Ground state energy and geodesics in a dynamic energy landscape

Cameron Martin

June 2, 2021

1 Discrete Last Passage Percolation

In discrete last passage percolation, we independently assign to each vertex v in $\mathbb{Z}^2$ a sample of some random variable w_v. A key example will be *Bernoulli* last passage percolation, where we assign 1 to each vertex with probability 1/2, and 0 otherwise. For any $n \in \mathbb{N}$, consider the space $\mathcal{P}_n$ of **upright paths** from $(0,0)$ to (n,n). An upright path is one which can only travel up or to the right. The **energy** of a path Γ is given by

$$E(\gamma) = \sum_{v \in \gamma} w_v.$$

The **ground state energy** for fixed n is defined to be the maximal energy of upright paths travelling from $(0,0)$ to (n,n):

$$M_n = \max_{\gamma \in \mathcal{P}_n} E(\gamma),$$

while the **geodesic** Γ_n is the energy-maximizing path. The geodesic is almost surely unique when the distribution of vertex weights has a continuous distribution function. In the Bernoulli case, when we say "the geodesic Γ_n", we are referring to the upper-left-most energy-maximizing path.

1.1 Dynamical Bernoulli Last Passage Percolation

We may introduce dynamics to the system as follows: assign to each vertex a Poisson clock of rate one, and when a given clock rings, re-sample a Bernoulli random variable at that vertex. In practice, this means that for a given vertex, the time until the next ring is exponentially distributed with mean one. Using the variable t to keep track of these dynamics, we consider the time t ground state energy M_n^t and geodesic Γ_n^t. We aim to understand the *time-scale of decorrelation*. That is, how long does it take for the time-t geodesic Γ_n^t to no longer resemble Γ_n^0? And how long does it take for M_n^t to decorrelate from M_n^0? To address the first question, we consider the **overlap** $\mathcal{O}_n\big(\Gamma_n^0, \Gamma_n^t\big)$, which is the one-dimensional Lebesgue measure of the intersection of the horizontal portions of these paths. To address the second, we consider the correlation

$$\mathrm{Corr}\big(M_n^0, M_n^t\big) = \frac{\mathrm{Cov}\big(M_n^0, M_n^t\big)}{\mathrm{Var} M_n^0}.$$

It is expected that there is a phase transition from "stability" to "chaos" in both the overlap of geodesics and correlation of time-t energies at $t = \Theta\big(n^{-1/3}\big)$.

2 Brownian Last Passage Percolation

It turns out that it is easier to say things about the "semi-discretized" version of this model, **Brownian last passage percolation** [2]. Consider n independent copies of Brownian motion $(B_1, B_2, \ldots, B_n)$. The energy of an upright path γ going from $(n,0)$ to $(1,1)$ is

$$E(\gamma) = \sum_{i=1}^{n} (B_i(x_i) - B_i(x_{i-1})),$$

where $x_1 \leq x_2 \leq \ldots \leq x_{n-1}$ are the "jump times" of the upright path, and $x_0 = 0, x_n = 1$. This can be viewed as either maximizing increments or minimizing "gaps" - see Fig. 1. The last passage value (ground state energy) is defined again by taking the supremum:

$$M_n = \sup_{\gamma:(n,0)\to(1,1)} E(\gamma) =: E(\Gamma_n).$$

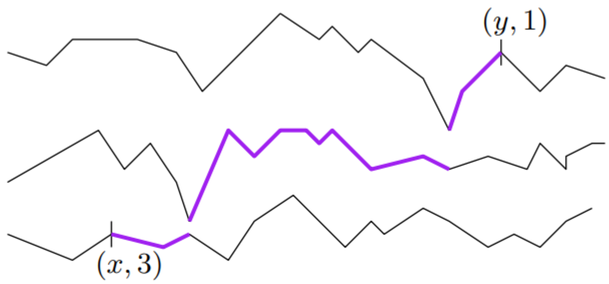

Figure 1: An example of last passage across three functions. The purple path is the last passage path from $(x, 3)$ to $(y, 1)$. It can be viewed as either maximizing the sum of increments along the path, or minimizing the sum of gaps. We will always think of our sequences of functions as being labelled so that f_i sits above f_{i+1}. Our notions of 'right' and 'left' in the paper are with respect to this picture.

Figure 1: Figure and caption shamelessly taken from [2].

2.1 Dynamical Brownian Last Passage Percolation

We introduce dynamics into this system by allowing the Brownian motions B_i to evolve according to an Ornstein-Uhlenbeck flow. That is, at time t each B_i is equal in distribution to

$$e^{-t} B_i(\cdot, t) + \left(1 - e^{-2t}\right)^{1/2} B_i'(\cdot),$$

where B_i' is an independent copy of standard Brownian motion. Using facts about the Ornstein-Uhlenbeck semigroup (see appendix), the covariance lemma provides a nice characterization of the correlation between M_n^0 and M_n^t in terms of the expected overlap. Considering the fully discrete model (with, say, Gaussian vertex weights), the ground state energy M_n is an (almost surely) differentiable function from $\mathbb{R}^{\Lambda_n}$ to $\mathbb{R}$, where $\Lambda_n = \{0, 1, \ldots, n\}^2$. The gradient is

$$\nabla M_n(x, y) = I((x, y) \in \Gamma_n).$$

Therefore

$$\nabla M_n^0 \cdot \nabla M_n^t = \left|\Gamma_n^0 \cap \Gamma_n^t\right|.$$

This is analogous to the overlap function $\mathcal{O}_n(t)$ defined earlier. It takes some more work to approximate the Brownian last passage model with these discrete models, but the upshot is that we have a covariance lemma which says that

$$\operatorname{Cov}\left(M_n^0, M_n^t\right) = \int_t^\infty e^{-s} \mathbb{E}\mathcal{O}_n(s) ds.$$

Supercritical low overlap follows from this lemma and the KPZ scaling $\operatorname{Var}(M_n) = \Theta\left(n^{2/3}\right)$. Ganguly and Hammond recently established high overlap and high correlation of M_n^0 and M_n^t for $t \ll n^{-1/3}$ [3]. See Fig. 2 for an illustration of the overlap phase transition. The question of supercritical energy decorrelation remains open.

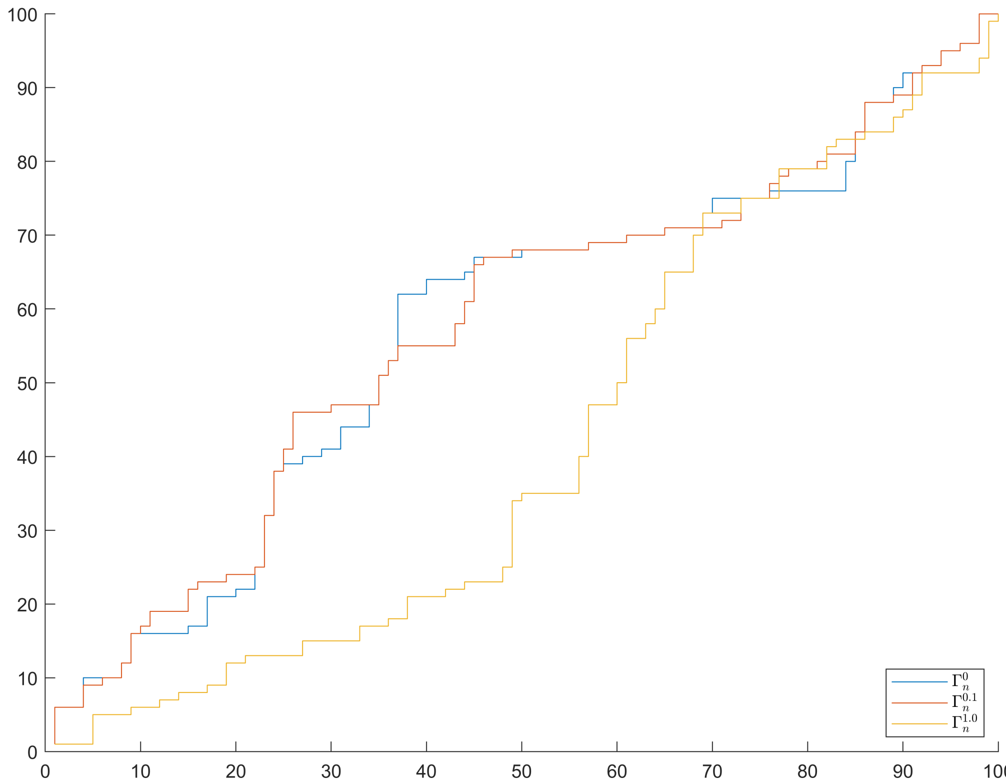

Figure 2: An illustration of subcritical and supercritical overlap in Bernoulli last passage. With $n = 100$, $n^{-1/3} \approx 0.22$. The geodesic at time 0.1 has high overlap with the time zero geodesic, while the time 1.0 geodesic bears little resemblance.

A Markov Semigroups and the Covariance Lemma

The material in this section is from [1]. A **semigroup** is a set Y together with a binary operation such that for every $a, b, c \in Y$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$. In our context $\cdot$ will be composition. Given a Markov process $(X_t)_{t \geq 0}$ with values in a state space $\mathcal{S}$, we can define the associated **Markov semigroup** of operators $(P_t)_{t \geq 0}$ defined by

$$P_t f(x) = \mathbb{E}\big(f(X_t)\big|X_0 = x\big)$$

for $f : S \to \mathbb{R}$. The **generator** of this semigroup is the operator L defined by

$$Lf = \lim_{t \to 0} \frac{P_t f - f}{t} = \partial_t P_t\big|_{t=0}.$$

The identity $\partial_t P_t = LP_t$ follows from the definition of L and the semigroup property, and is called the **heat equation** for the semigroup $(P_t)_{t \geq 0}$. Assuming the Markov process has a stationary distribution μ, we can also define an inner product on the space $L^2(\mu)$ by

$$\langle f, g \rangle = \int fg d\mu.$$

Finally, the **Dirichlet form** $\mathcal{E}$ is the bilinear form defined by

$$\mathcal{E}(f, g) = -\langle f, Lg \rangle.$$

The expectation, variance, and covariance are defined as expected in terms of the equilibrium measure μ:

$$\mathbb{E}_\mu(f) = \int f d\mu, \ \mathrm{Cov}_\mu(f, g) = \int fg d\mu - \int f d\mu \int g d\mu, \text{ and } \mathrm{Var}_\mu(f) = \mathrm{Cov}_\mu(f, f).$$

The following result (the *covariance lemma*) is quite useful.

Lemma 1. For any $f, g \in L^2(\mu)$,

$$\mathrm{Cov}_\mu(f, g) = \int_0^\infty \mathcal{E}(f, P_t g) dt.$$

A.1 The Ornstein-Uhlenbeck Semigroup

The **Ornstein-Uhlenbeck semigroup** is the semigroup $(P_t)_{t \geq 0}$ corresponding to the Markov process $(X_t)_{t \geq 0}$ which satisfies the stochastic differential equation

$$dX_t = -X_t dt + \sqrt{2} dB_t,$$

where $(B_t)_{t \geq 0}$ is a standard Brownian motion. Equivalently, one can write

$$X_t = e^{-t} X_0 + e^{-t} W_{e^{2t}-1},$$

where $(W_s)_{s \geq 0}$ is a standard Brownian motion. The n-dimensional version is simply n independent one-dimensional processes. Gathered here are some facts about the n-dimensional Ornstein-Uhlenbeck semigroup:

- $P_t f(x) = \mathbb{E}\big(f\big(e^{-t}x + \sqrt{1 - e^{-2t}} Z\big)\big)$, where Z is an n-dimensional standard Gaussian random vector;
- $Lf(x) = \Delta f(x) - x \cdot \nabla f(x)$;
- $\mathcal{E}(f, g) = \mathbb{E}_{\gamma^n}(\nabla f \cdot \nabla g)$, where γ^n is the n-dimensional standard Gaussian measure.

References

[1] Chatterjee, S. *Superconcentration and related topics.* Springer, 2016.

[2] Dauvergne, D., Ortmann, J., and Virag, B. The directed landscape, 2018. Preprint.

[3] Ganguly, S., and Hammond, A. Stability and chaos in dynamical last passage percolation, 2020. Preprint.