MATH 311, HW 5

Due 11:00pm on Thursday March 2, 2023

Remember that you must provide an explanation of how you computed the answers. Remember to scan/upload exactly the 6 pages of this template.

Problem 1. [20 pts] In the following we consider vectors $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ in $\mathbb{R}^3$. Determine (**with justification**) whether the following sets are subspaces of $\mathbb{R}^3$.

1. (10pts) $V_1 = \{x \in \mathbb{R}^3 \mid x_1 = x_2 = x_3\}$

Is it a subspace?
Yes ■ No ☐

Check C1: If $\alpha \in \mathbb{R}$ and $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in V$. Multiply all by α

results in $\alpha x_1 = \alpha x_2 = \alpha x_3$. So, vector $\alpha \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \alpha x_1 \\ \alpha x_2 \\ \alpha x_3 \end{pmatrix} \in V_1$ C1 holds

Explanation: For every $\alpha \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in V_1$ all holds because $x_1 = x_2 = x_3$ and $\alpha x_1 = \alpha x_2 = \alpha x_3$

Check C2: if $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \in V$, $x_1 = x_2 = x_3$ and $y_1 = y_2 = y_3$.

Add them together: $x_1 + y_1 = x_2 + y_2 = x_3 + y_3$. So, $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{pmatrix} \in V_1$

C2 HOLDS ✓

Because, $x_1 + y_1 = x_2 + y_2 = x_3 + y_3 \in V$
So C2 is true
and it is a subspace.

2. (10pts) $V_2 = \{x \in \mathbb{R}^3 \mid x_1 + x_3 = 1\}$

Is it a subspace?
Yes ☐ No ■

Check C1: If $\alpha \in \mathbb{R}$ and $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in V$. Multiply by α: $\alpha x_1 + \alpha x_3 = \alpha 1$

So does, $\alpha \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$?

Plug in numbers,

$\alpha \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ ✗

Explanation: It is false: for every $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in V_1$ does not equal $\alpha \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ in V_1

C1 does NOT pass

Problem 2. [15 pts] Recall that the space of 2×2 matrices $\mathbb{R}^{2\times 2}$ is a vector space with the usual addition and scalar multiplication. Fix some 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Determine (**with justification**) whether the following subsets of $\mathbb{R}^{2\times 2}$ are subspaces:

1. (5pts) $S_1 = \{M \in \mathbb{R}^{2\times 2} | MA = AM\}$. (The set of matrices that commute with A)

Is it a subspace?
Yes ☐ No ■

C1 $m \in S$, $A \in S$ $\quad \alpha \in \mathbb{R} \longrightarrow \alpha MA \overset{?}{\in} S$

$(\alpha m)A = m(\alpha A) = \alpha MA$ ✓

$(\alpha A)m = A(\alpha m) = \alpha AM$

Why?
For plugging in numbers,
it does not give the same result

$A\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in S$, $\alpha \in \mathbb{R}$

$m\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \in S$

$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

$\alpha MA = \alpha \begin{pmatrix} a+2c & b+2d \\ 3a+4c & 3b+4d \end{pmatrix} \neq \alpha AM = \alpha \begin{pmatrix} a+3b & 2a+4b \\ c+3d & 2c+4d \end{pmatrix}$

$\alpha MA = \alpha AM$ ✗ DOES NOT HOLD C1

2. (5pts) $S_2 = \{M \in \mathbb{R}^{2\times 2} | MA = O\}$. (The set of matrices that when multiplying with A on the rgiht yield the zero matrix $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$) $\quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ or $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ $\quad M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$

Is it a subspace?
Yes ■ No ☐

C1 $M = 0$, $A \in S$ $\quad \alpha \in \mathbb{R} \longrightarrow \alpha A \overset{?}{\in} S$

$M(\alpha A) \overset{?}{=} 0$

$M(\alpha A) = \alpha MA = \alpha \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ ✓

C2 $M, A \in S \rightarrow M + N \overset{?}{\in} S$

$MA = 0$ $\quad A(M+N) = 0$

$NA = 0$

$A(M+N) = AM + AN = 0 + 0 = 0$ ✓

3. (5pts) $S_3 = \{M \in \mathbb{R}^{2\times 2} | MA \neq AM\}$. (The set of matrices that do not commute with A)

Is it a subspace?
Yes ☐ No ●

C1 $M \in S$, $A \in S$, $\alpha \in \mathbb{R} \longrightarrow \alpha MA \overset{?}{\in} S$

$(\alpha M)A = M(\alpha A) = \alpha MA$ ✓

$(\alpha A)M = A(\alpha M) = \alpha AM$

$A\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in S$, $\alpha \in \mathbb{R}$

$M\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \in S$

TEST WITH NUMBERS

$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

$\alpha MA = \alpha \begin{pmatrix} a+2c & b+2d \\ 3a+4c & 3b+4d \end{pmatrix}$ $\alpha AM = \alpha \begin{pmatrix} a+3b & 2a+4b \\ c+3d & 2c+4d \end{pmatrix}$

$\alpha MA \neq \alpha AM$ × DOES HOLD C1

C2 $M, A \in S \rightarrow M+N \overset{?}{\in} S$

$MA = AM$ $A(M+N) = A(N+M)$

$NA = AN$

$A(M+N) = AM + AN \neq AN + AM = A(N+M)$ ✓

C2 DOES NOT HOLD

Because, it does equal to each other but they should not

Problem 3. [15pts] For each of the following, find the transition matrix corresponding to the change of basis from the standard basis $E = \{e_1, e_2\}$ to $\{u_1, u_2\}$: Only take the inverse

1. (5pts) $u_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $u_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Answer:	$\begin{bmatrix} 1/3 & 1/3 \\ 2/3 & -1/3 \end{bmatrix}$

$A = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$ if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then, $\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

$= \frac{1}{1(-1)-1(2)}\begin{bmatrix} 1 & -1 \\ -2 & -1 \end{bmatrix}$

$= \frac{1}{-1-2}\begin{bmatrix} 1 & -1 \\ -2 & -1 \end{bmatrix}$

$= \frac{1}{-3}\begin{bmatrix} 1 & -1 \\ -2 & -1 \end{bmatrix} = \begin{bmatrix} 1/3 & 1/3 \\ 2/3 & -1/3 \end{bmatrix}$

2. (5pts) $u_1 = \begin{pmatrix} 4 \\ 7 \end{pmatrix}$, $u_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

Answer:	$\begin{bmatrix} -3/2 & 1 \\ 7/2 & -2 \end{bmatrix}$

$A = \begin{pmatrix} 4 & 2 \\ 7 & 3 \end{pmatrix}$ if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then, $\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

$= \frac{1}{4(3)-2(7)}\begin{bmatrix} 3 & -2 \\ -7 & 4 \end{bmatrix}$

$= -\frac{1}{2}\begin{bmatrix} 3 & -2 \\ -7 & 4 \end{bmatrix} = \begin{bmatrix} -3/2 & 1 \\ 7/2 & -2 \end{bmatrix}$

3. (5pts) $u_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $u_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Answer:	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then, $\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

$= \frac{1}{0(0)-1(1)}\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

$= -1\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Problem 4. [40pts] Consider the vector space $\mathbb{R}^3$. In this vector space consider the vectors

$$u_1 = \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

1. (10pts) Find the transition matrix corresponding to the change of basis from the standard basis $E = \{e_1, e_2, e_3\}$ to the basis $U = \{u_1, u_2, u_3\}$.

Ans:	$\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ -1 & 0 & 2 \end{pmatrix}$

① Find $\mathcal{U}^{-1}$

$$\left(\begin{array}{ccc|ccc} 4 & 2 & 1 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 & 0 & 1 \end{array}\right)$$

$R_1 \to R_1/4$

$$\left(\begin{array}{ccc|ccc} 1 & 1/2 & 1/4 & 1/4 & 0 & 0 \\ 3 & 2 & 1 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 & 0 & 1 \end{array}\right)$$

$R_2 \to R_2 - 3R_1$: $3 - 3(1) = 0$, $2 - 3(1/2) = 1/2$, $1 - 3(1/4) = 1/4$, $0 - 3(1/4) = -3/4$, $1 - 3(0) = 1$, $0 - 3(0) = 0$

$R_3 \to R_3 - 2R_1$: $2 - 2(1) = 0$, $1 - 2(1/2) = 0$, $1 - 2(1/4) = 1/2$, $0 - 2(1/4) = -1/2$, $0 - 2(0) = 0$, $1 - 2(0) = 1$

$R_2 \to R_2 \cdot 2$

$$\left(\begin{array}{ccc|ccc} 1 & 1/2 & 1/4 & 1/4 & 0 & 0 \\ 0 & 1/2 & 1/4 & -3/4 & 1 & 0 \\ 0 & 0 & 1/2 & -1/2 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 1/2 & 1/4 & 1/4 & 0 & 0 \\ 0 & 1 & 1/2 & -3/2 & 2 & 0 \\ 0 & 0 & 1/2 & -1/2 & 0 & 1 \end{array}\right)$$

$R_1 \to R_1 - \frac{1}{2}R_2$

$1 - 1/2(0) = 1$, $1/2 - \frac{1}{2}(1) = 0$, $1/4 - 1/2(1/2) = 0$ | $1/4 - 1/2(-3/2) = 1$, $0 - 1/2(2) = -1$, $0 - 1/2(0) = 0$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1/2 & -3/2 & 2 & 0 \\ 0 & 0 & 1/2 & -1/2 & 0 & 1 \end{array}\right)$$

$R_3 \to R_3 \cdot 2$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1/2 & -3/2 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 & 2 \end{array}\right)$$

$R_2 \to R_2 - \frac{1}{2}R_3$

$0 - 1/2(0) = 0$, $1 - 1/2(0) = 1$, $\frac{1}{2} - 1/2(1) = 0$ | $-3/2 - 1/2(-1) = -1$, $2 - 1/2(0) = 2$, $0 - 1/2(2) = -1$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 & 2 & -1 \\ 0 & 0 & 1 & -1 & 0 & 2 \end{array}\right)$$

2. (5pts) Find the coordinates of $v_1 = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}$ with respect to the ordered basis $U = \{u_1, u_2, u_3\}$:

$[v_1]_U =$	$\begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$

$$\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ -1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}$$

$1(2) + -1(3) + 2(0)$
$2 + -3 + 0 = -1$

$-1(2) + 2(3) + -1(2)$
$-2 + 6 + -2 = \underline{2}$

$-1(2) + 0(3) + 2(2)$
$-2 + 0 + 4 = \underline{2}$

$= \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$

3. (5pts) Find the coordinates of $v_2 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ with respect to the ordered basis $U = \{u_1, u_2, u_3\}$:

$[v_2]_U =$	$\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ -1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

$1(2) + -1(1) + 0(1)$
$2 + -1 = \underline{1}$
$-1(2) + 2(1) + -1(1)$
$-2 + 2 + -1 = \underline{-1}$
$-1(2) + 0(1) + 2(1)$
$-2 + 2 = \underline{0}$

$= \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

4. (10pts) Consider the vectors $w_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$, $w_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $w_3 = \begin{pmatrix} 7 \\ 6 \\ 4 \end{pmatrix}$. Find the transition matrix from $W = \{w_1, w_2, w_3\}$ to $U = \{u_1, u_2, u_3\}$.

Ans:	$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -2 & -1 & 1 \end{pmatrix}$

$U^{-1} \cdot W$

$W \longrightarrow U$

$(u_1, u_2, u_3)^{-1}(w_1, w_2, w_3)$

$$\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ -1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 & 7 \\ 1 & 1 & 6 \\ 0 & 0 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 1(2) + -1(1) + 0(0) & 1(1) + -1(1) + 0(0) & 1(7) + -1(6) + 0(4) \\ -1(2) + 2(1) + -1(0) & -1(1) + 2(1) + -1(0) & -1(7) + 2(6) + -1(4) \\ -1(2) + 0(1) + 2(0) & -1(1) + 0(1) + 2(0) & -1(7) + 0(6) + 2(4) \end{pmatrix}$$

$\underline{\text{Ans}}$ $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -2 & -1 & 1 \end{pmatrix}$

5. (10pts) If $v = -4w_1 + 3w_2 + 2w_3$, determine the coordinates of v with respect to $U = \{u_1, u_2, u_3\}$. In other words, find $[v]_U$.

$[v]_U =$	$\begin{pmatrix} -2 \\ 5 \\ 7 \end{pmatrix}$

$(u^{-1}\omega)[v]_\mu$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -2 & -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} -4 \\ 3 \\ 2 \end{pmatrix} =$$

$1(-4) + 0(3) + 1(2) = -2$
$-4 + 2 = -2$
$0(-4) + 1(3) + 1(2) = 5$
$3 + 2 =$
$-2(-4) + -1(3) + 1(2) = 7$
$8 + -3 + 2$

$$\begin{pmatrix} -2 \\ 5 \\ 7 \end{pmatrix}$$

Problem 5. [10 pts] Find the dimension of the following vector space:

$$span \left\{ \begin{pmatrix} 1 \\ 3 \\ -4 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}, \begin{pmatrix} -5 \\ -1 \\ -8 \end{pmatrix} \right\}$$

REF
Destroy what has no pivots

Answer:	Span $\left\{ \begin{pmatrix} 1 \\ 3 \\ -4 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} \right\}$ $\mathbb{R}^2$

$R_3 \to R_3 + 4R_1$
$-4 + 4(1) = 0$
$4 + 4(1) = 8$
$-8 + 4(-5) = -28$

$R_2 \to R_2 - 3R_1$
$3 - 3(1) = 0$
$-1 - 3(1) = -4$
$-1 - 3(-5) = 14$

$$\text{Span} \begin{pmatrix} 1 & 1 & -5 \\ 3 & -1 & -1 \\ -4 & 4 & -8 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & -5 \\ 0 & -4 & 14 \\ 0 & 8 & -28 \end{pmatrix}$$

$R_3 \to R_3 + 2R_2$
$8 + 2(-4) = 0$
$-28 + 2(14) = 0$

$$\longrightarrow \begin{pmatrix} 1 & 1 & -5 \\ 0 & -4 & 14 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{R_2 + \frac{R_2}{-4}} \begin{pmatrix} \boxed{1} & 1 & -5 \\ 0 & \boxed{1} & -14/4 \\ 0 & 0 & 0 \end{pmatrix}$$