1 Introduction

posar refs It is a natrual question to ask whether there exist infinitely many prime numbers ending in 3. This can be translated into finding out if the arithmetic progression $a_n = 10n + 3$ for $n \geqslant 0$ contains infinitely many primes. The first terms of this sequence are

$$3, 13, 23, 33, 43, 53, 63, 73, 83, \ldots$$

and indeed $3, 13, 23, 43, 53, 73$ and 83 are primes. Our question is easily answered using Dirichlet's Theorem, that guarantees that such progression contains infinitely many primes since 10 and 3 are coprime. The theorem only requires the two integers defining the progression, say k and ℓ, to be coprime to guarantee the existence of inifinitely many primes in $a_n = kn + \ell$ for $n \geqslant 0$. The proof of this theorem involves studying advanced properties of L-functions and working with Dirichlet's characters (see [**Dirichlet**]).

However, proving the weaker claim that there are infinitely many primes is very straightforward, as basic divisibility properties are needed. Such is the simplicity, that this was first proved around year 300 BC by Euclid. One can go a little further and —using a proof that very much resembles Euclid's idea— prove that there exist infinitely many primes of the form $n + 1$. This can also be done with the progression $4n + 1$ and $4n + 3$ and was noted in 1856 [**Lebesgue**]. Thus, it is again natural to ask in what progressions $kn + \ell$ a "Euclidean proof" can be found to show that infinitely many primes exist in it.

In 1912, Issai Schur proved that if $\ell^2 \equiv 1 \pmod{k}$, then infinitely many primes could be found following Euclid's idea. In 1988, M. Ram Murty made the term "Euclidean proof" precise and further proved that this was the only case where a "Euclidean proof" could be established. Therefore, Dirichlet's Theorem cannot be fully proved *à la Euclid*. In fact, Schur and Murty show us that Euclidean proofs are rather restricted, being only available when $\ell^2 \equiv 1 \pmod{k}$. Although many proofs only using elemental mathematics exist for specific progressions, in this thesis we are only looking for proofs that mimic Euclid's idea.

The objective of this thesis will be to understand every detail that leads to Schur and Murty's theorems. Once their results stand in their full form we will use the ideas their ideas to build our own *automated proof generator*, which will consist of a code that returns a full Euclidean proof of the infinity of primes in $kn + \ell$. The values of k and ℓ will be supplied by the user, and a proof resembling that of Euclid will be automatically generated, specifically tailored for the given progression. Some cases have already been discussed in the literature. For instance, Euclidean proofs for every ℓ satisfying $\ell^2 \equiv 1 \pmod{24}$ is given in ... . However, giving a complete and systematic method to construct Euclidean proofs for the infinite family of artihmetic progressions satisfying $\ell^2 \equiv 1 \pmod{k}$ is something new to the literature.

This document will be structured as follows. In **??** we will briefly discuss the basic notions we will use to proof the main theorems. In **??** we will proof Schur's and Murty's Theorems in detail. In **??** we will use the ideas developed in the previous section to build Euclidean proofs for every possible value of k and ℓ satisfying Schur's and Murty's conditions. Conclusions and further work will be described in **??**.