A Mass-Conserving Toy Model of Blood Pulses in Arterial Networks

Adam Morgan

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Abstract

The generalized Benjamin-Bona-Mahony-Burgers equation (gBBMB) provides a toy model of blood flow through long, viscoelastic arteries. In this talk I will describe a formulation of gBBMB valid on networks with semi-infinite edges joined at a single junction, with the network's edges corresponding to a segment of the arterial tree. To reflect sudden changes in the material properties of blood vessels, the coefficients of gBBMB are allowed to take different values on each edge of the network; such changes physically represent the presence of arteriosclerosis or a stent. Critically, my formulation ensures that the total mass of blood in the network is constant in time. I will also establish local-in-time well-posedness of my formulation for sufficiently regular initial data. Then, I will use energy methods to establish global well-posedness, provided certain constraints are imposed on the parameters of the model PDE and the network. To build intuition for how waves scatter off the central junction of a network with two edges, I will demonstrate the results of some simple numerical simulations of the model. In particular, we shall see that wave reflection off the junction is indeed possible, provided the ratio of stiffness between network edges is sufficiently large.

1 Motivation

*** Most of the content of these notes will (likely) appear in expanded form in my doctoral thesis ***

Let $u(x,t)$ represent the deviation from equilibrium of the cross-sectional area of an artery conducting the flow of homogeneous blood, where x is the axial coordinate along the artery and t is time. We assume that the artery is impermeable and viscoelastic, and that the blood is inviscid. Additionally, we suppose the artery is very long so we can treat $x \in \mathbb{R}$. For real constants $\mu, \alpha, \nu \geq 0,\ \gamma \in [0,1]$, and $p \in \mathbb{N}$, we model $u(x,t)$ as the solution to the **generalized Benjamin-Bona-Mahony-Burgers equation (gBBMB)**,

$$\left(1 - \mu^2 \partial_{xx}\right) u_t + \partial_x \left(\alpha u + \frac{\gamma}{p+1} u^{p+1}\right) - \nu u_{xx} = 0. \tag{1}$$

The parameter μ represents the dispersive influence of the arterial wall's linear elasticity, α represents the influence of linear advection, γ represent the influence of nonlinear advection as well as nonlinear wall elasticity, and ν represents the influence of viscoelastic dissipation.

Think of gBBMB as a (dissipative) substitute for the generalized Korteweg-de Vries (KdV) equation [1], featuring only second-order space derivatives (so numerical discretization is way easier) and "finite speed of propagation" (more accurately, bounded group velocity). [5] provides a justification of KdV as a model for blood flow

Modelling blood flow in relatively large subsets of the circulatory system demands we account for the influence of bifurcations (trifurcations, et cetera) in the arterial tree. Accordingly, the suitable formulation of gBBMB on a **network**, loosely understood for now to mean a collection of subintervals of $\mathbb{R}$ joined together at various points, has scientific merit. BBM ($p = 1, \nu = 0$) has been studied on networks previously, most thoroughly in [3].

By allowing the coefficients of gBBMB to vary between edges of the network in question we can also investigate how a flow is altered when it moves between two vessels with different elastic properties. Physiologically, the elasticity of a vessel can change due to **arteriosclerosis**, or hardening of the arteries. Alternatively, the elastic properties of an unhealthy artery can be modified by inserting a small wire or polymer mesh called a **stent**, which keeps an unhealthy artery from becoming closed off by built-up plaques. Studying gBBMB on a network with variable coefficients can therefore also help us understand how stents affect flow in the arterial tree.

In order for dispersion to influence the movement of some material continuum, there must be "enough room" to let waves disperse. Thus we expect gBBMB to be a quality model of flow in the femoral artery, which is reasonably long on a physiological scale. Thus we can model the effects of bifurcations, arteriosclerosis, and stents in the femoral artery (and its subarteries) by analyzing solutions to gBBMB on a network. The use of stents in certain parts of the femoral artery is medically controversial (see [8] vs. [6,7]), so math modelling may help determine best practices. Of course, gBBMB is so simple all our predictions would have to be benchmarked against simulations of the primitive fluid-structure interaction equations before they could impact medical practice.

2 Problem Setup

Let X be a network with edges $e_i \simeq [0,\infty)$ $(i = 1, ..., N)$ glued together at 0, and assume we are given N functions $u_i(x,t)$, each solving gBBMB on e_i. We need N conditions, one per u_i, if we want any hope of well-posedness. Continuity, in the sense that for all i, j we have

$$u_i(0,t) = u_j(0,t), \tag{2}$$

is an obvious constraint for gBBMB. However, continuity only yields $n-1$ equations, so we need one more constraint.

We call a function $g(x,t,u,u_{i,x},u_{i,t},...)$ **globally conserved** under the evolution of gBBMB if

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_i \int_{e_i} g(x,t,u,u_{i,x},u_{i,t},...)\ \mathrm{d}x = 0.$$

Let

$$f_i(u_i) = \alpha_i u_i + \frac{\gamma_i}{p+1}u_i^{p+1} \tag{3}$$

denote the advective flux on the edge e_i. Then, by inspection, the solution $u(x,t)$ to gBBMB on X is globally conserved if and only if

$$\sum \left[-\mu_{\mathrm{in}}^2 u_{\mathrm{in},xt} + f_{\mathrm{in}}(u_{\mathrm{in}}) - \nu_{\mathrm{in}} u_{\mathrm{in},x}\right]_{x=0} = \sum \left[-\mu_{\mathrm{out}}^2 u_{\mathrm{out},xt} + f_{\mathrm{out}}(u_{\mathrm{out}}) - \nu_{\mathrm{out}} u_{\mathrm{out},x}\right]_{x=0}. \tag{4}$$

Since

$$\sum_i \int_{e_i} u(x,t)\ \mathrm{d}x$$

physically represents the "normalized" volume bounded by a network of elastic blood vessels and we assume the blood conducted by our artery has constant density, we can justifiably call (4) the **mass conservation** condition. This condition tells us that the amount of fluid contained in our system remains constant for all time, a critical constraint to impose from a physical perspective.

Thus we seek functions $u_i(x,t)$ defined for $(x,t) \in e_i \times [0,\infty)$ (with suitable regularity) satisfying the system

$$\begin{aligned}
0 &= \left(1-\mu_i^2\partial_x^2\right)u_{i,t} + \partial_x\left(\alpha_i u_i + \frac{\gamma_i}{p+1}u_i^{p+1}\right) - \nu_i u_{i,xx} \\
u_i|_{x=0} &= u_j|_{x=0}\ \forall\ i,j \text{ (continuity at junction)} \\
\sum \left[-\mu_{\mathrm{in}}^2 u_{\mathrm{in},xt} + f_{\mathrm{in}}(u_{\mathrm{in}}) - \nu_{\mathrm{in}} u_{\mathrm{in},x}\right]_{x=0} &= \sum \left[-\mu_{\mathrm{out}}^2 u_{\mathrm{out},xt} + f_{\mathrm{out}}(u_{\mathrm{out}}) - \nu_{\mathrm{out}} u_{\mathrm{out},x}\right]_{x=0} \text{ (mass conservation)} \\
&+ \text{ initial conditions.}
\end{aligned}$$

3 Results and proofs

3.1 Function Spaces

- $C_b^k(U)$ = real-valued functions on $U \subseteq \mathbb{R}^n$ whose derivatives up to order k are cts. and bounded; this becomes a Banach space when endowed with sup-norm
- Given any Banach space A, $C_b(0,T;A)$ denotes the Banach space of all continuous functions $u\colon [0,T] \to A$ equipped with the norm

$$\|u\|_{C_b(0,T;A)} = \sup_{[0,T]} \|u(t)\|_A. \tag{5}$$

- This less-common function space appears naturally when studying gBBMB on $[0,\infty)$:

$$\mathcal{B}_T^{k,\ell} \doteq \left\{ u \mid \forall\ i \in [0,k],\ j \in [0,\ell],\ \partial_t^k \partial_x^\ell u \in C_b\left([0,\infty) \times [0,T]\right) \right\}$$

- Also need function spaces on network X:

$$\begin{aligned} C_b(X) &\doteq \{(u_1, ..., u_N) \in \left(C_b[0,\infty)\right)^N \mid u_1(0) = \cdots = u_N(0)\}, \\ H^1(X) &\doteq \left(H^1(0,\infty)\right)^N \cap C_b(X). \end{aligned}$$

  $C_b(X)$ becomes a Banach space when equipped with the norm

$$\|u\|_{C_b(X)} \doteq \max_{i=1,2,3} \|u_i\|_{C_b[0,\infty)}.$$

  Additionally, $H^1(X)$ becomes a Hilbert space when equipped with the sum inner product.

Remark. *Using the Sobolev embedding $H^1(0,\infty) \hookrightarrow C_b[0,\infty)$, we obtain the network fnc. space embedding $H^1(X) \hookrightarrow C_b(X)$.*

3.2 Review of Fixed-Point Formulation of gBBMB on a Half-Line

In this subsection, we review some ideas behind the proof of local well-posedness of gBBMB posed on $(x,t) \in [0,\infty)^2$. The idea is to express the solution to gBBMB as the fixed point of a certain nonlinear integral operator on $\mathcal{B}_T^{0,0}$ for small enough $T > 0$. Throughout, we denote the advective flux in gBBMB by

$$f(u) = \alpha u + \frac{\gamma}{p+1} u^{p+1}. \tag{6}$$

We are interested in solving the following problem: for given $h(t) \in C[0,\infty)$, $\varphi(x) \in C_b[0,\infty)$, find $T>0$ and $u(x,t) \in \mathcal{B}_T^{1,2}$ such that

$$(1-\mu^2\partial_x^2)u_t + (f(u))_x - \nu u_{xx} = 0 \quad \forall\ (x,t) \in (0,\infty)^2, \tag{7a}$$

$$u(0,t) = h(t) \quad \forall\ t \in [0,\infty), \tag{7b}$$

$$u(x,0) = \varphi(x) \quad \forall\ x \in [0,\infty). \tag{7c}$$

At least formally, we can say

$$u_t = (1-\mu^2\partial_x^2)^{-1}\left[-(f(u))_x + \nu u_{xx}\right]. \tag{8}$$

As in [1], one treats the above expression as an ordinary differential equation (ODE) for u as a function of t. Once we find an explicit expression for $(1-\mu^2\partial_x^2)^{-1}$, we can solve the ODE and determine a nonlinear operator for which u arises as a fixed point. Computing $(1-\mu^2\partial_x^2)^{-1}$ is trivial given the following:

Lemma 1. *Let $\delta(x-y)$ denote the Dirac function centred at $y \in \mathbb{R}$. The function*

$$G(x,y) \doteq -\frac{1}{2\mu}\left(e^{\frac{-(x+y)}{\mu}} - e^{-\frac{|x-y|}{\mu}}\right) : [0,\infty)^2 \to \mathbb{R} \tag{9}$$

satisfies the PDE

$$\left(1-\mu^2\partial_x^2\right)G(x,y) = \delta(x-y) \quad \forall\ (x,y) \in (0,\infty)^2 \tag{10}$$

in the sense of distributions, with $G(0,y)=0$ and $\lim_{x\to\infty} G(x,y) = 0\ \forall\ y \in [0,\infty)$.

□

In light of Lemma 1, we may write

$$u_t = h'(t)e^{-\frac{x}{\mu}} + \int_0^\infty G(x,y)\ \left((-f(u(y,s)))_y + \nu u_{yy}(y,s)\right)\ \mathrm{d}y. \tag{11}$$

Integrating by parts and solving a linear first order ODE in time, we arrive at the expressions [4, Equations 3.3-3.8]

$$K(x,y) \doteq \frac{1}{2\mu^2}\left(e^{\frac{-(x+y)}{\mu}} + \mathrm{sgn}(x-y)\ e^{-\frac{|x-y|}{\mu}}\right), \tag{12a}$$

$$\mathbb{B}_{\mathrm{adv}}[u](x,t) \doteq \int_0^t \int_0^\infty e^{-\frac{\nu}{\mu^2}(t-s)}\ K(x,y)\ f(u(y,s))\ \mathrm{d}y\ \mathrm{d}s, \tag{12b}$$

$$\mathbb{B}_{\mathrm{visc}}[u](x,t) \doteq \frac{\nu}{\mu^2}\int_0^t \int_0^\infty e^{-\frac{\nu}{\mu^2}(t-s)}\ G(x,y)\ u(y,s)\ \mathrm{d}y\ \mathrm{d}s, \tag{12c}$$

$$u(x,t) = e^{-\frac{\nu t}{\mu^2}}\varphi(x) + \left(h(t) - h(0)e^{-\frac{\nu t}{\mu^2}}\right)e^{-\frac{x}{\mu}} + \mathbb{B}_{\mathrm{adv}}[u](x,t) + \mathbb{B}_{\mathrm{visc}}[u](x,t). \tag{12d}$$

I have chosen the notation $\mathbb{B}_{\text{adv}}$ and $\mathbb{B}_{\text{visc}}$ because, when viscoelasticity is ignored, $\nu = 0$ and $\mathbb{B}_{\text{visc}}[u] \equiv 0$. Additionally, $\mathbb{B}_{\text{adv}}$ contains all information on how advection affects the dynamics.

We conclude by stating the following lemma, which helps some calculations in the next subsection:

Lemma 2. *If $u(x,t)$ solves (7), then*

$$f(h(t)) - \left[\mu^2 u_{xt} + \nu u_x\right]_{x=0} = \mu h'(t) + \frac{\nu}{\mu} h(t) - \frac{1}{\mu}\int_0^\infty e^{\frac{-y}{\mu}} \left[f(u(y,t)) - \frac{\nu}{\mu} u(y,t)\right] \mathrm{d}y.$$

Proof. Differentiate both sides of (11) with respect to x, then integrate by parts to get rid of all the derivatives in the integrand. □

3.3 Fixed-Point Formulation of gBBMB on X

We now adapt the techniques from the previous subsection to prove local well-posedness of gBBMB on a star network X with N infinitely long edges e_i, with $e_1 = (-\infty, 0]$ (incoming edge) and $e_i = [0,\infty)$ for $i > 1$ (outgoing edges).Let

$$f_i(u_i) = \alpha_i u_i + \frac{\gamma_i}{p+1} u_i^{p+1} \tag{13}$$

denote the advective flux on the edge e_i. For $\varphi \in C_b(X)$ with $\varphi_i \doteq \varphi|_{e_i}$, our formulation of gBBMB on X then reads

$$\left(1 - \mu_1^2 \partial_x^2\right) u_{1,t} + \left(f_1(u_1)\right)_x - \nu_1 u_{1,xx} = 0 \quad \text{on } (x,t) \in (-\infty, 0) \times (0,\infty), \tag{14a}$$

$$\left(1 - \mu_i^2 \partial_x^2\right) u_{i,t} + \left(f_i(u_i)\right)_x - \nu_i u_{i,xx} = 0 \quad \text{on } (x,t) \in (0,\infty)^2,\ i = 2, ..., N, \tag{14b}$$

$$\mu_1^2\, u_{1,xt}(0,t) - f_1(u_1(0,t)) + \nu_1 u_{1,x}(0,t) = \sum_{i=2}^{N} \mu_i^2\, u_{i,xt}(0,t) - f_i(u_i(0,t)) + \nu_i u_{i,x}(0,t) \quad \forall\, t \in [0,\infty), \tag{14c}$$

$$u_i(0,t) = u_j(0,t) \quad \forall\, i,j = 1, ..., N,\ t \in [0,\infty), \tag{14d}$$

$$u_i(x,0) = \varphi_i(x) \quad \forall\, x \in e_i,\ i = 1, ..., N. \tag{14e}$$

Assuming a classical solution $(u_1, u_2, ..., u_N)$ to (14) exists for some time T, let

$$u(x,t) \in C_b\left(0, T, C_b(X)\right)$$

be defined by

$$u(x,t)|_{e_i} = u_i(x,t).$$

We attack this problem using the strategy of the previous subsection. Specifically, we cast (14) as an equivalent fixed-point problem on $C_b(0, T, C_b(X))$. To do this, we write out the integral form of gBBMB on each e_i in terms of the *a priori* unknown common junction value

$$h(t) \doteq u_1(0,t) = \cdots = u_N(0,t).$$

Then, we use the mass conservation condition to write out a linear initial-value problem for $h(t)$ with u-dependent forcing, which is trivially solvable in terms of u.

We start by changing variables $x \mapsto -x$ in e_1 to make sure all u_i's are defined on the same spatial domain $[0, \infty)$. Letting $\sigma_i = -1$ if $i = 1$ and $\sigma_i = 1$ otherwise, the integral form of gBBMB on e_i can be written as

$$u_i(x,t) = e^{-\frac{\nu_i t}{\mu_i^2}}\varphi_i(x) + \left(h(t) - h(0)e^{-\frac{\nu_i t}{\mu_i^2}}\right)e^{-\frac{x}{\mu_i}} + \sigma_i \mathbb{B}_{\mathrm{adv},i}[u_i](x,t) + \mathbb{B}_{\mathrm{visc},i}[u_i](x,t). \tag{15}$$

Adapting Lemma 2 gives

$$\sigma_i f_i(h(t)) - \left[\mu_i^2 u_{i,xt} + \nu_i u_{i,x}\right]_{x=0} = \mu_i h'(t) + \frac{\nu_i}{\mu_i}h(t) - \frac{1}{\mu_i}\int_0^\infty e^{\frac{-y}{\mu_i}}\left[\sigma_i f_i(u_i(y,t)) - \frac{\nu_i}{\mu_i}u_i(y,t)\right]\,\mathrm{d}y. \tag{16}$$

Now, after changing variables, we can write (14c) as

$$\sum_i \sigma_i f_i(h(t)) - \left[\mu_i^2 u_{i,xt} + \nu_i u_{i,x}\right]_{x=0} = 0. \tag{17}$$

Combining this with (16) and defining $\mu_* \doteq \sum_i \mu_i,\ \nu_* \doteq \sum_i \frac{\nu_i}{\mu_i}$, we get

$$h'(t) + \frac{\nu_*}{\mu_*}h(t) = \sum_i \frac{1}{\mu_i \mu_*}\int_0^\infty e^{\frac{-y}{\mu_i}}\left[\sigma_i f_i(u_i(y,t)) - \frac{\nu_i}{\mu_i}u_i(y,t)\right]\,\mathrm{d}y. \tag{18}$$

Finding the junction value $h(t)$ thus amounts to solving a linear, parameterized (by u) ODE (18) subject to the initial condition $h(0) = \varphi(0)$. This is trivial, however:

$$h(t) = \varphi(0)e^{-\frac{\nu_* t}{\mu_*}} + \sum_i \frac{1}{\mu_i\mu_*}\int_0^t\int_0^\infty e^{-\left(\frac{\nu^*}{\mu_*}(t-s) + \frac{y}{\mu_i}\right)}\left[\sigma_i f_i(u_i(y,s)) - \frac{\nu_i}{\mu_i}u_i(y,s)\right]\,\mathrm{d}y\,\mathrm{d}s. \tag{19}$$

Let

$$\Phi[u] \doteq h(t) - \varphi(0), \tag{20}$$

with $h(t)$ given by (19). Then, we may write the fixed-point formulation of gBBMB on X as follows: find u_i $(i = 1, ..., N)$ such that

$$u_i(x,t) = e^{-\frac{\nu_i t}{\mu_i^2}}\varphi_i(x) + \left(\Phi[u] + \varphi(0)\left(1 - e^{-\frac{\nu_i t}{\mu_i^2}}\right)\right)e^{-\frac{x}{\mu_i}} + \sigma_i\mathbb{B}_{\mathrm{adv},i}[u_i](x,t) + \mathbb{B}_{\mathrm{visc},i}[u_i](x,t). \tag{21}$$

Notice how coupling between individual edges is described entirely by $\Phi[u]$.

Now, we are at last ready to state and prove our main theorem for this section.

Theorem 1. *Given $\varphi \in C_b(X)$ with $\varphi_i \in C_b^2(0,\infty)$ for each i, there exists a $T > 0$ and a unique $u \in C_b(0,T;C_b(X))$ such that $u(x,t)$ is a classical solution to gBBMB on X satisfying the mass conservation condition (4). Further, $u|_{e_i} \in \mathcal{B}_T^{1,2}$ $\forall$ i, and u has Lipschitz dependence on the initial data φ.*

Proof. (Sketch) We begin by choosing any $T > 0$. For brevity, let us define

$$A \doteq C_b(0,T;C_b(X)).$$

We then pick any $R > 0$ and define

$$B \doteq B(0,R) \subseteq A; \tag{22}$$

correct choices of R and T emerge naturally in the course of the proof. Let $\Psi\colon A \to A$ be defined by

$$\Psi[u]|_{e_i} \doteq e^{-\frac{\nu_i t}{\mu_i^2}}\varphi_i(x) + \left(\Phi[u] + \varphi(0)\left(1 - e^{-\frac{\nu_i t}{\mu_i^2}}\right)\right)e^{-\frac{x}{\mu_i}} + \sigma_i \mathbb{B}_{\mathrm{adv},i}[u_i](x,t) + \mathbb{B}_{\mathrm{visc},i}[u_i](x,t). \tag{23}$$

It is easy to see that Ψ maps B to itself provided

$$\|\varphi\|_{C_b(X)} + TR(1 + R^p) \le c_1 R \tag{24}$$

for some constant c_1 depending only on the parameters $\mu_i, \alpha_i, \gamma_i$, and ν_i. Further, Ψ is a contraction mapping if

$$T\ \ (1 + R^p) < c_2 \tag{25}$$

where c_2 is a constant depending on $\mu_i, \alpha_i, \gamma_i$, and ν_i. The two constraints (24) and (25) are satisfied if we choose

$$R \ge \max\left\{1, \frac{2}{c_1}\right\}\ \|\varphi\|_{C_b(X)} \quad \text{and} \tag{26a}$$

$$T < \frac{\min\left\{\frac{c_1}{2}, c_2\right\}}{1 + R^p}. \tag{26b}$$

Now, we may apply the contraction mapping theorem to see that Ψ has a unique fixed point in B if R and T satisfy (26). Unconditional uniqueness of the fixed point can be established by a routine bootstrap argument as in [9], Ch.3. From the definition of Ψ, this fixed point has Lipschitz dependence on φ (one may have to shrink the existence time a bit to get this to work). Due to the nested integrals in the definition of Ψ, the claimed regularity of the fixed point given a smooth enough φ is also obvious. We conclude that the fixed point is actually a classical solution to gBBMB, and the proof is complete. □

3.4 Global Well-Posedness via Energy Methods

In this section, we show that the local solution from the previous section can be extended to exist for all time. First, we need a helpful lemma characterizing the far-field behaviour of solutions to (14). This result can be obtained by adapting the proof of Lemma 3 in [2]:

Lemma 3. *Assume that all φ_i's and their derivatives converge to* 0 *as $x \to \infty$. Then, the functions $u_i(x,t)$ and all their derivatives converge to* 0 *as $x \to \infty$, uniformly in t.*

□

We can now begin studying the H^1 theory of (14). Note that we use a model-dependent energy norm that is equivalent to the usual H^1 norm provided $\mu_i > 0 \quad \forall\, i$:

Definition 1. *The* ***energy*** $E(t)$ *of the solution $u(x,t)$ to* (14) *is defined by*

$$E(t) \doteq \frac{1}{2}\|u\|^2_{H^1(X)} = \frac{1}{2}\sum_i \int_0^\infty |u_i|^2 + \mu_i^2 \ |u_{i,x}|^2 \ \mathrm{d}x, \tag{27}$$

*provided all of the integrals are finite (**we've changed the defn of the H^1 norm a bit here**).*

Next, we exhibit conditions under which $u(x,t)$ lies in $H^1(X)$ for all time and determine the evolution of u's energy.

Theorem 2. *If $\varphi \in H^1(X) \cap \left(C_b^2[0,\infty)\right)^N$, then $u(\cdot,t) \in H^1(X) \ \ \forall\, t \in [0,T]$. Further, for such φ, we have that the energy of $u(x,t)$ satisfies*

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -h^2(t) \ \left[\sum_i \sigma_i \left(\frac{\alpha_i}{2} + \frac{\gamma_i}{(p+1)(p+2)} h^p(t)\right)\right] - \sum_i \nu_i \int_0^\infty |u_{i,x}|^2 \ \mathrm{d}x. \tag{28}$$

Proof. We follow the proof of [2, Lemma 4]. Pick any $L > 0$, then multiply both sides of gBBMB on each edge by $2u_i$ and integrate with respect to x over $[0,L]$ to see that

$$\begin{aligned} 0 &= \int_0^L \partial_t \left(|u_i|^2 + \mu_i^2|u_{i,x}|^2\right) + 2\sigma_i \ \partial_x \left(\frac{\alpha_i}{2}u_i^2 + \frac{\gamma_i}{p+2}u_i^{p+2}\right) + 2\nu_i \ |u_{i,x}|^2 \ \mathrm{d}x - 2\left[\mu_i^2 u_i u_{i,xt} + \nu_i u_i u_{i,x}\right]_0^L \\ &= \int_0^L \partial_t \left(|u_i|^2 + \mu_i^2|u_{i,x}|^2\right) + 2\nu_i \ |u_{i,x}|^2 \ \mathrm{d}x + 2\left[\sigma_i \left(\frac{\alpha_i}{2}u_i^2 + \frac{\gamma_i}{p+2}u_i^{p+2}\right) - u_i\left(\mu_i^2 u_{i,xt} + \nu_i u_{i,x}\right)\right]_0^L. \end{aligned}$$

Adding up the above expressions for $i = 1, ..., N$ and using the mass conservation junction condition (17), we obtain

$$\begin{aligned}\frac{1}{2}\sum_i \int_0^L \partial_t\left(|u_i|^2+\mu_i^2|u_{i,x}|^2\right)\,\mathrm{d}x &= h^2(t)\ \left[\sum_i \sigma_i\left(\frac{\alpha_i}{2}+\frac{\gamma_i}{p+2}h^p(t)\right)\right] \\ &\quad - h(t)\ \left[\sum_i \sigma_i\left(\alpha_i h(t)+\frac{\gamma_i}{p+1}h^{p+1}(t)\right)\right]-\sum_i\int_0^L \nu_i\ |u_{i,x}|^2\,\mathrm{d}x \\ &\quad -\sum_i\left\{\sigma_i\left(\frac{\alpha_i}{2}u_i^2+\frac{\gamma_i}{p+2}u_i^{p+2}\right)-u_i\left(\mu_i^2 u_{i,xt}+\nu_i u_{i,x}\right)\right\}\Big|_{x=L} \\ &= -h^2(t)\ \left[\sum_i \sigma_i\left(\frac{\alpha_i}{2}+\frac{\gamma_i}{(p+1)(p+2)}h^p(t)\right)\right]-\sum_i\int_0^L \nu_i\ |u_{i,x}|^2\,\mathrm{d}x \\ &\quad -\sum_i\left\{\sigma_i\left(\frac{\alpha_i}{2}u_i^2+\frac{\gamma_i}{p+2}u_i^{p+2}\right)-u_i\left(\mu_i^2 u_{i,xt}+\nu_i u_{i,x}\right)\right\}\Big|_{x=L}.\end{aligned} \tag{29}$$

Since u is bounded on $[0,T]\times X$, $h(t)=u(0,t)$ is bounded on $[0,T]$. Additionally, by Lemma 3 all of the terms in curly braces in (29) vanish as $L\to\infty$ uniformly in t. Consequently, (29) indicates that there exists $C\geq 0$ depending on T, $\sup_{[0,T]}|h(t)|\ \leq \|u\|_{C_b(0,T,C_b(X))}$, and the coefficients of the PDE such that

$$\lim_{L\to\infty}\sum_i\int_0^L |u_i(x,t)|^2+\mu_i^2|u_{i,x}(x,t)|^2\,\mathrm{d}x \leq \|\varphi\|^2_{H^1(X)}+C, \tag{30}$$

By hypothesis, $u(\cdot,t)\in H^1(X)\ \forall\, t\in[0,T]$. Accordingly, we can go back to (29) and take $L\to\infty$ to obtain the formula (28). □

Physically, (28) tells us that any change in the solution's energy is due to either viscoelastic damping or movement through the central junction. For special parameter values and networks we can guarantee that, at the very least, energy is never gained at the junction. Further, this is enough to obtain a solution to gBBMB on X valid for all time. We state these global well-posedness results in the next two corollaries:

Corollary 1. *If* $\sum_i \sigma_i\alpha_i,\ \sum_i\sigma_i\gamma_i\geq 0$, *and p is even, then the solution to* (14) *valid up to time T has non-increasing energy, and can be extended to a unique global-in-time solution* $u\in C_b\left([0,\infty),H^1(X)\right)$.

Proof. Applying Theorem 2, we see that

$$\mathrm{d}E/\mathrm{d}t\leq 0$$

if the parameters of the problem are chosen according to the hypothesis. Therefore, we can extend the solution out to a further time $T' > T$ by defining new initial conditions $\tilde{\varphi} \doteq u(x,T) \in H^1(X) \cap \left(C_b^2(0,\infty)\right)^N$ and applying our local well-posedness result once more. Of course, this extended solution has the same regularity as the solution on $[0,T]$, by all the theory we have built so far. How can we be sure that we do not have $T' = T$? Since $\mathrm{d}E/\mathrm{d}t \leq 0$, Sobolev embeddings imply

$$\|u(x,T)\|_{C_b(X)} \lesssim \|\varphi\|_{H^1(X)} < \infty$$

hence

$$\left(1 + \|u(x,T)\|_{C_b(X)}\right)^{-p} \gtrsim \left(1 + \|\varphi\|_{H^1(X)}\right)^{-p} > 0. \tag{31}$$

Now, recalling that the existence time guaranteed by local theory when the initial conditions are taken to be $\|u(x,T)\|_{C_b(X)}$ is on the order of the leftmost side of the above inequality, we know $T' > T$.

Next, observe that the energy of the solution thus obtained remains non-increasing, so we may iterate the procedure described above as much as we like. To complete the proof of global existence, it suffices to show the sequence of extension times does not tend to 0. However, this follows immediately from changing the meaning of T in (31), allowing it to be some arbitrary time for which we know a solution exists.

Uniqueness of the global solution follows from an energy argument along the lines of [1] §4, using the junction conditions as in Theorem 2. □

Corollary 2. *If* $\sum_i \sigma_i \alpha_i,\ \sum_i \sigma_i \gamma_i = 0$, *then the solution to* (14) *valid up to time* T *has non-increasing energy, and can be extended to a unique global-in-time solution* $u \in C_b\left([0,\infty), H^1(X)\right)$.

Proof. Apply the same arguments used to prove Corollary 1. □

I emphasize that Corollary 2 holds regardless of the value of p. Note also that, if viscoelasticity is ignored ($\nu_i = 0\ \forall\ i$), then energy is actually conserved if the conditions of Corollary 2 are met.

Are the parameter restrictions imposed by the above corollaries physically meaningful? As we shall see in the next section, $\alpha_i > 0$ is necessary to ensuring long linear waves always move towards $+\infty$ on each edge. Since blood pressure waves are indeed long waves, $\alpha_i > 0$ is a suitable physical restriction. We also demand $\gamma_i \geq 0$, following Erbay et al. [5]. Since we do not expect scleroses or stents to cause large changes in the coefficients α_i and γ_i between edges, the constraints $\sum_i \sigma_i \alpha_i \geq 0$, $\sum_i \sigma_i \gamma_i \geq 0$ seem to be perfectly reasonable from a scientific standpoint. In particular, these constraints are satisfied in the simple case

$\alpha_i \equiv \alpha > 0$ and $\gamma_i \equiv \gamma \leq 1$. Also, in light of the aforementioned work of Erbay et al., $p = 1$ and $p = 2$ both correspond to valid asymptotic models of pulsatile flow in viscoelastic tubes. In fact, according to Erbay et al., choosing $p = 2$ may in fact be more physically relevant: compared to the $p = 1$ model, the $p = 2$ model captures genuinely nonlinear behaviour in a wider variety of viscoelastic materials. Therefore, the hypotheses of both corollaries are definitely of physical relevance.

3.5 Numerical Experiments

See slides!

4 Brief Summary of Talk

- gBBMB on a network can model blood flow in arterial tree, may give modelling insights into arteriosclerosis or stents in the femoral artery
- Formulated gBBMB to guarantee mass conservation in full generality: impose a "flux-balancing" condition at the network junction
- Proved local well-posedness, can often be extended to global by routine energy methods
- Numerical simulations of nonlinear scattering with and without viscoelasticity lead to interesting questions (see slides!)

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