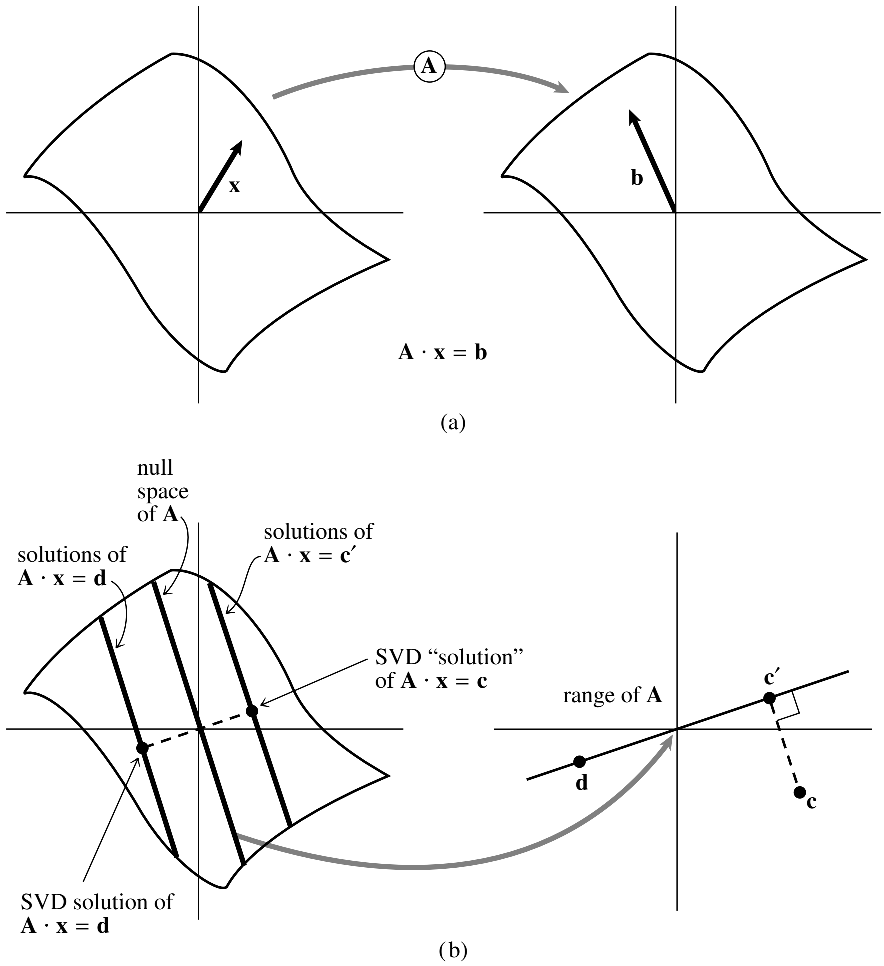

Figure 2.6.1. (a) A nonsingular matrix $\mathbf{A}$ maps a vector space into one of the same dimension. The vector $\mathbf{x}$ is mapped into $\mathbf{b}$, so that $\mathbf{x}$ satisfies the equation $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$. (b) A singular matrix $\mathbf{A}$ maps a vector space into one of lower dimensionality, here a plane into a line, called the “range” of $\mathbf{A}$. The “nullspace” of $\mathbf{A}$ is mapped to zero. The solutions of $\mathbf{A} \cdot \mathbf{x} = \mathbf{d}$ consist of any one particular solution plus any vector in the nullspace, here forming a line parallel to the nullspace. Singular value decomposition (SVD) selects the particular solution closest to zero, as shown. The point $\mathbf{c}$ lies outside of the range of $\mathbf{A}$, so $\mathbf{A} \cdot \mathbf{x} = \mathbf{c}$ has no solution. SVD finds the least-squares best compromise solution, namely a solution of $\mathbf{A} \cdot \mathbf{x} = \mathbf{c}'$, as shown.

In the discussion since equation (2.6.6), we have been pretending that a matrix either is singular or else isn’t. That is of course true analytically. Numerically, however, the far more common situation is that some of the w_j’s are very small but nonzero, so that the matrix is ill-conditioned. In that case, the direct solution methods of LU decomposition or Gaussian elimination may actually give a formal solution to the set of equations (that is, a zero pivot may not be encountered); but the solution vector may have wildly large components whose algebraic cancellation, when multiplying by the matrix $\mathbf{A}$, may give a very poor approximation to the right-hand vector $\mathbf{b}$. In such cases, the solution vector $\mathbf{x}$ obtained by *zeroing* the