Scattering of Small Solutions to the Generalized Benjamin-Bona-Mahony Equation

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Abstract

When dealing with any nonlinear dispersive PDE, it is natural to ask when we can think of the nonlinear term as a "small perturbation". Specifically, we often want to identify when the nonlinear perturbation is so tiny as to have a negligible effect on the PDE's solution, at least on very large time scales: if this holds, we say the solution *scatters*. In this talk, I'll discuss some results (due to Albert and Dziubański & Karch) on the scattering of small solutions to the generalized Benjamin-Bona-Mahony equation (GBBM), a classical model for long wave propagation in ideal fluids. We shall see that solutions to GBBM do scatter, provided the initial data is very small and the nonlinear term satisfies some growth condition. Time-permitting, I'll also describe some of my recent attempts to prove scattering with a weaker condition on the nonlinear term.

1 Motivation

We consider the Cauchy problem for the **generalized Benjamin-Bona-Mahony equation (GBBM)**: for $p \in \mathbb{N}$, this takes the form

$$\begin{cases} u_t - u_{xxt} + u_x + u^p u_x = 0 \quad \forall\ (t,x) \in \mathbb{R} \times \mathbb{R} \\ u|_{t=0}(x) = u_0(x) \quad \forall\ x \in \mathbb{R}. \end{cases} \tag{1.1}$$

For $p = 1, 2$, this equation provides a simplified model for the propagation of long waves through channels of water or perfectly elastic blood vessels; we have already studied the latter application in my presentation on January 22.

It is easy to see that (1.1) admits a classical global-in-time solution for

$$u_0 \in C^2_x(\mathbb{R}) \cap H^1_x(\mathbb{R})$$

(for instance, see Souganidis & Strauss [18] or perform an easy modification of the arguments of Benjamin et al. [4]). Further, it is trivial to check this classical solution obeys the energy conservation rule

$$\|u(x,t)\|_{H^1_x} = \|u_0(x)\|_{H^1_x}. \tag{1.2}$$

Indeed, energy conservation is the key to going from local-in-time well-posedness to global well-posedness.

Intuitively, energy conservation implies that if $u_0(x)$ is suitably small, then the solution $u(x,t)$ to GBBM should remain small for all time. In particular, the nonlinear term $u^p u_x$ should be negligible for small $u_0(x)$ and sufficiently large p: for large p and small $|z|$, we have $|z|^p \ll |z|$. Thus, for such $u_0(x)$ and p, $u(x,t)$ should be "close to" a solution to LBBM, at least in the long-time limit after dispersion has mollified the nonlinearity even more. In the jargon of PDE theory, we can formulate this conjecture by saying that, for $p >> 1$ and small enough u_0, the solution to GBBM *scatters* to a linear solution.

One can make the intuitive argument above rigorous using a theorem of Dziubański & Karch [6]. This result is inspired by earlier work from Albert [2, 3]; see also Strauss [19] for the original appearance of the main techniques applied to the simpler case of semilinear Schrödinger equations. Here is a loose statement of the theorem (we use the notation $\langle z\rangle = \sqrt{1+z^2}$):

Theorem (Scattering in H^1, [6])**.** *Suppose $s \geq 7/2$ and $p > 4$. Let $u(x,t)$ denote the solution to GBBM with initial state $u_0(x)$. Then, we can find $0 < \delta \ll 1$ such that*

$$\|u_0\|_{L^1_x} + \|u_0\|_{H^s_x} < \delta$$

implies there exist functions $u_\pm(x,t) \in C^1_t\left(\mathbb{R}; H^s_x\right)$ satisfying the following:

1. *$u_\pm$ both provide classical solutions to LBBM and*
2. *$\lim_{t\to\pm\infty} \|u_\pm(x,t) - u(x,t)\|_{H^1_x} = 0$.*

The purpose of this talk is to prove the above scattering result.

Naturally, to show that nonlinear solutions eventually start to look like linear solutions, we need to learn something about linear solutions. Accordingly, in the first part of the talk we'll build up plenty of intuition for dealing with the linearized version of GBBM, called the **linearized Benjamin-Bona-Mahony equation (LBBM)**. Along the way, we'll review some tools for oscillatory integrals, and use these tools to get insight into the basic physics of dispersive waves. With this intuition taken care of, we move on to rigorously establishing the **dispersive estimate** for LBBM: this is a critical estimate giving a worst-case bound for the time decay of a solution to LBBM, and a key ingredient to eventually proving scattering for GBBM. We then show that for small enough initial data and large enough p, the solution to GBBM actually satisfies the dispersive estimate as well. In other words, sometimes solutions of GBBM exhibit the same time decay as solutions of LBBM. This turns out to be enough to prove the main theorem on scattering in H^1.

2 Background

2.1 Review of Stationary Phase Asymptotics

Definition 2.1. *Let $f, g\colon \mathbb{R} \to \mathbb{C}$, and let $t_0 \in \mathbb{R} \cup \{\pm\infty\}$. We say that f is* ***asymptotically equivalent to*** g ***as*** $t \to t_0$ *(or simply that f is* ***asymptotic to*** g ***as*** $t \to t_0$*) if*

$$\lim_{t \to t_0} \frac{f(t)}{g(t)} = 1,$$

in which case we write $f \sim g$.

Throughout this section, we often abuse notation and write (implicitly)

$$f(\pm\infty) = 0$$

to mean

$$\lim_{t \to \pm\infty} f(t) = 0.$$

Proposition 2.2. *(Non-stationary Phase Estimate, 1D Case) Let $a, b \in \mathbb{R} \cup \{\pm\infty\}$ satisfy $a < b$. Suppose $\phi\colon [a, b] \to \mathbb{R}$ is smooth and $\phi'(\xi) \neq 0 \quad \forall\ \xi \in [a, b]$. Also, let $f \in C^k([a, b])$ satisfy*

$$f^{(\ell)}(x) \in L^1([a, b]) \quad \forall\ \ell = 0, 1, ..., k$$

and

$$f(a) = f(b) = 0,$$

If one of a or b is infinite, additionally assume that f vanishes outside a closed, bounded subinterval of $[a, b]$. Then, there exists a constant C such that, as $t \to \infty$,

$$\int_a^b f(\xi) e^{i\phi(\xi)t}\ \mathrm{d}\xi \sim C\ t^{-k}. \tag{2.1}$$

Proof. The result for $k = 1$ follows immediately upon integrating by parts, taking

$$C = i \int_a^b \left(\frac{f}{\phi'(\xi)}\right)'\ e^{i\phi(\xi)t}\ \mathrm{d}\xi.$$

If we assume that f admits more derivatives, we could integrate by parts again and again to show that the integral decays as a higher power of t, thus completing the proof. □

In particular, if f is smooth, then the integral in the proposition must go to 0 faster than any rational functional as $t \to \infty$.

Theorem 2.3. *(Stationary Phase Estimate, 1D Case) Let $a, b \in \mathbb{R} \cup \{\pm\infty\}$ satisfy $a < b$. Suppose $\phi\colon [a,b] \to \mathbb{R}$ is smooth, $\xi_0 \in (a,b)$ is the only zero of $\phi'(\xi)$, and there exists a natural number N such that*

$$\phi^{(n)}(\xi_0) = 0 \text{ for } n = 1, 2, 3, ..., N-1.$$

Next, suppose that $f\colon [a,b] \to \mathbb{C}$ is continuous with

$$f(a) = f(b) = 0.$$

If one of a or b is infinite, additionally assume that f vanishes outside a closed, bounded subinterval of $[a,b]$. Then, as $t \to \infty$,

$$\int_a^b f(\xi) e^{i\phi(\xi)t} \, \mathrm{d}\xi \sim \frac{2\Gamma\left(\frac{1}{N}\right)}{N\left(\frac{|\phi^{(N)}(\xi_0)|}{N!}\right)^{\frac{1}{N}}} f(\xi_0) e^{i\left(\phi(\xi_0)t + \frac{\pi}{2N}\operatorname{sgn}\left[\phi^{(N)}(\xi_0)\right]\right)} t^{-\frac{1}{N}}, \tag{2.2}$$

where $\Gamma(z)$ denotes the Gamma function.

Proof. See Ablowitz & Fokas [1] or my own notes [16]: there are too many prerequisite lemmas to give a complete proof without bloating the presentation. □

In the sequel, we shall actually need an improved variant of the non-stationary phase estimate arising from the follow lemma.

Lemma 2.4 (van der Corput)**.** *Let $a, b \in \mathbb{R}$ satisfy $a < b$. Suppose $\phi\colon [a,b] \to \mathbb{R}$ is smooth. Pick some $k = 1, 2, 3, ...$ and suppose that*

$$m \doteq \min_{\xi \in [a,b]} \left|\phi^{(k)}(\xi)\right| > 0.$$

If $k = 1$, additionally assume that $\phi'(\xi)$ is monotonic on $[a,b]$. Then, we have

$$\left| \int_a^b e^{it\phi(\xi)} \, \mathrm{d}\xi \right| \lesssim_k (tm)^{-1/k} \quad \forall \ t > 0.$$

Proof. See Linares & Ponce [15] §1.4: our result follows from simply rescaling theirs. □

Note that the same bound still applies if the phase $\phi(\xi)$ does not have a monotonic first derivative but still has only a finite number of inflection points in $[a,b]$: this follows from partitioning $[a,b]$ into finitely many pieces, with $\phi'(\xi)$ monotonic on each piece.

2.2 Introduction to LBBM & Intuition for Dispersive Estimate

Consider the **linear Benjamin-Bona-Mahony equation (LBBM)** on the real line:

$$\begin{cases} u_t - u_{xxt} + u_x = 0 \quad \forall \ (t,x) \in \mathbb{R} \times \mathbb{R} \\ u|_{t=0}(x) = u_0(x) \quad \forall \ x \in \mathbb{R}. \end{cases} \tag{2.3}$$

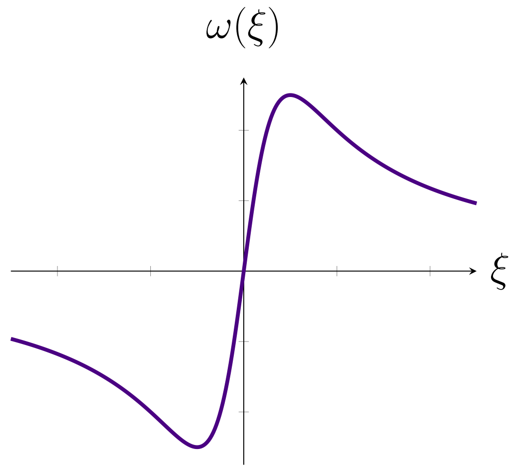

Figure 1: Dispersion relation for linearized BBM.

If we define the elliptic operator $M = 1 - \partial_x^2$, then LBBM can be written in the form

$$M\partial_t u = -\partial_x (u) . \tag{2.4}$$

Now, an easy computation shows M is invertible in the sense that it admits a Green's function. Specifically, if δ denotes the Dirac distribution then

$$M\left(e^{-|x-y|}\right) = \delta(x-y)$$

holds in the sense of distributions. Therefore, we can just as well write (2.4) as

$$u_t = -M^{-1}\partial_x (u) . \tag{2.5}$$

Immediately, we know the symbol of $\frac{1}{i}M^{-1}\partial_x$, denoted $\omega(\xi)$, is given by

$$\omega(\xi) = \frac{\xi}{1+\xi^2} = \frac{\xi}{\langle\xi\rangle^2}, \tag{2.6}$$

where we recall that $\langle\xi\rangle = \sqrt{1+\xi^2}$. See Figure 1 for a plot of $\omega(\xi)$. Notice by the quotient rule for derivatives that $\omega(\xi)$ lives in the Hörmander symbol class $S^{-1}_{1,0}$, so $\omega\left(\frac{1}{i}\partial_x\right) = \frac{1}{i}M^{-1}\partial_x$ is a pseudodifferential operator of order -1. In other words, applying $M^{-1}\partial_x$ gives us a gain of one derivative. Thus we expect that the flow of (2.3) induces a great deal of smoothing on the initial data. In physical language, $\omega(\xi)$ can be thought of as the temporal frequency of waves evolving according to LBBM. (2.6) is called the **dispersion relation**, allowing us to express temporal frequency as a function of *spatial* frequency (or "wavenumber").

By Fourier-transforming LBBM (written in the form (2.5)) with respect to x and solving the resulting ODE, we find that the solution of (2.3) can be written as an oscillatory integral

$$u(x,t) = e^{tM^{-1}\partial_x}u_0 \doteq \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\left(\xi x - \frac{\xi t}{\langle\xi\rangle^2}\right)}\ \widehat{u_0}(\xi)\ \mathrm{d}\xi = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i(\xi x - \omega(\xi)t)}\ \widehat{u_0}(\xi)\ \mathrm{d}\xi. \tag{2.7}$$

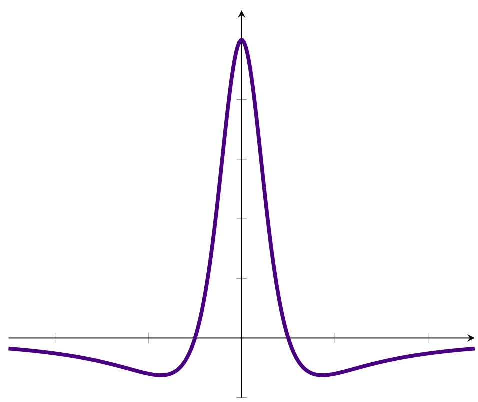

Figure 2: Plot of $\omega'(\xi)$.

When u_0 is Schwartz, we can think of the above solution as describing a **wavepacket**, or weighted sum of normal modes: the function

$$e^{-i\omega(\xi)t}\widehat{u_0}(\xi)$$

is Schwartz in ξ, so its inverse Fourier transform $u(x,t)$ will be Schwartz (and therefore "localized") in x at each time t. Thus the solution resembles a little spatially localized bulge modulating some sinusoidal signal, hence the name "wavepacket". As we shall see below, all the sinusoidal signals in fact travel at different speeds, so the wavepacket only remains coherent on a certain timescale.

For future reference, we take a moment to record the derivatives of $\omega(\xi)$:

$$\omega'(\xi) = \frac{1-\xi^2}{\langle\xi\rangle^4} = \textbf{ group velocity}, \tag{2.8a}$$

$$\omega''(\xi) = \frac{2\xi\left(\xi^2-3\right)}{\langle\xi\rangle^6}, \tag{2.8b}$$

$$\omega'''(\xi) = \frac{-6\left(\xi^2-2\xi-1\right)\left(\xi^2+2\xi-1\right)}{\langle\xi\rangle^8}. \tag{2.8c}$$

Also, we shall need the bound

$$-\frac{1}{8} \leq \omega'(\xi) \leq 1, \tag{2.9}$$

which is easily verified using Figure 2 and elementary calculus. Note then that the group velocity for LBBM is bounded, in marked contrast to other dispersive equations like the linearized Korteweg-de Vries equation. Additionally, the LBBM group velocity may be either positive or negative. We shall see the physical significance of the group velocity presently.

We start by studying $u(x,t)$ on spacetime rays $\Gamma_c = \{x = ct\}$. Given a fixed ray slope c, define the **LBBM phase** by

$$\phi(\xi) = c\xi - \omega(\xi)$$

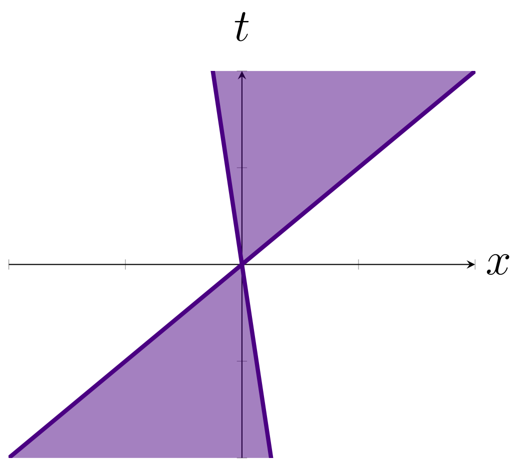

Figure 3: The shaded region denotes the LBBM lightcone. The boundary of the lightcone is marked in a darker colour. Outside the shaded region, the solution to LBBM decays faster than any rational function of t.

so we may write

$$u(x,t) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{i\phi(\xi)t}\ \widehat{u_0}(\xi)\ \mathrm{d}\xi.$$

Thus the dominant asymptotic contribution to $u(x,t)$ comes from those c-values for which there exists at least one $\xi_0 \in \mathbb{R}$ with

$$c = \omega'(\xi_0). \tag{2.10}$$

Of course, if $\widehat{u_0}(\xi)$ is supported away from ξ_0, we ignore this leading-order behaviour. Thus, the wavepacket remains (mostly) spacetime-localized along any rays with slope given by the group velocity corresponding to any normal modes contained in the initial state. More concretely, if $\widehat{u_0}$ is tightly localized near some ξ_0, then the wavepacket $u(x,t)$ remains localized in spacetime along the ray $\Gamma_{\omega'(\xi_0)}$. Thus we discover that Fourier-localized wavepackets roughly travel at the group velocity. This is the main physical idea to keep in mind whenever handling dispersive waves. Indeed, in physics the definition of a dispersive wave is a wave for which the frequency $\omega(\xi)$ satisfies

$$\frac{\omega(\xi)}{\xi} \neq \omega'(\xi),$$

which essentially means group velocity is non-constant. If $\widehat{u_0}$ is more spread out, all of its component normal modes have different group velocities, meaning wave packet “disperses” into a bunch of separated normal modes as $t \to \infty$.

Now, (2.10) can only be satisfied provided

$$-\frac{1}{8} \leq c \leq 1.$$

The set of all such Γ_c is shown in Figure 3. Note that this picture implies the following “finite speed of propagation” property: the intersection of the shaded region with any line

$\{t = \text{constant}\}$ is always a bounded interval or a singleton. In other words, a disturbance initially localized around the origin at time 0 only affects the values of the solution in a bounded set up to any fixed time (modulo some rapidly decaying correction outside the shaded region) . By analogy with the linear wave equation, we call the shaded region in Figure 3 the **LBBM lightcone**. Of course, one must take this terminology with a grain of salt: depending on the support of $\widehat{u_0}$, not all possible group velocities for LBBM may be observed in our solution, while for the linear wave equation

$$u_{tt} - u_{xx} = 0$$

we always observe both possible group velocities, namely ± 1. Thus, the "lightcone" for a particular solution may actually be only a subset of the shaded region in Figure 3.

Using Figure 2, we find there are between one and four solutions to $c = \omega'(\xi_0)$ for $c \in \left[-\frac{1}{8}, 1\right]$. Thus, we split our asymptotic analysis of $u(x,t)$ into several special cases:

CASE 1 : $\mathbf{c = 1}$

In this case, the phase only has a single critical point $\xi = 0$. Using our expressions for $\omega''(\xi)$ and $\omega'''(\xi)$, we find that $\phi''(0) = 0$ but $\phi'''(0) = 6$. Additionally, $\phi(0) = 0$, so the stationary phase estimate tells us that, along the ray Γ_1, we have

$$u(x,t) \sim \frac{2\Gamma\left(\frac{1}{3}\right) e^{\frac{i\pi}{6}}}{3\sqrt{2\pi}} \; \widehat{u_0}(0) \; t^{-1/3}.$$

CASE 2 : $\mathbf{c \in [0, 1)}$

In this case, there are two critical points for the phase:

$$\pm\xi_0 = \begin{cases} \pm\sqrt{\frac{-(2c+1)+\sqrt{8c+1}}{2c}} & c \neq 0, \\ \pm 1 & c = 0. \end{cases} \in [-1, 1] - \{0\}$$

Since these critical points can never equal 0 or $\pm\sqrt{3}$, they are never degenerate. Using that $\phi(\xi)$ is odd then gives

$$u(x,t) \sim \frac{2\mathrm{Re}\left[\widehat{u_0}\left(\xi_0\right) \;\; e^{i\left(\phi(\xi_0)t + \frac{\pi}{4}\right)}\right]}{\sqrt{|\phi''(\xi_0)|}} t^{-1/2}.$$

CASE 3 : $\mathbf{c \in \left(-\frac{1}{8}, 0\right)}$

For such ξ, Figure 2 tells us that there are four critical points:

$$\pm\xi_0 = \pm\sqrt{\frac{-(2c+1) + \sqrt{8c+1}}{2c}} \in (-\sqrt{3}, -1) \cup (1, \sqrt{3})$$

and

$$\pm\tilde{\xi_0} = \pm\sqrt{\frac{\xi_0^2+3}{\xi_0^2-1}} \in (-\infty, -\sqrt{3}) \cup (\sqrt{3}, \infty).$$

Essentially, one wants to think of the correspondence $\xi_0 \mapsto \tilde{\xi_0}$ as reflection about $\xi = \sqrt{3}$ on the graph of $\omega'(\xi)$. None of these critical points are degenerate, so the stationary phase estimate yields

$$u(x,t) \sim 2\left[\frac{\mathrm{Re}\left[\widehat{u_0}\left(\xi_0\right)\ e^{i\left(\phi(\xi_0)t+\frac{\pi}{4}\right)}\right]}{\sqrt{|\phi''(\xi_0)|}} + \frac{\mathrm{Re}\left[\widehat{u_0}\left(\tilde{\xi_0}\right)\ e^{i\left(\phi(\tilde{\xi_0})t-\frac{\pi}{4}\right)}\right]}{\sqrt{\left|\phi''\left(\tilde{\xi_0}\right)\right|}}\right] t^{-1/2}.$$

CASE 4 : $\mathbf{c = -\frac{1}{8}}$

In this case, the critical points are $\pm\sqrt{3}$. These are both degenerate, but $\left|\phi'''\left(\pm\sqrt{3}\right)\right| > 0$ so we can still apply the method of stationary phase. Since

$$\phi\left(\pm\sqrt{3}\right) = \pm\sqrt{3}\left(c - \frac{1}{4}\right),$$

the stationary phase estimate gives

$$u(x,t) \sim \left[\frac{4\Gamma\left(\frac{1}{3}\right)e^{\frac{i\pi}{6}}}{3\sqrt{2\pi}\sqrt[3]{\frac{\left|\phi'''\left(\sqrt{3}\right)\right|}{6}}}\ \mathrm{Re}\left(\widehat{u_0}\left(\sqrt{3}\right)e^{i\left(\sqrt{3}\left(x-\frac{t}{4}\right)\right)}\right)\right] t^{-1/3}.$$

Thus the weakest decay corresponds to $c = -\frac{1}{8}$ and $c = 1$. In other words, the solution decays slowest along the boundary of the lightcone.

Putting all of our stationary phase estimates for LBBM together, we find that

$$\|u\|_{L^\infty} \lesssim \|u_0\|_{L^1}\ t^{-1/3}, \quad t \gg 1, \tag{2.11}$$

where we have used that the Fourier transform is bounded on $L^1 \to L^\infty$. (2.11) is an example of a **dispersive estimate**, a quantitative expression of the balance between mass conservation and dispersion (variable group velocity): as time evolves, all the normal modes in the initial state begin to radiate off to $\pm\infty$ at their respective group velocities, but by mass conservation this means that the amplitude of the wavepacket must decrease over time. The dispersive estimate (2.11) gives us the worst-case scenario for wavepacket decay (ie. the slowest possible loss of amplitude).

Naturally, we can conjecture that the above dispersive estimate holds irrespective of the size of t. Proving this assertion (well, at least a modified version of this assertion!) rigorously is our first goal in the next section.

3 Results and Proofs

3.1 Rigorous Proof of the LBBM Dispersive Estimate

In this section we summarize the work of Albert [3], Souganidis & Strauss [18], and Dziubański & Karch [6] on establishing the LBBM dispersive estimate rigorously. In going through the proof, we shall find that the right-hand side of (2.11) must take into account a sufficiently high Sobolev norm of the Cauchy data.

Before proving the dispersive estimate proper, we'll need a bound on the "high-frequency-truncated" fundamental solution of LBBM. The necessity of this endeavour is easy to see using the picture of $\omega(\xi)$ in Figure 1: formally, the fundamental solution of LBBM can be written as

$$\left(e^{-i\omega(\xi)(t)}\right)^{\vee},$$

but since $\omega(\xi) \to 0$ for $|\xi| \gg 1$ we find the integrand in this inverse Fourier transform stops oscillating and the integral must diverge. Accordingly, we can only make sense of the fundamental solution as a distribution, so asking for a pointwise bound on the entire fundamental solution is nonsense. However, if we only allow our frequencies to get so large, there should be no issues.

Proposition 3.1 (Bound on Low-Frequency Part of Fundamental Solution, [3, 18])**.** *Let $\omega(\xi) = \xi/\langle\xi\rangle^2$. There exists $n_0 \in \mathbb{N}$ such that, for all $t > 0$ and $n > n_0$ we have*

$$\sup_{c\in\mathbb{R}} \left| \int_{-n}^{n} e^{it(c\xi-\omega(\xi))}\, \mathrm{d}\xi \right| \lesssim \left(t^{-1/3} + t^{-1/2} n^{3/2}\right).$$

Proof. Let $\xi_0 = 0$, $\xi_1 = \sqrt{3}$, and $\xi_2 = -\xi_1$. Thus for all $j = 0, 1, 2$ we have

$$\omega''(\xi_j) = 0.$$

Recall also that ξ_j are the *only* roots of $\omega''(\xi)$. Now, pick $n_0 = 2$ so that $|\xi_j| < n$. Also, pick $\epsilon \ll 1$ (say, $\epsilon = 10^{-3}$) and define

$$A_\epsilon = \{\xi \in [-n, n] \mid |\xi - \xi_j| \geq \epsilon \quad \forall\, j = 0, 1, 2\}.$$

Now, for $|\xi| \geq 1$, we clearly have $|\omega''(\xi)| \simeq |\xi|^{-3}$. Thus by increasing n_0 if necessary (in response to the choice of ϵ), we can ensure that the minimizer of $\omega''(\xi)$ on A_ϵ is achieved at the extreme endpoints $\pm n$. In particular, this implies

$$\min_{\xi\in A_\epsilon} |\omega''(\xi)| \simeq |n|^{-3}.$$

Then, by van der Corput's lemma, we have that for any $c \in \mathbb{R}$.

$$\left| \int_{A_\epsilon} e^{it(c\xi-\omega(\xi))} \right| \lesssim t^{-1/2} n^{3/2}. \tag{3.1}$$

Now, we turn to estimating the contributions outside of A_ϵ. The strategy is to localize around the "bad points" ξ_j on a t-dependent scale. To this end, define six sets $B^j_{\epsilon,t}, C^j_t$ by

$$B^j_{\epsilon,t} = \left\{t^{-1/3} < |\xi - \xi_j| < \epsilon\right\},$$
$$C^j_t = \left\{t^{-1/3} \geq |\xi - \xi_j|\right\}$$

so that

$$[-n, n] = A_\epsilon \bigcup_{j=0}^{2} B^j_{\epsilon,t} \bigcup_{j=0}^{2} C^j_t.$$

We obviously have

$$\left| \int_{C^j_t} e^{it(c\xi - \omega(\xi))} \right| \lesssim t^{-1/3}. \tag{3.2}$$

As for the contributions from $B^j_{\epsilon,t}$, we know by Taylor expansion of $\omega''(\xi)$ that

$$|\omega''(\xi)| \gtrsim_{j,\epsilon} |\xi - \xi_j| \quad \forall\ \xi \in B^j_{\epsilon,t}.$$

By definition of $B^j_{\epsilon,t}$ this tells us

$$\left(\min_{\xi \in B^j_{\epsilon,t}} |\omega''(\xi)| \right)^{-1/2} \lesssim t^{1/6},$$

hence by van der Corput we get

$$\left| \int_{B^j_{\epsilon,t}} e^{it(c\xi - \omega(\xi))} \right| \lesssim t^{-1/3}. \tag{3.3}$$

Combining (3.1), (3.2), and (3.3), the proof is complete. □

Corollary 3.2 (LBBM Dispersive Estimate, [3, 6])**.** *Let $u_0(x) \in \mathcal{S}(\mathbb{R})$, so $e^{tM^{-1}\partial_x} u_0$ is a classical solution of* (2.3)*. Then, for any $s \geq \frac{7}{2}$, we have*

$$\left\| e^{tM^{-1}\partial_x} u_0 \right\|_{L^\infty} \lesssim \left(\|u_0\|_{L^1} + \|u_0\|_{H^s} \right) \langle t \rangle^{-1/3}, \quad \forall\ t > 0.$$

Proof. Set $u(x,t) \doteq e^{tM^{-1}\partial_x} u_0$. Let n_0 be obtained from proposition 3.1. Clearly it suffices to prove the theorem for $t \geq t_0$ for any fixed t_0; we choose $t_0 = n_0^9$ for convenience. From (2.7) and the triangle inequality, we have for any $n \geq n_0$ that

$$\|u(x,t)\|_{L^\infty} \lesssim \int_{|\xi|>n} |\widehat{u_0}(\xi)|\ \mathrm{d}\xi + \sup_{c\in\mathbb{R}} \left| \int_{-n}^{n} e^{it(c\xi - \omega(\xi))}\ \widehat{u_0}(\xi)\ \mathrm{d}\xi \right|$$

Our strategy is control each of the above integrals separately, then choose an appropriate n to complete the proof.

To bound the first integral, observe by Cauchy-Schwarz that

$$\begin{aligned}\int_{|\xi|>n} |\widehat{u_0}(\xi)| \ \mathrm{d}\xi &= \int_{|\xi|>n} \left(\langle\xi\rangle^s \ |\widehat{u_0}(\xi)|\right) \ \langle\xi\rangle^{-s} \ \mathrm{d}\xi \\ &\leq \|\langle\xi\rangle^s \widehat{u_0}(\xi)\|_{L^2_\xi} \left(\int_{|\xi|>n} \langle\xi\rangle^{-2s} \ \mathrm{d}\xi\right)^{1/2} \\ &\lesssim_s \|u_0\|_{H^s_x} n^{\frac{1}{2}-s}. \end{aligned} \tag{3.4}$$

As for the second integral, let's start by defining

$$q(\xi, t) = e^{-i\omega(\xi)t} 1_{[-n,n]}(\xi) \in L^1_\xi(\mathbb{R}).$$

Then, by the convolution theorem for Fourier transforms,

$$\int_{-n}^{n} e^{it(c\xi-\omega(\xi))} \ \widehat{u_0}(\xi) \ \mathrm{d}\xi = \int_{-\infty}^{\infty} e^{i\xi ct} q(\xi, t) \ \widehat{u_0}(\xi) \ \mathrm{d}\xi \simeq \left(\check{q}(\cdot, t) * u_0(\cdot)\right)(ct).$$

Applying Young's inequality for convolutions, we get

$$\sup_{c\in\mathbb{R}} \left| \int_{-n}^{n} e^{it(c\xi-\omega(\xi))} \ \widehat{u_0}(\xi) \ \mathrm{d}\xi \right| \lesssim \|\check{q}(x, t)\|_{L^\infty_x} \ \|u_0\|_{L^1_x}.$$

Proposition 3.1 then gives a bound on $\|\check{q}(x, t)\|_{L^\infty_x}$:

$$\sup_{c\in\mathbb{R}} \left| \int_{-n}^{n} e^{it(c\xi-\omega(\xi))} \ \widehat{u_0}(\xi) \ \mathrm{d}\xi \right| \lesssim \left(t^{-1/3} + t^{-1/2}n^{3/2}\right) \ \|u_0\|_{L^1_x}. \tag{3.5}$$

Combining (3.4) and (3.5) yields

$$\|u(x, t)\|_{L^\infty} \lesssim \left(n^{\frac{1}{2}-s} + t^{-1/3} + t^{-1/2}n^{3/2}\right) \left(\|u_0\|_{L^1_x} + \|u_0\|_{H^s_x}\right).$$

Now, $t \geq t_0 = n_0^9$ hence we may choose $n = n(t) = t^{1/9}$ to obtain the required bound and finish the proof. □

Note in particular that the choice $s \geq 7/2$ is necessary to ensure that the high-frequency "tail" contribution to $e^{tM^{-1}\partial_x} u_0$ decays at least as quickly as $t^{-1/3}$.

One may ask why we need a Sobolev norm of the initial data in our dispersive estimate for LBBM. The answer lies in the high-frequency disparity between these two dispersive equations: since the high-frequency part of the fundamental solution to LBBM is poorly behaved, obtaining a dispersive estimate independent of $\|u_0\|_{H^s_x}$ seems unlikely, while for KdV the fundamental solution is so nice we have nothing to worry about. To get some intuition for why this is the case, think in terms of wavepacket velocities. The group velocity of a wavepacket is miniscule for $|\xi| \gg 1$, so if the initial state consists of a high-frequency wave spatially localized around the origin, then even after a long time we will still see a high-frequency waves spatially localized near the origin. Therefore, the L^∞-norm of our solution to LBBM must necessarily be affected by these high-frequency waves that just don't want to move. In other words, any dispersive estimate on LBBM should depend on a norm that weighs high frequencies heavily: a Sobolev norm is built to do just this. For linearized Korteweg-de Vries, high-frequency wavepackets shoot off to $-\infty$ extremely quickly, so this high-frequency issue does not occur.

Remark 3.3. *For $s \geq \frac{7}{2}$, we may use the Sobolev embedding*

$$W^{s+\frac{1}{2},1}(\mathbb{R}) \hookrightarrow H^s(\mathbb{R})$$

to write the dispersive estimate for LBBM with a right-hand side involving only L^1-based norms:

$$\|u\|_{L^\infty} \lesssim \left(\|u_0\|_{W^{s+\frac{1}{2},1}}\right) \langle t\rangle^{-1/3}, \quad \forall\, t > 0.$$

3.2 Application of Dispersive Estimate to Nonlinear Scattering

Now, we apply our work so far to address scattering for GBBM. We begin by collecting some useful bounds.

Lemma 3.4. *Pick any $s > 0$. For all $u \in L^1(\mathbb{R}) \cap H^s(\mathbb{R})$ and any $p \in \mathbb{N}$, we have*

$$\left\|M^{-1}\partial_x u\right\|_{L^1} \lesssim \|u\|_{L^1}, \tag{3.6a}$$

$$\left\|M^{-1}\partial_x u\right\|_{H^s} \lesssim \|u\|_{H^s}, \quad \textit{and} \tag{3.6b}$$

$$\left\|u^{p+1}\right\|_{H^s} \lesssim \|u\|_{L^\infty}^p \|u\|_{H^s}. \tag{3.6c}$$

Proof. The first bound follows by expressing M^{-1} as convolution with $e^{-|x|}$, integrating by parts, and using Young's Inequality for convolutions. The second bound is a trivial consequence of $M^{-1}\partial_x \in S^{-1}_{1,0}$ and Stein's Theorem: $M^{-1}\partial_x$ has order -1 as a Fourier multiplier, so in particular it obeys an order 0 type bound (one could also avoid this heavy machinery and prove the bound directly without much trouble, since $M^{-1}\partial_x$ has a bounded symbol). Finally, the last expression follows directly from using induction and the product estimate for Sobolev spaces. □

Theorem 3.5 (Nonlinear Solutions with Linear Decay, [3,6])**.** *Let $s \geq \frac{7}{2}$ and suppose $u_0 \in L^1(\mathbb{R}) \cap H^s(\mathbb{R})$. If $p > 4$ then there exists $\delta > 0$ such that*

$$\|u_0\|_{L^1} + \|u_0\|_{H^s} < \delta$$

implies $u(x,t) \in H^s_x\ \forall\, t \in \mathbb{R}$ and

$$\|u(x,t)\|_{L^\infty} \lesssim_{u_0} \langle t\rangle^{-1/3} \quad \forall\, t \in \mathbb{R}.$$

Proof. For any $t \geq 0$ (of course, the case $t \leq 0$ is similar), let us define

$$q(t) \doteq \sup_{\tau\in[0,t]} \left[\|u(x,\tau)\|_{L^\infty_x} \langle\tau\rangle^{1/3} + \|u(x,\tau)\|_{H^s_x}\right]. \tag{3.7}$$

By definition of the supremum, to prove the claim it suffices to prove that $q(t)$ is bounded. In turn, to show that $q(t)$ is bounded, it's enough to establish that there is some $C > 0$ (independent of x, t, and u_0) such that

$$q(t) \leq C\left(\|u_0\|_{L^1} + \|u_0\|_{H^s} + q(t)^{p+1}\right). \tag{3.8}$$

To see why (3.8) implies that $q(t)$ is bounded, we use a bootstrap argument (also known as a "continuity method"), see [20] Ch. 1 for an introduction to this technique. Let $A > 1$ satisfy

$$\|\varphi\|_{L^\infty} \leq A\|\varphi\|_{H^s} \quad \forall\, \varphi \in H^s.$$

Pick $\eta \ll 1$ so that

$$\eta > C\,(3A\eta)^{p+1}.$$

Then, pick $\delta < \eta$ such that

$$\eta \geq C\left(\delta + (3A\eta)^{p+1}\right).$$

Having picked δ, we now suppose

$$\|u_0\|_{L^1} + \|u_0\|_{H^s} < \delta$$

as in the statement of the claim. Observe first that by Sobolev embedding

$$q(0) \leq (1 + A)\delta < 2A\eta.$$

Additionally, if we assume $q(t) \leq 3A\eta$ for some particular t, then by (3.8) we have

$$q(t) \leq C\left(\delta + (3A\eta)^{p+1}\right) \leq \eta < 2A\eta$$

for that same particular t. Since $q(t)$ is continuous, the bootstrap principle then implies that $q(t) \leq 2A\eta$ for all $t \geq 0$, and we have shown that $q(t)$ is bounded.

Now, we turn to actually obtaining the bound (3.8). We start by writing GBBM in the form

$$u_t = -M^{-1}\partial_x u - M^{-1}\partial_x f(u),$$

where $f(u) = \frac{u^{p+1}}{p+1}$. Duhamel's Principle then yields

$$u(x,t) = e^{tM^{-1}\partial_x}u_0 - \int_0^t e^{(t-\tau)M^{-1}\partial_x}M^{-1}\partial_x f(u(x,\tau))\ \mathrm{d}\tau. \tag{3.9}$$

To control $q(t)$, observe first that by the above Duhamel formula and the LBBM dispersive estimate we have

$$\begin{aligned}\langle t\rangle^{1/3}\|u(x,t)\|_{L^\infty} &\lesssim \|u_0\|_{L^1} + \|u_0\|_{H^s} \\ &\quad + \langle t\rangle^{1/3}\int_0^t \langle t-\tau\rangle^{-1/3}\left(\left\|M^{-1}\partial_x f(u(x,\tau))\right\|_{L^1_x} + \left\|M^{-1}\partial_x f(u(x,\tau))\right\|_{H^s_x}\right)\ \mathrm{d}\tau.\end{aligned}$$

We need to bound the term in parentheses above. Using all three parts of Lemma 3.4 and the easy bound

$$\left\|u^{p+1}\right\|_{L^1} \leq \|u\|_{L^\infty}^{p-1}\|u\|_{L^2}^2 \leq \|u\|_{L^\infty}^{p-1}\|u\|_{H^s}^2,$$

we find that

$$\left\|M^{-1}\partial_x f(u(x,\tau))\right\|_{L^1_x} + \left\|M^{-1}\partial_x f(u(x,\tau))\right\|_{H^s_x} \lesssim \|u\|_{L^\infty}^{p-1}\|u\|_{H^s}^2 + \|u\|_{L^\infty}^p\|u\|_{H^s}.$$

Since $\langle\tau\rangle \geq 1$, judicious multiplication by 1 and an application of the Binomial Theorem together yield

$$\left\|M^{-1}\partial_x f(u(x,\tau))\right\|_{L^1_x} + \left\|M^{-1}\partial_x f(u(x,\tau))\right\|_{H^s_x} \lesssim \langle\tau\rangle^{(1-p)/3} q(\tau)^{p+1}. \tag{3.10}$$

Thus, we know that

$$\langle t\rangle^{1/3}\|u(x,t)\|_{L^\infty} \lesssim \|u_0\|_{L^1} + \|u_0\|_{H^s} + q(t)^{p+1}\left(\langle t\rangle^{1/3}\int_0^t \langle t-\tau\rangle^{-1/3}\langle\tau\rangle^{(1-p)/3}\,\mathrm{d}\tau\right).$$

Since $\langle t\rangle \sim 1+t$,

$$\langle t-\tau\rangle^{-1/3}\langle\tau\rangle^{(1-p)/3} \lesssim \langle\ t\rangle^{-1/3}(1+\tau)^{(1-p)/3}.$$

Therefore, for $p > 4$

$$\langle t\rangle^{1/3}\int_0^t \langle t-\tau\rangle^{-1/3}\langle\tau\rangle^{(1-p)/3}\,\mathrm{d}\tau \lesssim \langle t\rangle^{(4-p)/3} \leq 1.$$

We conclude that

$$\langle t\rangle^{1/3}\|u(x,t)\|_{L^\infty} \lesssim \|u_0\|_{L^1} + \|u_0\|_{H^s} + q(t)^{p+1}. \tag{3.11}$$

Note that if $p \leq 4$ we would either get a positive power or a $\log\langle t\rangle$ when integrating, which would prevent us from obtaining a uniform-in-time bound on $\langle t\rangle^{1/3}\|u(x,t)\|_{L^\infty}$.

Next, we turn to bounding the other part of $q(t)$. By taking the H^s-norm of both sides of (3.9) and using

$$\left\|e^{tM^{-1}\partial_x}u_0\right\|_{H^s_x} = \left\|e^{-i\omega(\xi)t}\langle\xi\rangle^s\widehat{u_0}(\xi)\right\|_{L^2_\xi} = \|u_0\|_{H^s_x},$$

we find

$$\|u(x,t)\|_{H^s} \lesssim \|u_0\|_{H^s_x} + \int_0^t \left\|u^{p+1}(x,\tau)\right\|_{H^s_x}\,\mathrm{d}\tau$$

Using the product estimate in Lemma 3.4 and the Binomial Theorem as above, we find that

$$\begin{aligned}
\int_0^t \left\|u^{p+1}(x,\tau)\right\|_{H^s_x}\,\mathrm{d}\tau &\lesssim \int_0^t \|u(x,\tau)\|_{L^\infty_x}^p\|u(x,\tau)\|_{H^s_x}\,\mathrm{d}\tau \\
&= \int_0^t \left(\langle\tau\rangle^{1/3}\|u(x,\tau)\|_{L^\infty_x}\right)^p\|u(x,\tau)\|_{H^s_x}\,\langle\tau\rangle^{-p/3}\,\mathrm{d}\tau \\
&\lesssim q(t)^{p+1}\int_0^t \langle\tau\rangle^{-p/3}\,\mathrm{d}\tau.
\end{aligned}$$

Since $p > 4$, the integral above is bounded above by a constant. Thus we have proven

$$\|u(x,t)\|_{H^s} \lesssim \|u_0\|_{H^s_x} + q(t)^{p+1}. \tag{3.12}$$

Combining (3.11) and (3.12), we obtain (3.8) and the proof is done. □

Remark 3.6. *The u_0-dependence of the constant in the decay estimate is a bit subtle, so we should take some time to explain where it actually comes in. As we make u_0 smaller in the right norms, we are allowed to pick our η, δ to be smaller as well. Therefore, since $q(t) \lesssim \eta$, as we shrink u_0 we can also shrink the constant in the decay estimate.*

Using the decay estimate for the nonlinear equation, we can easily show that solutions of GBBM converge to solutions of LBBM (in the energy norm) as $t \to \pm\infty$. In the language of dispersive PDEs, we then say that small-data solutions to GBBM with $p > 4$ **scatter** to solutions to LBBM in the long-time limit.

Corollary 3.7 (Scattering in H^1, [6])**.** *Under the same hypotheses as Theorem 3.5, there exist functions $u_\pm(x,t) \in C_t^1(\mathbb{R}; H_x^s)$ such that*

1. *$u_\pm$ both provide classical solutions to LBBM and*
2. *we have*
$$\|u_\pm(x,t) - u(x,t)\|_{H_x^1} \lesssim \langle t\rangle^{1-p/3}.$$
In particular,
$$\lim_{t\to\pm\infty} \|u_\pm(x,t) - u(x,t)\|_{H_x^1} = 0.$$

Proof. Again, let $f(u) \doteq \frac{u^{p+1}}{p+1}$. We need to prove that

$$I \doteq \int_{-\infty}^{\infty} \left| e^{-\tau M^{-1}\partial_x} M^{-1}\partial_x f(u(x,\tau)) \right| \, \mathrm{d}\tau < \infty$$

in order to sensibly define $u_\pm$ as a continuous function. The LBBM dispersive estimate gives

$$I \lesssim \int_{-\infty}^{\infty} \langle\tau\rangle^{-1/3} \left(\left\|M^{-1}\partial_x f(u(x,\tau))\right\|_{L_x^1} + \left\|M^{-1}\partial_x f(u(x,\tau))\right\|_{H_x^s} \right) \, \mathrm{d}\tau.$$

Using (3.10) and recalling that $q(t)$ can bounded independently of x, t, and u_0, the above becomes

$$I \lesssim \int_{-\infty}^{\infty} \langle\tau\rangle^{-1/3} \left(\langle\tau\rangle^{(1-p)/3}\right) \, \mathrm{d}\tau = \int_{-\infty}^{\infty} \langle\tau\rangle^{-p/3} \, \mathrm{d}\tau < \infty$$

since $p > 4$. A similar argument using that $e^{-\tau M^{-1}\partial_x}$ preserves all Sobolev norms implies we in fact have $I \in H_x^s$.

Now that we know $I < \infty$, we may define

$$u_+(x,t) \doteq e^{tM^{-1}\partial_x} \left(u_0 - \int_0^{\infty} e^{-\tau M^{-1}\partial_x} M^{-1}\partial_x f(u(x,\tau)) \, \mathrm{d}\tau \right),$$

which is obviously a solution to LBBM living in H^s_x at each t. Then, using (3.9), we know for all $t > 0$ that

$$\|u_+ - u\|_{H^1_x} \lesssim \int_t^\infty \left\| e^{(t-\tau)M^{-1}\partial_x} M^{-1}\partial_x f(u(x,\tau)) \right\|_{H^1_x} \, \mathrm{d}\tau.$$

Since the LBBM flow $e^{tM^{-1}\partial_x}$ preserves the H^1 norm, we in fact have

$$\|u_+ - u\|_{H^1_x} \lesssim \int_t^\infty \left\| M^{-1}\partial_x f(u(x,\tau)) \right\|_{H^1_x} \, \mathrm{d}\tau.$$

Applying Lemma 3.4, this becomes

$$\|u_+ - u\|_{H^1_x} \lesssim \int_t^\infty \|u(x,\tau)\|^p_{L^\infty_x} \, \|u(x,\tau)\|_{H^1_x} \, \mathrm{d}\tau.$$

Using (1.2) and Theorem 3.5, the above implies

$$\|u_+ - u\|_{H^1_x} \lesssim_{u_0} \int_t^\infty \langle \tau \rangle^{-p/3} \, \mathrm{d}\tau.$$

Since $p > 4$, we conclude that

$$\|u_+ - u\|_{H^1_x} \lesssim_{u_0} \langle t \rangle^{1-p/3} \to 0 \text{ as } t \to +\infty.$$

By defining

$$u_- \doteq e^{tM^{-1}\partial_x} \left(u_0 + \int_{-\infty}^0 e^{-\tau M^{-1}\partial_x} M^{-1}\partial_x f(u(x,\tau)) \, \mathrm{d}\tau \right)$$

and following a similar argument, the proof is complete. □

Using the embedding $H^1 \hookrightarrow L^\infty$, the above result gives a pointwise error estimate

$$|u_\pm(x,t) - u(x,t)| \lesssim \langle t \rangle^{1-p/3} \quad \text{uniformly for all } x \in \mathbb{R}.$$

So, for $p > 4$, small solutions to GBBM eventually become indistinguishable from solutions to LBBM.

3.3 Related Results. Future Directions.

Note that Ponce & Vega [17] proved scattering result similar to corollary 3.7 for the **generalized Korteweg-de Vries equation (GKdV)**

$$u_t + u_{xxx} + u^p u_x = 0$$

provided $p \geq 4$. Actually, in this paper the authors allowed non-integer p and could take $p > p_0 \approx 3.4$. Subsequently, Christ & Weinstein [5] improved on this result by proving scattering for $p > p_0 \approx 2.9$, and Hayashi & Naumkin [10] were able to get the power down to $p > 2$. Accordingly, we (reasonably) conjecture that the GBBM scattering result proven above may be sharpened to cover the case $p > 2$. Indeed, even a result for $p = 3, 4$ would be novel, and I am currently investigating the $p = 3$ case.

For GKdV with $p = 1, 2$, soliton theory implies that we can no longer expect scattering to a linear solution. However, one can still develop precise asymptotics and "modified scattering" for small initial data. In the past two decades, there has been plenty of work in this regard: see Hayashi & Naumkin [11, 12], Harrop-Griffiths [8], Germain et al. [7], and Ifrim et al. [13]. It would be interesting to test whether or not any of the methods from these sources can be adapted to prove modified scattering for GBBM with $p \in [1, 2]$, but we first must address the $p \in (2, 4]$ case.

To get a brief feel for one of the more modern tools I am using to study GBBM scattering under less stringent constraints, we'll now review (with some modifications) a formal calculation presented by J. Kato & Pusateri [14] (see also Hayashi & Naumkin [9]). This computation neatly summarizes the main ideas of their proof of modified scattering for the **cubic nonlinear Schrödinger equation (NLS)**:

$$\begin{cases} iu_t - \dfrac{1}{2}u_{xx} + \lambda |u|^2 u = 0 \quad \forall\ (t, x) \in \mathbb{R} \times \mathbb{R} \\ \qquad\qquad u|_{t=0}(x) = u_0(x) \quad \forall\ x \in \mathbb{R} \end{cases} \tag{3.13}$$

where $u(x, t) \colon \mathbb{R} \times \mathbb{R} \to \mathbb{C}$. The goal of my current work is to adapt the following argument (and its rigorous completion) to GBBM with $p = 3$. Of course, NLS is very different from GBBM, but the computation I present here extends naturally (though not easily!) to the setting of GBBM as well. I'm choosing to describe the method using NLS rather than GBBM since NLS is vastly simpler, and also the work for GBBM has yet to be completed!

We start by recalling that the solution to the Cauchy problem for the linear Schrödinger equation

$$\begin{cases} iu_t - \dfrac{1}{2}u_{xx} = 0 \quad \forall\ (t, x) \in \mathbb{R} \times \mathbb{R} \\ \quad u|_{t=0}(x) = u_0(x) \quad \forall\ x \in \mathbb{R} \end{cases} \tag{3.14}$$

can be written as

$$e^{\frac{it}{2}\partial_{xx}} u_0 \doteq \frac{1}{\sqrt{2\pi}} \int_{\infty}^{\infty} e^{i\left(\xi x - \frac{t}{2}\xi^2\right)}\ \widehat{u_0}(\xi)\ \mathrm{d}\xi.$$

Then, if $u(x, t)$ satisfies (3.13), we define its **profile** by

$$f(x, t) = e^{-\frac{it}{2}\partial_{xx}} u(x, t).$$

Note that the profile of a linear solution would simply be $f(x, t) = u_0(x)$. Thus, if we find that the profile of a solution $u(x, t)$ to NLS is asymptotically constant, then we have good reason to believe solutions to this PDE scatter. Conversely, if the profile does not asymptote to a constant, then the particular form of the nonlinear term is working some magic to keep the solution "coherent" and prevent scattering.

By a straightforward calculation, the profile obeys the following ODE in Fourier space:

$$\partial_t \hat{f} = -i\lambda \, e^{\frac{it}{2}\partial_{xx}} \left(|u|^2 u\right)^{\wedge}. \tag{3.15}$$

We can simplify the right-hand side using the convolution theorem and the identity $\hat{\overline{u}}(\xi) = \overline{\hat{u}}(-\xi)$:

$$\begin{aligned}
\left(|u|^2 u\right)^{\wedge}(\xi) &= \frac{1}{\sqrt{2\pi}}\left((\overline{u}u)^{\wedge} * \hat{u}\right) \\
&= \frac{1}{2\pi}\hat{\overline{u}} * \hat{u} * \hat{u} \\
&= \frac{1}{2\pi}\int \mathrm{d}\eta \int \mathrm{d}\sigma \; \hat{u}\left(\xi-\eta-\sigma\right)\; \hat{u}(\eta)\; \hat{\overline{u}}(\sigma) \\
&= \frac{1}{2\pi}\int \mathrm{d}\eta \int \mathrm{d}\sigma \; \exp\left(\frac{it}{2}\left[-\eta^2+\sigma^2-(\xi-\eta-\sigma)^2\right]\right)\; \hat{f}\left(\xi-\eta-\sigma\right)\; \hat{f}(\eta)\; \overline{\hat{f}}(-\sigma)
\end{aligned}$$

If we define a phase function by

$$\Phi(\eta,\sigma;\xi) \doteq \frac{1}{2}\left(\xi^2-\eta^2+\sigma^2-(\xi-\eta-\sigma)^2\right), \tag{3.16}$$

then the work above may be combined with (3.15) to obtain the clearer ODE

$$\partial_t \hat{f} = -\frac{i\lambda}{2\pi}\int \mathrm{d}\eta \int \mathrm{d}\sigma \; \exp\left(it\Phi\left(\eta,\sigma;\xi\right)\right)\; \hat{f}\left(\xi-\eta-\sigma\right)\; \hat{f}(\eta)\; \overline{\hat{f}}(-\sigma). \tag{3.17}$$

We can roughly think of the integral above as describing how the nonlinearity causes all the different normal modes in our solution to interact with one another. In the long run, the dominant contribution to this integral should thus come from the modes with wavenumbers (η, σ) that have "resonant interactions" (interfere constructively during their interaction). We use a two-dimensional version of the stationary phase estimate (see [16] for discussion) to make this discussion of resonances more precise. This estimate tells us that the dominant contributions to the integral as $t \to \infty$ arise from those (η, σ) for which

$$0 = \partial_\eta \Phi(\eta,\sigma;\xi) = \xi - 2\eta - \sigma \tag{3.18a}$$

$$0 = \partial_\sigma \Phi(\eta,\sigma;\xi) = \xi - \eta. \tag{3.18b}$$

This is a simple linear system of algebraic equations parameterized by $\xi \in \mathbb{R}$. The solution is

$$(\eta_0, \sigma_0) = (\xi, -\xi).$$

Note that

$$\mathrm{detHess}\Phi\big|_{(\eta_0,\sigma_0)} = -1,$$

so this critical point is nondegenerate. Further,

$$\Phi(\eta_0, \sigma_0; \xi) = 0.$$

Plugging into the 2D stationary phase estimate tells us that (3.17) can be approximated for large t by

$$\partial_t \hat{f}(\xi,t) \approx -\frac{i\lambda}{\sqrt{2\pi}} t^{-1} \left|\hat{f}(\xi,t)\right|^2 \hat{f}(\xi,t), \quad t \gg 1. \tag{3.19}$$

A bit of easy manipulation shows that the above implies

$$\partial_t |\hat{f}|^2 \approx 0, \quad t \gg 1$$

hence we expect that $|\hat{f}(\xi,t)|^2 \to |F(\xi)|^2$ as $t \to \infty$ for some fixed asymptotic profile $F(\xi)$. Thus (3.19) becomes

$$\partial_t \hat{f}(\xi,t) \approx -\frac{i\lambda}{\sqrt{2\pi}} t^{-1} |F(\xi)|^2 \hat{f}(\xi,t), \quad t \gg 1. \tag{3.20}$$

This separable ODE is readily integrated to yield

$$\hat{f}(\xi,t) \approx e^{-\frac{i\lambda}{\sqrt{2\pi}}|F(\xi)|^2 \log t} \, F(\xi).$$

So, we have shown that the profile does not asymptote to a constant, hence on the formal level we conclude that scattering cannot occur for NLS. Instead, we find that the nonlinearity eventually gives rise to a logarithmic correction to the frequency of our solution. That is, by looking at the frequency of waves that have gone out to infinity, we can tell whether or not they evolved according to linear Schrödinger or NLS based on if we detect the logarithmic correction! Thus this approach gives us a little bit more than just telling us scattering cannot occur.

For GBBM, attempting a similar calculation is much more complex. This is largely because the system of equations determining the critical points is no longer linear, and indeed it can have solutions that cannot be written down explicitly. Actually, the number of solutions varies with the parameter ξ! Additionally, for GBBM the phase function Φ has many degenerate critical points, and this also causes trouble. Despite the awkwardness of this stationary phase approach for GBBM, however, it does appear to be the most promising candidate for developing more powerful scattering theory. Other relevant techniques like the method of vector fields (see for example [10]) do not apply to GBBM since this PDE lacks scaling or Galilean symmetries, so the vector field propagating superlinear decay along the LBBM flow cannot be estimated through the vector field generating one of these nice symmetries.

4 Brief Summary of Talk

- Basic intuition says that small solutions to "sufficiently nonlinear" PDEs (including GBBM with nonlinear term $u^p u_x$, $p \gg 1$) should scatter to linear solutions, at least after a very long time has passed. Can prove this rigorously (with $p > 4$) via careful analysis of linear solutions.

- The key tool underlying most of the hard analysis was the dispersive estimate for LBBM. One can nearly guess the correct dispersive estimate using intuition from the method of stationary phase.
- Proving scattering is easy once one shows that initially small (in the right norm!) nonlinear solutions obey the same dispersive estimate as solutions to LBBM.
- Methods of proof treat the nonlinearity as perturbative via Duhamel's Principle. To get scattering for lower p, one may need more powerful modern methods (ie. studying resonant interactions between normal modes in detail)

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