Non-local interactions exhibiting dichotomy

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February 19

Introduction

We'll be looking at the problem of minimizing the energy

$$\mathcal{E}[\rho] = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} K(|x-y|)\rho(x)\rho(y)\,dx\,dy$$

given by an interaction kernel K over the set

$$\mathcal{A}_m = \{\rho \in L^1(\mathbb{R}^N) \ : \ 0 \le \rho \le 1, \ ||\rho||_{L^1(\mathbb{R}^N)} = m\}$$

of densities having total mass m. In this talk, I will be discussing my recent work regarding kernels having a particular shape, a sketch of which is shown in figure 1. The key features of these kernels are that they are attractive at short distances, followed by a repulsive "barrier," followed by decay at infinity. It turns out, for large mass, minimizers do not exist, but we can construct a minimizing sequence that consists of indicators of finitely many disjoint balls that become increasingly far apart.

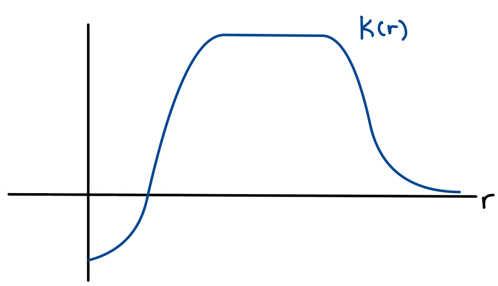

Figure 1: $K : [0, \infty) \to \mathbb{R}$

First, I'll discuss the concentration compactness principle, which is an important tool that allows us to deal with the lack of compactness. In particular, this will create a framework for dealing with the fact that a minimizing sequence may split into many pieces, whose support's become infinitely far apart from one another. Such sequences are referred to as "dichotomous." I'll then give a sketch of how to construct a minimizing sequence for our problem.

1 Motivation

Our problem involves minimizing a non-local interaction energy over the subset $\mathcal{A}_m$. Recall that

$$\mathcal{A}_m = \{\rho \in L^1(\mathbb{R}^N) \ : \ 0 \leq \rho \leq 1, \ ||\rho||_{L^1(\mathbb{R}^N)} = m\}.$$

A natural approach to dealing with such a problem is to consider a minimizing sequence and hope that it has a subsequence that converges to some reasonable limit. But, we are lacking in compactness! If I have a sequence $\{\rho_k\}$ in $\mathcal{A}_m$ which converges weakly in some L^q, its limit may not belong to $\mathcal{A}_m$ if mass "spills out at infinity." The way we will deal with this problem is through Lions' concentration compactness principle, which allows us to pass to a subsequence whose behaviour we can understand.

2 Concentration Compactness

In order to deal with the lack of compactness, we will make use of the following Lemma, first proved by Lions in [4].

Lemma 1 (Concentration Compactness Principle). *Let $\{\rho_n\}$ be a sequence in $L^1(\mathbb{R}^N)$ satisfying*

$$\rho_n \geq 0, \quad \text{and} \quad \int_{\mathbb{R}^N} \rho_n(x)\, dx = m,$$

where $m > 0$ is fixed. Then there exists a subsequence $\{\rho_{n_k}\}$ that satisfies one of the following three properties:

1. *(Tightness up to translation) There exists $y_k \in \mathbb{R}^N$ such that for every $\epsilon > 0$ there exists an $R > 0$ such that*

$$\int_{B(y_k,R)} \rho_{n_k}(x)\, dx \geq m - \epsilon,$$

for all k.

2. *(Vanishing) For all* $R > 0$

$$\lim_{k\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B(y,R)} \rho_{n_k}(x)\, dx = 0.$$

3. *(Dichotomy) There exists an* $0 < \alpha < m$*, such that for any* $\epsilon > 0$*, there exist* $k_0 \geq 1$*,* $y_k \in \mathbb{R}^N$*, and radii* $R > 0$ *and* $R_k \to \infty$ *as* $k \to \infty$ *such that*

$$\rho_k^1 = \rho_{n_k}|_{B(y_k,R)}, \quad \text{and} \ \ \rho_k^2 = \rho_{n_k}|_{\mathbb{R}^N\setminus B(y_k,R_k)}$$

satisfy

$$\begin{aligned} &||\rho_{n_k} - (\rho_k^1 + \rho_k^2)||_{L^1(\mathbb{R}^N)} \leq \epsilon, \\ &||\rho_k^1||_{L^1(\mathbb{R}^N)} - \alpha \leq \epsilon, \quad \text{and} \\ &||\rho_k^2||_{L^1(\mathbb{R}^N)} - (m-\alpha) \leq \epsilon \end{aligned}$$

for $k \geq k_0$*.*

The statement for the dichotomy case here appears differently from the statement in [4], but the modification reflects the construction of ρ_k^1, and ρ_k^2 in [4]. Also note there are other versions of the concentration compactness principle, for example there is a version for probability measures.

Proof of Lemma 1. We follow Lion's proof. For each n, consider

$$Q_n(t) = \sup_{y\in\mathbb{R}^N} \int_{B(y,t)} \rho_n(x)\, dx,$$

the concentration function of ρ_n. Then, $\{Q_n\}$ is a sequence of non-decreasing, non-negative, uniformly bounded functions on the positive real numbers. Also notice that

$$\lim_{t\to\infty} Q_n(t) = m$$

for each n. By Helly's selection theorem, there exists a subsequence $\{Q_{n_k}\}$ and a non-decreasing function $Q : [0,\infty) \to \mathbb{R}$ such that

$$Q_{n_k}(t) \to Q(t) \tag{1}$$

as $k \to \infty$ for all t. Note that since Q is bounded and monotone, the limit

$$\lim_{t\to\infty} Q(t) = \alpha \tag{2}$$

must exist. Moreover, $0 \leq \alpha \leq m$. To identify the three cases in the Lemma, we must consider the different possible values of α.

Case 1: $\alpha = m$. This corresponds to the tight up to translation case.
Let $0 < \epsilon < \dfrac{m}{2}$. First, we will argue that there exists an $R > 0$ and a sequence $\{y_k\}$ of points in $\mathbb{R}^N$ such that

$$\int_{B(y_k,R)} \rho_{n_k} > m - \epsilon.$$

for all k. Note that the sequence $\{y_k\}$ depends on ϵ. By (2) there exists $R > 0$ such that

$$Q(R) > m - \frac{\epsilon}{2}.$$

Then, by (1) we have that

$$Q_{n_k}(R) > m - \epsilon \tag{3}$$

for large k. After possibly enlarging R, (3) holds for all k, and the claim follows from the definition of Q_{n_k}.

Following the previous reasoning, there must exist a radius $R' > 0$ and a sequence $\{x_k\}$ of points in $\mathbb{R}^N$ such that

$$\int_{B(x_k,R')} \rho_{n_k} > \frac{m}{2}$$

for all k. Note that the sequence $\{x_k\}$ does not depend on ϵ. Then, we have

$$\int_{B(y_k,R)} \rho_{n_k} + \int_{B(x_k,R')} \rho_{n_k} > m - \epsilon + \frac{m}{2} > m = \int_{\mathbb{R}^N} \rho_{n_k}.$$

Thus,

$$B(y_k, R) \cap B(x_k, R') \neq \emptyset,$$

and so in particular $B(y_k, R) \subset B(x_k, R' + 2R)$, and

$$\int_{B(x_k,R'+2R)} \rho_{n_k} > m - \epsilon,$$

as desired.

Case 2: $\alpha = 0$. This corresponds to the vanishing case.
Since Q is monotone, with $Q(0) = 0$ and $\lim\limits_{t\to\infty} Q(t) = 0$, Q is identically 0, and so

$$\lim_{k\to\infty} Q_{n_k}(R) = 0$$

for any $R > 0$, which is the statement for the vanishing case.

Case 3: $0 < \alpha < m$. This corresponds to the dichotomy case.
Let $\epsilon > 0$ and choose $R > 0$ such that $Q(R) > \alpha - \epsilon$. Then for large k we must also have $Q_{n_k}(R) > \alpha - \epsilon$, and we can find a sequence $R_k \to \infty$ such that $Q_{n_k}(R_k) < \alpha + \epsilon$. By the definition of Q_{n_k}, there exist $\{y_k\}$ such that

$$\int_{B(y_k,R)} \rho_{n_k} > \alpha - \epsilon.$$

The fact that $Q_{n_k}(R_k) < \alpha + \epsilon$ means in particular that

$$\int_{B(y_k,R_k} \rho_{n_k} < \alpha + \epsilon.$$

Finally, set

$$\rho_k^1 = \rho_{n_k}|_B(y_k, R), \quad \text{and} \quad \rho_k^2 = \rho_{n_k}|_{\mathbb{R}^N \setminus B(y_k,R_k)}.$$

□

3 Applying concentration compactness to a nonlocal aggregation problem

3.1 Definitions

Recall that, given a kernel K, we define the interaction energy

$$\mathcal{E}[\rho] = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} K(|x-y|)\rho(x)\rho(y)\, dx\, dy$$

and seek to minimize this energy over the set of densities

$$\mathcal{A}_m = \{\rho \in L^1(\mathbb{R}^N) \;:\; 0 \leq \rho \leq 1,\; ||\rho||_{L^1(\mathbb{R}^N)} = m\},$$

for a given $m > 0$. $\mathcal{E}[\rho]$ measures the energy associated with ρ interacting with itself. Given $\rho, \eta \in \mathcal{A}_m$, define the pairwise interaction energy

$$\mathcal{E}[\rho, \eta] = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} K(|x-y|)\rho(x)\eta(y)\, dx\, dy.$$

Remark. *This energy is translation invariant. That is,*

$$\mathcal{E}[\rho] = \mathcal{E}[\rho(\cdot + y)]$$

for any $y \in \mathbb{R}^N$.

3.2 Some more set up - what does our K actually look like?

Throughout the rest of this talk, we will consider an interaction kernel $K : [0,\infty) \to \mathbb{R}$ which satisfies the following

1. $K(0) = -d$, $K(w_1) = 0$
2. K is increasing on $[0, a]$
3. $K \geq h$ on $[a, a + w_2]$
4. K is decreasing on $[a + w_2, \infty)$ and $\lim_{r\to\infty} K(r) = 0$

The parameters $d, w_1 > 0$ represent the depth and width of the attractive "well" near 0. The parameters $h, w_2 > 0$ represent the height and width of the repulsive "barrier." I won't get into the details of these parameters in this talk, but we need to assume that the repulsive barrier is high and wide enough relative to the attract well. Basically, the attraction at short distances means that mass wants to cluster together into a droplet, but the repulsive barrier means that the mass does not want to cluster together in one big droplet, but rather in several droplets which are far away from one another.

3.3 Existence Of Minimizers for Truncated Kernel

To understand what happens for our more general problem, we consider the truncated kernel

$$\bar{K} = K\mathbb{1}_{[0,a+w_2]},$$

and consider the minimization problem for the truncated energy

$$\bar{\mathcal{E}}[\rho] = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \bar{K}(|x-y|)\rho(x)\rho(y)\,dx\,dy.$$

The reason for this is that minimizers exist for the truncated problem, and moreover that minimizers are always of the form $\mathbb{1}_{B_1\cup\ldots B_l}$, where $B_1, \ldots, B_l$ are balls in $\mathbb{R}^N$ that are far (enough) away from each other. From here, we can construct a minimizing sequence for our original problem by translating these balls so that they all become "infinitely far apart" from one another. The reason this will give us a minimizing sequence is that $\bar{K} \leq K$ implies

$$\bar{\mathcal{E}}[\rho] \leq \mathcal{E}[\rho]$$

for all densities ρ.

In these notes, I will only focus on the proof of existence of minimizers for the truncated problem, because this is where the concentration compactness is used. The decomposition of minimizers into separated balls comes from the competition between the attractive well and the repulsive barrier. Essentially, these create a "forbidden" distance for points in the support of a minimizing density.

The rest of this section will be dedicated to proving the existence of minimizers for the truncated problem.

Theorem 1 (Minimizers exist for the truncated problem)**.** *For any* $m > 0$*, there exists a* $\rho \in \mathcal{A}_m$ *such that*

$$\bar{\mathcal{E}}[\rho] = \inf_{\eta \in \mathcal{A}_m} \bar{\mathcal{E}}[\eta].$$

To prove this, we will start with a minimizing sequence and iteratively apply the concentration compactness principle, eventually passing to a subsequence which we can decompose into many "tight up to translation" pieces. To know that this process must end after finitely many steps, we will need the following theorem, which states that balls are minimizers for small mass.

Theorem 2 (Balls minimizes the energy for small mass)**.** *Let* $m_0 = |B(0, w_1/2)|$*. Then, for* $m \leq m_0$*,*

$$\mathcal{E}[\mathbb{1}_{B(0,r)}] = \inf_{\rho \in \mathcal{A}_m} \mathcal{E}[\rho].$$

where r *is the radius of the ball with volume* m*.*

Note that this Theorem is also true for the truncated energy.

Proof of theorem 2. This follows from the Riesz Rearrangement Inequality, and the bathtub principle. Note that $K_- = \min\{K, 0\}$ is a non-positive, non-decreasing function. So, the Riesz Rearrangement Inequality says that

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} K_-(|x-y|)\rho(x)\rho(y)\,dx\,dy \geq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} K_-(|x-y|)\rho^*(x)\rho^*(y)\,dx\,dy,$$

where ρ^* is the symmetric decreasing rearrangement of ρ. Then, by the bathtub principle

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} K_-(|x-y|)\rho^*(x)\rho^*(y)\,dx\,dy \geq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} K_-(|x-y|)\mathbb{1}_{B(0,r)}(x)\mathbb{1}_{B(0,r)}(y)\,dx\,dy.$$

Note that if $|x|, |y| \leq r$, then $|x-y| \leq 2r$. So

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} K_-(|x-y|)\mathbb{1}_{B(0,r)}(x)\mathbb{1}_{B(0,r)}(y)\,dx\,dy. = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} K(|x-y|)\mathbb{1}_{B(0,r)}(x)\mathbb{1}_{B(0,r)}(y)\,dx\,dy.$$

Also, $K_- \leq K$. Altogether, this means that for any $\rho \in \mathcal{A}_m$,

$$\mathcal{E}[\rho] \geq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} K_-(|x-y|) \mathbb{1}_{B(0,r)}(x) \mathbb{1}_{B(0,r)}(y)\, dx\, dy = \mathcal{E}[\mathbb{1}_{B(0,r)}].$$

□

I am glossing over some background on rearrangement here. Essentially, the idea is that since K_- is non-decreasing, the mass wants to concentrate together as much as possible, subject to the density constraint. If the mass is small enough, then a ball sees no difference between K and K_-, thanks to the attractive well. The lecture notes [1] are a good resource if you are interested in learning more about rearrangement inequalities. Additionally the bathtub principle can be found in section 1.14 of [3].

Next, we will need a series of three Lemmas, each to deal with a case arising from the concentration compactness Lemma. The first lemma is used to extract droplets from the "tight up to translation" case:

Lemma 2. *Suppose $\{\rho_k\}$ is a sequence of densities in $\mathcal{A}_m$. If $\{\rho_k\}$ is tight up to translation, then there is a $\rho \in \mathcal{A}_m$ such that*

$$\lim_{k\to\infty} \mathcal{E}[\rho_k] = \mathcal{E}[\rho].$$

Proof. Follows similar argument as in [2] By tightness up to translation, there exist $y_k \in \mathbb{R}^N$ such that for any $\epsilon > 0$, there is a radius $R > 0$ such that

$$\int_{B(y_k,R)} \rho_k(y)\, dy \geq m - \epsilon \tag{4}$$

for all k. By the translation invariance of the energy, without loss of generality we can take $y_k = 0$ for all k. Fix any $1 < q < \infty$, then $\{\rho_k\}$ is a bounded sequence in $L^q(\mathbb{R}^N)$, so there is a $\rho \in L^q(\mathbb{R}^N)$ such that ρ_k converges to ρ weakly in L^q.

First, we will verify that $\rho \in \mathcal{A}_m$. By (4),

$$\int_{\mathbb{R}^N} \rho(y)\, dy = m.$$

To see that $\rho \geq 0$, consider the set $A = \{\rho < 0\}$. Suppose $|A| > 0$, and if $|A| = \infty$, replace A with a subset that has finite measure. Then, by weak convergence we have

$$0 \leq \lim_{k\to\infty} \int_A \rho_k(y)\, dy = \int_A \rho(y)\, dy < 0,$$

which is a contradiction. Therefore $|A| = 0$. Similarly, if we let $B = \{\rho > 1\}$, and assume $|B| > 0$, we can compute

$$|B| \geq \lim_{k\to\infty} \int_B \rho_k(y)\,dy = \int_B \rho(y)\,dy > |B|.$$

Combined, this means that $\rho \in \mathcal{A}_m$

Next, we will prove that $\lim_{k\to\infty} \mathcal{E}[\rho_k] = \mathcal{E}[\rho]$. Let

$$\begin{aligned} G_k(x) &= K * \rho_k(x), \ \text{ and} \\ G(x) &= K * \rho(x). \end{aligned}$$

By (4), ρ_k also converges to ρ weakly in $L^1(\mathbb{R}^N)$. Then since K is bounded, this means that G_k converges to G pointwise.

Fix $\epsilon > 0$, and pick $R > 0$ such that

$$\int_{\mathbb{R}^N \setminus B(0,R)} \rho_k(y)\,dy \leq \epsilon \tag{5}$$

for all k. Then compute

$$\begin{aligned} \mathcal{E}[\rho_k] - \mathcal{E}[\rho] &= \int_{\mathbb{R}^N} G_k(x)\rho_k(x)\,dx - \int_{\mathbb{R}^N} G(x)\rho(x)\,dx \\ &= \int_{B(0,R)} (G_k(x) - G(x))\rho_k(x)\,dx + \int_{B(0,R)} G(x)(\rho_k(x) - \rho(x))\,dx \\ &\quad + \int_{\mathbb{R}^N \setminus B(0,R)} G_k(x)\rho_k(x)\,dx. - \int_{\mathbb{R}^N \setminus B(0,R)} G(x)\rho(x)\,dx. \end{aligned} \tag{6}$$

The first term in (6) converges to 0 as $k \to \infty$ by the bounded convergence theorem. The second term also converges to 0 as $k \to \infty$ since $G|_{B(0,R)}$ is an admissible test function, noting that $||G||_{L^\infty(\mathbb{R}^N)} \leq m||K||_{L^\infty(\mathbb{R}^N)}$ Thus, for large enough k we have

$$|\mathcal{E}[\rho_k] - \mathcal{E}[\rho]| \leq 2\epsilon + 2m||K||_{L^\infty(\mathbb{R}^N)}\epsilon.$$

□

The next lemma will be used to rule out the vanishing case:

Lemma 3. *Suppose $\{\rho_k\}$ is a sequence in $\mathcal{A}_m$ that satisfies the vanishing property in the concentration compactness lemma. Then,* $\lim_{k\to\infty} \mathcal{E}[\rho_k] = 0$.

Proof. Fix any $R > 0$. Then for any $\epsilon > 0$, using the fact that $\{\rho_k\}$ is vanishing we have for large k

$$\int_{B(y,R)} \rho(x)dx < \epsilon$$

for any $y \in \mathbb{R}^N$. We can now compute

$$\begin{aligned}
|\mathcal{E}[\rho_k]| &\leq \left| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} K(|x-y|)\mathbb{1}_{|x-y|\leq R}\rho_k(x)\rho_k(y)\; dx\; dy \right| + \left| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} K(|x-y|)\mathbb{1}_{|x-y|>R}\rho_k(x)\rho_k(y)\; dx\; dy \right| \\
&\leq ||K||_{L^\infty(\mathbb{R}^N)} \int_{\mathbb{R}^N} \rho_k(y) \int_{B(y,R)} \rho_k(x)\; dx\; dy + m^2 \sup_{r\geq R} K(r) \\
&\leq ||K||_{L^\infty(\mathbb{R}^N)} m\epsilon + m^2 \sup_{r\geq R} K(r)
\end{aligned}$$

The second term of this final sum can be made arbitrarily small since $\lim_{r\to\infty} K(r) = 0$. $\square$

Finally, this next lemma will allow us to separate a dichotomous minimizing sequence into multiple pieces.

Lemma 4. *Suppose $\{\rho_k\}$ is a dichotomous sequence in $\mathcal{A}_m$. Then, for some $0 < \alpha < m$ there are sequences $\{\rho_k^1\}$ and $\{\rho_k^2\}$ in $\mathcal{A}_\alpha$ and $\mathcal{A}_{m-\alpha}$, respectively, such that*

$$\lim_{k\to\infty} \left(\mathcal{E}[\rho_k] - \mathcal{E}[\rho_k^1] - \mathcal{E}[\rho_k^2]\right) = 0. \tag{7}$$

Moreover if ρ_k is a minimizing sequence for $\mathcal{E}$, then

$$\liminf_{k\to\infty} \mathcal{E}[\rho_k^1] = \inf_{\rho\in\mathcal{A}_\alpha} \mathcal{E}[\rho], \tag{8}$$

and

$$\liminf_{k\to\infty} \mathcal{E}[\rho_k^2] = \inf_{\rho\in\mathcal{A}_{m-\alpha}} \mathcal{E}[\rho]. \tag{9}$$

Proof. By dichotomy, after perhaps passing to a subsequence, we can find radii $R_k > 0$ and points $y_k \in \mathbb{R}^N$ so that the sequences

$$\rho_k^1 = \rho_k|_{B(y_k,R_k)}, \quad \text{and} \quad \rho_k^2 = \rho_k|_{\mathcal{R}^N\setminus B(y_k,R_k+\tilde{R})} \tag{10}$$

satisfy

$$\lim_{k\to\infty} \int \rho_k^1 = \alpha, \quad \text{and} \tag{11}$$

$$\lim_{k\to\infty} \int \rho_k^2 = m - \alpha \tag{12}$$

for some $0 < \alpha < m$. Note that since $K(r) = 0$ for $r \geq R$,

$$\mathcal{E}[\rho_k^1, \rho_k^2] = 0.$$

So,

$$\mathcal{E}[\rho_k] = \mathcal{E}[\rho_k^1] + \mathcal{E}[\rho_k^2] + \mathcal{E}[\rho - \rho_k^1 - \rho_k^2] + 2\mathcal{E}[\rho_k^1 + \rho_k^2, \rho - \rho_k^1 - \rho_k^2] \tag{13}$$

Note that as $k \to \infty$ the last two terms approach 0 since

$$\lim_{k\to\infty} \int_{\mathbb{R}^N} \rho - \rho_k^1 - \rho_k^2 = m - \alpha - (m - \alpha) = 0.$$

Next, we modify these sequences so that they have constant mass. To do so for ρ_k^1, we either decrease the radius R_k if ρ_k^1 has too much mass, or add a small piece far away if ρ_k^1 has too little mass. For each k, let

$$\rho_k^1 = \rho_k|_{B(y_k, R_k^1)} + \mathbb{1}_{B(x_k, r_k^1)},$$

where $R_k^1 \leq R_k$, $r_k^1 \geq 0$, and $x_k \in \mathbb{R}^N$ are chosen so that $\rho_k^1 \in \mathcal{A}_\alpha$, and $|x_k - y_k| > R_k + r_k^1 + \tilde{R}$, and $r_k^1 \to 0$ as $k \to \infty$. Similarly for ρ_k^2, we either increase the radius R_k to decrease the mass, or add a small ball centered at y_k to increase the mass. That is we let

$$\rho_k^2 = \rho_k|_{\mathbb{R}^N \setminus B(y_k R_k^2 + \tilde{R})} + \mathbb{1}_{B(y_k, r_k^2)},$$

where $R_k^2 \geq R_k$, and $r_k^2 \geq 0$ are chosen so that $\rho_k^2 \in \mathcal{A}_{m-\alpha}$, and $r_k^2 \to 0$ as $k \to \infty$. It is straightforward to check, using the fact that $||\rho_k^i - \rho_k^i||_{L^1(\mathbb{R}^N)} \to 0$ as $k \to \infty$ for $i = 1, 2$ that

$$\lim_{k\to\infty} \left(\mathcal{E}[\rho_k^i] - \mathcal{E}[\rho_k^i]\right) = 0.$$

Then, combining this with (13),

$$\lim_{k\to\infty} \left(\mathcal{E}[\rho_k] - \mathcal{E}[\rho_k^1] - \mathcal{E}[\rho_k^2]\right) = 0. \tag{14}$$

Now, suppose ρ_k is a minimizing sequence. Note that the function

$$E(m) = \inf_{\rho \in \mathcal{A}_m} \mathcal{E}[\rho].$$

is subadditive. So, we have

$$\lim_{k\to\infty} \mathcal{E}[\rho_k] = \inf_{\rho\in\mathcal{A}_m} \mathcal{E}[\rho] \leq \inf_{\rho\in\mathcal{A}_\alpha} \mathcal{E}[\rho] + \inf_{\rho\in\mathcal{A}_{m-\alpha}} \mathcal{E}[\rho].$$

By (14)

$$\begin{aligned}\lim_{k\to\infty} \mathcal{E}[\rho_k] &= \lim_{k\to\infty} \left(\mathcal{E}[\rho_k^1] + \mathcal{E}[\rho_k^2]\right) \\ &\geq \liminf_{k\to\infty} \mathcal{E}[\rho_k^1] + \liminf_{k\to\infty} \mathcal{E}[\rho_k^2].\end{aligned}$$

Then we are done since

$$\liminf_{k\to\infty} \mathcal{E}[\rho_k^1] \geq \inf_{\rho\in\mathcal{A}_\alpha} \mathcal{E}[\rho], \quad \text{and} \quad \liminf_{k\to\infty} \mathcal{E}[\rho_k^1] \geq \inf_{\rho\in\mathcal{A}_{m-\alpha}} \mathcal{E}[\rho].$$

□

Finally, we are ready to prove the existence of minimizers for the truncated problem!

Proof of Theorem 1. Let $\{\rho_k\}$ be a minimizing sequence. Then by concentration compactness (Lemma 1), we can pass to a subsequence to a obtain a minimizing sequence which, abusing notation slightly, I will also label $\{\rho_k\}$ that is either tight up to translation, vanishing, or dichotomous. By lemma [3] a minimizing sequence cannot be vanishing since we know $\inf_{\rho\in\mathcal{A}_m} \mathcal{E}[\rho] < 0$.

If the sequence is tight up to translation, we can apply Lemma 2, and conclude that there is a $\rho \in \mathcal{A}_m$ such that $\lim_{k\to\infty} \mathcal{E}[\rho_k] = \mathcal{E}[\rho]$, and so ρ is a minimizer and we are done.

If the sequence is dichotomous, then by Lemma 4 there are $\rho_k \in \mathcal{A}_{\alpha_1}$, and $\rho_k \in \mathcal{A}_{\alpha_2}$, with $\alpha_1 + \alpha_2 = m$, and

$$\lim_{k\to\infty} \mathcal{E}[\rho_k] = \lim_{k\to\infty} \left(\mathcal{E}[\rho_k^1] + \mathcal{E}[\rho_k^2]\right).$$

We then apply concentration compactness principle to $\{\rho_k^1\}$ and $\{\rho_k^2\}$, noting that neither can have a vanishing subsequence. Applying lemmas 2 and 4, and then concentration compactness again iteratively, we eventually obtain (after passing to subsequences, and some relabelling) sequences $\rho_k^1, \rho_k^2, \ldots \rho_k^l$, which are tight up to translation with $\rho_k^i \in \mathcal{A}_{\alpha_i}$, $\alpha_1 + \cdots + \alpha_l = m$, and

$$\lim_{k\to\infty} \mathcal{E}[\rho_k] = \lim_{k\to\infty} \left(\mathcal{E}[\rho_k^1] + \cdots + \mathcal{E}[\rho_k^l]\right).$$

Note that this process of applying concentration compactness iteratively must end after finitely many steps since we know for $\alpha \leq m_0$ that $\inf_{\rho\in\mathcal{A}_\alpha} \mathcal{E}[\rho]$ is attained by the indicator of a ball, and so if $\{\eta_k\}$ is a dichotomous sequence in $\mathcal{A}_\alpha$, then

$$\liminf_{k\to\infty} \mathcal{E}[\eta_k] > \inf_{\rho\in\mathcal{A}_\alpha} \mathcal{E}[\rho],$$

which contradicts Lemma 4. To conclude, by Lemma 2 there exist $\rho^1 \in \mathcal{A}_{\alpha_1}, \ldots \rho^l \in \mathcal{A}_{\alpha_l}$ such that

$$\lim_{k\to\infty} \mathcal{E}[\rho_k] = \mathcal{E}[\rho^1] + \cdots + \mathcal{E}[\rho^l].$$

Each ρ^i is a minimizer of $\mathcal{E}$ in $\mathcal{A}_{\alpha_i}$. I haven't proven it here, but in particular this means each ρ^i has compact support. So, we can construct a minimizer for the truncated problem by letting $\rho = \rho(z_1 + \cdot) + \ldots \rho^l(z_l + \cdot)$ for suitably chosen $z_1, \ldots, z_l \in \mathbb{R}^N$ □

4 Brief Summary of Talk

1. Motivation: The concentration compactness principle can be a useful tool for dealing with the lack of compactness of an unbounded domain.
2. Sometimes, it essentially gives us compactness (the "tight up to translation" case). It can also help us to deal with problems where we expect a minimizing sequence to break off into many distant pieces.
3. In fact, this is what happens when we consider the problem of minimizing the interaction energy given by kernels having a particular shape.

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