Nonlinear Filtering with Lévy Noise

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1 Motivation

Given a signal process X_t and an observation process Y_t, the goal of filtering is to find $\mathbb{E}(\varphi(X_t)|Y_t)$, where φ is a smooth function.
More precisely, X_t and Y_t are often vector-valued processes given by a stochastic dynamical system. Then all our information is given by $\mathcal{F}_t^Y = \sigma(\{Y_s : s \in [0,t]\})$, the *history* of Y_t, and we are looking for S(P)DEs that help us in determining $\mathbb{E}(\varphi(X_t)|\mathcal{F}_t^Y)$, which for a time t is almost surely the conditional expectation of X_t given Y_t.
Writing, for a finite measure μ on $\mathbb{R}^d$ and $\varphi \in C_0^\infty(\mathbb{R}^d)$,

$$\mu_t(\varphi) := \int_{\mathbb{R}^d} \varphi(x)\mu_t(dx), \quad t \in [0,T].$$

the standard approaches are to find S(P)DEs for the measures μ_t, P_t or the density $u(x)$ of μ_t satisfying in some sense

- $P_t(\varphi) = \mathbb{E}(\varphi(X_t)|\mathcal{F}_t^Y)$,

- $\mu_t(\varphi) = \mathbb{E}_Q(\gamma_t^{-1}\varphi(X_t)|\mathcal{F}_t^Y)$,
- $(P_t)_{t\in[0,T]} = \mu_t(\varphi)/\mu(\mathbf{1})$,

where γ_t is the Girsanov exponent and $dQ = \gamma_T dP$.
In this talk, we will find an SPDE for μ_t and establish the existence of a solution.
This is joint work with Istvàn Gyöngy.
All of the content of this talk is taken from [1]-[6].

2 Some preliminaries

Throughout this article we will consider a complete filtered probability space $(\Omega, \mathcal{F}_t, P)$ which means in particular that all P-zero sets are included in our filtration $\mathcal{F}_t$.

2.1 Noise processes and random measures

Definition 1 (Wiener process).
We call W_t an $\mathcal{F}_t$-Wiener process if

1. $W_0 = 0$ almost surely,
2. W_t has independent, normally distributed, stationary increments, meaning that $W_t - W_s$ is independent of the σ-algebra $\mathcal{F}_s$, for $0 \leq s \leq t$ and has the same distribution as W_{t-s}, which is a Gaussian distribution with mean 0 and variance $(t-s)$,
3. The sample paths of W_t, in other words the functions $W_t(\omega)$, for any fixed ω, are continuous.

Definition 2 (Lévy process).
X is a *Lévy process* if

1. $X(0) = 0$ a.s.
2. X has independent and stationary increments
3. X is stochastically continuous, i.e. for all $a > 0$ and $s \geq 0$

   $$\lim_{t\to s} P(|X(t) - X(s)| > a) = 0,$$

   which with the first two properties is equivalent to

   $$\lim_{t\to 0^+} P(|X(t)| > a) = 0.$$

While Wiener processes are a standard choice (no pun intended) as disturbance processes, let us get some intuition for Lévy processes. We may remember that a Wiener process can be regarded as the distributional limit of a random walk (see Donsker's theorem). A similar procedure can be applied to obtain a Lévy process.

Example 1. A process $N_t \sim \pi(t\lambda)$ taking values in $\mathbb{N}_0$, i.e.

$$P(N_t = n) = \frac{(t\lambda)^n}{n!} e^{-t\lambda},$$

is called a *Poisson process with intensity* λ.
It can be shown that the stopping times

$$T_n = \inf\{t \geq 0 : N_t = n\}$$

are gamma distributed, which follows from the fact that the inter-arrival times $X_i = T_i - T_{i-1}$ follow an exponential distribution with parameter λ. i.e. mean $\frac{1}{\lambda}$. The sample paths of N_t are piecewise constant with jumps of height 1 at each of the time T_n. Let now $\{Z_n\}_{n\in\mathbb{N}}$ be a sequence of i.i.d. random variables taking values in $\mathbb{R}^d$ with common law μ_Z. Let N_t be a Poisson process with intensity λ independent of the Z_i. The *compound Poisson process* is

$$Y_t = Z(1) + \cdots + Z(N_t)$$

so that $Y_t \sim \pi(t\lambda, \mu_Z)$. Then Y_t is a Lévy process.

Definition 3.
Let X_t be a Lévy process. The *jump process* is

$$\Delta X_t = X_t - X_{t-}.$$

Exercise: If $N_t \sim \pi(t\lambda)$ then ΔN_t is not a Lévy process.
Exercise2: Show that $\sum_{0\leq s\leq t} |\Delta X_s| < \infty$ a.s. if X is a compound Poisson process.

Before we go on, remember that if a process is said to be cádlág (right-continuous with left limits, aka. *continue à droite, limit à gauche*), this actually means that there is Ω_0, with $P(\Omega_0) = 0$, such that the map $t \mapsto X_t(\omega)$ is càdlàg for all $\omega \in \Omega \setminus \Omega_0$.

Definition 4.
Let $t \in \mathbb{R}_+$ and $A \in \mathcal{B}(\mathbb{R}^d \setminus \{0\})$. Define

$$N(t, A)(\omega) = \#\{0 \leq s \leq t : \Delta X_s(\omega) \in A\} = \sum_{0\leq s\leq t} \mathbb{1}_A(\Delta X_s(\omega)).$$

For each $\omega \in \Omega \setminus \Omega_0$ and $t \geq 0$ the set function

$$A \mapsto N_t(A)(\omega)$$

is a *counting measure*, meaning it takes values in $\mathbb{N}$, and furthermore

$$\mathbb{E}(N(t,A)) = \int N(t,A)(\omega)dP(\omega)$$

is a Borel measure on $\mathcal{B}(\mathbb{R}^d \setminus \{0\})$. The measure

$$\mu(\cdot) = \mathbb{E}(N(1,\cdot))$$

is referred to as *intensity measure* associated with X_t.

Question: why did we have to exclude 0 from the Borel sets above?

Lemma 1.
If A is *bounded below*, meaning that $0 \notin \bar{A}$, then $N(t,A) < \infty$ a.s. for all t.

Proof. Exercise. $\square$

Theorem 1.
If A is bounded below, then $N(t,A)$ is a Poisson process with intensity $\mu(A)$.
Also, for disjoint sets A_i, that are bounded below, and distinct times t_i the random variables $N(t_i, A_i)$ are independent.

The above construction is a natural example for a so-called random measure. The following is a formal definition, albeit adapted to our cause. The theory on random measures is rich and beautiful, one can easily get lost in them. However, elaborating more on them would exceed the scope here. Instead let us look at what may be the most useful, or common one.

Definition 5 (Poisson random measure)**.** A Poisson random measure $N(dt, dz)$ on $\mathbb{R}_+ \times \mathbb{R}^d$ with intensity measure $dt \otimes \nu(dz)$ is a family of random variables $N(A)$, $A \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R}^d)$ such that

1. for each $A \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R}^d)$, the random variable $N(A)$ is Poisson distributed with intensity $(dt \otimes \nu(dz))(A)$,
2. For any disjoint sets $A_1, \dots, A_n$ the random variables $N(A_i)$, $i = 1, \dots, n$ are independent,
3. for all $\omega \in \Omega$, $N(\cdot)(\omega)$ is a σ-finite measure on $(\mathbb{R}_+ \times \mathbb{R}^d, \mathcal{B}(\mathbb{R}_+ \times \mathbb{R}^d))$.

Theorem 2.
Given a σ-finite measure λ on a mesurable space $(S, \mathcal{S})$, there exists a Poisson raondom measure M on a probability space $(\Omega, \mathcal{F}, P)$ such that $\lambda(A) = \mathbb{E}(M(A))$ for all $A \in \mathcal{S}$.

Definition 6 (Poisson point process and martingale measure)**.**
Let $S = \mathbb{R}_+ \times U$ where U is equipped with $\mathcal{U}$ and $\mathcal{S} = \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{U}$. Let p_t be an adapted process taking values in U such that

$$M([0,t) \times A) = \#\{0 \leq s \leq t : p_s \in A\}.$$

Then p_t is called a *Poisson point process* and M is its associated Poisson random measure.
If for all A the process $M_t(A)$ is a martingale then M is called a *martingale-valued* measure.

Definition 7 (compensated Poisson random measure)**.**
Let $U = \mathbb{R}^d \setminus \{0\}$ and $\mathcal{U} = \mathcal{B}(U)$. Let X_t be a Lévy process. Then ΔX_t is a Poisson point process and N its associated random measure. The *compensated Poisson random measure* is the martingale-valued measure given by

$$\tilde{N}(t, A) = N(t, A) - t\mu(A).$$

2.2 Stochastic integration

Definition 8 (adaptedness)**.**
A process X is adapted to a filtration $\mathcal{F}$ if for each fixed t the random variable X_t is measurable with respect to $\mathcal{F}_t$.

Finally comes the Wiener integral, albeit only for simple random variables. Recall that f_t is called a *simple* process if for some $0 \leq t_0 \leq \cdots \leq t_N$ and random variables $Z_i, i = 0, \ldots, N-1$, measurable with respect to $\mathcal{F}_{t_i}$ respectively,

$$f_t = \sum_{i=0}^{N-1} \mathbb{1}_{(t_i, t_{i+1}]}(t) Z_i.$$

Let W_t be a Wiener process adapted to a filtration $\mathbb{F}$ and f_t be a simple process, adapted as well, we can define

$$I(f) = \int_0^\infty f_t dW_t := \sum_{i=0}^{N-1} Z_i \Delta W_{t_i}$$

where the $\Delta W_{t_i} = W_{t_{i+1}} - W_{t_i}$.

Proposition 1.
Within the setting defined above

- $\mathbb{E}(f_t) = 0$
- $\mathbb{E}(I^2(f)) = \mathbb{E}\int_0^\infty f_t^2 dt$, called *Itô's isometry*. It extends to:
- $\mathbb{E}(I(f)I(g)) = \mathbb{E}\int_0^\infty f_t g_t dt$.

Then, for general f satisfying $\mathbb{E}\int_0^\infty f_t^2 dt < \infty$ there exists a sequence of simple functions f^n such that $\mathbb{E}\int_0^\infty |f_t - f_t^n|^2 dt \to 0$. Then we can define

$$I(f) = \int_0^\infty f_t dW_t := \lim_n \int_0^\infty f_t^n dW_t$$

as the mean-square limit.
Now, let N be a Poisson random measure on $\mathbb{R}^+ \times (\mathbb{R}^d \setminus \{0\})$ with intensity $dt \otimes \nu$. Let $\tilde{N}$ be the compensated random measure and let $K(t,x)$ be (predictable and) such that

$$P\Big(\int_0^T \int_E |K(t,x)|^2 \nu(dx) dt < \infty\Big) = 1.$$

Let $A \in \mathcal{B}(\mathbb{R}^d \setminus \{0\})$ be bounded below and introduce $P_t = \int_A x N(t, dx)$. Then define the integral against the Poisson random measure as

$$\int_0^T \int_A K(t,x) N(dt, dx) = \sum_{0 \le u \le T} K(u, \Delta P_u) \mathbb{1}_A(\Delta P_u),$$

i.e. as random finite sum. If $K = (K_1, \ldots, K_n)$, then

$$\int_0^T \int_A K_i(t,x) \tilde{N}(dt, dx) = \int_0^T \int_A K_i(t,x) N(dt, dx) - \int_0^T \int_A K_i(t,x) \nu(dx) dt.$$

One last important result, and maybe the most important one in stochastic calculus: Itô's formula. It can be stated in various generality and found in any textbook, so we will give a more informal version here.

Theorem 3. Let $W_t = W_t^i$, $i = 1, \ldots, d$ be a d-dimensional Wiener process and consider adapted processes $\mu_t = \mu_t^i$ and $\sigma_t = \sigma_t^{ij}$ that are almost surely square integrable in each component. Let, for $i,j = 1, \ldots, d$

$$X_t^i = X_0^i + \int_0^t \mu_s^i ds + \int_0^t \sigma_s^{ij} dW_s^j.$$

Then for any $\varphi \in C^2(\mathbb{R}^d)$, the process $vp(X_t)$ satisfies

$$d\varphi(X_t) = D_i \varphi(X_t) \mu_t^i dt + D_i \varphi(X_t) \sigma_t^{ij} dW_t^j + \frac{1}{2} D_{ij} \sigma_t^{ik} \sigma_t^{jk} dt.$$

3 The Zakai equation

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete filtered probability space, carrying a $d_1 + d'$-dimensional $\mathcal{F}_t$-Wiener process $(W_t, V_t)_{t\geq 0}$ and an independent $\mathcal{F}_t$-Poisson martingale measure $\tilde{N}(dz, dt) = N(dz, dt) - \nu(dz)dt$ on $\mathbb{R}_+ \times (\mathbb{R}^{d'} \setminus \{0\})$ with a σ-finite intensity measure ν. We consider the signal and observation model

$$\begin{aligned} dX_t &= b(t, X_t, Y_t)dt + \sigma(t, X_t, Y_t)dW_t + \rho(t, X_t, Y_t)dV_t \\ dY_t &= B(t, X_t, Y_t)dt + dV_t + \int_{\mathbb{R}^{d'}} z\tilde{N}(dz, dt), \end{aligned} \tag{1}$$

where $b = (b^i)$, $B = (B^i)$, $\sigma = (\sigma^{ij})$ and $\rho = (\rho^{il})$ are Borel functions on $\mathbb{R}_+ \times \mathbb{R}^{d+d'}$, with values in $\mathbb{R}^d$, $\mathbb{R}^{d'}$, $\mathbb{R}^{d\times d_1}$ and $\mathbb{R}^{d\times d'}$, respectively.

Assumption 1. (i) For $v_j = (x_j, y_j) \in \mathbb{R}^{d+d'}$ $(j = 1, 2)$, $t \geq 0$,

$$|b(t, v_1)-b(t, v_2)|+|B(t, v_1)-B(t, v_2)|+|\sigma(t, v_1)-\sigma(t, v_2)|+|\rho(t, v_1)-\rho(t, v_2)| \leq L|v_1-v_2|,$$

(ii) For all $v = (x, y) \in \mathbb{R}^{d+d'}$, $t \geq 0$ and some $K_0, K_1 > 0$ we have

$$|b(t, v)| + |\sigma(t, v)| + |\rho(t, v)| + |B(t, v)| \leq K_0 + K_1|v|.$$

(iii) The initial condition $Z_0 = (X_0, Y_0)$ is an $\mathcal{F}_0$-measurable random variable with values in $\mathbb{R}^{d+d'}$ such that $K_1(\mathbb{E}|X_0|^2 + \mathbb{E}|Y_0|^2) < \infty$.

By a well-known theorem of Itô it then follows that there exists a unique solution $(X_t, Y_t)_{t\geq 0}$ to (1) for any given $\mathcal{F}_0$-measurable initial value $U_0 = (X_0, Y_0)$, and for every $T > 0$,

$$\mathbb{E} \sup_{t\leq T}(|X_t|^2 + |Y_t|^2) \leq N(1 + \mathbb{E}|X_0|^2 + \mathbb{E}|Y_0|^2) \tag{2}$$

holds with a constant N depending only on T, K_0, K_1, d, d'.

Assumption 2. We have $\mathbb{E}\gamma_T = 1$, where

$$\gamma_t = \exp\left(-\int_0^t B(s, X_s, Y_s)\, dV_s - \tfrac{1}{2}\int_0^t |B(s, X_s, Y_s)|^2\, ds\right), \quad t \in [0, T]. \tag{3}$$

Definition 9. A process M_t is a martingale (with respect to $\mathcal{F}_t$) if it is integrable and for all $s \leq t$, $\mathbb{E}(M_t|\mathcal{F}_s) = M_s$.

Lemma 2. Under Assumption 2, the process γ_t is a martingale.

Proof. First note that γ satisfies

$$d\gamma_t = -B_t \gamma_t dV_t.$$

Define the stopping time

$$\tau_n := \inf\{t \geq 0 : \int_0^t |B_s|^2 \gamma_s^2 ds \geq n\}.$$

Then, $\gamma_t \mathbb{1}_{[t \leq \tau_n]}$ is square integrable and we know that

$$\gamma_{t \wedge \tau_n} = 1 + \int_0^t \gamma_s \mathbb{1}_{[s \leq \tau_n]} dV_s$$

is a martingale, meaning that for all n, and $t_2 \geq t_1$,

$$\mathbb{E}(\gamma_{t_2 \wedge \tau_n} | \mathcal{F}_{t_1}) = \gamma_{t_1 \wedge \tau_n}.$$

As $n \to \infty$ we have $\tau_n \to \infty$ as well, which in particular implies $\tau_n \wedge t \to t$, so that by Fatou's lemma, almost surely

$$\mathbb{E}(\gamma_{t_2} | \mathcal{F}_{t_1}) \leq \gamma_{t_1}.$$

Recalling that γ is positive as well as Assumption 2 finishes the proof. □

With this, we can define a new probability measure.

Theorem 4 (Girsanov)**.** Let $T \in [0, \infty)$ and let Assumption 2 hold. Then $Q = \gamma_T P$ defines an equivalent probability measure. Moreover, under Q, the process

$$\tilde{V}_t := \int_0^t B_s ds + V_t$$

is a martingale.

Proof. Assumption 2 directly gives that Q is an equivalent probability measure. It remains to check that $\tilde{V}$ is a $(Q, \mathcal{F}_t)$-martingale. For that it suffices to check that the finite-dimensional distributions of V under P are the same as those of $\tilde{V}$ under Q. To show that, let $\lambda_j \in \mathbb{R}^d$, $j = 1 \ldots, n$ and define the function

$$\lambda_t := \sum_j \lambda_j \mathbb{1}_{(t_j, t_{j+1}]}(t).$$

Then

$$\mathbb{E}_Q \exp\left(i \sum_j \lambda_j (\tilde{V}_{t_{j+1}} - \tilde{V}_{t_j})\right) = \mathbb{E} \exp\left(\int_0^T \lambda_s dV_s + \int_0^T \lambda_s B_s ds\right) \gamma_T$$

$$= \mathbb{E}\exp\left(\int_0^T (\lambda_s - B_s)dV_s + \frac{1}{2}\int_0^T |\lambda_s - B_s|^2 ds\right) e^{\frac{1}{2}\int_0^T |\lambda_s|^2 ds} = \exp\left(\frac{1}{2}\int_0^T |\lambda_s|^2 ds\right),$$

which concludes the proof. □

With the same method the following can be shown.

Lemma 3. Under Q the Wiener process $\tilde{V}$ and the Poisson martingale measure $\tilde{N}$ are independent.

Next let us show that the filtration generated by Y allows for a certain decomposition.

Lemma 4. For every $t \geq 0$ we have

$$\mathcal{F}_t^Y = \mathcal{F}_0^Y \vee \mathcal{F}_t^{\tilde{V}} \vee \mathcal{F}_t^{\tilde{N}},$$

where $\mathcal{F}_0^Y \vee \mathcal{F}_t^{\tilde{V}} \vee \mathcal{F}_t^{\tilde{N}}$ denotes the P-completion of the smallest σ-algebra containing $\mathcal{F}_0^Y$, $\mathcal{F}_t^{\tilde{V}}$ and $\mathcal{F}_t^{\tilde{N}}$.

The following lemma is an essential tool for calculating conditional expectations of stochastic integrals of simple processes under Q given $\mathcal{F}_t^Y$.

Lemma 5. Let X and Y be random variables such that $E|X| < \infty$, $E|Y| < \infty$ and $E|XY| < \infty$. Let $\mathcal{G}^1$, $\mathcal{G}^2$ and $\mathcal{G}$ be σ-algebras of events such that $\mathcal{G}^1 \subset \mathcal{G}$, $\mathcal{G}^2$ is independent of $\mathcal{G}$, X is $\mathcal{G}$-measurable and Y is independent of $\mathcal{G} \vee \mathcal{G}^2 := \sigma(\mathcal{G}, \mathcal{G}^2)$. Then

$$E(XY|\mathcal{G}^1 \vee \mathcal{G}^2) = E(X|\mathcal{G}^1)EY.$$

Lemma 6.
Let $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ be a filtered probability space and let f_t be predictable with respect to $\mathcal{F}_t$ such that for all $T > 0$

$$\|f\|_{2,T}^2 = \mathbb{E}\Big(\int_0^T |f_s|^2 \,\mathrm{ds}\Big) < \infty.$$

Then the following equations hold:

$$\mathbb{E}\Big(\int_0^t f_s d\tilde{V}_s \Big| \mathcal{F}_t^Y\Big) = \int_0^t \mathbb{E}(f_s|\mathcal{F}_s^Y)d\tilde{V}_s. \tag{4}$$

$$\mathbb{E}\Big(\int_0^t f_s dW_s \Big| \mathcal{F}_t^Y\Big) = 0 \tag{5}$$

$$\mathbb{E}\Big(\int_0^t f_s \,\mathrm{ds} \Big| \mathcal{F}_t^Y\Big) = \int_0^t \mathbb{E}(f_s|\mathcal{F}_s^Y)\,\mathrm{ds} \tag{6}$$

Proof. We will begin by showing equation (4) for simple processes of the form

$$f_t^{(n)} = \sum_{i=0}^{n-1} f_i \mathbb{1}_{(t_i,t_{i+1}]}(t)$$

where all f_i are $\mathcal{F}_{t_i}$-adapted and where $0 = t_0 \leq \cdots \leq t_n = T$. Define $\Delta\tilde{V}_{t_i} = \tilde{V}_{t_{i+1}} - \tilde{V}_{t_i}$ and $\mathcal{F}_{s,t}^Y = \bigvee_{\tau\in(s,t]} \mathcal{F}_\tau^Y$. Then clearly

$$\begin{aligned}
\mathbb{E}\Big(\int_0^t f_s^{(n)} d\tilde{V}_s \Big| \mathcal{F}_t^Y \Big) &= \mathbb{E}\Big(\sum_{i=0}^{n-1} f_i \Delta\tilde{V}_{t_i} \Big| \mathcal{F}_t^Y \Big) \\
&= \sum_{i=0}^{n-1} \mathbb{E}(f_i | \mathcal{F}_{t_i}^Y \vee \mathcal{F}_{t_i,t}^Y) \Delta\tilde{V}_{t_i} \\
&= \sum_{i=0}^{n-1} \mathbb{E}(f_i | \mathcal{F}_{t_i}^Y) \Delta\tilde{V}_{t_i} \\
&= \int_0^t \mathbb{E}(f_s^{(n)} | \mathcal{F}_s^Y) d\tilde{V}_s,
\end{aligned}$$

where we used that $\mathcal{F}_{t_i,t}^Y$ is independent of $\mathcal{F}_t \vee \mathcal{F}_{t_i}^Y \supset \sigma(f_i) \vee \mathcal{F}_{t_i}^Y$. Due to the continuity of the map

$$\begin{cases} \mathbb{I} : f_t & \mapsto \int_0^t f_s d\tilde{V}_s \\ L^2(\Omega, \mathcal{F}, \mathcal{F}_t, P) & \to M_2^c(\Omega, \mathcal{F}, \mathcal{F}_t, P), \end{cases}$$

where M_2^c denotes the space of continuous local martingales, it remains to show that if $f_t^{(n)} \to f_t$ in L^2 then $\mathbb{E}(f_t^{(n)} | \mathcal{F}_t^Y) \to \mathbb{E}(f_t | \mathcal{F}_t^Y)$ in L^2. However, as for all t the map $\mathbb{E}(\cdot | \mathcal{F}_t^Y) : L^2 \to L^2$ is continuous, this is satisfied. To prove equation (5), observe that

$$\mathbb{E}\Big(\int_0^t f_s^{(n)} dW_s \Big| \mathcal{F}_t^Y \Big) = \sum_{i=1}^{n-1} \mathbb{E}(f_i \Delta W_{t_i} | \mathcal{F}_t^Y) = \sum_{i=1}^{n-1} \mathbb{E}(f_i | \mathcal{F}_t^Y) \mathbb{E}(\Delta W_{t_i}) = 0,$$

as ΔW_{t_i} is independent of $\mathcal{F}_{t_i} \vee \mathcal{F}_t^Y$. □

We introduce the random differential operators

$$\mathcal{L}_t = a_t^{ij}(x) D_{ij} + b_t^i(x) D_i, \quad \mathcal{M}_t^k = \rho_t^{ik}(x) D_i + B_t^k(x), \quad k = 1, 2, ..., d',$$

where

$$a_t^{ij}(x) := \tfrac{1}{2} \sum_{k=1}^{d_1} (\sigma^{ik}\sigma^{jk})(t, x, Y_t) + \tfrac{1}{2} \sum_{l=1}^{d_2} (\rho^{il}\rho^{jl})(t, x, Y_t), \quad \rho_t^{il}(x) := \rho^{il}(t, x, Y_t),$$

$$b_t^i(x) := b^i(t,x,Y_t), \quad B_t^k(x) := B^k(t,x,Y_t)$$

for $\omega \in \Omega$, $t \geq 0$, $x = (x^1, ..., x^d) \in \mathbb{R}^d$, and $D_i = \partial/\partial x^i$, $D_{ij} = \partial^2/(\partial x^i \partial x^j)$ for $i, j = 1, 2..., d$.

Proposition 2. For the stochastic differential of $\gamma_t^{-1}\varphi(X_t)$ we have

$$d\big(\gamma_t^{-1}\varphi(X_t)\big) = \gamma_t^{-1}\mathcal{L}_t\varphi(X_t)\,dt + \gamma_t^{-1}\mathcal{M}_t^l\varphi(X_t)\,d\tilde{V}_t^l + \gamma_t^{-1}\sigma_t^i(X_t)D_i\varphi(X_t)\,dW_t \tag{7}$$

Proof. Apply Itô's formula and the stochastic differential rule for products,

$$d(\gamma_t^{-1}\varphi(X_t)) = \gamma_t^{-1}d\varphi(X_t) + \varphi(X_{t-})\,d\gamma_t + d\gamma_t^{-1}d\varphi(X_t),$$

with

$$d\gamma_t^{-1}d\varphi(X_t) = \gamma_t^{-1}\rho_t^{il}B_t^l(X_t)D_i\varphi(X_t)\,dt,$$

we obtain (7). □

Let $\mathbb{M}$ be the set of finite Borel measures on $\mathbb{R}^d$ and let $\mathcal{G}_t \subset \mathcal{F}_t$ be a filtration. We say μ_t is a weakly cádl'ag, $\mathcal{G}_t$-adapted, $\mathbb{M}$-valued process is for all $\varphi \in C_0\infty(\mathbb{R}^d)$ the process

$$\mu_t(\varphi) := \int_{\mathbb{R}^d}\varphi(x)\mu_t(dx)$$

is c'adlàg almost surely and $\mathcal{G}_t$-adapted. The following is one of two filtering equations, from which the filtering measure can be characterized completely.

Theorem 5. Under Assumptions 1 and 2 there exists a weakly càdlàg, $\mathcal{F}_t^Y$-adapted, $\mathbb{M}$-valued process μ_t such that

$$\mu_t(\varphi) = \mathbb{E}_Q(\gamma^{-1}\varphi(X_t)|\mathcal{F}_t^Y)$$

for all $\varphi \in C_0\infty(\mathbb{R}^d)$ and such that μ_t satisfies almost surely

$$\mu_t(\varphi) = \mu_0(\varphi) + \int_0^t \mu_s(\mathcal{L}_s\varphi)ds + \int_0^t \mu_s(\mathcal{M}_s^k\varphi)d\tilde{V}_s^k + \int_0^t \mu_s(\sigma_s^i D_i\varphi)dW_s$$

for all $t \in [0, T]$.

4 Summary

In this talk we

1. motivated Poisson random measures as generalization of Wiener process noise,

2. established the necessary components of stochastic calculus to treat such terms,
3. proved that under a change of measure the dift term of a stochastic differential can be 'swallowed' be the Wiener process noise, making it a partingale,
4. showed that, as a result, the observation filtration allows for a specific, useful decomposition,
5. proved a projection lemma for conditional expectations of integral processes, and finally
6. derived the Zakai equation for the evolution of the unnormalized conditional measure process.

References

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