Wave Equation on Spherically Symmetric Spacetimes

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Abstract

The wave equation is an important tool to understand various phenomena in General Relativity. In this talk I will discuss the late-time asymptotics of solutions to wave equation on symmetric, stationary spacetimes. I will show that the leading order term in the asymptotic expansion is related to the existence of the Newman-Penrose quantities at null infinity. I will derive similar asymptotics for solutions to the wave equation on extremal Reissner-Nordstrom black holes, highlighting the differences from the previous derivation, particularly the contribution of the Horizon (Aretakis) charge. Finally, I will talk about the applications of this work; how it potentially violates the famous No Hair Theorem in general relativity as well as potentially giving observational signatures to determine the "extremality" of black holes.

1 Motivation

The Einstein Field equations describe how our universe evolves. It can be shown that they take the form (after some clever manipulations) of a nonlinear wave equation. These equations, however have been very difficult to understand. Even showing stability has only be done for the trivial solution, the Minkowski spacetime. In order to understand various spacetimes better, we fix certain known solutions to the Einstein equations as a "background" and see how various types of matter propagate through it. This is a much simpler problem, since these equations are linear, and provide deep insight. We will be focused on studying how scalar waves (which are mathematically the simplest form of matter to test) on a fixed Reissner-Nordstrom background, a spherically symmetric spacetime consisting of a single charged black hole. The Einstein Field Equations are given as follows

$$R_{\mu\nu} - \frac{1}{2}R(g)g_{\mu\nu} = 2\mathrm{T}_{\mu\nu}$$

2 Definitions

General Relativity postulates that space and time are combined to form unified entities known as Lorentzian manifolds. A *Lorentzian manifold* $(\mathcal{M}, g)$ is a differentiable manifold of dimension $n+1$, endowed with a *Lorentzian metric* g, namely a differentiable assignment of a symmetric, non-degenerate bilinear form g_x with signature $(-,+,...,+)$ in $T_x\mathcal{M}$ at each point $x \in \mathcal{M}$.

The fundamental aspect of Lorentzian metrics is that g_x is not positive definite definite on $T_x\mathcal{M}$. We say a vector $X \in T_x\mathcal{M}$ is:

1. *spacelike*, if $\quad g(X,X) > 0$
2. *null*, if $\quad g(X,X) = 0$
3. *timelike*, if $\quad g(X,X) < 0$

All null vector space a *double cone* C_x in $T_x\mathcal{M}$ with its vertex at x. The interior of the cones consists of all the timelike vectors at x and the exterior all spacelike vectors. We can also classify submanifolds as well. Let $\mathcal{N}$ be a submanifold of $\mathcal{M}$. Then we say $\mathcal{N}$ is

1. *spacelike*, if the induced metric $\quad g|_{T_x\mathcal{M}} \quad$ is positive definite
2. *timelike*, if the induced metric $\quad g|_{T_x\mathcal{M}} \quad$ has signature $(-,+,+)$
3. *null*, if the induced metric $\quad g|_{T_x\mathcal{M}} \quad$ is degenerate for all x

2.1 Metric Connection and Curvature

An affine connection on $(\mathcal{M}, g)$ is a bi-linear map

$$\nabla : \mathfrak{X}(\mathcal{M}) \times \mathfrak{X}(\mathcal{M}) \to \mathfrak{X}(\mathcal{M})$$

which satisfies the following properties

1. $\nabla_{fX+gY}Z = f\nabla_X Z + g\nabla_Y Z$

2. $\nabla_X(Y+Z) = \nabla_X Y + \nabla_X Z$
3. $\nabla_X(fY) = f\nabla_X Y + X(f)Y$

where $X, Y, Z \in \mathfrak{X}(\mathcal{M})$ and f, g are smooth functions. A particular manifold can admit many affine connection bet we now will define the Levi-Civita Connection, which is the unique affine connection which further satisfies the following

1. $Xg(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$
2. $\nabla_X Y - \nabla_Y X = [X,Y]$

Finally, we can define curvature, the central part of Einsteins Equations. The Riemann Curvature is a tensor associates to every pair of vector fields X, Y a mapping $R(X,Y) : \mathfrak{X}(\mathcal{M}) \to \mathfrak{X}(\mathcal{M})$ given by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z + \nabla_{[X,Y]} Z$$

We can define the curvature components as follows. For a given basis on our mainfold

$$R_{ijkl} = g(\nabla_{\partial i} \nabla_{\partial j} \partial_k - \nabla_{\partial j} \nabla_{\partial i} \partial_k, \partial_l)$$

From the Riemann Curvature tensor, we can define the Ricci Curvature, given a basis.

$$R_{\mu\nu} = \sum_{i,k} g_{ik} R_{i\mu k\nu}$$

where $g_{ij} = g(\partial_i, \partial_j)$. Finally, we define the scalar curvature as follows

$$R = \sum_{i,j} g_{ij} R_{ij}$$

Now we can introduce the Einstein Equations.

2.2 Einsteins equation

The Einstein Field Equations are given as follows

$$R_{\mu\nu} - \frac{1}{2} R(g) g_{\mu\nu} = 2\mathrm{T}_{\mu\nu}$$

Where $T_{\mu\nu}$ denotes the energy momentum tensor of matter fields. For any (reasonable) model of matter, one can construct an energy momentum tensor. It is also important to note that the energy-momentum tensor has trace zero and is divergence free. If the right-hand side of the equation is zero i.e. there is no matter present, the equations reduce to the following.

$$R_{\mu\nu}(g) = 0$$

These are called the Einstein Vacuum Equation. One matter model we are interested in is the Electromagnetic Field. The Einstein equations for the this model, also called the Einstein-Maxwell equations are given as follows

$$R_{\mu\nu} - \frac{1}{2}R(g)g_{\mu\nu} = F_\mu^\rho F_{\nu\rho} - \frac{1}{4}g_{\mu\nu}F^{ab}F_{ab} \tag{1}$$

$$\nabla^\mu F_{\mu\nu} = 0 \tag{2}$$

$$dF = 0 \tag{3}$$

We are interested in a particular family of solutions to this equation, namely the Reissner-Nordstrom spacetimes.

2.3 Reissner-Nordstrom spacetimes

The Reissner-Nordstrom (RN) family $(\mathcal{M}_{M,e}, g_{M,e})$ forms the unique family of spherically symmetric solutions to the Einstein-Maxwell equation. The spacetimes have two parameters, the mass M and the (electromagnetic) charge e. The Extremal Reissner-Nordstrom (ERN) corresponds to $M = |e|$. The RN metric in (t, r, θ, ϕ) is given as follows

$$g = -Ddt^2 + \frac{1}{D}dr^2 + r^2 d\Omega$$

where

$$D = D(r) = 1 - \frac{2M}{r} + \frac{e^2}{r^2}$$

and $d\Omega = \sin\theta d\theta d\phi$ is the standard metric on $\mathbb{S}^2$. For ERN, D takes the form

$$D = D(r) = \left(1 - \frac{M}{r}\right)^2$$

Note that the metric has components that are singular at $r = 0$ and at $r = M$. One can show that as $r \to \infty$, the curvature of the metric blows up, making the singularity *essential* (cannot be removed via change of variables), however, points where $r = M$ which can be eliminated by introducing so call tortoise coordinate r^*

$$\frac{\partial r^*(r)}{\partial r} = \frac{1}{D}$$

We can see that

$$r^*(r) = r + 2M \ln |r - M| - \frac{M^2}{r - M} - 2M \ln M - M$$

We then define the *ingoing Eddington-Finkelstein coordinates* (v, r) where

$$v = t + r^*(r)$$

In these coordinates, metric is given by

$$g = -Ddv^2 + 2dvdr + r^2 d\Omega$$

In these coordinates, we no longer have a singularity at $r = M$. We also can also define the *outgoing Eddington-Finkelstein coordinates* (u, r) where we let

$$u = t - r^*(r)$$

In these coordinates, metric is given by

$$g = -Ddu^2 - 2dvdr + r^2 d\Omega$$

The final set of useful coordinates we will define are the *double null corrdiante sytem* (u, v), where we take u an v to be the same as before and with respect to which the ERN metric is given by

$$g = -Ddudv + r^2 d\Omega$$

In this stem, ∂_u and ∂_v are both future directed null vector fields.

3 Results and proofs

3.1 Wave equation

Let us now introduce the Cauchy problem for the wave equation on the space time region we defined in **??**. The following provides the global existence and uniqueness statement for our problem

Theorem 1. *Let* $\Psi \in C^\infty(\Sigma_0), \Psi' \in C^\infty(\Sigma_0$*. Then there exists a smooth function* $\psi : \mathcal{R} \to \mathbb{R}$ *satisfying*

$$\Box_g \psi = 0,$$

with initial data

$$\psi|_{\Sigma_0} = \Psi, \tag{4}$$

$$n_{\Sigma_0}(\psi)|_{\Sigma_0} = \Psi' \tag{5}$$

In the ingoing Eddingtion-Finkelstein (v, r, θ, ϕ) coordinates takes the form

$$\Box_g \psi = D\partial_r\partial_r\psi + 2\partial_v\partial_r\psi + \frac{2}{r}\partial_v\psi + R\partial_r\psi + \not{\Delta}\psi = 0$$

where $\not{\Delta} = \frac{1}{r^2}\not{\Delta}_{\mathbb{S}^2}$ is the standard Laplacian on the 2 sphere and $R(r) = \frac{dD}{dr} + \frac{2D}{r}$.

3.2 Newman-Penrose, Aretakis Charge

Let ψ be a solution to the wave equation **??** emanating from initial data given as in Theorem 1.

We define $I_0[\psi](u)$ to be a function in u given by

$$I_0[\psi](u) == \frac{1}{4\pi}\int_{\mathbb{S}^2} r^2\partial_r(r\psi)(u, r, \omega)d\omega$$

We call the above quantity the *first Newman-Penrose quantity*

Theorem 2. *Let ψ be a solution to Eq.1. Then the first Newman-Penrose quantity defined in 3.2 is independent of u.*

If we have compactly supported initial data, we can see that I_0 vanishes. In this case we can define the *time inverted Newman-Penrose* constant for initial data prescribed on the initial hypersurface $t = 0$.

$$I_0^{(1)}[\psi] = \frac{M}{4\pi}\int_{t=0} \frac{1}{1 - \frac{2M}{r}}\partial_t\psi r^2 dr d\Omega$$

Generally speaking, these constants tell you about your initial data at infinity. We can also define the following quantity

$$H_0[\psi](v) = -\frac{1}{4\pi}\int_{S_v} \partial_r(r\psi)M^2 d\Omega$$

Similar to the previous theorem, this quantity is independent of the v coordinate and is a constant. Here S_v refers to the sphere at $r = M$ at the specific v value in Eddington - Finkelstein coordinate. We call this the *Horizon Charge.*

3.3 Asymptotics of the wave propagation

Theorem 3. *(Estimates for the asymptotics)*