Online Appendix

A Non-linear characteristics model

A.1 Consumer problem in the non-linear model

This paper focuses on the linear characteristics model, $\boldsymbol{z}_t = \boldsymbol{A}\boldsymbol{x}_t$. However, most of our analysis generalises to a non-linear characteristics setting where we assume $\boldsymbol{z}_t = \boldsymbol{F}(\boldsymbol{x}_t)$ for some concave, increasing function $\boldsymbol{F} : \mathbb{R}^K_{\geq 0} \to \mathbb{R}^J_{\geq 0}$. Here, we provide an analogue notion of consistency and the relevant Afriat Theorem for such non-linear technologies. For convenience, we take $\boldsymbol{F}$ to be differentiable, with an associated (denominator layout) $K \times J$ matrix derivative at $\boldsymbol{x}_t$ given by,

$$\nabla \boldsymbol{F}(\boldsymbol{x}_t) := \begin{pmatrix} \frac{\partial \boldsymbol{z}^c_t}{\partial \boldsymbol{x}_t} & | & \frac{\partial \boldsymbol{z}^a_t}{\partial \boldsymbol{x}_t} \end{pmatrix} = \begin{pmatrix} \frac{\partial z^c_{1,t}}{\partial x_{1,t}} & \cdots & \frac{\partial z^c_{J_1,t}}{\partial x_{1,t}} & \frac{\partial z^a_{1,t}}{\partial x_{1,t}} & \cdots & \frac{\partial z^a_{J_2,t}}{\partial x_{1,t}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial z^c_{1,t}}{\partial x_{K,t}} & \cdots & \frac{\partial z^c_{J_1,t}}{\partial x_{K,t}} & \frac{\partial z^a_{1,t}}{\partial x_{K,t}} & \cdots & \frac{\partial z^a_{J_2,t}}{\partial x_{K,t}} \end{pmatrix}.$$

If instead $\boldsymbol{F}$ is only superdifferentiable, one must replace all gradients with the superdifferential.

Use the augmented notation for $\tilde{\boldsymbol{x}}_t$ and $\tilde{\boldsymbol{z}}_t$ from Section 3.1. In addition, define the augmented technology function $\tilde{\boldsymbol{F}} : \mathbb{R}^{2K}_{\geq 0} \to \mathbb{R}^{J+J2}_{\geq 0}$ via,

$$\tilde{\boldsymbol{F}}(\tilde{\boldsymbol{x}}_t) := \begin{pmatrix} \boldsymbol{F}(\boldsymbol{x}_t) \\ \boldsymbol{F}^a(\boldsymbol{x}_{t-1}) \end{pmatrix}, \tag{20}$$

where $\boldsymbol{F}^a : \mathbb{R}^{2K}_{\geq 0} \to \mathbb{R}^{J2}_{\geq 0}$ is the function defined by taking the last J_2 rows of $\boldsymbol{F}$. Note that the derivative of this extended vector-valued function is a $2K \times (J + J_2)$ matrix given by,

$$\nabla \tilde{\boldsymbol{F}}(\tilde{\boldsymbol{x}}_t) = \begin{pmatrix} \frac{\partial \boldsymbol{z}^c_t}{\partial \boldsymbol{x}_t} & | & \frac{\partial \boldsymbol{z}^a_t}{\partial \boldsymbol{x}_t} & | & \boldsymbol{0}_{K\times J_2} \\ \boldsymbol{0}_{K\times J_1} & | & \boldsymbol{0}_{K\times J_2} & | & \frac{\partial \boldsymbol{z}^a_{t-1}}{\partial \boldsymbol{x}_{t-1}} \end{pmatrix} = \begin{pmatrix} \nabla \boldsymbol{F}(\boldsymbol{x}_t) & | & \boldsymbol{0}_{K\times J_2} \\ \boldsymbol{0}_{K\times J} & | & \nabla \boldsymbol{F}^a(\boldsymbol{x}_{t-1}) \end{pmatrix} \tag{21}$$

where $\nabla \boldsymbol{F}^a(\boldsymbol{x}_{t-1})$ is the $K \times J_2$ submatrix found by taking the last J_2 columns of $\nabla \boldsymbol{F}(\boldsymbol{x}_{t-1})$. and we use the fact that $\frac{\partial \boldsymbol{z}^a_{t-1}}{\partial \boldsymbol{x}_t} = \boldsymbol{0}_{K\times J_2}$ and $\begin{pmatrix} \frac{\partial \boldsymbol{z}^c_t}{\partial \boldsymbol{x}_{t-1}} & | & \frac{\partial \boldsymbol{z}^a_t}{\partial \boldsymbol{x}_{t-1}} \end{pmatrix} = \begin{pmatrix} \boldsymbol{0}_{K\times J_1} & | & \boldsymbol{0}_{K\times J_2} \end{pmatrix}$.

Using this augmented notation, our model of interest is,

$$\max_{\{\boldsymbol{x}_t\}} \sum_{t=1}^{T} \beta^{t-1} u\left(\tilde{\boldsymbol{z}}_t\right) \text{ subject to } \sum_{t=1}^{T} \boldsymbol{\rho}'_t \boldsymbol{x}_t = W,\ \tilde{\boldsymbol{z}}_t = \tilde{\boldsymbol{F}}(\tilde{\boldsymbol{x}}_t), \tag{22}$$

where we use $\{\boldsymbol{x}_t\}$ as short-hand notation for the set of T K-vectors $\{\boldsymbol{x}_t\}_{t=1}^T$.

A.2 Consistency in the non-linear model

We now formalise the notion of "*consistency*" when the transformation technology is non-linear. As in Section 3.1, this amounts to solving the consumer's constrained maximisation problem defined in (22). Indeed, substituting the technology constraint $\tilde{\boldsymbol{z}}_t = \boldsymbol{F}(\tilde{\boldsymbol{x}}_t)$ into the objective function, the consumer's problem becomes,

$$\max_{\{\boldsymbol{x}_t\}} \sum_{t=1}^{T} \beta^{t-1} u(\tilde{\boldsymbol{F}}(\tilde{\boldsymbol{x}}_t)) \quad \text{subject to} \quad \sum_{t=1}^{T} \boldsymbol{\rho}_t' \boldsymbol{x}_t = W,$$

which has the associated Lagrangian,

$$\mathcal{L}(\{\boldsymbol{x}_t\}, \lambda) = \sum_{t=1}^{T} \beta^{t-1} u(\tilde{\boldsymbol{F}}(\tilde{\boldsymbol{x}}_t)) - \lambda \left\{ \sum_{t=1}^{T} \boldsymbol{\rho}_t' \boldsymbol{x}_t - W \right\}. \tag{23}$$

The first-order necessary conditions then follow analogously to Section 3.1, except that the derivative of the augmented characteristic vector with respect to the augmented market goods vector is now,

$$\frac{\partial \tilde{\boldsymbol{z}}_t}{\partial \tilde{\boldsymbol{x}}_t} = \frac{\partial \tilde{\boldsymbol{F}}(\tilde{\boldsymbol{x}}_t)}{\partial \tilde{\boldsymbol{x}}_t} = \boldsymbol{\nabla} \tilde{\boldsymbol{F}}(\tilde{\boldsymbol{x}}_t),$$

as defined in (21). Hence, following the same simplification steps as in Section 3.1, the first-order conditions reduce to,

$$\boldsymbol{\rho}_t = \frac{\beta^{t-1}}{\lambda} \left(\boldsymbol{\nabla} \boldsymbol{F}(\boldsymbol{x}_t) \begin{bmatrix} \partial_{\boldsymbol{z}_t^c} u(\tilde{\boldsymbol{z}}_t) \\ \partial_{\boldsymbol{z}_t^a} u(\tilde{\boldsymbol{z}}_t) \end{bmatrix} + \beta \boldsymbol{\nabla} \boldsymbol{F}^a(\boldsymbol{x}_t) \left[\partial_{\boldsymbol{z}_t^a} u(\tilde{\boldsymbol{z}}_{t+1}) \right] \right), \tag{24}$$

where $\boldsymbol{\nabla} \boldsymbol{F}^a(\boldsymbol{x}_t)$ is the $K \times J_2$ submatrix found by taking the last J_2 columns of $\boldsymbol{\nabla} \boldsymbol{F}(\boldsymbol{x}_t)$. [I was perhaps anticipating $\boldsymbol{\nabla} \boldsymbol{F}^a(\boldsymbol{x}_{t+1})$ in the second term here rather than $\boldsymbol{\nabla} \boldsymbol{F}^a(\boldsymbol{x}_t)$. I have followed it through though and am confident the math tells me the latter. Do you agree?] This gives rise to our formal definition of "*consistency*" in the non-linear model as follows:

Definition A.1. The data $\{i_t, \boldsymbol{p}_t; \boldsymbol{x}_t\}_{t \in \{1,\ldots,T\}}$ *are consistent with* the non-linear one-lag habits-over-characteristics model given the increasing, concave technology $\boldsymbol{F}$ if there exists a locally non-satiated, superdifferentiable, and concave utility/felicity function $u(.)$ and positive constants λ and β such that for all $t \in \{1, \ldots, T-1\}$,

$$\boldsymbol{\rho}_t \geq \frac{\beta^{t-1}}{\lambda} \left(\boldsymbol{\nabla} \boldsymbol{F}(\boldsymbol{x}_t) \begin{bmatrix} \partial_{\boldsymbol{z}_t^c} u(\tilde{\boldsymbol{z}}_t) \\ \partial_{\boldsymbol{z}_t^a} u(\tilde{\boldsymbol{z}}_t) \end{bmatrix} + \beta \boldsymbol{\nabla} \boldsymbol{F}^a(\boldsymbol{x}_t) \left[\partial_{\boldsymbol{z}_t^a} u(\tilde{\boldsymbol{z}}_{t+1}) \right] \right), \tag{25}$$

with equality for all k such that $x_t^k > 0$, and where $\tilde{\boldsymbol{z}}_t = \tilde{\boldsymbol{F}}(\tilde{\boldsymbol{x}}_t)$ for all $t \in \{1, \ldots, T\}$ and $\rho_t^j = p_t^j / \prod_{s=1}^{s=t} (1 + i_s)$.

It follows by substituting in the shadow discounted prices defined in (SP_0) and (SP_1) that Definition A.1 can be equivalently stated as follows:

Definition A.2. The data $\{i_t, \boldsymbol{p}_t; \boldsymbol{x}_t\}_{t\in\{1,\ldots,T\}}$ *are consistent with* the non-linear one-lag habits-over-characteristics model given the increasing, concave technology $\boldsymbol{F}$ if there exists a locally non-satiated, superdifferentiable, and concave utility/felicity function $u(.)$ and positive constants λ and β such that for all $t \in \{1, \ldots, T-1\}$,

$$\boldsymbol{\rho}_t \geq \boldsymbol{\nabla F}(\boldsymbol{x}_t)\boldsymbol{\pi}_t^0 + \boldsymbol{\nabla F}^a(\boldsymbol{x}_t)\boldsymbol{\pi}_{t+1}^1, \qquad (\star_N)$$

with equality for all k such that $x_t^k > 0$, and where $\tilde{\boldsymbol{z}}_t = \tilde{\boldsymbol{F}}(\tilde{\boldsymbol{x}}_t)$ for all $t \in \{1, \ldots, T\}$ and $\rho_t^j = p_t^j / \prod_{s=1}^{s=t}(1+i_s)$.

Clearly, Definitions A.1 and A.2 nest the linear one-lag habits-over-characteristics model when $\boldsymbol{F}(\boldsymbol{x}_t) = \boldsymbol{A}\boldsymbol{x}_t$. The key difference when $\boldsymbol{F}(\boldsymbol{x}_t) \neq \boldsymbol{A}\boldsymbol{x}_t$ is that the marginal product of the market goods in terms of the characteristics is no longer independent of demand. This of course also means that the price of a market good is no longer a linear combination of the shadow discounted prices. Rather, prices become a non-linear function of shadow prices that vary with demand. Using this more general, non-linear notion of consistency, we now derive testable empirical conditions involving only observables.

A.3 Afriat conditions in the non-linear model

There are two ways to present this Theorem (they differ in $(B2_N)$). The first follows directly from the consistency definition, hence an identical proof to the main Afriat Theorem 3.1. The second is the analogue of the statement from Blow et al. (2008), which doesn't follow as directly so I have provided a proof (which I am not totally convinced by...). Not sure which is better? I wondered why Blow et al. (2008) for example chose to state this condition differently to the main theorem? Am I missing something?

A.3.1 APPROACH 1

Theorem A.1. The following statements are equivalent:

(A_N) The data $\{i_t, \boldsymbol{p}_t; \boldsymbol{x}_t\}_{t\in\{1,\ldots,T\}}$ are consistent with the one-lag habits model given the increasing, concave technology $\boldsymbol{F}$.

(B_N) There exist T J-vector shadow discounted prices $\{\boldsymbol{\pi}_t^0\}_{t\in\{1,\ldots,T\}}$, T J_2-vector shadow discounted prices $\{\boldsymbol{\pi}_t^1\}_{t\in\{1,\ldots,T\}}$ and positive constants β and λ such that,

$$0 \leq \sum_{\forall s,t\in\sigma} \tilde{\boldsymbol{\pi}}_s' (\tilde{\boldsymbol{z}}_t - \tilde{\boldsymbol{z}}_s) \qquad \forall\, \sigma \subseteq \{1, \ldots, T\} \qquad (B1_N)$$

$$\rho_t^k \geq [\boldsymbol{\nabla F}(\boldsymbol{x}_t)]_k \boldsymbol{\pi}_t^0 + [\boldsymbol{\nabla F}^a(\boldsymbol{x}_t)]_k \boldsymbol{\pi}_{t+1}^1 \qquad \forall\, k, t \in \{1, \ldots, T-1\} \qquad (B2_N)$$

$$\rho_t^k = [\boldsymbol{\nabla F}(\boldsymbol{x}_t)]_k \boldsymbol{\pi}_t^0 + [\boldsymbol{\nabla F}^a(\boldsymbol{x}_t)]_k \boldsymbol{\pi}_{t+1}^1 \qquad \text{if } x_t^k > 0,\ \forall\, k, t \in \{1, \ldots, T-1\} \qquad (B3_N)$$

where $[\boldsymbol{\nabla F}(\boldsymbol{x}_t)]_k$ is the J-column vector corresponding to the k-th row of $\boldsymbol{\nabla F}(\boldsymbol{x}_t)$, $[\boldsymbol{\nabla F}^a(\boldsymbol{x}_t)]_k$ is the J_2-column vector corresponding to the k-th row of $\boldsymbol{\nabla F}^a(\boldsymbol{x}_t)$, and $\tilde{\boldsymbol{\pi}}_t := \frac{\lambda}{\beta^{t-1}} \left[\boldsymbol{\pi}_t^{0\prime}, \boldsymbol{\pi}_t^{1\prime}\right]'$.

Proof. Identical to Theorem 3.1 using the updated notion of consistency given in Definition A.2. □

A.3.2 APPROACH 2

Theorem A.2. The following statements are equivalent:

(A_N) The data $\{i_t, \boldsymbol{p}_t; \boldsymbol{x}_t\}_{t\in\{1,\ldots,T\}}$ are consistent with the one-lag habits model given the increasing, concave technology $\boldsymbol{F}$.

(B_N) There exist T J-vector shadow discounted prices $\{\boldsymbol{\pi}_t^0\}_{t\in\{1,\ldots,T\}}$, T J_2-vector shadow discounted prices $\{\boldsymbol{\pi}_t^1\}_{t\in\{1,\ldots,T\}}$ and positive constants β and λ such that,

$$0 \le \sum_{\forall s,t\in\sigma} \tilde{\boldsymbol{\pi}}_s' \left(\tilde{\boldsymbol{z}}_t - \tilde{\boldsymbol{z}}_s\right) \qquad \forall\, \sigma \subseteq \{1,\ldots,T\} \qquad (B1_N)$$

$$\boldsymbol{\rho}_t'(\boldsymbol{x}_s - \boldsymbol{x}_t) \ge \boldsymbol{\pi}_t^{0\prime}(\boldsymbol{z}_s - \boldsymbol{z}_t) + \boldsymbol{\pi}_{t+1}^{1\prime}(\boldsymbol{z}_s^a - \boldsymbol{z}_t^a) \quad \forall\, s,t \in \{1,\ldots,T-1\} \qquad (B2_N)$$

where $\tilde{\boldsymbol{\pi}}_t := \frac{\lambda}{\beta^{t-1}}\left[\boldsymbol{\pi}_t^{0\prime}, \boldsymbol{\pi}_t^{1\prime}\right]'$.

Proof.

$(A_N) \Rightarrow (B_N)$: Assume the data $\{i_t, \boldsymbol{p}_t; \boldsymbol{x}_t\}_{t\in\{1,\ldots,T\}}$ are consistent. The proof that (A_N) implies $(B1_N)$ comes immediately from Theorem 3.1, since this portion of the proof was independent of the technology function $\boldsymbol{F}$. Hence, it remains to show that (A_N) implies $(B2_N)$. By concavity of the technology $\boldsymbol{F}$, we have for all $s,t \in \{1,\ldots,T\}$ that,

$$\boldsymbol{z}_s \le \boldsymbol{z}_t + \boldsymbol{\nabla F}(\boldsymbol{x}_t)'(\boldsymbol{x}_s - \boldsymbol{x}_t) \qquad (26)$$

and,

$$\boldsymbol{z}_s^a \le \boldsymbol{z}_t^a + \boldsymbol{\nabla F}^a(\boldsymbol{x}_t)'(\boldsymbol{x}_s - \boldsymbol{x}_t). \qquad (27)$$

Pre-multiplying Equations (26) and (27) by some $\boldsymbol{\pi}_t^{0\prime} \ge 0$ and $\boldsymbol{\pi}_{t+1}^{1\prime} \ge 0$ respectively yields,

$$\boldsymbol{\pi}_t^{0\prime}\boldsymbol{z}_s \le \boldsymbol{\pi}_t^{0\prime}\boldsymbol{z}_t + \boldsymbol{\pi}_t^{0\prime}\boldsymbol{\nabla F}(\boldsymbol{x}_t)'(\boldsymbol{x}_s - \boldsymbol{x}_t) \qquad (28)$$

and,

$$\boldsymbol{\pi}_{t+1}^{1\prime}\boldsymbol{z}_s^a \le \boldsymbol{\pi}_{t+1}^{1\prime}\boldsymbol{z}_t^a + \boldsymbol{\pi}_{t+1}^{1\prime}\boldsymbol{\nabla F}^a(\boldsymbol{x}_t)'(\boldsymbol{x}_s - \boldsymbol{x}_t), \qquad (29)$$

for all $s,t \in \{1,\ldots,T-1\}$. Adding expressions (28) and (29) together and rearranging gives,

$$\boldsymbol{\pi}_t^{0\prime}\boldsymbol{z}_s + \boldsymbol{\pi}_{t+1}^{1\prime}\boldsymbol{z}_s^a \le \boldsymbol{\pi}_t^{0\prime}\boldsymbol{z}_t + \boldsymbol{\pi}_{t+1}^{1\prime}\boldsymbol{z}_t^a + \left[\boldsymbol{\pi}_t^{0\prime}\boldsymbol{\nabla F}(\boldsymbol{x}_t)' + \boldsymbol{\pi}_{t+1}^{1\prime}\boldsymbol{\nabla F}^a(\boldsymbol{x}_t)'\right](\boldsymbol{x}_s - \boldsymbol{x}_t), \qquad (30)$$

for all $s,t \in \{1,\ldots,T-1\}$. But by consistency of the data, we have that,

$$\boldsymbol{\rho}_t' \ge \boldsymbol{\pi}_t^{0\prime}\boldsymbol{\nabla F}(\boldsymbol{x}_t)' + \boldsymbol{\pi}_{t+1}^{1\prime}\boldsymbol{\nabla F}^a(\boldsymbol{x}_t)' \qquad \forall\, t \in \{1,\ldots,T-1\}.$$

Substituting this into Equation (30) delivers,

$$\boldsymbol{\pi}_t^{0\prime}\boldsymbol{z}_s + \boldsymbol{\pi}_{t+1}^{1\prime}\boldsymbol{z}_s^a \le \boldsymbol{\pi}_t^{0\prime}\boldsymbol{z}_t + \boldsymbol{\pi}_{t+1}^{1\prime}\boldsymbol{z}_t^a + \boldsymbol{\rho}_t'(\boldsymbol{x}_s - \boldsymbol{x}_t), \qquad \forall\, s,t \in \{1,\ldots,T-1\}$$

which upon rearranging yields,

$$\boldsymbol{\rho}_t'(\boldsymbol{x}_s - \boldsymbol{x}_t) \geq \boldsymbol{\pi}_t^{0\prime}(\boldsymbol{z}_s - \boldsymbol{z}_t) + \boldsymbol{\pi}_{t+1}^{1\prime}(\boldsymbol{z}_s^a - \boldsymbol{z}_t^a) \quad \forall\, s, t \in \{1, \ldots, T-1\}$$

which is exactly ($B2_N$).

$(B_N) \Rightarrow (A_N)$: [This is the direction I'm less convinced by...]

From ($B2_N$) we have,

$$\left[\boldsymbol{\pi}_t^{0\prime}\boldsymbol{z}_t + \boldsymbol{\pi}_{t+1}^{1\prime}\boldsymbol{z}_t^a\right] - \boldsymbol{\rho}_t'\boldsymbol{x}_t \geq \left[\boldsymbol{\pi}_t^{0\prime}\boldsymbol{z}_s + \boldsymbol{\pi}_{t+1}^{1\prime}\boldsymbol{z}_s^a\right] - \boldsymbol{\rho}_t'\boldsymbol{x}_s \quad \forall\, s, t \in \{1, \ldots, T-1\}.$$

It follows that for all $t \in \{1, \ldots, T-1\}$,

$$\boldsymbol{\rho}_t'\boldsymbol{x}_t \geq \boldsymbol{\rho}_t'\mathring{\boldsymbol{x}}_t \quad \Rightarrow \quad \left[\boldsymbol{\pi}_t^{0\prime}\boldsymbol{z}_t + \boldsymbol{\pi}_{t+1}^{1\prime}\boldsymbol{z}_t^a\right] \geq \left[\boldsymbol{\pi}_t^{0\prime}\mathring{\boldsymbol{z}}_t + \boldsymbol{\pi}_{t+1}^{1\prime}\mathring{\boldsymbol{z}}_t^a\right].$$

Since this holds for all $t \in \{1, \ldots, T-1\}$, we can take the summation to obtain,

$$\sum_{t=1}^{T-1} \boldsymbol{\rho}_t'\boldsymbol{x}_t \geq \sum_{t=1}^{T-1} \boldsymbol{\rho}_t'\mathring{\boldsymbol{x}}_t \quad \Rightarrow \quad \sum_{t=1}^{T-1} \left[\boldsymbol{\pi}_t^{0\prime}\boldsymbol{z}_t + \boldsymbol{\pi}_{t+1}^{1\prime}\boldsymbol{z}_t^a\right] \geq \sum_{t=1}^{T-1} \left[\boldsymbol{\pi}_t^{0\prime}\mathring{\boldsymbol{z}}_t + \boldsymbol{\pi}_{t+1}^{1\prime}\mathring{\boldsymbol{z}}_t^a\right]. \tag{31}$$

Now, let us turn to condition ($B1_N$). Notice ($B1_N$) is identical (after accounting for the difference in technology, $\boldsymbol{F}$) to (B1). Hence, assuming ($B1_N$), it follows from the proof of Theorem 3.1 that there exists a locally non-satiated, superdifferentiable, and concave utility function $u(\tilde{\boldsymbol{z}})$ such that,

$$\tilde{\boldsymbol{\pi}}_t = \partial u\left(\tilde{\boldsymbol{z}}_t\right)$$

for all $t \in \{1, \ldots, T\}$. That is to say, ($B1_N$) is equivalent to their existing a well-behaved utility function $u(\tilde{\boldsymbol{z}})$ such that the data $\{\tilde{\boldsymbol{\pi}}_t, \tilde{\boldsymbol{z}}_t\}_{t=1,\ldots,T}$ are consistent with maximising $U(\tilde{\boldsymbol{z}}_t) = \sum_{t=1}^{T} \beta^{t-1} u\left(\tilde{\boldsymbol{z}}_t\right)$ subject to the lifetime budget constraint, $\sum_{t=1}^{T} \boldsymbol{\rho}_t'\boldsymbol{x}_t = W$. Put differently, for any $\mathring{\boldsymbol{z}}_t = \boldsymbol{F}(\mathring{\boldsymbol{x}}_t) = (\mathring{\boldsymbol{z}}_t^c, \mathring{\boldsymbol{z}}_t^a, \mathring{\boldsymbol{z}}_{t-1}^a)$ such that,

$$\sum_{t=1}^{T-1} \left[\boldsymbol{\pi}_t^{0\prime}\boldsymbol{z}_t + \boldsymbol{\pi}_{t+1}^{1\prime}\boldsymbol{z}_t^a\right] \geq \sum_{t=1}^{T-1} \left[\boldsymbol{\pi}_t^{0\prime}\mathring{\boldsymbol{z}}_t + \boldsymbol{\pi}_{t+1}^{1\prime}\mathring{\boldsymbol{z}}_t^a\right],$$

it holds,

$$U(\tilde{\boldsymbol{z}}_t) \geq U(\mathring{\boldsymbol{z}}_t). \tag{32}$$

Combining results (31) and (32), we conclude that for any $\mathring{\boldsymbol{z}}_t = \boldsymbol{F}(\mathring{\boldsymbol{x}}_t)$ such that,

$$\sum_{t=1}^{T-1} \boldsymbol{\rho}_t'\boldsymbol{x}_t \geq \sum_{t=1}^{T-1} \boldsymbol{\rho}_t'\mathring{\boldsymbol{x}}_t,$$

it holds that,

$$U(\tilde{\boldsymbol{z}}_t) \geq U(\mathring{\boldsymbol{z}}_t),$$

meaning the data $\{i_t, \boldsymbol{p}_t; \boldsymbol{x}_t\}_{t\in\{1,\ldots,T\}}$ are consistent with the one-lag habits model given the increasing, concave technology $\boldsymbol{F}$. $\square$

B Extended Lag Model

B.1 Consumer problem in the extended model

The key results in this paper focus on a simple case of the "short memory habits" (SMH) model in which the effects of addictive characteristics persist for only one period. Here, we show that the results extend easily to a more general L-lag SMH model.

Defining $L \in \mathbb{N}$ to be the length of habit persistence, our model of interest becomes,

$$\max_{\{\boldsymbol{x}_t\}_{t=1}^T} \sum_{t=1}^{T} \beta^{t-1} u\left(\boldsymbol{z}_t^c, \boldsymbol{z}_t^a, \boldsymbol{z}_{t-1}^a, \ldots, \boldsymbol{z}_{t-L}^a\right) \text{ subject to } \sum_{t=1}^{T} \boldsymbol{\rho}_t' \boldsymbol{x}_t = W,\ \boldsymbol{z}_t = \boldsymbol{A}\boldsymbol{x}_t,$$

where $\rho_t^j = p_t^j / \prod_{s=1}^{s=t} (1 + i_s)$ denote discounted prices, $\beta = 1/(1+\delta)$ where $\delta \in [0, \infty)$ is the consumer's rate of time preference, and W is the present value of the consumer's lifetime wealth.

With this extended lag dependency, we redefine the augmented vectors and matrices via:

$$\tilde{\boldsymbol{z}}_t := \begin{pmatrix} \boldsymbol{z}_t^c \\ \boldsymbol{z}_t^a \\ \boldsymbol{z}_{t-1}^a \\ \vdots \\ \boldsymbol{z}_{t-L}^a \end{pmatrix} \quad \tilde{\boldsymbol{x}}_t := \begin{pmatrix} \boldsymbol{x}_t \\ \boldsymbol{x}_{t-1} \\ \vdots \\ \boldsymbol{x}_{t-L} \end{pmatrix} \quad \tilde{\boldsymbol{A}} := \begin{pmatrix} \boldsymbol{A} & \boldsymbol{0}_{J\times K} & \cdots & \boldsymbol{0}_{J\times K} \\ \boldsymbol{0}_{J_2\times K} & \boldsymbol{A}^a & \cdots & \boldsymbol{0}_{J_2\times K} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{0}_{J_2\times K} & \boldsymbol{0}_{J_2\times K} & \cdots & \boldsymbol{A}^a \end{pmatrix}, \tag{33}$$

so that $\tilde{\boldsymbol{z}}_t$ is now a $J + LJ_2$ column vector, $\tilde{\boldsymbol{x}}_t$ is a $(L+1)K$ column vector, and $\tilde{\boldsymbol{A}}$ is a $(J + LJ_2) \times (L+1)K$ block matrix. Using this augmented notation, the general model of interest can be more compactly expressed via,

$$\max_{\{\boldsymbol{x}_t\}} \sum_{t=1}^{T} \beta^{t-1} u\left(\tilde{\boldsymbol{z}}_t\right) \text{ subject to } \sum_{t=1}^{T} \boldsymbol{\rho}_t' \boldsymbol{x}_t = W,\ \tilde{\boldsymbol{z}}_t = \tilde{\boldsymbol{A}}\tilde{\boldsymbol{x}}_t. \tag{34}$$

Notice that by setting $L = 1$, we recover the basic model analysed in Section 3.1.

B.2 Consistency in the extended model

The Lagrangian for the constrained optimisation problem associated with our extended lag model is identical to before:

$$\mathcal{L}(\{\boldsymbol{x}_t\}, \lambda) = \sum_{t=1}^{T} \beta^{t-1} u(\tilde{\boldsymbol{A}}\tilde{\boldsymbol{x}}_t) - \lambda \left\{ \sum_{t=1}^{T} \boldsymbol{\rho}_t' \boldsymbol{x}_t - W \right\}. \tag{35}$$

The associated first-order necessary conditions follow as before using the chain rule, noting we now have the following changes in dimensionality:

$$\underbrace{\frac{\partial u(\tilde{\boldsymbol{A}}\tilde{\boldsymbol{x}}_t)}{\partial \boldsymbol{x}_t}}_{(K\times 1)} = \underbrace{\frac{\partial \tilde{\boldsymbol{x}}_t}{\partial \boldsymbol{x}_t}}_{(K\times(L+1)K)} \underbrace{\frac{\partial u(\tilde{\boldsymbol{z}}_t)}{\partial \tilde{\boldsymbol{x}}_t}}_{((L+1)K\times 1)} = \underbrace{\frac{\partial \tilde{\boldsymbol{x}}_t}{\partial \boldsymbol{x}_t}}_{(K\times(L+1)K)} \underbrace{\frac{\partial \tilde{\boldsymbol{z}}_t}{\partial \tilde{\boldsymbol{x}}_t}}_{((L+1)K\times(J+LJ_2))} \underbrace{\frac{\partial u(\tilde{\boldsymbol{z}}_t)}{\partial \tilde{\boldsymbol{z}}_t}}_{((J+LJ_2)\times 1)} \tag{36}$$

where, using our notation defined in (33) we have,

$$\frac{\partial \tilde{\boldsymbol{x}}_t}{\partial \boldsymbol{x}_t} = \left[\boldsymbol{I}_{K\times K} \,\middle|\, \mathbf{0}_{K\times LK}\right],$$

$$\frac{\partial \tilde{\boldsymbol{z}}_t}{\partial \tilde{\boldsymbol{x}}_t} = \frac{\partial(\tilde{\boldsymbol{A}}\tilde{\boldsymbol{x}}_t)}{\partial \tilde{\boldsymbol{x}}_t} = \tilde{\boldsymbol{A}}',$$

$$\frac{\partial u(\tilde{\boldsymbol{z}}_t)}{\partial \tilde{\boldsymbol{z}}_t} = \partial u(\tilde{\boldsymbol{z}}_t) := \left[\partial_{\boldsymbol{z}_t^c} u(\tilde{\boldsymbol{z}}_t)', \partial_{\boldsymbol{z}_t^a} u(\tilde{\boldsymbol{z}}_t)', \partial_{\boldsymbol{z}_{t-1}^a} u(\tilde{\boldsymbol{z}}_t)', \ldots, \partial_{\boldsymbol{z}_{t-L}^a} u(\tilde{\boldsymbol{z}}_t)'\right]',$$

where $\big|$ denotes the horizontal concatenation of the $K \times K$ identity matrix and the $K \times LK$ matrix of zeros, and $\partial u(\tilde{\boldsymbol{z}})$ denotes the superderivative of u at $\tilde{\boldsymbol{z}}$. Repeating the chain rule exercise in (36), except this time differentiating with respect to the l-period lag of market goods, $l \in \{1, \ldots, L\}$, we have,

$$\underbrace{\frac{\partial u(\tilde{\boldsymbol{A}}\tilde{\boldsymbol{x}}_t)}{\partial \boldsymbol{x}_{t-l}}}_{(K\times 1)} = \underbrace{\frac{\partial \tilde{\boldsymbol{x}}_t}{\partial \boldsymbol{x}_{t-l}}}_{(K\times(L+1)K)} \underbrace{\frac{\partial u(\tilde{\boldsymbol{z}}_t)}{\partial \tilde{\boldsymbol{x}}_t}}_{((L+1)K\times 1)} = \underbrace{\frac{\partial \tilde{\boldsymbol{x}}_t}{\partial \boldsymbol{x}_{t-l}}}_{(K\times(L+1)K)} \underbrace{\frac{\partial \tilde{\boldsymbol{z}}_t}{\partial \tilde{\boldsymbol{x}}_t}}_{((L+1)K\times(J+LJ_2))} \underbrace{\frac{\partial u(\tilde{\boldsymbol{z}}_t)}{\partial \tilde{\boldsymbol{z}}_t}}_{((J+LJ_2)\times 1)} \tag{37}$$

where the only new term is,

$$\frac{\partial \tilde{\boldsymbol{x}}_t}{\partial \boldsymbol{x}_{t-l}} = \left[\mathbf{0}_{K\times lK} \,\middle|\, \boldsymbol{I}_{K\times K} \,\middle|\, \mathbf{0}_{K\times(L-l)K}\right].$$

It follows from these intermediate calculations of the vector derivatives that,

$$\frac{\partial u(\tilde{\boldsymbol{A}}\tilde{\boldsymbol{x}}_t)}{\partial \boldsymbol{x}_t} = \left[\boldsymbol{I}_{K\times K} \,\middle|\, \mathbf{0}_{K\times KL}\right] \tilde{\boldsymbol{A}}' \partial u(\tilde{\boldsymbol{z}}_t)$$

and for all $l \in \{1, \ldots, L\}$,

$$\frac{\partial u(\tilde{\boldsymbol{A}}\tilde{\boldsymbol{x}}_{t+l})}{\partial \boldsymbol{x}_t} = \left[\mathbf{0}_{K\times lK} \,\middle|\, \boldsymbol{I}_{K\times K} \,\middle|\, \mathbf{0}_{K\times(L-l)K}\right] \tilde{\boldsymbol{A}}' \partial u(\tilde{\boldsymbol{z}}_{t+L}).$$

The first-order necessary conditions associated with the Lagrangian in (35) now follow immediately as,

$$\partial_{\boldsymbol{x}_t}\mathcal{L} = 0 \quad \Rightarrow \quad \lambda \boldsymbol{\rho}_t = \beta^{t-1}\left[\boldsymbol{I}_{K\times K} \,\middle|\, \mathbf{0}_{K\times LK}\right] \tilde{\boldsymbol{A}}' \partial u(\tilde{\boldsymbol{z}}_t) + \sum_{l=1}^{L} \beta^{t-1+l} \left[\mathbf{0}_{K\times lK} \,\middle|\, \boldsymbol{I}_{K\times K} \,\middle|\, \mathbf{0}_{K\times(L-l)K}\right] \tilde{\boldsymbol{A}}' \partial u(\tilde{\boldsymbol{z}}_{t+l})$$

$$\Rightarrow \quad \boldsymbol{\rho}_t = \frac{\beta^{t-1}}{\lambda}\left(\left[\boldsymbol{I}_{K\times K} \,\middle|\, \mathbf{0}_{K\times LK}\right] \tilde{\boldsymbol{A}}' \partial u(\tilde{\boldsymbol{z}}_t) + \sum_{l=1}^{L} \beta^{l} \left[\mathbf{0}_{K\times lK} \,\middle|\, \boldsymbol{I}_{K\times K} \,\middle|\, \mathbf{0}_{K\times(L-l)K}\right] \tilde{\boldsymbol{A}}' \partial u(\tilde{\boldsymbol{z}}_{t+l})\right). \tag{38}$$

But, just as in the simple case of $L = 1$, these first-order conditions can be substantially simplified. Multiplying the conformable block matrices as in Section 3.1 the first-order conditions in (38) reduce to:

$$\boldsymbol{\rho}_t = \frac{\beta^{t-1}}{\lambda}\left(\boldsymbol{A}'\begin{bmatrix}\partial_{\boldsymbol{z}_t^c} u(\tilde{\boldsymbol{z}}_t)\\ \partial_{\boldsymbol{z}_t^a} u(\tilde{\boldsymbol{z}}_t)\end{bmatrix} + \sum_{l=1}^{L}\beta^l(\boldsymbol{A}^a)'\left[\partial_{\boldsymbol{z}_t^a} u(\tilde{\boldsymbol{z}}_{t+l})\right]\right). \tag{39}$$

This gives rise to our formal definition of "*consistency*" in the extended SMH model as follows:

Definition B.1. The data $\{i_t, \boldsymbol{p}_t; \boldsymbol{x}_t\}_{t\in\{1,\ldots,T\}}$ *are consistent with* the L-lag habits-over-characteristics model for given technology $\boldsymbol{A}$ if there exists a locally non-satiated, superdifferentiable, and concave utility/felicity function $u(.)$ and positive constants λ and β such that for all $t \in \{1,\ldots,T-L\}$,

$$\boldsymbol{\rho}_t \geq \frac{\beta^{t-1}}{\lambda}\left(\boldsymbol{A}'\begin{bmatrix}\partial_{\boldsymbol{z}_t^c} u(\tilde{\boldsymbol{z}}_t)\\ \partial_{\boldsymbol{z}_t^a} u(\tilde{\boldsymbol{z}}_t)\end{bmatrix} + \sum_{l=1}^{L}\beta^l(\boldsymbol{A}^a)'\left[\partial_{\boldsymbol{z}_t^a} u(\tilde{\boldsymbol{z}}_{t+l})\right]\right), \tag{40}$$

with equality for all k such that $x_t^k > 0$, and where $\tilde{\boldsymbol{z}}_t = \tilde{\boldsymbol{A}}\tilde{\boldsymbol{x}}_t$ for all $t \in \{1,\ldots,T\}$ and $\rho_t^j = p_t^j / \prod_{s=1}^{s=t}(1+i_s)$.

To interpret this relationship more meaningfully, we define the extended lag shadow prices of characteristics naturally as,

$$\boldsymbol{\pi}_t^l = \frac{\beta^{t-1}}{\lambda}\left[\partial_{\boldsymbol{z}_{t-l}^a} u(\tilde{\boldsymbol{z}}_t)\right], \tag{SP_l}$$

where λ denotes the marginal utility of lifetime wealth. We can interpret $\boldsymbol{\pi}_t^l$ as the discounted willingness-to-pay for the consumption of habit-forming characteristics l periods ago.

Using this extended lag shadow price notation, along with the shadow discounted price of current characteristics, $\boldsymbol{\pi}_t^0$, from Equation (SP_0), Definition B.1 can be equivalently stated as follows:

Definition B.2. The data $\{i_t, \boldsymbol{p}_t; \boldsymbol{x}_t\}_{t\in\{1,\ldots,T\}}$ *are consistent with* the L-lag habits-over-characteristics model for given technology $\boldsymbol{A}$ if there exists a locally non-satiated, superdifferentiable, and concave utility/felicity function $u(.)$ and positive constants λ and β such that for all $t \in \{1,\ldots,T-L\}$,

$$\boldsymbol{\rho}_t \geq \boldsymbol{A}'\boldsymbol{\pi}_t^0 + \sum_{l=1}^{L}(\boldsymbol{A}^a)'\boldsymbol{\pi}_{t+l}^l, \tag{$\star_L$}$$

with equality for all k such that $x_t^k > 0$, and where $\tilde{\boldsymbol{z}}_t = \tilde{\boldsymbol{A}}\tilde{\boldsymbol{x}}_t$ for all $t \in \{1,\ldots,T\}$ and $\rho_t^j = p_t^j / \prod_{s=1}^{s=t}(1+i_s)$.

Clearly, Definitions B.1 and B.2 nest the simple one-lag habits-over-characteristics model when $L = 1$. Indeed, the latter definition is the natural (dynamic) extension of the hedonic pricing equation when habits persist for exactly L periods. It tells us that the discounted prices $\boldsymbol{\rho}_t$ of goods today depend on current discounted shadow prices of the characteristics *as well as* the discounted shadow price of habit-forming characteristics tomorrow, $\boldsymbol{\pi}_{t+1}^1$, the next day, $\boldsymbol{\pi}_{t+2}^2$, and up to L periods in the future, $\boldsymbol{\pi}_{t+3}^3, \ldots, \boldsymbol{\pi}_{t+L}^L$. This is because today's consumption of goods (and the habit-forming characteristics contained therein) affects the agent's marginal utility L periods in the future by building up a habit. A characterisation of this notion of consistency in the extended SMH model follows naturally from Theorem 3.1.

B.3 Afriat conditions in the extended model

Theorem B.1. The following statements are equivalent:

(A_L) The data $\{i_t, \boldsymbol{p}_t; \boldsymbol{x}_t\}_{t\in\{1,\dots,T\}}$ are consistent with the L-lag habits model for given technology $\boldsymbol{A}$.

(B_L) There exist T J-vector shadow discounted prices $\left\{\boldsymbol{\pi}_t^0\right\}_{t\in\{1,\dots,T\}}$, T LJ_2-vector shadow discounted prices $\left\{\boldsymbol{\pi}_t^1,\dots,\boldsymbol{\pi}_t^L\right\}_{t\in\{1,\dots,T\}}$ and positive constants β and λ such that,

$$0 \le \sum_{\forall s,t\in\sigma} \tilde{\boldsymbol{\pi}}_s' \left(\tilde{\boldsymbol{z}}_t - \tilde{\boldsymbol{z}}_s\right) \qquad \forall\, \sigma \subseteq \{1,\dots,T\} \tag{$B1_L$}$$

$$\rho_t^k \ge \boldsymbol{a}_k' \boldsymbol{\pi}_t^0 + \sum_{l=1}^{L} \boldsymbol{a}_k^{a\prime} \boldsymbol{\pi}_{t+l}^l \qquad \forall\, k,t \in \{1,\dots,T-L\} \tag{$B2_L$}$$

$$\rho_t^k = \boldsymbol{a}_k' \boldsymbol{\pi}_t^0 + \sum_{l=1}^{L} \boldsymbol{a}_k^{a\prime} \boldsymbol{\pi}_{t+l}^l \qquad \text{if } x_t^k > 0,\ \forall\, k,t \in \{1,\dots,T-L\} \tag{$B3_L$}$$

where $\boldsymbol{a}_k$ is the J-vector corresponding to the k-th column of $\boldsymbol{A}$, $\boldsymbol{a}_k^a$ is the J_2-vector corresponding to the last J_2 rows of the k-th column of $\boldsymbol{A}$, and $\tilde{\boldsymbol{\pi}}_t := \frac{\lambda}{\beta^{t-1}}\left[\boldsymbol{\pi}_t^{0\prime}, \boldsymbol{\pi}_t^{1\prime}, \dots, \boldsymbol{\pi}_t^{L\prime}\right]'$.

Proof. Identical to Theorem 3.1 with extended lag notation. □

C Solving the consumer's constrained maximisation problem

To define "*consistency*" more formally, we must set-up and solve the consumer's constrained maximisation problem defined in (2).

Substituting the technology constraint $\tilde{\boldsymbol{z}}_t = \tilde{\boldsymbol{A}}\tilde{\boldsymbol{x}}_t$ into the objective function, the consumer's problem becomes,

$$\max_{\{\boldsymbol{x}_t\}} \sum_{t=1}^{T} \beta^{t-1} u(\tilde{\boldsymbol{A}}\tilde{\boldsymbol{x}}_t) \quad \text{subject to} \quad \sum_{t=1}^{T} \boldsymbol{\rho}_t' \boldsymbol{x}_t = W$$

where $\boldsymbol{x}_0 \in \mathbb{R}_+^k$ is taken to be some fixed parameter.

The Lagrangian for this constrained optimisation problem is thus,

$$\mathcal{L}(\{\boldsymbol{x}_t\}, \lambda) = \sum_{t=1}^{T} \beta^{t-1} u(\tilde{\boldsymbol{A}}\tilde{\boldsymbol{x}}_t) - \lambda \left\{ \sum_{t=1}^{T} \boldsymbol{\rho}_t' \boldsymbol{x}_t - W \right\}. \tag{41}$$

To define the first-order necessary conditions for an interior solution to this constrained optimisation problem, we require several vector derivatives. Applying the chain rule for both scalar and vector functions (Felippa, 2004) and using the "denominator layout" as our notational choice we have,

$$\underbrace{\frac{\partial u(\tilde{\boldsymbol{A}}\tilde{\boldsymbol{x}}_t)}{\partial \boldsymbol{x}_t}}_{(K\times 1)} = \underbrace{\frac{\partial \tilde{\boldsymbol{x}}_t}{\partial \boldsymbol{x}_t}}_{(K\times 2K)} \underbrace{\frac{\partial u(\tilde{\boldsymbol{z}}_t)}{\partial \tilde{\boldsymbol{x}}_t}}_{(2K\times 1)} = \underbrace{\frac{\partial \tilde{\boldsymbol{x}}_t}{\partial \boldsymbol{x}_t}}_{(K\times 2K)} \underbrace{\frac{\partial \tilde{\boldsymbol{z}}_t}{\partial \tilde{\boldsymbol{x}}_t}}_{(2K\times(J+J_2))} \underbrace{\frac{\partial u(\tilde{\boldsymbol{z}}_t)}{\partial \tilde{\boldsymbol{z}}_t}}_{((J+J_2)\times 1)} \tag{42}$$

where, recalling our notation defined in (3) we have,

$$\frac{\partial \tilde{\boldsymbol{x}}_t}{\partial \boldsymbol{x}_t} = \left[\boldsymbol{I}_{K\times K} \,\middle|\, \boldsymbol{0}_{K\times K} \right],$$

$$\frac{\partial \tilde{\boldsymbol{z}}_t}{\partial \tilde{\boldsymbol{x}}_t} = \frac{\partial (\tilde{\boldsymbol{A}}\tilde{\boldsymbol{x}}_t)}{\partial \tilde{\boldsymbol{x}}_t} = \tilde{\boldsymbol{A}}',$$

$$\frac{\partial u(\tilde{\boldsymbol{z}}_t)}{\partial \tilde{\boldsymbol{z}}_t} = \partial u(\tilde{\boldsymbol{z}}_t) := \left[\partial_{\boldsymbol{z}_t^c} u(\tilde{\boldsymbol{z}}_t)', \partial_{\boldsymbol{z}_t^a} u(\tilde{\boldsymbol{z}}_t)', \partial_{\boldsymbol{z}_{t-1}^a} u(\tilde{\boldsymbol{z}}_t)' \right]',$$

where $\big|$ denotes the horizontal concatenation of the $K \times K$ identity matrix and the $K \times K$ matrix of zeros, and $\partial u(\tilde{\boldsymbol{z}})$ denotes the superderivative of u at $\tilde{\boldsymbol{z}}$. Repeating the chain rule exercise in (42), except this time differentiating with respect to the one period lag of market goods, we have,

$$\underbrace{\frac{\partial u(\tilde{\boldsymbol{A}}\tilde{\boldsymbol{x}}_t)}{\partial \boldsymbol{x}_{t-1}}}_{(K\times 1)} = \underbrace{\frac{\partial \tilde{\boldsymbol{x}}_t}{\partial \boldsymbol{x}_{t-1}}}_{(K\times 2K)} \underbrace{\frac{\partial u(\tilde{\boldsymbol{z}}_t)}{\partial \tilde{\boldsymbol{x}}_t}}_{(2K\times 1)} = \underbrace{\frac{\partial \tilde{\boldsymbol{x}}_t}{\partial \boldsymbol{x}_{t-1}}}_{(K\times 2K)} \underbrace{\frac{\partial \tilde{\boldsymbol{z}}_t}{\partial \tilde{\boldsymbol{x}}_t}}_{(2K\times(J+J_2))} \underbrace{\frac{\partial u(\tilde{\boldsymbol{z}}_t)}{\partial \tilde{\boldsymbol{z}}_t}}_{((J+J_2)\times 1)} \tag{43}$$

where the only new term is,

$$\frac{\partial \tilde{\boldsymbol{x}}_t}{\partial \boldsymbol{x}_{t-1}} = \left[\mathbf{0}_{K\times K} \,\middle|\, \boldsymbol{I}_{K\times K}\right].$$

It follows from these intermediate calculations of the vector derivatives that,

$$\frac{\partial u(\tilde{\boldsymbol{A}}\tilde{\boldsymbol{x}}_t)}{\partial \boldsymbol{x}_t} = \left[\boldsymbol{I}_{K\times K} \,\middle|\, \mathbf{0}_{K\times K}\right] \tilde{\boldsymbol{A}}' \partial u(\tilde{\boldsymbol{z}}_t)$$

and,

$$\frac{\partial u(\tilde{\boldsymbol{A}}\tilde{\boldsymbol{x}}_{t+1})}{\partial \boldsymbol{x}_t} = \left[\mathbf{0}_{K\times K} \,\middle|\, \boldsymbol{I}_{K\times K}\right] \tilde{\boldsymbol{A}}' \partial u(\tilde{\boldsymbol{z}}_{t+1}).$$

The first-order necessary conditions associated with the Lagrangian[21] in (41) now follow immediately as,

$$\begin{aligned}
\partial_{\boldsymbol{x}_t}\mathcal{L} = 0 \quad &\Rightarrow \quad \lambda \boldsymbol{\rho}_t = \beta^{t-1}\left[\boldsymbol{I}_{K\times K} \,\middle|\, \mathbf{0}_{K\times K}\right] \tilde{\boldsymbol{A}}' \partial u(\tilde{\boldsymbol{z}}_t) + \beta \left[\mathbf{0}_{K\times K} \,\middle|\, \boldsymbol{I}_{K\times K}\right] \tilde{\boldsymbol{A}}' \partial u(\tilde{\boldsymbol{z}}_{t+1}) \\
&\Rightarrow \quad \boldsymbol{\rho}_t = \frac{\beta^{t-1}}{\lambda}\left(\left[\boldsymbol{I}_{K\times K} \,\middle|\, \mathbf{0}_{K\times K}\right] \tilde{\boldsymbol{A}}' \partial u(\tilde{\boldsymbol{z}}_t) + \beta \left[\mathbf{0}_{K\times K} \,\middle|\, \boldsymbol{I}_{K\times K}\right] \tilde{\boldsymbol{A}}' \partial u(\tilde{\boldsymbol{z}}_{t+1})\right). \qquad (44)
\end{aligned}$$

But recall from (3) that $\tilde{\boldsymbol{A}}$ is a $2K \times (J + J_2)$ block matrix. Hence, these first-order conditions can be substantially simplified. Indeed, since we have conformable partitions of the block matrices,

$$\begin{aligned}
\left[\boldsymbol{I}_{K\times K} \,\middle|\, \mathbf{0}_{K\times K}\right] \tilde{\boldsymbol{A}}' &= \left[\boldsymbol{I}_{K\times K} \,\middle|\, \mathbf{0}_{K\times K}\right] \begin{bmatrix} \boldsymbol{A}' & \mathbf{0}_{K\times J_2} \\ \mathbf{0}_{K\times J} & (\boldsymbol{A}^a)' \end{bmatrix} \\
&= \left[\boldsymbol{I}_{K\times K}\boldsymbol{A}' + \mathbf{0}_{K\times K}\mathbf{0}_{K\times J} \,\middle|\, \boldsymbol{I}_{K\times K}\mathbf{0}_{K\times J_2} + \mathbf{0}_{K\times K}(\boldsymbol{A}^a)'\right] \\
&= \left[\boldsymbol{A}' \,\middle|\, \mathbf{0}_{K\times J_2}\right]
\end{aligned}$$

Analogously,

$$\begin{aligned}
\beta\left[\mathbf{0}_{K\times K} \,\middle|\, \boldsymbol{I}_{K\times K}\right] \tilde{\boldsymbol{A}}' &= \beta\left[\mathbf{0}_{K\times K} \,\middle|\, \boldsymbol{I}_{K\times K}\right] \begin{bmatrix} \boldsymbol{A}' & \mathbf{0}_{K\times J_2} \\ \mathbf{0}_{K\times J} & (\boldsymbol{A}^a)' \end{bmatrix} \\
&= \beta\left[\mathbf{0}_{K\times K}\boldsymbol{A}' + \boldsymbol{I}_{K\times K}\mathbf{0}_{K\times J} \,\middle|\, \mathbf{0}_{K\times K}\mathbf{0}_{K\times J_2} + \boldsymbol{I}_{K\times K}(\boldsymbol{A}^a)'\right] \\
&= \beta\left[\mathbf{0}_{K\times J} \,\middle|\, (\boldsymbol{A}^a)'\right].
\end{aligned}$$

Inserting these simplifications, the first-order conditions in (44) reduce to:

$$\boldsymbol{\rho}_t = \frac{\beta^{t-1}}{\lambda}\left(\left[\boldsymbol{A}' \,\middle|\, \mathbf{0}_{K\times J_2}\right] \partial u(\tilde{\boldsymbol{z}}_t) + \beta\left[\mathbf{0}_{K\times J} \,\middle|\, (\boldsymbol{A}^a)'\right] \partial u(\tilde{\boldsymbol{z}}_{t+1})\right). \qquad (45)$$

[21] Wait for Ian's feedback: may change notation slightly here to stress that the the RHS of the Lagrangian contains a superderivative, so the FONC is really a set equality.

But again, since the supergradient $\partial u(\tilde{\boldsymbol{z}}_t)$ can be partitioned as a $J + J_2$ block vector,

$$\partial u(\tilde{\boldsymbol{z}}_t) := \begin{bmatrix} \begin{bmatrix} \partial_{\boldsymbol{z}_t^c} u(\tilde{\boldsymbol{z}}_t) \\ \partial_{\boldsymbol{z}_t^a} u(\tilde{\boldsymbol{z}}_t) \end{bmatrix} \\ \\ \begin{bmatrix} \partial_{\boldsymbol{z}_{t-1}^a} u(\tilde{\boldsymbol{z}}_t) \end{bmatrix} \end{bmatrix},$$

the first-order conditions in (45) further simplify to,

$$\begin{aligned} \boldsymbol{\rho}_t &= \frac{\beta^{t-1}}{\lambda} \left(\begin{bmatrix} \boldsymbol{A}' \,\big|\, \boldsymbol{0}_{K\times J_2} \end{bmatrix} \begin{bmatrix} \begin{bmatrix} \partial_{\boldsymbol{z}_t^c} u(\tilde{\boldsymbol{z}}_t) \\ \partial_{\boldsymbol{z}_t^a} u(\tilde{\boldsymbol{z}}_t) \end{bmatrix} \\ \begin{bmatrix} \partial_{\boldsymbol{z}_{t-1}^a} u(\tilde{\boldsymbol{z}}_t) \end{bmatrix} \end{bmatrix} + \beta \begin{bmatrix} \boldsymbol{0}_{K\times J} \,\big|\, (\boldsymbol{A}^a)' \end{bmatrix} \begin{bmatrix} \begin{bmatrix} \partial_{\boldsymbol{z}_{t+1}^c} u(\tilde{\boldsymbol{z}}_{t+1}) \\ \partial_{\boldsymbol{z}_{t+1}^a} u(\tilde{\boldsymbol{z}}_{t+1}) \end{bmatrix} \\ \begin{bmatrix} \partial_{\boldsymbol{z}_t^a} u(\tilde{\boldsymbol{z}}_{t+1}) \end{bmatrix} \end{bmatrix} \right) \\ &= \frac{\beta^{t-1}}{\lambda} \left(\boldsymbol{A}' \begin{bmatrix} \partial_{\boldsymbol{z}_t^c} u(\tilde{\boldsymbol{z}}_t) \\ \partial_{\boldsymbol{z}_t^a} u(\tilde{\boldsymbol{z}}_t) \end{bmatrix} + \beta (\boldsymbol{A}^a)' \begin{bmatrix} \partial_{\boldsymbol{z}_t^a} u(\tilde{\boldsymbol{z}}_{t+1}) \end{bmatrix} \right). \end{aligned} \tag{46}$$

Since we assume u to be concave, solutions to this first-order necessary conditions are also sufficient for a global maximum. Hence, we can replace this final equality with an inequality to obtain the full-set of price and data pairs $\{i_t, \boldsymbol{p}_t; \boldsymbol{x}_t\}_{t\in\{1,\ldots,T\}}$ consistent with an interior or corner solution to the consumer's maximisation problem.[22] This gives rise to our formal definition of "*consistency*" given in Definition 3.1.

[22] I've never seen authors write this out explicitly. They seem to just "do it". Is this a rigorous enough argument?

D Testing model consistency via linear programming

Theorem 3.1 is the characteristics model analogue of Theorem 1 in the habits-over-goods model by Crawford (2010). As discussed in Section 3.4, theoretical consistency of the data reduces to a linear programming problem when one commits to a grid search over the discount factor, β. However, in its current form, condition (B) has some disadvantages in its practical implementation. Indeed, since (B1) must hold for all $\sigma \subseteq \{1, \ldots, T\}$, this condition alone requires testing $2^T - 1$ inequalities. This becomes very computationally expensive with the length of panel. To address this issue, we have derived the following equivalent statement to be used when implementing the test for model consistency.

Theorem D.1. The following statements are equivalent:

(A) The data $\{i_t, \boldsymbol{p}_t; \boldsymbol{x}_t\}_{t\in\{1,\ldots,T\}}$ are consistent with the one-lag habits model for given technology $\boldsymbol{A}$.

(L) There exist T numbers $\{V_t\}_{t=1,\ldots,T}$, T J-vector shadow discounted prices $\left\{\boldsymbol{\pi}_t^0\right\}_{t\in\{1,\ldots,T\}}$, T J_2-vector shadow discounted prices $\left\{\boldsymbol{\pi}_t^1\right\}_{t\in\{1,\ldots,T\}}$ and a positive constant β such that,

$$V_s - V_t - \frac{1}{\beta^{t-1}}\left[\boldsymbol{\pi}_t^{0\prime}, \boldsymbol{\pi}_t^{1\prime}\right](\tilde{\boldsymbol{z}}_s - \tilde{\boldsymbol{z}}_t) \leq 0 \quad \forall\, s,t \in \{1,\ldots,T\} \tag{L1}$$

$$\left[\boldsymbol{A}' \,\middle|\, (\boldsymbol{A}^a)'\right]\begin{bmatrix}\boldsymbol{\pi}_t^0\\ \boldsymbol{\pi}_{t+1}^1\end{bmatrix} \leq \boldsymbol{\rho}_t \quad \forall\, t \in \{1,\ldots,T-1\} \tag{L2}$$

$$\left[\boldsymbol{B}_t' \,\middle|\, (\boldsymbol{B}_t^a)'\right]\begin{bmatrix}\boldsymbol{\pi}_t^0\\ \boldsymbol{\pi}_{t+1}^1\end{bmatrix} = \boldsymbol{\rho}_t^+ \quad \forall\, t \in \{1,\ldots,T-1\} \tag{L3}$$

where $\boldsymbol{\rho}_t^+$ is a K_t^+ vector equal to the sub-vector of period t prices for which demands are positive, and $\boldsymbol{B}_t$ and $\boldsymbol{B}_t^a$ are the corresponding $J \times K_t^+$ and $J_2 \times K_t^+$ sub-matrices matrices of $\boldsymbol{A}$ and $\boldsymbol{A}^a$, respectively (as introduced in Section 3.6).

Notice that the original (B1) has been converted to the equivalent constraint (L1), which requires testing only $(T-1)^2$ inequalities. This is a strict improvement on the $2^T - 1$ inequalities involved in testing constraint (B1).

Proof.

Set $\lambda = 1$ in Theorem 3.1 without loss of generality (see Section 3.4). Then, condition (A) is identical to that in Theorem 3.1. Accounting for notational differences, conditions (L2) and (L3) are also identical to conditions (B2) and (B3) in Theorem 3.1, respectively. Hence, the proof reduces to showing that condition (L1) is equivalent to condition (B1) in Theorem 3.1.

$(B1) \Rightarrow (L1)$: Assume (B1) holds. This means that for the specific choice of $\sigma = \{s, t\}$, where $s, t \in \{1, \ldots, T\}$ it holds,

$$0 \leq \tilde{\boldsymbol{\pi}}_s'(\tilde{\boldsymbol{z}}_t - \tilde{\boldsymbol{z}}_s) + \tilde{\boldsymbol{\pi}}_t'(\tilde{\boldsymbol{z}}_s - \tilde{\boldsymbol{z}}_t) = \tilde{\boldsymbol{\pi}}_s'\tilde{\boldsymbol{A}}\tilde{\boldsymbol{x}}_t - \tilde{\boldsymbol{\pi}}_s'\tilde{\boldsymbol{A}}\tilde{\boldsymbol{x}}_s + \tilde{\boldsymbol{\pi}}_t'(\tilde{\boldsymbol{z}}_s - \tilde{\boldsymbol{z}}_t). \tag{47}$$

Now, define Afriat numbers,

$$V_t := \tilde{\boldsymbol{\pi}}_s' \tilde{\boldsymbol{A}} \tilde{\boldsymbol{x}}_t, \quad \forall\, t \in \{1, \ldots T\}.$$

Then, (47) is equivalent to,

$$0 \leq V_t - V_s + \tilde{\boldsymbol{\pi}}_t'(\tilde{\boldsymbol{z}}_s - \tilde{\boldsymbol{z}}_t), \quad \forall\, s, t \in \{1, \ldots T\}.$$

Substituting in the definition for $\tilde{\boldsymbol{\pi}}_t$ from Theorem 3.1 this gives us,

$$V_s - V_t - \frac{1}{\beta^{t-1}} \left[\boldsymbol{\pi}_t^{0\prime},\, \boldsymbol{\pi}_t^{1\prime} \right] (\tilde{\boldsymbol{z}}_s - \tilde{\boldsymbol{z}}_t) \leq 0 \quad \forall\, s, t \in \{1, \ldots, T\},$$

which is constraint (L1).

$(L1) \Rightarrow (B1)$: Assume (L1) holds. Using the augmented notation, this means that

$$0 \leq V_t - V_s + \tilde{\boldsymbol{\pi}}_t'(\tilde{\boldsymbol{z}}_s - \tilde{\boldsymbol{z}}_t), \quad \forall\, s, t \in \{1, \ldots T\}.$$

Summing both sides of the inequality over any choice of $\sigma \subseteq \{1, \ldots T\}$ sees the Afriat numbers cancelling to leave us with,

$$0 \leq \sum_{\forall s, t \in \sigma} \tilde{\boldsymbol{\pi}}_s' \left(\tilde{\boldsymbol{z}}_t - \tilde{\boldsymbol{z}}_s \right) \quad \forall\, \sigma \subseteq \{1, \ldots, T\},$$

which is constraint (B1). $\square$