of the following causal systems. Plot the pole–zero patterns and determine which of the systems are stable.

(a) $y(n) = \frac{3}{4}y(n-1) - \frac{1}{8}y(n-2) + x(n)$

(b) $y(n) = y(n-1) - 0.5y(n-2) + x(n) + x(n-1)$

(c) $H(z) = \frac{z^{-1}(1+z^{-1})}{(1-z^{-1})^3}$

(d) $y(n) = 0.6y(n-1) - 0.08y(n-2) + x(n)$

(e) $y(n) = 0.7y(n-1) - 0.1y(n-2) + 2x(n) - x(n-2)$

(b) $y(n) = Y(z), \; y(n-1) = z^{-1}Y(z), \; y(n-2) = z^{-2}Y(z), \; x(n) = X(z)$

$$Y(z) = z^{-1}Y(z) - \frac{1}{2}z^{-2}Y(z) + X(z) + z^{-1}X(z)$$

$$Y(z)\left(1 - z^{-1} + \frac{1}{2}z^{-2}\right) = X(z)(1+z^{-1})$$

$$H(z) = \frac{1+z^{-1}}{1 - z^{-1} + \frac{1}{2}z^{-2}} = \frac{1+z^{-1}}{(1-\frac{1}{2}z^{-1})^2} \qquad z = -1$$

$$z = \frac{1 \pm \sqrt{(-1)^2 - 4 \cdot 1 \cdot 1/2}}{2} = \frac{1 \pm j}{2}, \quad z = \frac{1+j}{2}, \; z = \frac{1-j}{2}$$

$$H(z) = \frac{z+1}{\frac{1}{2} - z + z^2} = \frac{A}{z - \frac{1+j}{2}} + \frac{B}{z - \frac{1-j}{2}} = A\left(z - \frac{1-j}{2}\right) + B\left(z - \frac{1+j}{2}\right) = z+1$$

$$z+1 = (A+B)z + \left(-\frac{A}{2} - \frac{B}{2}\right) + \left(\frac{Aj}{2} - \frac{Bj}{2}\right)$$

$A = \frac{1}{2} + \frac{1}{4}j$

$B = \frac{1}{2} - \frac{1}{4}j$

$$\frac{\frac{1}{2} + \frac{1}{4}j}{z - \frac{1+j}{2}} + \frac{\frac{1}{2} - \frac{1}{4}j}{z - \frac{1-j}{2}}$$

$$h(z) = \left(\frac{1}{\sqrt{2}}\right)^n \left[\cos \frac{\pi}{4}n + j \sin \frac{\pi}{4}n\right] u(n)$$

(d)

$$Y(z)\left[1-0.6z^{-1}+0.08z^{-2}\right] = X(z)$$

$$\rightarrow Y(z) = \frac{X(z)}{1-0.6z^{-1}+0.08z^{-2}} = (1-0.2z^{-1})(1-0.4z^{-1})$$

$$\frac{1}{(1-0.2z^{-1})(1-0.4z^{-1})} = \frac{A}{1-0.2z^{-1}} + \frac{B}{1-0.4z^{-1}} \Rightarrow 1 = A(1-0.4z^{-1})+B(1-0.2z^{-1})$$

$$z^{-1} = 2.5$$

$$\rightarrow 1 = B(1-\tfrac{1}{2}) = \tfrac{1}{2}B$$

$$\underline{B=2}$$

$$z^{-1} = 5 \qquad A = -1$$

$$H(z) = \frac{-1}{1-0.2z^{-1}} + \frac{2}{1-0.4z^{-1}}$$

$$\rightarrow h(n) = -(0.2)^n u(n) + 2(0.4)^n u(n)$$

STEP

$$\frac{1}{(1-0.2z^{-1})(1-0.4z^{-1})(1-z^{-1})} = \frac{A}{1-0.2z^{-1}} + \frac{B}{1-0.4z^{-1}} + \frac{C}{1-z^{-1}}$$

$$= A(1-0.4z^{-1})(1-z^{-1}) + B(1-0.2z^{-1})(1-z^{-1}) + C(1-0.2z^{-1})(1-0.4z^{-1})$$

$$A = 2.08 = 25/12$$

$$B = 0.25 = 1/4$$

$$C = -1.333 = -4/3$$

$$y(n) = \frac{25}{12}u(n) + \frac{1}{4}\left(\frac{1}{5}\right)^n u(n) - \frac{4}{3}\left(\frac{2}{5}\right)u(n)$$

P4.12

$$b_0 = A_c$$

$$b_1 = r\left[A_s \sin(\pi\nu_0) - A_c \cos(\pi\nu_0)\right]$$

$$a_1 = -2r\cos(\pi\nu_0)$$

$$a_2 = r^2$$

$$r = \sqrt{a_2}\ , \quad \cos(\pi\nu_0) = \frac{-a_1}{2r}\ , \quad \sin(\pi\nu_0) = \sqrt{1-\cos^2(\pi\nu_0)}$$

$$A_s = \frac{b_1 + rA_c\cos(\pi\nu_0)}{r\sin(\pi\nu_0)}\ , \quad A_c = b_0$$