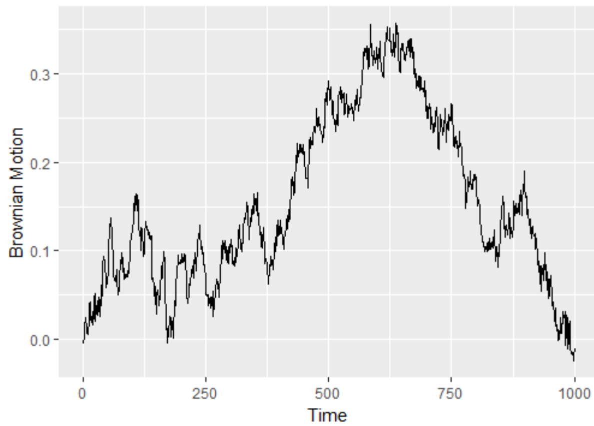
Introduction to stochastic analysis and finance



1 Introduction

1.1 About this notes

In this notes I will give a brief introduction to stochastic analysis, aiming to understand the fundamentals behind stochastic pricing in finance. Even though tools from Stochastic Analysis are often very technical, the aim of this notes is to give a broad understanding of the subject with just the right amount of details.
This notes are aimed to students with fair knowledge of measure theory and some functional analysis. The goal is not only to expose the topics but also to deepen my understanding of the subject.

1.2 What is stochastic analysis and why is it useful?

Stochastic analysis aims to understand randomness related with time, we look at a process, analyze it and we study functions of that process. We define the correct ways to understand measurability and we will ultimately learn about random integrals, all this concepts mature the idea of "random change". We may be studying something that may look like it has random effects but also some well defined deterministic variation, this is usually the case in applications.
The biggest examples of the use of stochastic analysis are financial markets, where we have the price of a stock and it depends in so many factors that we cannot fully model it. Here is where randomness comes into play, even if we cannot model it, we can still understand it via stochastic analysis.

2 Definitions and conditional expectation

Let $(\Omega, \mathcal{S}, \mathbb{P})$ be a probability space.
Meaning that Ω is a set, $\mathcal{S}$ is a σ-algebra defined on Ω and $\mathbb{P}$ is a measure with $\mathbb{P}(\Omega) = 1$.
A process is a set of indexed random variables $(X_t)_{t \in I}$.
(Note that it is not necessary for all the variables on the process to share the σ-algebra).

2.1 Conditional Expectation

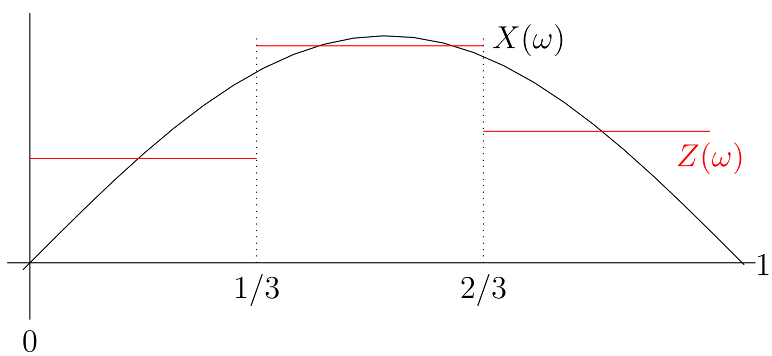
In the first courses of probability one learns the conditional probability $\mathbb{P}(X = x|Y = y)$, and learns to compute expectations of the random variable $X|Y = y$, this concept needs some maturing, and one may ask what happens if $\mathbb{P}(Y = y) = 0$. What happens to a stock price X in the case of default of the market $Y = y$? This leads to a more sofisticated definition of conditional expectation, one with respect to σ algebra, instead of random variables:

Definition 1. *(Conditional Expectation) Let $X : \Omega \to \mathbb{R} \cup \{\infty, -\infty\}$, be a random variable and let $\mathcal{F}$ be a sub-σ-algebra of $\mathcal{S}$ then we say that a **version** of the conditional expectation of X given the σ-algebra $\mathcal{F}$ is a random variable Z such that*

- *Z is integrable ($\int Z d\mathbb{P} < \infty$).*
- *Z is $\mathcal{F}$-measurable*
- *For every $A \in \mathcal{F}$ we have $\int \mathbf{1}_A X d\mathbb{P} = \int \mathbf{1}_A Z d\mathbb{P}$*

Observation 1. *Z can be understood as the best possible averages (in terms of information) with the knowledge only of $\mathcal{F}$. As $\mathcal{F}$ is a sub-σ-algebra of $\mathcal{S}$, it has less sets, meaning that it is more difficult to understand randomness through the eyes of $\mathcal{F}$ than through the eyes of $\mathcal{S}$. This can be understood as $\mathcal{F}$ giving us less information than $\mathcal{S}$.*

Example 1. *Suppose that $\mathbb{P} = \lambda$ (Lebesgue measure) $\mathcal{S} = \mathbb{B}([0, 1])$ (borelians, generated by half open sets), and that*
$\mathcal{F} = \{0, [0, 1], [0, 1/3), [1/3, 2/3), [2/3, 1)\}$



This means that in terms of all information (sets) on $\mathcal{F}$, Z realizes the same averages than X. The distinction is that Z is only $\mathcal{F}$ measurable.

Notation 1. *If such a variable Z exists, we usually denote it by $\mathbf{E}[X|\mathcal{F}]$*

Observation 2. *Observe that $\mathbf{E}[X|\mathcal{F}]$ is a random variable, meaning that we can evaluate $\mathbf{E}[X|\mathcal{F}](\omega)$ for fixed ω*

Exercise 1. *Verify existence of conditional expectation.*
(Hint: Consider $\mu(A) = \int_A X d\mathbb{P}$ and use Radon-Nykodin theorem).
Verify that the map $X \to \mathbf{E}[X|\mathcal{F}]$ is linear and monotone.

Properties 1. *(Of conditional expectation)*

1. *(Linearity)* $\mathbf{E}[\alpha X + Y|\mathcal{F}] = \alpha\mathbf{E}[X|\mathcal{F}] + \mathbf{E}[Y|\mathcal{F}]$ *a.s*

2. *(Monotonicity)* $X \geq 0$ *then* $\mathbf{E}[X|\mathcal{F}] \geq 0$ *a.s*

3. *(Tower property) If mathcalG* $\subseteq \mathcal{F}$ *is another sub* σ *algebra then*

$$\mathbf{E}[\mathbf{E}[X|\mathcal{F}]|\mathcal{G}] = \mathbf{E}[X|\mathcal{G}] \ a.s$$

4. *(Independence of* σ*algebra) If X is independent of $\mathcal{F}$, meaning that for every $A \in \sigma(X), B \in \mathcal{F}$ we have $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$, then*

$$\mathbf{E}[X|\mathcal{F}] = \mathbf{E}[X] \ a.s$$

5. *(Taking out) If Y is $\mathcal{F}$ measurable, then*

$$\mathbf{E}[XY|\mathcal{F}] = Y\mathbf{E}[X|\mathcal{F}] \ a.s$$

6. *(Monotone Convergence) If $0 \leq X_n$ is an **increasing** sequence then*

$$\sup_n \mathbf{E}[X_n|\mathcal{F}] = \mathbf{E}[\sup_n X_n|\mathcal{F}]$$

7. *(Jensen's inequality) If φ is a convex function with $\varphi(X) \in \mathcal{L}^1(X)$ then*

$$\varphi(\mathbf{E}[X|\mathcal{F}]) \leq \mathbf{E}[\varphi(X)|\mathcal{F}]$$

Observation 3. *Analyze the complexity of this equations. For example, most of them are equalities **between** random variables. This condition holds for every $\omega \in \Omega$. It is not a relationship of real numbers, it is a relation between functions (random variables).*

Exercise 2. *Choose 2 properties from 1-6 and prove them.*

One of the fundamental spaces in analysis (and statistics) is the $\mathbf{L}^2(X, \mathcal{S}, \mathbb{P})$ space, we know this is a Hilbert space with it's usual product. Some specific properties of this space are important and should be remarked.
Recall the projection theorem " In a Hilbert space, you can always project into closed convex sets".

Exercise 3. *(The L^2 case)*
Show that $L^2(X, \mathcal{F}, \mathbb{P})$ is a closed convex subset of $L^2(\Omega, \mathcal{S}, \mathbb{P})$.
Use the projection theorem and show that the projector of X into $L^2(X, \mathcal{F}, \mathbb{P})$ is in fact $\mathbf{E}[X|\mathcal{F}]$.
Explain what does this mean in terms of statistics.

3 Martingales

It is common when teaching stochastic analysis, to first explain all the concepts in the discrete case $(X_n)_{n\in\mathbb{N}}$ and then reproducing them in continuous case $(X_t)_{t\in\mathbb{R}^+\cup\{0\}}$, here we will look for a different approach. We will be doing both ambiances together. When definitions have a ".D" this means they are in the discrete setting and ".C" for the continuous one.
The next concept we need is the idea of martingales, it answers the question: How can we model a process for which the best possible information at time t is what we have seen up to time t?. For this to make sense we need to understand what the information up to time t is, in terms of mathematical rigour.

Definition 2. *(Filtration .D)*
A filtration$(\mathcal{F}_n)_{n\in\mathbb{N}}$ for our ambient space is an increasing sequence of sub σ-algebras of $\mathcal{S}$:

$$\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \cdots$$

If $(\mathcal{F}_n)$ is a filtration we say that $(\Omega, \mathcal{S}, (\mathcal{F}_n), \mathbb{P})$ is a filtered probability space.
(X_n) is said to be adapted to $(\mathcal{F}_n)$ if X_n is $\mathcal{F}_n$-measurable for all n

Definition. *(Filtration .C)*
A filtration$(\mathcal{F}_t)_{t\geq 0}$ for our ambient space is an increasing family of sub σ-algebras of $\mathcal{S}$:

$$\mathcal{F}_s \subseteq \mathcal{F}_t \qquad \forall s \leq t$$

If $(\mathcal{F}_t)$ is a filtration we say that $(\Omega, \mathcal{S}, (\mathcal{F}_t), \mathbb{P})$ is a filtered probability space.
(X_t) is said to be adapted to $(\mathcal{F}_t)$ if X_t is $\mathcal{F}_t$-measurable for all t

In both cases, the canonical filtration of (X_s) is the one generated by the process, this means $\mathcal{F}_s^X = \sigma(\{X_i\}_{i\leq s})$

Observation 4. *It follows from the definitions that (X_s) is $(\mathcal{F}_s)$ adapted iff $\mathcal{F}_s^X \subseteq \mathcal{F}_s$ for every s.*