What does maximise a Strichartz estimate?

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Abstract

In dispersive PDE, Strichartz estimates are a fundamental tool in understanding the evolution of waves. The search for extremizers in the corresponding inequalities is an active area of research, and is intimately related with the study of the Fourier extension operator from certain hypersurfaces.

In this talk, I introduce Strichartz estimates for the Schrödinger equation. We will see how some of these inequalities are related to Fourier extension from the paraboloid.

Then, I discuss a sharp Strichartz estimate for the *fourth order* Schrödinger equation in one spatial dimension:

$$i\partial_t u + \partial_x^4 u = 0.$$

A careful analysis of the convolution measure on the curve $\{y = \xi^4\}$ shows that extremizers for this inequality do exist, resolving the dichotomy in [JPS10a].

This talk is based on a joint work with Diogo Oliveira e Silva and René Quilodrán.

Motivation

How much can an operator stretch vectors?

For (sub)linear operators, the "maximal stretch" is the norm of the operator. Vectors who achive this maximal lenght (called *maximisers*) are often special, as they provide particular solutions to problems related to the operator in question; consider for example the operator that maps the initial data to the corresponding solution of a dispersive equation.

The "maximal stretch" is also the minimal (and so the best) constant in the inequality

$$\|Tf\|_Y \leqslant C\|f\|_X$$

where X, Y are normed vector spaces and $T\colon X \to Y$.

Sharp inequalities can be compared.

We consider the homogeneous Schrödinger equation in $\mathbb{R}^d$:

$$\begin{cases} i\partial_t u - \triangle u = 0 \\ u(0,x) = u_0(x)\,, \quad u_0 \in \mathscr{S}(\mathbb{R}^d). \end{cases} \tag{1}$$

The solution is given by

$$u(t,x) = e^{-it\triangle}u_0 := (e^{it|\xi|^2}\widehat{u_0}(\xi))\check{\ },$$

where $\widehat{u_0}$ and $(u_0)\check{\ }$ are the Fourier Transform and the Inverse Fourier Transform on $\mathbb{R}^d$.

Scaling If u is a solution of (1) with initial data u_0, then $u_\lambda(t,x) = u(\lambda^2 t, \lambda x)$ is a solution with initial data $(u_0)_\lambda(x) = u_0(\lambda x)$.

1 Restriction theory

Look closer at the solution of Equation (1):

$$u(t,x) = e^{-it\triangle}u_0 = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{i(x\cdot\xi + t|\xi|^2)}\widehat{u_0}(\xi)\,\mathrm{d}\xi.$$

We interpret the above display inequality as an inverse space-time ($\mathbb{R}^{d+1}$) Fourier Transform:

$$u(t,x) = \mathcal{F}^{-1}(v(\tau,\xi)) = \frac{1}{(2\pi)^{d+1}}\int_{\mathbb{R}^{d+1}} e^{i(t,x)\cdot(\tau,\xi)}v(\tau,\xi)\,\mathrm{d}\tau\,\mathrm{d}\xi,$$

from which:

$$v(\tau,\xi) = 2\pi\,\widehat{u_0}(\xi)\delta(\tau - |\xi|^2),$$

where $\delta(\tau - |\xi|^2)$ is the measure on the paraboloid $\Sigma = \{(\tau,\xi) \in \mathbb{R}^{d+1}, \tau = |\xi|^2\}$.

Definizione 1. Let $\mathcal{M} \subset \mathbb{R}^{d+1}$ be a d-dimensional manifold and μ a smooth measure supported on it. We define the following operators

$$\begin{array}{ll} \textit{Restriction operator} & \textit{Extension operator} \\ \mathcal{R}\colon L^p(\mathbb{R}^{d+1}) \to L^2(\mathcal{M},\mu) & \mathcal{R}^\star\colon L^2(\mathcal{M},\mu) \to L^{p'}(\mathbb{R}^{d+1}) \\ F \mapsto (\mathcal{F}F)_{\restriction\mathcal{M}} & g \mapsto \mathcal{F}^{-1}(g\,\mu) \end{array}$$

Thus, the solution of the Schrödinger equation (1) is given by applying the extension operator $\mathcal{R}^\star$ to the function $\widehat{u_0}$ when $\mathcal{M}$ is the paraboloid $\Sigma = \{(\tau,\xi) \in \mathbb{R}^{d+1}, \tau = |\xi|^2\}$ with the measure $\delta(\tau - |\xi|^2)$.

Theorem 1 (Tomas-Stein)**.** Let $\mathcal{M} \subset \mathbb{R}^{d+1}$ a *compact*[1] d-dimensional manifold with non vanishing Gaussian curvature, and $f \in L^p(\mathbb{R}^{d+1})$, then

$$\|\mathcal{R}f\|_{L^2(\mathcal{M})} \lesssim \|f\|_{L^p(\mathbb{R}^{d+1})} \quad \text{holds for} \quad 1 \leqslant p \leqslant \frac{2(d+2)}{d+4}.$$

The dual statement for the extension operator reads:

Theorem 2 (Dual Tomas-Stein)**.** Let $\mathcal{M} \subset \mathbb{R}^{d+1}$ a *compact* d-dimensional manifold with non vanishing Gaussian curvature, and $g \in L^2(\mathcal{M})$, then

$$\|\mathcal{R}^\star g\|_{L^{p'}(\mathbb{R}^{d+1})} \lesssim \|g\|_{L^2(\mathcal{M})} \quad \text{holds for} \quad p' \geqslant 2 + \frac{4}{d}. \tag{2}$$

Remark 1. The operator $e^{-it\triangle}$ is the composition of $\mathcal{R}^\star$ with the spatial Fourier Transform.

[1] or $\mathcal{M}$ is a hypersurface with a compactly supported measure μ.

Remark 2. The Tomas-Stein inequality (2) holds on *compact* hypersurface. We can get rid of this assumption via rescaling. Consider $u_0 \in L^2(\mathbb{R}^d)$ such that

$$\operatorname{supp}(\widehat{u_0}) \subseteq \mathbf{B}_1^d = \{\xi \in \mathbb{R}^d : |\xi| \leqslant 1\}.$$

Rescaling u_0 with $\lambda > 0$, the Fourier Transform changes with the dual scaling:

$$(u_0)_\lambda(x) = u_0(\lambda x) \quad \Rightarrow \quad \widehat{(u_0)_\lambda}(\xi) = \lambda^{-d}\widehat{u_0}(\xi/\lambda) = \widehat{u_0}^\lambda(\xi),$$

then $\widehat{u_0}^\lambda$ is supported on $\mathbf{B}_\lambda^d = \{\xi \in \mathbb{R}^d : |\xi| \leqslant \lambda\}$. The rescaled extension inequality (2):

$$\left\|\mathcal{R}^\star\widehat{u_0}^\lambda\right\|_{L^{p'}(\mathbb{R}^{d+1})} = \lambda^{-\frac{d+2}{p'}}\left\|\mathcal{R}^\star\widehat{u_0}\right\|_{L^{p'}(\mathbb{R}^{d+1})} \leqslant C\lambda^{-\frac{d}{2}}\left\|\widehat{u_0}\right\|_{L^2(\mathcal{M})} = \left\|\widehat{u_0}^\lambda\right\|_{L^2(\mathcal{M})}$$

holds with the constant $C_\lambda = C\lambda^{-\frac{d}{2}+\frac{d+2}{p'}}$. In particular, for the value $p' = 2 + \frac{4}{d}$ we have $C_\lambda = C$ for every $\lambda > 0$. From Theorem 2, letting $\lambda \to \infty$ we obtain the bound for the whole paraboloid Σ. Since functions with compactly supported Fourier Transform are dense in L^2, with a limiting argument we obtain the extension inequality for all initial data in L^2.

2 Strichartz estimates for Schrödinger equation

Restriction theory gives estimates in time and space only on isotropic Lebesgue space (on $L_t^q(\mathbb{R})L_x^p(\mathbb{R}^d)$ when $q = p$). The paraboloid is invariant under anisotropic scaling

$$(x, t) \mapsto (\lambda x, \lambda^2 t)$$

so it is reasonable to study restriction and extension on anisotropic spaces ($q \neq p$):

$$\left\|e^{-it\triangle}u_0\right\|_{L_t^qL_x^p(\mathbb{R}\times\mathbb{R}^d)} \lesssim \left\|u_0\right\|_{L^2(\mathbb{R}^d)}. \tag{3}$$

Proving this inequality is equivalent to showing either of the following:

- $T := e^{-it\triangle} \quad : L^2(\mathbb{R}^d) \longrightarrow L_t^qL_x^p(\mathbb{R}\times\mathbb{R}^d)$ is bounded,
- $T^\star := (e^{-it\triangle})^\star\colon L_t^{q'}L_x^{p'}(\mathbb{R}\times\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is bounded.

The composition $TT^\star$:

$$e^{-it\triangle}(e^{-is\triangle})^\star\colon L_t^{q'}L_x^{p'}(\mathbb{R}\times\mathbb{R}^d) \to L_t^qL_x^p(\mathbb{R}\times\mathbb{R}^d) \quad \text{is a bounded operator.}$$

We will prove the last bound for $TT^\star$ and, by Hölder and duality, the previous follow.

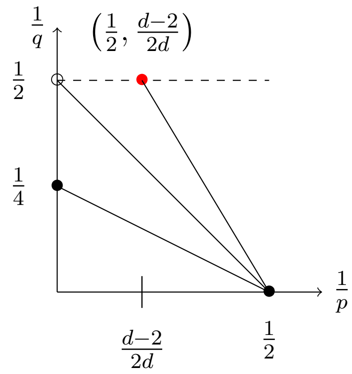
Theorem 3 (Nonendpoint estimates)**.** The operator $TT^\star$ is given by $u \mapsto \int_{-\infty}^{+\infty} e^{-i(t-s)\triangle}u\,\mathrm{d}s$ and the following inequality:

$$\left\|\int_{-\infty}^{\infty} e^{-i(t-s)\triangle}F(s)\,\mathrm{d}s\right\|_{L_t^qL_x^p(\mathbb{R}\times\mathbb{R}^d)} \lesssim \|F\|_{L_t^{q'}L_x^{p'}(\mathbb{R}\times\mathbb{R}^d)} \tag{4}$$

holds true for

$$\frac{2}{q} + \frac{d}{p} = \frac{d}{2} \quad \text{and} \quad \begin{cases} p \in [2, \infty] & \text{if } d = 1 \\ p \in [2, \infty) & \text{if } d = 2 \\ p \in \left[2, \frac{2d}{d-2}\right) & \text{if } d \geqslant 3 \end{cases}$$

Remark 3. The relation between q, p and d can be obtained by scaling (3).
In $d = 2$ the endpoint $(q, p) = (2, \infty)$ has been proved false by Montgomery-Smith [Mon97] with a counterexample involving Brownian motion.
For $d \geqslant 3$, the endpoint $(q, p) = \left(2, \frac{2d}{d-2}\right)$ has been proved by Keel and Tao [KT98].



We start proving L^p-bounds for the kernel in (4):

Lemma 4. We have the following estimates:

$$\underbrace{\left\| e^{-it\triangle} v \right\|_{L^2} = \|v\|_{L^2}}_{\text{Energy estimate}} \qquad \underbrace{\left\| e^{-it\triangle} v \right\|_{L^\infty} \leqslant (4\pi |t|)^{-\frac{d}{2}} \|v\|_{L^1}}_{\text{Decay estimate}}.$$

Interpolating between them for $2 \leqslant p \leqslant \infty$ we obtain:

$$\left\| e^{-it\triangle} v \right\|_{L^p} \leqslant (4\pi |t|)^{-d\left(\frac{1}{2} - \frac{1}{p}\right)} \|v\|_{L^{p'}}.$$

Proof of Theorem 3. From Lemma 4 applied to (4) we have:

$$\left\| \int_{-\infty}^{\infty} e^{-i(t-s)\triangle} F(s) \, \mathrm{d}s \right\|_{L^p_x} \leqslant \int_{-\infty}^{\infty} (4\pi |t-s|)^{-d\left(\frac{1}{2} - \frac{1}{p}\right)} \|F(s)\|_{L^{p'}_x} \, \mathrm{d}s.$$

The right hand side (RHS) of the above can be expressed as a convolution: call $f(t) = \|F(t)\|_{L^{p'}_x}$ and $g(t) = (4\pi |t|)^{-d\left(\frac{1}{2} - \frac{1}{p}\right)}$, then we can estimate the left hand side (LHS) as

$$\|\mathrm{LHS}\|_{L^q_t(\mathbb{R})} \lesssim \|f * g\|_{L^q(\mathbb{R})}.$$

Using *weak Young inequality* for $r > 1$:

$$\|f * g\|_{L^q} \leqslant \|f\|_s \|g\|_{r,\infty} \qquad \text{for all } \ (s, r) \ : \ \frac{1}{s} + \frac{1}{r} = 1 + \frac{1}{q}.$$

In our case $g \in L^{r,\infty}(\mathbb{R})$ where $\dfrac{1}{r} = d\left(\dfrac{1}{2} - \dfrac{1}{p}\right)$. Notice that, by scaling, $\dfrac{1}{q} = \dfrac{d}{2}\left(\dfrac{1}{2} - \dfrac{1}{p}\right)$, then $\dfrac{2}{q} = \dfrac{1}{r}$, which implies $s = q'$, and

$$\|\mathrm{LHS}\|_{L^q_t(\mathbb{R})} \lesssim \|f * g\|_{L^q(\mathbb{R})} \lesssim \|f\|_{q'} \|g\|_{r,\infty} = \|F\|_{L^{q'}_t L^{p'}_x(\mathbb{R}\times\mathbb{R}^d)}.$$

This proves the estimate apart from the endpoint. □

Strichartz estimates are a family of inequalties for dispersive equations in which the norm of the solution to (1) is taken in mixed Lebesgue space $L^q_t(\mathbb{R})L^p_x(\mathbb{R}^d)$:

$$\left\| e^{-it\triangle} u_0 \right\|_{L^q_t(\mathbb{R})L^p_x(\mathbb{R}^d)} \leqslant \mathcal{C} \|u_0\|_{L^2(\mathbb{R}^d)},$$

where the constant $\mathcal{C} = \mathcal{C}(q,p,d)$ depends on exponents and dimension, which have to satisfy:

$$\frac{2}{q}+\frac{d}{p}=\frac{d}{2} \quad \text{with} \quad \begin{cases} p\in[2,\infty] & \text{if } d=1 \\ p\in[2,\infty) & \text{if } d=2 \\ p\in\left[2,\frac{2d}{d-2}\right] & \text{if } d\geqslant 3 \end{cases}.$$

For a given dimension d, a pair (q,p) satisfying the above relation is called admissible.

3 Extremizers for Strichartz estimates

The sharp constant in the inequality (3) is defined by

$$\mathcal{C} := \sup_{u_0\in L^2(\mathbb{R}^d)\setminus\{0\}} \frac{\|e^{-it\triangle}u_0\|_{L^q_t(\mathbb{R})L^p_x(\mathbb{R}^d)}}{\|u_0\|_{L^2(\mathbb{R}^d)}}.$$

Definizione 2. A function f in $L^2(\mathbb{R}^d)$ is an *extremizer* for the inequality (3) if

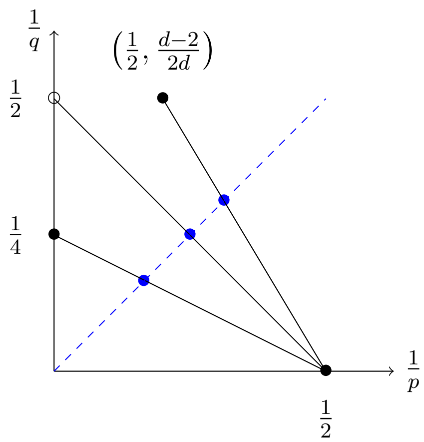
$$\|e^{-it\triangle}f\|_{L^q_t(\mathbb{R})L^p_x(\mathbb{R}^d)} = \mathcal{C}\,\|f\|_{L^2(\mathbb{R}^d)}\,.$$

A sequence $\{f_n\}_{n\in\mathbb{N}}$ in $\mathbf{B}=\{f\in L^2 : \|f\|\leqslant 1\}$ is an *extremizing sequence* for (3) if

$$\|e^{-it\triangle}f\|_{L^q_t(\mathbb{R})L^p_x(\mathbb{R}^d)} \to \mathcal{C} \qquad \text{as } n\to\infty.$$

Remark 4. Due to the several symmetries of the solution, even if extremizing sequences exist, they could not converge to an extremizer in the strong topology! So we can calculate the sharp constant, but find extremizers (when they exist) is a more subtle task.

Remark 5. The figure shows admissible pairs of exponent (q,p) for different dimensions. The blue dots on the diagonal represent the symmetric exponents (q,q), for which we can use results from restriction theory (i.e. Tomas-Stein). The existence of extremizers in dimension $d=1$ was proven by Kunze [Kun03] exploiting concentration-compactness principle. Then Foschi [Fos04] showed that in dimensions $d=1$ and 2, all extremizers are Gaussians. This means that the first two blue dots are solved. Then Shao [Sha08] proved the existence of extremizers in *all* dimensions, and not only for the symmetric case but for *any* non-endpoint admissible pair (q,p).



Lets focus on the symmetric case, when $p=q$ and the inequality (3) becomes:

$$\|e^{-it\triangle}u_0\|_{L^{q(d)}(\mathbb{R}^{d+1})} \leqslant \mathcal{C}\,\|u_0\|_{L^2(\mathbb{R}^d)}\,, \qquad q(d)=2+\frac{4}{d}. \tag{5}$$

It is possible to write the solution using the adjoint Fourier restriction operator:

$$u(t,x)=\int_{\mathbb{R}^d} e^{i(x\cdot\xi+t|\xi|^2)}\widehat{u_0}(\xi)\,\text{đ}\xi = \int_{\mathbb{R}^{d+1}} e^{i(t,x)\cdot(\tau,\xi)}\,2\pi\,\widehat{u_0}(\xi)\,\delta(\tau-|\xi|^2)\,\text{đ}\tau\,\text{đ}\xi,$$

where $đ\xi$ and $đ\tau$ indicate the measures $\frac{d\xi}{(2\pi)^d}$ and $\frac{d\tau}{2\pi}$. Then the solution is an extension of a measure from the paraboloid $\Sigma = \{(\tau, \xi) \in \mathbb{R}^{d+1}, \tau = |\xi|^2\}$ and we can write as

$$u = \mathcal{F}^{-1}(\underbrace{2\pi\, \widehat{u}_0}_{f(\xi)}\ \underbrace{\delta(\tau - |\xi|^2)}_{\sigma_0(\xi,\tau)}) = \mathcal{F}^{-1}(f\,\sigma_0). \tag{6}$$

In dimension $d = 1, 2$ the Strichartz exponent $q(d)$ is a even number, so we can exploit Plancherel in L^2. In dimension $d = 2$ for example:

$$\|u\|_{L^4(\mathbb{R}^3)} = \|u \cdot u\|^{\frac{1}{2}}_{L^2(\mathbb{R}^3)} = \|\mathcal{F}(u) * \mathcal{F}(u)\|^{\frac{1}{2}}_{L^2(\mathbb{R}^3,\ đ\xi)} \overset{(6)}{=} \|f\sigma_0 * f\sigma_0\|^{\frac{1}{2}}_{L^2(\mathbb{R}^3,\ đ\xi)}\,. \tag{7}$$

Then the problem of estimating the L^4-norm of the solution u reduces to estimate the L^2-norm of the convolution of a measure with itself. From now on we will focus only on this 2-dimensional case.

3.1 The Foschi's approach [Fos04]

Using δ-calculus it is possible to write the convolution above as

$$(f\sigma * f\sigma)(x,t) = \iint\limits_{\mathbb{R}^2\times\mathbb{R}^2} f(z)f(y)\,\delta(t - |y|^2 - |z|^2)\delta(x - y - z)\,\mathrm{d}z\,\mathrm{d}y.$$

Writing the expression as a L^2-product with respect the measure $\mu(x,t) = \delta(t - |y|^2 - |z|^2)\delta(x - y - z)\,\mathrm{d}z\,\mathrm{d}y$, and applying Cauchy-Schwarz we obtain:

$$\langle f\otimes f, 1\otimes 1,\rangle\, L^2(\mu(x,t)) \leqslant \|f\otimes f\|_{L^2(\mu(x,t))}\,\|1\otimes 1\|_{L^2(\mu(x,t))}\,,$$

that is

$$|f\sigma_0 * f\sigma_0|^2\,(x,t) \leqslant (|f|^2\,\sigma_0 * |f|^2\,\sigma_0)(x,t)\,(\sigma_0 * \sigma_0)(x,t).$$

To study the convolution of the singular measure $\sigma_0(t,x) = \delta(t - |x|^2)$, we will use some special symmetries of the problem:

$$\begin{aligned}
(\sigma_0 * \sigma_0)(t,x) = \iint\limits_{\mathbb{R}^2\times\mathbb{R}} \delta(t - \tau - |x-y|^2)\,\delta(\tau - |y|^2)\,\mathrm{d}\tau\,\mathrm{d}y \quad &= \int_{\mathbb{R}^2} \delta(t - |y|^2 - |x-y|^2)\,\mathrm{d}y \\
\text{Using \textit{Galilean invariance}:}\quad (t,x) \mapsto (t - |x|^2, 0) \quad &= \int_{\mathbb{R}^2} \delta(t - |x|^2 - 2\,|y|^2)\,\mathrm{d}y \\
\text{and \textit{parabolic dilation}:}\quad (t,x) \mapsto (\lambda^2 t, \lambda x) \quad &= \int_{\mathbb{R}^2} \delta(1 - 2\,|y|^2)\,\mathrm{d}y \\
\text{changing in polar coordinates:}\quad y \mapsto r\omega,\ \omega\in\mathbb{S}^1 \quad &= \int_{\mathbb{S}^1}\int_0^\infty \delta(1 - 2r^2)\,r\,\mathrm{d}r\,\mathrm{d}\omega = \frac{\pi}{2}.
\end{aligned} \tag{8}$$

The convolution $\sigma_0 * \sigma_0$ turn out to be *constant* in every point (t,x) of its support, so

$$\|f\sigma_0 * f\sigma_0\|^2_{L^2(\mathbb{R}^3)} \leqslant \left(\sup_{(\xi,\tau)\in\mathbb{R}^3} |\sigma_0 * \sigma_0|\,(\xi,\tau)\right)\,\|f\|^4_{L^2(\mathbb{R}^2)} = \frac{\pi}{2}\,\|f\|^4_{L^2(\mathbb{R}^2)}\,.$$

The constant above was proven sharp and the extremizers are the functions that realize equality in the Cauchy-Schwarz. Solving a functional equation they are showed to be Gaussians.

4 A fourth order Schrödinger equation

We consider the following fourth order differential equation:

$$\begin{cases} i\partial_t u + \triangle^2 u = 0 & x, t \in \mathbb{R} \\ u(0,x) = f(x) \in L^2(\mathbb{R}). \end{cases} \tag{9}$$

The solution is given by

$$S_0(t)f := e^{it\triangle^2} f = \left(e^{i(x\xi+t\xi^4)}\widehat{f}(\xi)\right)^{\check{}} = \int_{\mathbb{R}} e^{i(x\xi+t\xi^4)}\widehat{f}(\xi)\, đ\xi\,. \tag{10}$$

For the solution we have the corresponding Strichartz estimate:

$$\left\| D_0^{\frac{1}{3}} e^{it\triangle^2} f \right\|_{L^6_{t,x}(\mathbb{R}\times\mathbb{R})} \leqslant \mathbf{S}\, \|f\|_{L^2(\mathbb{R})}\,, \tag{11}$$

where the operator $D_0^{\frac{1}{3}}$ is defined as

$$D_0^{\frac{1}{3}} f(x) := \int_{\mathbb{R}} e^{ix\xi} \left|6\xi^2\right|^{\frac{1}{6}} \widehat{f}(\xi)\, \mathrm{d}\xi.$$

We indicate with $T(t)$ the propagator given by the composition $D_0^{\frac{1}{3}} S_0(t) = D_0^{\frac{1}{3}} e^{it\triangle^2}$. This is defined as

$$T(t)f(x) := \int_{\mathbb{R}} e^{ix\xi} 6^{\frac{1}{6}} \sqrt{w(\xi)} e^{it\xi^4} \widehat{f}(\xi)\, đ\xi, \qquad w(\xi) = |\xi|^{\frac{2}{3}}\,. \tag{12}$$

4.1 Existence of extremisers

Theorem 1. *There exist a maximizer for the Strichartz inequality (11).*

Then, the best constant in (11) is given by

$$\mathbf{S} = \sup_{\substack{f\in L^2(\mathbb{R}),\\ f\neq 0}} \frac{6^{\frac{1}{6}} \left\| |\nabla|^{\frac{1}{3}} e^{it\triangle^2} f \right\|_{L^6_{t,x}(\mathbb{R}\times\mathbb{R})}}{\|f\|_{L^2(\mathbb{R})}} = \max_{\substack{f\in L^2(\mathbb{R}),\\ f\neq 0}} \frac{6^{\frac{1}{6}} \left\| |\nabla|^{\frac{1}{3}} e^{it\triangle^2} f \right\|_{L^6_{t,x}(\mathbb{R}\times\mathbb{R})}}{\|f\|_{L^2(\mathbb{R})}}.$$

In [JPS10b, Theorem 1.8] the authors proved a dichotomy result for extremizers of our Strichartz estimate:

Theorem 2 (Dichotomy, [JPS10b]). *Either*

(i) $\mathbf{S} = \mathrm{S}_{\mathrm{Schr}}$ *and there exist* $f \in L^2(\mathbb{R})$ *and a sequence* $\{a_n\}_{n\in\mathbb{N}}$ *going to infinity as* $n \to \infty$, *such that* $\{e^{ixa_n} f\}_{n\in\mathbb{N}}$ *is an extremizing sequence for* (11), or

(ii) $\mathbf{S} \neq \mathrm{S}_{\mathrm{Schr}}$ *and extremizers for* (11) *exist.*

A solution to the extremizing problem (11) is related to the one for the classical Schrödinger equation. We recall that the sharp constant for the Strichartz estimate for the free propagator $e^{-it\triangle}$ is

$$\mathrm{S}_{\mathrm{Schr}} := \sup_{v \in L^2(\mathbb{R})\setminus\{0\}} \frac{\left\| e^{-it\triangle} v \right\|_{L^6_{t,x}(\mathbb{R}\times\mathbb{R})}}{\|v\|_{L^2(\mathbb{R})}} = \left(\frac{1}{12}\right)^{\frac{1}{12}}. \tag{13}$$

This constant was calculated by Foschi, see [Fos04, Theorem 1.1].

We study the convolution measure on the quartic. The following properties hold.

Proposition 1. *Let $w(\xi) = |\xi|^{\frac{2}{3}}$ and ν be the measure defined by*

$$\nu(\xi,\tau) = \delta(\tau - |\xi|^4)\,\mathrm{d}\xi\,\mathrm{d}\tau.$$

Then the following properties hold for the convolution measure $w\nu * w\nu * w\nu$.

(a) It is absolutely continuous with respect the Lebesgue measure on $\mathbb{R}^2$.

(b) Its support is given by

$$E = \{(\xi,\tau) \in \mathbb{R}^2 \,:\, \tau \geqslant 3^{-3}\,|\xi|^4\}.$$

(c) It is radial *and* homogeneous *of degree zero in ξ, the sense that:*

$$(w\nu * w\nu * w\nu)(\lambda\xi, \lambda^4\tau) = (w\nu * w\nu * w\nu)(\xi,\tau), \quad \text{for every } \lambda > 0.$$

$$\text{and} \quad (w\nu * w\nu * w\nu)(-\xi,\tau) = (w\nu * w\nu * w\nu)(\xi,\tau) \quad \text{for every } \xi \in \mathbb{R}.$$

Remark 1. As we saw in Proposition 1 the value of the convolution measure depends only on one parameter. This because $\nu * \nu * \nu$ is radial and it is constant along branches of the quartic $\tau = \alpha\xi^4$. Let $\alpha(t) = t^{-4}$ the amplitude of the quartic $\tau = \alpha(t)\xi^4$. When t ranges in $(0, 3^{\frac{3}{4}}]$, $\alpha(t)$ gives all possible amplitudes of quartic in the support of $\nu * \nu * \nu$.

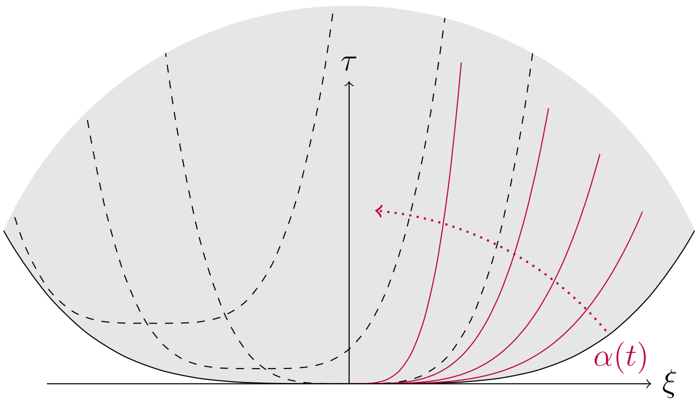


Figure 1: Support of the measure $w\nu * w\nu * w\nu$. Its value in a point (ξ,τ) depends only on $\alpha(t)$.

For a particular function, this allows to reduce the matter to approximating the L^2 nomr of the convolution measure $w\nu*w\nu*w\nu$. We can do so by using a basis of polynomial on $L^2([-1,1])$, to sow that $\mathbf{S} > \mathrm{S}_{\mathrm{Schr}}$, so maximisers to (11) do exist.

5 Brief Summary

1. Maximisers provide special solution in different problems
2. Some dispersive estimates can be seen as Fourier extension from hypersurfaces
3. This new insight leads to studying convolution of measures supported on these hypersurfaces to understand maximising sequences

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