# Blow-Up in Fluid Equations

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## 1 Introduction

Consider the incompressible Euler equations, given by

$$\partial_t u + u \cdot \nabla u + \nabla p = 0$$
$$\nabla \cdot u = 0$$

where $u : \mathbb{R}^d \times [0, T) \to \mathbb{R}^d$ represents the velocity field of an incompressible fluid. In 2 dimensions ($d = 2$) the system is globally well-posed and solutions exist for all time (one can choose $T = \infty$). In dimension $d = 3$, the problem of global well-posedness remains one of the more famous open problems in math.

## 2 The operator $D^2\Delta^{-1}$

Taking a derivative of the Euler equation and considering the equation along characteristics,

$$\partial_t(\nabla u \circ \Phi) + (\nabla u \circ \Phi)^2 + (D^2 p \circ \Phi) = 0.$$

If $D^2 p(x)$ was determined by $\nabla u(x)$ pointwise as $D^2 p(x) = F(\nabla u(x))$ then the above yields an ODE for $\nabla u(x)$ which would be easily studied. Of course, the pressure is not determined locally by $\nabla u(x)$. Indeed, taking the divergence of the Euler equations and recalling that $u$ is divergence free

$$\nabla \cdot (u \cdot \nabla u) = -\Delta p$$

so that $D^2 p = D^2(-\Delta)^{-1}(\nabla \cdot (u \cdot \nabla u)$. However, $(-\Delta)^{-1}$ is a non-local operator. Explicitly $(-\Delta)^{-1}$ is given by convolution against a singular integral kernel. This leads us to the study of the operator $D^2\Delta^{-1}$. In particular, we are interested in boundedness of $D^2\Delta^{-1}$. Fix $f \in L^\infty$ with compact support and consider the equation

$$\Delta\psi = f$$

on $\mathbb{R}^2$. The problem of studying boundedness of $D^2\Delta^{-1}$ is then about the boundedness of $D^2\psi$. For $p < \infty$, we have boundedness as $\psi \in W^{2,p}_{loc}$. For $p = \infty$ however, boundedness fails and it is not the case that $D^2\psi \in L^\infty_{loc}$. Indeed, $D^2\Delta^{-1}$ has symbol $\frac{-i\xi_i}{|\xi|}\frac{-i\xi_j}{|\xi|}$ and is therefore a composition of Riesz transforms, $D^2\psi_{ij} = R_i R_j f$ and hence bounding $D^2\psi$ by $\Delta\psi$ is equivalent to studying boundedness of Riesz transforms $R_i R_j$ on $L^\infty$. The unboundedness can be seen by considering the example $f(x_1, x_2) = x_1 x_2 \log(x_1^2 + x_2^2)\phi(x_1, x_2)$ where $\phi$ is a smooth cutoff function which is 1 near the origin or $f = \chi_{[0,1]^2}$. If $f$ has additional symmetries however, the situation improves. We now outline a result from [1].

Lemma: Let $g \in L^\infty(\mathbb{R}^2)$ and suppose $g$ has the 4-fold symmetry $g(x) = g(x^\perp)$ for all $x \in \mathbb{R}^2$. Let $\psi \in L^\infty_{loc}(\mathbb{R}^2)$ solve

$$\Delta\psi = g$$

on $\mathbb{R}^2$. Then, we have the bound

$$\sup_{x \in B_1(0)} \frac{|\nabla\psi(x)|}{|x|} < \infty.$$

The proof is a direct computation using the Green's function of the Laplacian on $\mathbb{R}^2$ which we outline below. First, using standard arguments it suffices to consider the case where $g$ has compact support. Using the Green's function representation, we have that

$$\nabla\psi(x) = \int_{\mathbb{R}^2} \frac{x - y}{|x - y|^2} g(y)\, dy.$$

From the symmetry of $g$,

$$\nabla\psi(x) = \int_{\mathbb{R}^2} \frac{x-y}{|x-y|^2} g(y)\ dy = \int_{\mathbb{R}^2} \frac{x-y^\perp}{|x-y^\perp|^2} g(y)\ dy = \int_{\mathbb{R}^2} \frac{x+y}{|x+y|^2} g(y)\ dy = \int_{\mathbb{R}^2} \frac{x+y^\perp}{|x+y^\perp|^2} g(y)\ dy$$

and consequently,

$$\nabla\psi(x) = \frac{1}{4}\int_{\mathbb{R}^2} \left[\frac{x-y}{|x-y|^2} + \frac{x-y^\perp}{|x-y^\perp|^2} + \frac{x+y}{|x+y|^2} + \frac{x+y^\perp}{|x+y^\perp|^2}\right] g(y)\ dy = (K * g)(x).$$

We now simplify the integral kernel $K$. First,

$$\begin{aligned}
\frac{x-y}{|x-y|^2} + \frac{x+y}{|x+y|^2} &= \frac{(x-y)|x+y|^2 + (x+y)|x-y|^2}{|x-y|^2|x+y|^2} \\
&= \frac{(x-y)(|x|^2+|y|^2-2\langle x,y\rangle) + (x+y)(|x|^2+|y|^2-2\langle x,y\rangle)}{|x-y|^2|x+y|^2} \\
&= \frac{x(2|x|^2+2|y|^2) - 4\langle x,y\rangle y}{|x-y|^2|x+y|^2}
\end{aligned}$$

and similarly,

$$\frac{x-y^\perp}{|x-y^\perp|^2} + \frac{x+y^\perp}{|x-y^\perp|^2} = \frac{x(2|x|^2+2|y|^2) - 4\langle x,y^\perp\rangle y^\perp}{|x-y^\perp|^2|x+y^\perp|^2}.$$

Combining both terms,

$$K(x-y) = 2x(|x|^2+|y|^2)\left[\frac{1}{|x-y|^2|x+y|^2} + \frac{1}{|x-y^\perp|^2|x+y^\perp|^2}\right] - \frac{4y\langle x,y\rangle}{|x-y|^2|x+y|^2} - \frac{4y^\perp\langle x,y^\perp\rangle}{|x-y^\perp|^2|x+y^\perp|^2}$$

Obtaining a common denominator for the first two terms, we note that

$$\begin{aligned}
|x-y|^2|x+y|^2 + |x-y^\perp|^2|x+y^\perp|^2 &= (|x|^2 - 2\langle x,y\rangle + |y|^2)(|x|^2+|y|^2+2\langle x,y\rangle) \\
&\quad + (|x|^2+|y|^2-2\langle x,y^\perp\rangle)(|x|^2+|y|^2+2\langle x,y^\perp\rangle) \\
&= 2|x|^4 + 2|y|^4 + 4|x|^2|y|^2 - 4\langle x,y\rangle^2 - 4\langle x,y^\perp\rangle^2 \\
&= 2(|x|^4+|y|^4) + 4|x|^2|y|^2 - 4|x|^2|y|^2 \\
&= 2(|x|^4+|y|^4).
\end{aligned}$$

Obtaining a common denominator for the remaining two terms, we note that

$$\begin{aligned}
&4y\langle x,y\rangle|x-y^\perp|^2|x+y^\perp| + 4y^\perp\langle x,y^\perp\rangle|x-y|^2|x+y|^2 \\
&= 4y^\perp\langle x,y^\perp\rangle\left[(|x|^2+|y|^2-2\langle x,y\rangle)(|x|^2+|y|^2+2\langle x,y\rangle)\right] \\
&+ 4y\langle x,y\rangle\left[(|x|^2+|y|^2-2\langle x,y^\perp\rangle)(|x|^2+|y|^2+2\langle x,y^\perp\rangle)\right] \\
&= 4y^\perp\langle x,y^\perp\rangle\left[(|x|^2+|y|^2)^2 - 4\langle x,y\rangle^2\right] + 4y\langle x,y\rangle\left[(|x|^2+|y|^2)^2 - 4\langle x,y^\perp\rangle^2\right].
\end{aligned}$$

Putting everything together, we obtain

$$K(x-y) = \frac{4x(|x|^2+|y|^2)(|x|^4+|y|^4 - 4y^\perp\langle x,y^\perp\rangle\left[(|x|^2+|y|^2)^2 - 2\langle x,y\rangle^2\right] + 4y\langle x,y\rangle\left[(|x|^2+|y|^2)^2 - 2\langle x,y^\perp\rangle^2\right]}{|x-y|^2|x+y|^2|x-y^\perp|^2|x+y^\perp|^2}.$$

Next, we collect terms according to their order in $x$. Note that there are no even ordered terms in $x$. At first order in $x$, we obtain

$$4x|y|^6 - 4y^\perp\langle x,y^\perp\rangle|y|^4 - 4y\langle x,y\rangle|y|^4 = 4x|y|^6 - 4|y|^4|y|^2\left[\frac{\langle x,y^\perp\rangle}{|y|^2}y^\perp + \frac{\langle x,y\rangle}{|y|^2}y\right] = 4x|y|^6 - 4|y|^6x = 0.$$

At third order,

$$\begin{aligned}
&4x|x|^2|y|^4 - 4y^\perp \left\langle x, y^\perp \right\rangle 2|x|^2|y|^2 - 4y^\perp \left\langle x, y^\perp \right\rangle (-2) \left\langle x, y \right\rangle^2 - 4y \left\langle x, y \right\rangle 2|x|^2|y|^2 - 4y \left\langle x, y \right\rangle (-2) \left\langle x, y^\perp \right\rangle^2 \\
&= 4x|x|^2|y|^4 + 8y^\perp \left\langle x, y^\perp \right\rangle \left\langle x, y \right\rangle^2 + 8y \left\langle x, y \right\rangle \left\langle x, y^\perp \right\rangle - 8|x|^2|y|^2 \left[ y^\perp \left\langle x, y^\perp \right\rangle + y \left\langle x, y \right\rangle \right] \\
&= -4x|x|^2|y|^4 + 8y^\perp \left\langle x, y^\perp \right\rangle \left\langle x, y \right\rangle^2 + 8y \left\langle x, y \right\rangle \left\langle x, y^\perp \right\rangle
\end{aligned}$$

At fifth order,

$$4x|y|^2|x|^4 - 4y^\perp \left\langle x, y^\perp \right\rangle |x|^4 - 4y \left\langle x, y \right\rangle |x|^4 = 0.$$

Finally, at seventh order,

$$4x|x|^2|x|^4 = 4x|x|^6.$$

Thus, we obtain the kernel,

$$K(x-y) = \frac{4x|x|^6 - 4x|x|^2|y|^4 + 8y^\perp \left\langle x, y^\perp \right\rangle \left\langle x, y \right\rangle^2 + 8y \left\langle x, y \right\rangle \left\langle x, y^\perp \right\rangle}{|x-y|^2|x+y|^2|x-y^\perp|^2|x+y^\perp|^2}.$$

We then have that

$$\begin{aligned}
|\nabla \psi(x)| &\le \int_{B_R(0)} \left| \frac{4x|x|^6 - 4x|x|^2|y|^4 + 8y^\perp \left\langle x, y^\perp \right\rangle \left\langle x, y \right\rangle^2 + 8y \left\langle x, y \right\rangle \left\langle x, y^\perp \right\rangle}{|x-y|^2|x+y|^2|x-y^\perp|^2|x+y^\perp|^2} g(y) \right| \, dy \\
&\le \|g\|_{L^\infty} \int_{B_R(0)} \left| \frac{4x|x|^6 - 4x|x|^2|y|^4 + 8y^\perp \left\langle x, y^\perp \right\rangle \left\langle x, y \right\rangle^2 + 8y \left\langle x, y \right\rangle \left\langle x, y^\perp \right\rangle}{|x-y|^2|x+y|^2|x-y^\perp|^2|x+y^\perp|^2} \right| \, dy.
\end{aligned}$$

Re-scaling the integral, we change varaibles $z = |x|y$ to obtain

$$\begin{aligned}
&\int_{B_R(0)} \left| \frac{4x|x|^6 - 4x|x|^2|y|^4 + 8y^\perp x y^\perp \left\langle x, y \right\rangle^2 + 8y \left\langle x, y \right\rangle \left\langle x, y^\perp \right\rangle}{|x-y|^2|x+y|^2|x-y^\perp|^2|x+y^\perp|^2} \right| \, dy \\
&= |x|^2 \int_{B_{R/|x|}(0)} \frac{\left| 4x|x|^6 - 4x|x|^6|z|^4 + 8z^\perp |x|^4 \left\langle x, z^\perp \right\rangle \left\langle x, z \right\rangle^2 + 8z|x|^4 \left\langle x, z \right\rangle \left\langle x, z^\perp \right\rangle^2 \right|}{|x - |x|z||^2|x + |x|z|^2|x - |x|z^\perp|^2|x + |x|z^\perp|^2} \, dz \\
&= |x| \int_{B_{R/|x|}(0)} \frac{\left| 4\frac{x}{|x|} - 4\frac{x}{|x|}|z|^4 + 8z^\perp \left\langle \frac{x}{|x|}, z^\perp \right\rangle \left\langle \frac{x}{|x|}, z \right\rangle^2 + 8z \left\langle \frac{x}{|x|}, z \right\rangle \left\langle \frac{x}{|x|}, z^\perp \right\rangle^2 \right|}{\left| \frac{x}{|x|} - z \right|^2 \left| \frac{x}{|x|} + z \right|^2 \left| \frac{x}{|x|} - z^\perp \right|^2 \left| \frac{x}{|x|} + z^\perp \right|^2} \, dz
\end{aligned}$$

Splitting the integral into the regions $|z| < 2$ and $|z| > 2$, the latter integral is bound uniformly in $|x|$ as it decays like $|z|^4/|z|^8 = |z|^{-4}$. To bound the former integral, note that at the (isolated) singularities $z = \pm x/|x|, \pm x^\perp/|x|$ the numerator vanishes linearly in $x/|x|$ and consequently, the integrand behaves like $\frac{1}{|x/|x|-z|}$ which is integrable independent of $x$ in $\mathbb{R}^2$. Consequently, we conclude that

$$|\nabla \psi(x)| \le \|g\|_{L^\infty} |x| C$$

for some constant $C$, completing the proof of the lemma.

# 3 The Boussinesq System

We now turn to the Boussinesq equations

$$\begin{aligned} \partial_t u + u \cdot \nabla u + \nabla p &= (-\rho, 0) \\ \partial_t \rho + u \cdot \nabla \rho &= 0 \\ \nabla \cdot u &= 0 \end{aligned}$$

where $u : \Omega \times \mathbb{R} \to \mathbb{R}^2$ is again the velocity of an incompressible fluid, and $\rho$ represents the density of the fluid which is transported by the velocity. Note that for $\rho \equiv 0$, the Boussinesq system reduces to the Euler equation. The Boussinesq system is similar in form to the axi-symmetric $3D$ Euler equations and is often expected to exhibit similar behaviour to the $3D$ Euler equation away from the symmetry axis. In [2] solutions which blow-up in finite time are constructed on the domain

$$\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq \gamma |x_2|\}$$

for any $\gamma > 0$. Taking $\gamma \to 0$, the solutions can be constructed on domains arbitrarily close to the half-plane. The construction makes use of scale invariant solutions of the Boussinesq equation. Indeed, note that if $(u, \rho)$ solve the Boussinesq system, then for any $\lambda > 0$, $\lambda^{-1}(u(t, \lambda x), \rho(t, \lambda x))$ is also a solution of the Boussinesq system. Thus, if $(u_0, \rho_0)$ are scale invariant as above, then $u, \rho$ will also be scale invariant (provided we have appropriate existence/uniqueness of solutions). To accommodate these scale-invariant solutions, a weighted Hölder space is introduced,

$$\|f\|_{\dot{C}^\alpha} := \|f\|_{L^\infty} + \sup_{x \neq y} \frac{||x|^\alpha f(x) - |y|^\alpha f(y)|}{|x - y|^\alpha}.$$

This space is large enough to contain scale-invariant solutions, yet small enough to have local well-posedness. To prove local well-posedness in $\dot{C}^\alpha$, requires sharp elliptic estimates which rely on the symmetry of the domain.

# 4 Summary

1. Understanding $D^2\Delta^{-1}$ is important to understanding blow-up
2. $D^2\Delta^{-1}$ is unbounded on $L^\infty$, but symmetries help
3. Scale invariance can be used to construct solutions of the Boussinesq system which form singularities

# References

[1] Tarek M. Elgindi. Remarks on Functions with Bounded Laplacian, 2016.

[2] Tarek M. Elgindi and In-Jee Jeong. Finite-Time Singularity Formation for Strong Solutions to the Boussinesq System. *Annals of PDE*, 6(1), 2020.