Long-time Asymptotics for the Cubic Nonlinear Schrödinger Equation: Approach via the Method of Space-Time Resonances

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Abstract

When analyzing any nonlinear dispersive PDE, it's useful to quantify the long-time behaviour of solutions: this is often a baby step towards understanding the stability of solitons or other special solutions. In this talk I will describe one approach (due to J. Kato and Pusateri) for developing rigorous long-time asymptotic approximations to small solutions of the cubic nonlinear Schrödinger equation (NLS). This approach is based on the method of space-time resonances, a relatively recent addition to the dispersive PDE toolkit. After reviewing some basic properties of the linear Schrödinger equation, I'll go through the formal computation underlying the space-time resonance point-of-view on long-time asymptotics for NLS. Then, I'll explain how to upgrade this formal computation to obtain rigorous error estimates using a bootstrap argument and a clever analogy with the method of integrating factors from kindergarten ODE theory.

1 Introduction

1.1 Background on the Model PDE and Main Results

We study the Cauchy problem for the **cubic nonlinear Schrödinger equation (NLS)**:

$$\begin{cases} iu_t + \dfrac{1}{2}\Delta u + \lambda |u|^2 u = 0 \quad \forall\ (t,x) \in \mathbb{R}\times\mathbb{R}^d \\ u|_{t=0}(x) = u_0(x) \quad \forall\ x \in \mathbb{R}^d \end{cases} \tag{1.1}$$

where $u(t,x)\colon \mathbb{R}\times\mathbb{R}^d \to \mathbb{C}$, $u_0\colon \mathbb{R}^d \to \mathbb{C}$, and $\lambda = \pm 1$. NLS is a universal model for the evolution of wavepackets in semilinear dispersive systems [13, 16], and it also appears in the mean-field description of spin-0 Bose-Einstein condensates [14]. Critically, NLS is a completely integrable system and so can be solved exactly using inverse scattering theory, at least formally. As a completely integrable system, NLS possesses special particle-like **soliton** solutions that are of great interest in many different areas of physics including nonlinear optics: in this particular context (assuming $\lambda > 0$), the solitons correspond to localized pulses of light travelling at constant speed (see again [13, 16] for details).

For the rest of these notes, we take $d = 1$ for simplicity. In this case, global-in-time well-posedness for (1.1) in L^2_x (that is, existence and uniqueness of a weak solution in L^2_x, with the correspondence $u_0 \mapsto u$ being continuous) was established in 1987 by Y. Tsutsumi [17]. Given that a global solution exists, we can naturally ask the following question: how does this solution behave for very large times? Does the nonlinear term eventually become negligible, or does the solution always carry with it some signature of the cubic self-interaction? Such questions on the long-time asymptotic behaviour of solutions are prerequisite to understanding the stability of solitons or other special solutions. Typically, when investigating soliton stability, one wants to decompose a solution of NLS into one or more solitons plus some small "radiation" on long time scales: to properly quantify what we expect the radiation to look like we therefore need satisfying asymptotics, for small solutions at least. So, even if one is only interested in NLS because of the siren call of its solitons, one must have a solid understanding of long-time asymptotics.

The purpose of these notes is to prove the following famous result on the long-time asymptotic behaviour of small solutions to NLS:

Theorem 1.1 (Main Theorem, Loose Version)**.** *Suppose*

$$\|u_0(x)\|_{H^1_x} + \|xu_0(x)\|_{L^2_x} \leq \epsilon \ll 1.$$

Then, the unique global-in-time solution to NLS satisfies the following asymptotics: there exists some small $\beta > 0$ and two (unique) bounded, real-valued functions $F(\xi), \phi(\xi)$ such that

$$u(t,x) = (it)^{-1/2}\, F(x/t)\, \exp\left(\frac{i|x|^2}{2t} + i\,|F(x/t)|^2 \log t + i\phi(x/t)\right) + \mathcal{O}\left(t^{-1/2-\beta}\right).$$

Later, we shall state the main theorem more precisely. Before discussing our approach to the proof, a few remarks are in order.

- First, notice how the hypothesis of the main theorem involves a *weighted norm* of the initial data. We shall see that this restriction is essentially inevitable when studying long-time asymptotics.

- Additionally, the use of a weighted norm prevents *a priori* relaxation of the smallness assumption by scaling (we shall discuss this later).

- Finally, after some review of the linear Schrödinger equation, we will see that the $\log t$ appearing inside the exponential represents a deviation from linear behaviour: in other words, the logarithmic correction to the temporal frequency is the distinctive scent of nonlinearity in the asymptotic solution.

1.2 Our Approach: Method of Space-Time Resonances

Our proof of the main theorem is taken from a 2011 paper of J. Kato and Pusateri [10], where the authors use the **method of space-time resonances** (STR). STR is a relatively young technique, introduced by Germain, Masmoudi, and Shatah in 2009 [3] to understand asymptotics for a *quadratic*-nonlinear Schrödinger equation in *three* space dimensions. Since its introduction, the method has also been applied to many difficult problems including long-time regularity for the 3D gravity-capillary water waves system [1].

Broadly speaking, STR provides a framework, rather than an algorithm, for establishing rigorous long-time asymptotics for nonlinear dispersive equations. In applying STR, one identifies what pairs (or triples or quadruples depending on the PDE) of wavepacket-like solutions interact "resonantly": these resonant interactions contribute to the main deviations from linearity. Identifying resonances can be done using a nice heuristic computation relying on the classical **stationary phase lemma**. Then, you can zero in on the resonant cases and estimate all non-resonant contributions away: essentially, the bulk of a typical proof involves justifying all the steps of the aforementioned heuristic computation.

Of course, STR is not a magic bullet and may need to be combined with alternative approaches such as the **method of vector fields** in order to produce satisfying results (see for example [4]). For a gentle overview of the main ideas of STR as applied to a general dispersive PDE, see [2]. For more history on the method and connections to other approaches, see again [3] and the references therein.

1.3 Summary of Other Approaches to Long-Time Asymptotics

Of course, the main theorem was well known before the work of Kato and Pusateri, and there are many different approaches to proving it. To my knowledge, the first rigorous proof is due to Hayashi and Naumkin [7], who used the method of vector fields to obtain some requisite estimates on weighted norms, then gave a general argument for obtaining asymptotics once those estimates were in hand. In particular, the only difference between the proof of Kato and Pusateri and the proof of Hayashi and Naumkin is how the required weighted norms are controlled (admittedly this is a very big difference!). Additionally, in 2003 Deift and Zhou provided a proof of the main theorem using inverse scattering theory: interestingly, though this approach is not applicable to non-integrable PDE, it does allow the hypothesis of small initial data to be relaxed. In 2006 Lindblad and Soffer [12] came up with yet another proof using an inspired ansatz for the asymptotic behaviour and some energy estimates. Also, in 2015 Ifrim and Tataru [8] developed a new proof using their **method of testing by wave packets**, which has also been successfully applied to problems such as the modified KdV equation [6] and 2D capillary water waves [9]. Finally, I remark that there is no hope of trying to establish the main theorem using the perturbative tools developed by Strauss in 1981 [15]: the power in the nonlinear term is too small to close the necessary bootstrap estimates.

2 Essential Properties of the Linear Schrödinger Equation

Before diving into the nonlinear problem (1.1) proper, we'll need some background on the Cauchy problem for the linear Schrödinger equation,

$$\begin{cases} iu_t + \dfrac{1}{2}u_{xx} = 0 \quad \forall\ (t,x) \in \mathbb{R} \times \mathbb{R} \\ \quad u|_{t=0}(x) = u_0(x) \quad \forall\ x \in \mathbb{R}. \end{cases} \tag{2.1}$$

By Fourier-transforming $u(t,x) \mapsto \hat{u}(t,\xi)$, the solution to (2.1) can be written as

$$u(t,x) = e^{\frac{it}{2}\partial_x^2}u_0 \doteq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\left(\xi x - \frac{t}{2}\xi^2\right)}\ \widehat{u_0}(\xi)\ \mathrm{d}\xi.$$

This should be thought of a superposition of **normal modes** $e^{i(\xi x - \omega t)}$ with wavenumber (or spatial frequency) ξ and temporal frequency $\omega = \frac{1}{2}\xi^2$. Additionally, $u_0 \mapsto e^{\frac{it}{2}\partial_x^2}u_0$ is sometimes called the **Schrödinger flow** or **(free) Schrödinger propagator**. Note that $e^{\frac{it}{2}\partial_x^2}$ is invertible with inverse $e^{-\frac{it}{2}\partial_x^2}$.

Immediately from the above Fourier representation of our solution and Plancherel's theorem, we obtain

Lemma 2.1 (Conservation of L^2-norm)**.**

$$\left\| e^{\frac{it}{2}\partial_x^2}u_0 \right\|_{L^2_x} = \|u_0\|_{L^2_x}.$$

□

While the Fourier representation is certainly useful, we shall also need to write the linear solution in purely spatial terms. After computing a Gaussian integral involving contours in $\mathbb{C}$ (see [11, Ch. 4] for all the details), we can also write the Schrödinger flow as a convolution in the spatial variable:

$$e^{\frac{it}{2}\partial_x^2}u_0 = (2\pi i t)^{-1/2} \int_{-\infty}^{+\infty} \mathrm{d}y\ \exp\left(\frac{i|x-y|^2}{2t}\right)\ u_0(y). \tag{2.2}$$

Actually, since we are primarily interested in $t \gg 1$, we can deal with a simplified version of the above. To establish this simplified representation, we need a little lemma first:

Lemma 2.2. *For all $z \in \mathbb{R}$ and $\kappa \in \left(0, \frac{1}{2}\right)$ we have*

$$\left|e^{iz} - 1\right| \lesssim_\kappa |z|^\kappa.$$

Proof. By direct calculation, we find that

$$\left|e^{iz} - 1\right|^2 = 2\left(1 - \cos z\right) \tag{2.3}$$

Now, we work by splitting the proof into cases based on the size of z. If $|z|\geq 2^{1/2\kappa}$ then

$$|1-\cos z| \leq 2 \leq z^{2\kappa}.$$

Conversely, if $|z|< 2^{1/2\kappa}$ then by Taylor expansion we get

$$\begin{aligned}|1-\cos z| &= \sup_{z_0} |\sin z_0| \ \ |z| \\ &\leq |z| \\ &= |z|^{2\kappa}|z|^{-2\kappa+1} \\ &< 2^{-1+1/2\kappa}|z|^{2\kappa}.\end{aligned}$$

Then, picking

$$c_\kappa \geq \max\left\{2^{\frac{1}{2}}, 2^{\frac{1}{2}\left(-1+\frac{1}{\kappa}\right)}\right\} = 2^{\frac{1}{2}\left(-1+\frac{1}{\kappa}\right)},$$

we find by (2.3) that

$$\left|e^{iz}-1\right| \leq c_\kappa |z|^\kappa$$

as required. □

Now, we can state the aforementioned simplified long-time asymptotic expansion of solutions to (2.1):

Lemma 2.3. *Suppose $u(t,x) = e^{\frac{it}{2}\partial_x^2} f(t,x)$. Then,*

$$u(t,x) = (it)^{-1/2}\, e^{\frac{ix^2}{2t}} \hat{f}\left(t, \frac{x}{t}\right) + r(t,x). \tag{2.4}$$

and there exists $\kappa > 0$ so that

$$\|r(t,x)\|_{L_x^\infty} \lesssim t^{-1/2-\kappa} \|xf(t,x)\|_{L_x^2}. \tag{2.5}$$

In particular, the solution to (2.1) can be written in the form

$$u(t,x) = (it)^{-1/2}\, e^{\frac{ix^2}{2t}} \widehat{u_0}\left(\frac{x}{t}\right) + \mathcal{O}\left(t^{-1/2-\kappa}\right). \tag{2.6}$$

Proof. Using the representation of $e^{\frac{it}{2}\partial_x^2}$ as a spatial convolution (2.2) gives

$$u(t,x) = (2\pi i t)^{-1/2} \int_{-\infty}^{+\infty} \mathrm{d}y\ \exp\left(\frac{i|x-y|^2}{2t}\right)\ f(t,y).$$

Some judicious re-arrangement of the above yields

$$\begin{aligned}u(t,x) &= (2\pi i t)^{-1/2} \int_{-\infty}^{+\infty} \mathrm{d}y\ \exp\left(\frac{ix^2}{2t} - \frac{ixy}{t} + \frac{iy^2}{2t}\right)\ f(t,y) \\ &= \frac{\exp\left(\frac{ix^2}{2t}\right)}{(it)^{1/2}} \hat{f}\left(t, \frac{x}{t}\right) + \left[\frac{\exp\left(\frac{ix^2}{2t}\right)}{(2\pi i t)^{1/2}} \int_{-\infty}^{+\infty} \mathrm{d}y\ f(t,y)\ e^{-ixy/t}\left(e^{iy^2/2t}-1\right)\right].\end{aligned}$$

If we define $r(t,x)$ by the term in square brackets above, to complete the proof all we have to do is establish (2.5). We start by picking some small $\kappa \ll 1/2$, then lemma 2.2 gives

$$\left|e^{iy^2/2t} - 1\right| \lesssim_\kappa t^{-\kappa} \langle y \rangle^{2\kappa},$$

where we recall that $\langle y \rangle = \sqrt{1+y^2}$. Then, we have for any $\zeta > 1/2$ that

$$\begin{aligned} \|r(t,x)\|_{L^\infty_x} &\lesssim t^{-\frac{1}{2}-\kappa} \int_{-\infty}^{+\infty} \mathrm{d}y \ \langle y \rangle^{2\kappa} \ |f(t,y)| \\ &\lesssim t^{-\frac{1}{2}-\kappa} \left\| \langle x \rangle^{2\kappa+\zeta} f(t,x) \right\|_{L^2_x}. \end{aligned}$$

By shrinking κ and ζ if necessary, we can guarantee $2\kappa + \zeta < 1$. This in turn allows us to write

$$\|r(t,x)\|_{L^\infty_x} \lesssim t^{-\frac{1}{2}-\kappa} \|xf(t,x)\|_{L^2_x}, \tag{2.7}$$

completing the proof. □

This result is the key to understanding why estimating weighted norms is non-negotiable: the only way to get any mileage with the nice asymptotic expansion above is to deal with the error in terms of a weighted L^2-norm. Additionally, the above trivially implies

$$\|u(t,x)\|_{L^\infty_x} \lesssim_{u_0} \langle t \rangle^{-1/2}.$$

In the sequel, we spend a great deal of time proving that the above also holds for small solutions of NLS.

3 Long-Time Asymptotics by the Space-Time Resonance Approach

3.1 An Instructive Heuristic

Now that we have some tools from the linear problem at our disposal, we can start to attack the problem of long-time dynamics for the nonlinear problem (1.1). If $u(t,x)$ satisfies NLS, we define its **profile** by

$$f(t,x) = e^{-\frac{it}{2}\partial_x^2} u(t,x).$$

The profile of a linear solution would simply be $f(t,x) = u_0(x)$. Thus, if we find that the profile of a solution $u(t,x)$ to NLS is asymptotically constant, then we have good reason to believe solutions to this PDE scatter tp to linear solutions for large t. Conversely, if the profile does not asymptote to a constant, then the nonlinear term is working some magic to keep the solution "coherent" and prevent scattering to linear solutions.

By a straightforward calculation, the profile obeys the following ODE in Fourier space:

$$\partial_t \hat{f} = i\lambda\, e^{\frac{it}{2}\xi^2} \left(|u|^2 u\right)^\wedge . \tag{3.1}$$

Using the convolution theorem and the identity

$$\hat{\overline{u}}(\xi) = \overline{\hat{u}}(-\xi)$$

we can simplify the right-hand side:

$$\begin{aligned}
\left(|u|^2 u\right)^\wedge (\xi) &= \frac{1}{\sqrt{2\pi}} \left(\left(u^2\right)^\wedge * \hat{\overline{u}}\right) \\
&= \frac{1}{2\pi} \hat{u} * \hat{u} * \hat{\overline{u}} \\
&= \frac{1}{2\pi} \int \mathrm{d}\eta \int \mathrm{d}\sigma\ \hat{u}\left(\xi - \eta\right)\ \hat{u}(\sigma)\ \hat{\overline{u}}(\eta - \sigma) \\
&= \frac{1}{2\pi} \int \mathrm{d}\eta \int \mathrm{d}\sigma\ \hat{u}\left(\xi - \eta\right)\ \hat{u}(\sigma)\ \overline{\hat{u}}(\sigma - \eta) \\
&= \frac{1}{2\pi} \int \mathrm{d}\eta \int \mathrm{d}\sigma\ \exp\left(\frac{it}{2}\left[-(\xi-\eta)^2 - \sigma^2 + (\sigma-\eta)^2\right]\right)\ \hat{f}\left(\xi - \eta\right)\ \hat{f}(\sigma)\ \overline{\hat{f}}(\sigma - \eta) \\
&= \frac{1}{2\pi} \int \mathrm{d}\eta \int \mathrm{d}\sigma\ \exp\left(\frac{it}{2}\left[-\xi^2 + 2\eta(\xi-\sigma)\right]\right)\ \hat{f}\left(\xi - \eta\right)\ \hat{f}(\sigma)\ \overline{\hat{f}}(\sigma - \eta)
\end{aligned}$$

If we define a phase function by

$$\Phi(\eta, \sigma; \xi) \doteq \eta(\xi - \sigma), \tag{3.2}$$

then the work above may be combined with (3.1) to obtain the clearer ODE

$$\partial_t \hat{f} = \frac{i\lambda}{2\pi} \int \mathrm{d}\eta \int \mathrm{d}\sigma\ \exp\left(it\Phi\left(\eta, \sigma; \xi\right)\right)\ \hat{f}\left(\xi - \eta\right)\ \hat{f}(\sigma)\ \overline{\hat{f}}(\sigma - \eta). \tag{3.3}$$

We can roughly think of the integral above as describing how the nonlinearity causes all the different normal modes in our solution to interact with one another. In the long run, the dominant contribution to this integral should thus come from the modes with wavenumbers (η, σ) that have interfere constructively. These are also called **resonant interactions**. We use a two-dimensional version of the stationary phase estimate to make this discussion of interference and resonances more precise. Stationary phase tells us that the dominant contributions to the integral as $t \to \infty$ arise from those (η, σ) for which

$$0 = \partial_\eta \Phi(\eta, \sigma; \xi) = \xi - \sigma \tag{3.4a}$$

$$0 = \partial_\sigma \Phi(\eta, \sigma; \xi) = -\eta. \tag{3.4b}$$

Any pair (η_0, σ_0) solving the above linear system of algebraic equations parameterized by $\xi \in \mathbb{R}$ is called a **space resonance** for NLS. The only space resonance is

$$(\eta_0, \sigma_0) = (0, \xi)\,.$$

Note that

$$\det \mathrm{Hess}\Phi|_{(\eta_0,\sigma_0)} = -1,$$

so this critical point is nondegenerate. Further,

$$\Phi(\eta_0, \sigma_0; \xi) = 0,$$

so we also call (η_0, σ_0) a **time resonance**: since (η_0, σ_0) is both a space resonance and a time resonance, we call this point a **space-time resonance** (for more general definitions of all these different flavours of resonance, see [2]). Plugging into the 2D stationary phase estimate (see for example the relevant appendix of [5]) tells us that (3.3) can be approximated for large t by

$$\partial_t \hat{f}(t,\xi) \approx i\lambda t^{-1} \left|\hat{f}(t,\xi)\right|^2 \hat{f}(t,\xi), \quad t \gg 1. \tag{3.5}$$

A bit of easy manipulation shows that the above implies

$$\partial_t |\hat{f}|^2 \approx 0, \quad t \gg 1$$

hence we expect that $|\hat{f}(t,\xi)|^2 \to |F(\xi)|^2$ as $t \to \infty$ for some fixed asymptotic profile $F(\xi)$. Thus (3.5) becomes

$$\partial_t \hat{f}(t,\xi) \approx i\lambda t^{-1} |F(\xi)|^2 \hat{f}(t,\xi), \quad t \gg 1. \tag{3.6}$$

This separable ODE is readily integrated to yield

$$\hat{f}(t,\xi) \approx e^{i\lambda |F(\xi)|^2 \log t}\, F(\xi). \tag{3.7}$$

So, we have shown that the profile does not asymptote to a constant, hence on the formal level we conclude that *scattering to a linear solution cannot occur for NLS*. Instead, we find that the nonlinearity eventually gives rise to a logarithmic correction to the frequency of our solution. That is, by looking at the frequency of waves that have gone out to infinity, we can tell whether or not they evolved according to linear Schrödinger or NLS based on if we detect the logarithmic correction! Thus this approach gives us a little bit more than just telling us scattering to a linear solution cannot occur.

3.2 Rigorous Statement and Discussion of the Main Theorem

Now, we turn to developing a rigorous refinement of the heuristic computation from the previous section. The focus of our analysis is still the profile ODE (3.3). Of course, since integrals are easier to control than derivatives, we prefer to work with the integral form of this ODE instead:

$$\hat{f}(t,\xi) = \hat{f}(1,\xi) + \frac{i\lambda}{2\pi} \int_1^t \mathrm{d}s \int \mathrm{d}\eta \int \mathrm{d}\sigma\; e^{is\eta(\xi-\sigma)}\; \hat{f}(\xi-\eta)\; \hat{f}(\sigma)\; \overline{\hat{f}}(\sigma-\eta) \tag{3.8}$$

So, we can estimate various norms of the profile by estimating an oscillatory integral. This can essentially be accomplished by rigorously re-proving the stationary phase estimate. However, we shall also need to use a special **null structure** in the nonlinear term to show that waves initially localized near the origin remain "localized" in some other suitable sense (read: has a small weighted norm).

We start by defining some weighted Sobolev spaces appearing in our estimates:

Definition 3.1.

$$H_x^{s,\ell} = \left\{ u \in \mathcal{S}'(\mathbb{R}) \;\middle|\; \|u\|_{H_x^{s,\ell}} \doteq \left\| \langle x \rangle^{\ell} \langle \partial_x \rangle^{s} u \right\|_{L_x^2} < \infty \right\}.$$

Similarly, we have

$$\dot{H}_x^{s,\ell} = \left\{ u \in \mathcal{S}'(\mathbb{R}) \;\middle|\; \|u\|_{\dot{H}_x^{s,\ell}} \doteq \left\| \langle x \rangle^{\ell} \left|\partial_x\right|^{s} u \right\|_{L_x^2} < \infty \right\}.$$

Now, we can state the main theorem we focus on for the remainder of this section:

Theorem 3.2. *Suppose $u_1 \in H_x^{1,0} \cap H_x^{0,1}$ with*

$$\|u_1\|_{H_x^{1,0}} + \|u_1\|_{H_x^{0,1}} \leq \epsilon$$

where ϵ is sufficiently small. Then, there exists a unique

$$u(t,x) \in C\left(\mathbb{R}; H_x^{1,0} \cap H_x^{0,1}\right)$$

weakly solving NLS such that

1. $\|u(t,x)\|_{L_x^\infty} \lesssim \epsilon \langle t \rangle^{-1/2}$,
2. *there is a unique asymptotic profile $F(\xi) \in L_\xi^\infty$ and some $\rho \in (0, 1/2)$ such that for all $t \geq 1$ we have*
$$\left\| \hat{f}(\xi) \exp\left(-i\lambda \int_1^t |\hat{u}(s,\xi)|^2 \, \frac{\mathrm{d}s}{s} \right) - F(\xi) \right\|_{L_\xi^\infty} \lesssim t^{-\rho}, \quad \textit{and} \tag{3.9}$$
3. *there is some small $\beta > 0$ and a real-valued $\phi(\xi) \in L_\xi^\infty$ uniquely determine by $F(\xi)$ such that we have the leading-order asymptotics*
$$u(t,x) = (it)^{-1/2} F(x/t) \exp\left(\frac{i|x|^2}{2t} + i\left|F(x/t)\right|^2 \log t + i\phi(x/t) \right) + \mathcal{O}\left(t^{-1/2-\beta}\right). \tag{3.10}$$

Note that (3.10) is the "proper version" of the expression (3.7) we derived earlier. To prove this theorem, we use a bootstrap argument to establish the dispersive decay estimate

$$\|u(t,x)\|_{L_x^\infty} \lesssim \epsilon \langle t \rangle^{-1/2}.$$

Most of the rest of the notes (and most of Kato and Pusateri's original paper [10]) is devoted to going through the details of this argument. Once we have this bound and some weighted-norm bounds in hand, we can follow the path laid out by Hayashi and Naumkin [7] to establish existence F and ϕ and conclude (3.10).

One may wonder why the smallness assumption on the initial data is required: since NLS is L^2-subcritical in one space dimension (that is, the L^2_x norm can be scaled to be arbitrarily small), why don't we just shrink the L^2 norm of any given solution by scaling? To address this concern, notice that we're really dealing with a *weighted* L^2 norm of the initial data here. When we apply a scaling transformation

$$u(t,x) \mapsto u_\mu(t,x) = \mu^{1/2} u(\mu t, \mu^{1/2} x)$$

we find that

$$\|u_\mu(t,x)\|_{L^2_x} = \mu^{1/2} \|u(\mu t, x)\|_{L^2_x}$$

while

$$\|u_\mu(t,x)\|_{H^{0,1}_x} = \|x\ u_\mu(t,x)\|_{L^2_x} = \mu^{-1/2} \|x\ u(\mu t, x)\|_{L^2_x}.$$

So, if we shrink μ to decrease $\|u_\mu(t,x)\|_{L^2_x}$, we also increase $\|u_\mu(t,x)\|_{H^{0,1}_x}$. So, we cannot use scale freedom to weaken the hypotheses of the theorem.

3.3 Proof of the Main Theorem

3.3.1 Notation and Preliminaries

Our proof of the main theorem requires us to use a bootstrap space defined below.

Definition 3.3. *For $T > 0$ and some small $\alpha > 0$*

$$X_T \doteq \left\{ u(t,x)\colon [0,T]\times\mathbb{R} \to \mathbb{R} \;\middle|\; \|u\|_{X_T} \doteq \left\|t^{1/2} u\right\|_{L^\infty_t L^\infty_x} + \left\|t^{-\alpha} u\right\|_{L^\infty_t \dot{H}^{1,0}_x} + \left\|t^{-\alpha} f\right\|_{L^\infty_t H^{0,1}_x} + \|u\|_{L^\infty_t L^2_x} < \infty \right\}$$

So, functions in X_T may have spatial moments with mild growth in time. Critically, any function that lives in X_T automatically satisfies the same time decay estimate satisfied by solutions to the linear Schrödinger equation:

$$\|u(t,x)\|_{L^\infty_x} \lesssim \langle t \rangle^{-1/2}.$$

We shall see over the course of the next few sections how all of the terms in the X_T-norm appear naturally when trying to control $\hat{f}$ using the integral equation (3.8).

We begin by stating a local existence result that one can easily establish using a routine fixed-point argument.

Proposition 3.4 (Local Existence)**.** *Given $\epsilon > 0$ sufficiently small and a function $u_1(x)$ satisfying*

$$\|u_1\|_{H^{1,0}_x \cap H^{0,1}_x} \le \epsilon,$$

there exists $T > 1$ and a solution

$$u \in C\left([0,T]; H_x^{1,0}(\mathbb{R}) \cap H_x^{0,1}(\mathbb{R})\right)$$

to NLS satisfying

$$\|u\|_{X_T} \lesssim \epsilon.$$

□

Note how we have tacitly switched to prescribing the Cauchy data at $t = 1$ in the above: by shrinking the "real" Cauchy data given at time 0 if necessary, prescribing data at $t = 1$ is perfectly okay, and indeed doing so helps us avoid issues like blowup of the logarithm at 0.

3.3.2 Bootstrapping Linear Decay 1: L_x^2-based Estimates

First, we control the L_x^2 parts of $\|u\|_{X_T}$. Of course, the pure L_x^2 part

$$\|u\|_{L_t^\infty L_x^2}$$

is trivial to bound in terms of u_1, since NLS conserves the L_x^2 norm (as the reader may easily verify).

Definition 3.5. *The function $\mathcal{N}[u](t,x)$ is defined by*

$$\widehat{\mathcal{N}[u]}(t,\xi) \doteq \frac{i\lambda}{2\pi}\int_1^t \mathrm{d}s \int \mathrm{d}\eta \int \mathrm{d}\sigma\ e^{is\eta(\xi-\sigma)}\ \hat{f}(\xi-\eta)\ \hat{f}(\sigma)\ \overline{\hat{f}}(\sigma-\eta).$$

By changing variables, we can also write

$$\widehat{\mathcal{N}[u]}(t,\xi) = \frac{-i\lambda}{2\pi}\int_1^t \mathrm{d}s \int \mathrm{d}\eta \int \mathrm{d}\sigma\ e^{is\eta\sigma}\ \hat{f}(\xi-\eta)\ \hat{f}(\xi-\sigma)\ \overline{\hat{f}}(\xi-\sigma-\eta). \tag{3.11}$$

Lemma 3.6 (L_x^2-type bounds on Nonlinear Term)**.** *If $u(t,x)$ is the local solution to NLS constructed in proposition 3.4, then we have*

$$\|\mathcal{N}[u]\|_{\dot{H}_x^{1,0}\cap H_x^{0,1}} \lesssim t^\alpha \|u\|_{X_T}^3 \tag{3.12}$$

Proof. It is enough to estimate $\|\partial_x\mathcal{N}[u]\|_{L_x^2}$ and $\|x\mathcal{N}[u]\|_{L_x^2}$ separately. To start with, we easily have

$$\begin{aligned}\|\partial_x\mathcal{N}[u]\|_{L_x^2} &= \left\|\langle\xi\rangle\widehat{\mathcal{N}[u]}(t,\xi)\right\|_{L_\xi^2} \\ &\lesssim \int_1^t \mathrm{d}s\ \left\|\langle\xi\rangle\ \int \mathrm{d}\eta \int \mathrm{d}\sigma\ e^{is\eta(\xi-\sigma)}\ \hat{f}(\xi-\eta)\ \hat{f}(\sigma)\ \overline{\hat{f}}(\sigma-\eta)\right\|_{L_\xi^2}.\end{aligned}$$

Now, by reversing the convolution theorem computations leading to (3.3), we know

$$\int \mathrm{d}\eta \int \mathrm{d}\sigma\ e^{is\eta(\xi-\sigma)}\ \hat{f}(\xi-\eta)\ \hat{f}(\sigma)\ \overline{\hat{f}}(\sigma-\eta) \simeq \left(|u|^2u\right)^\wedge,$$

so the above implies

$$\begin{aligned}\|\partial_x \mathcal{N}[u]\|_{L^2_x} &\lesssim \int_1^t \mathrm{d}s \ \left\| \langle \xi \rangle \left(|u|^2 u \right)^\wedge \right\|_{L^2_\xi} \\ &\lesssim \int_1^t \mathrm{d}s \ \left\| \partial_x \left(|u|^2 u \right) \right\|_{L^2_\xi} \\ &\lesssim \int_1^t \mathrm{d}s \ \left\| u^2 u_x \right\|_{L^2_\xi} \\ &\leq \int_1^t \mathrm{d}s \ \|u(s,x)\|^2_{L^\infty_x} \|u_x(s,x)\|_{L^2_\xi} .\end{aligned}$$

By definition of the X_T norm, we know that

$$\begin{aligned}\|u(s,x)\|_{L^\infty_x} &\lesssim s^{-1/2} \|u\|_{X_T} \quad \forall \ s \in [0,T] \quad \text{and} \\ \|u_x(s,x)\|_{L^2_x} &\lesssim s^{\alpha} \|u\|_{X_T} \quad \forall \ s \in [0,T]\end{aligned}$$

so

$$\|\partial_x \mathcal{N}[u]\|_{L^2_x} \lesssim \|u\|^3_{X_T} \int_1^t \mathrm{d}s \ s^{\alpha-1} \lesssim t^{\alpha} \|u\|^3_{X_T} .$$

This takes us halfway there. Now, we need to deal with the weighted norm. This requires more care. For starters, we know

$$\|x\mathcal{N}[u]\|_{L^2_x} = \left\| \partial_\xi \widehat{\mathcal{N}[u]}(t,\xi) \right\|_{L^2_\xi} .$$

Thus, we need to differentiate $\mathcal{N}[u]$ in Fourier space. However, this presents a problem: when ∂_ξ hits the exponential term $e^{is\eta(\xi-\sigma)}$ in the integrand of $\widehat{\mathcal{N}[u]}$, we will gain an extra factor of s. Since our goal is to control $\mathcal{N}[u]$ in terms of a small power of s, we obviously have quite a problem on our hands. However, recall that by changing variables $\sigma \mapsto \xi - \sigma$ we can re-write $\mathcal{N}[u]$ in the form (3.11), where the exponential term *no longer depends on* ξ (this is entirely due to the special null structure of the nonlinear term!). Therefore, we really have

$$\begin{aligned}\partial_\xi \widehat{\mathcal{N}[u]} = \frac{-i\lambda}{2\pi} \int_1^t \mathrm{d}s \int \mathrm{d}\eta \int \mathrm{d}\sigma \ e^{is\eta\sigma} \ &\Big\{ \partial_\xi \hat{f}(\xi-\eta) \ \hat{f}(\xi-\sigma) \ \overline{\hat{f}}(\xi-\sigma-\eta) \\ &+ \hat{f}(\xi-\eta) \ \partial_\xi \hat{f}(\xi-\sigma) \ \overline{\hat{f}}(\xi-\sigma-\eta) \\ &+ \hat{f}(\xi-\eta) \ \hat{f}(\xi-\sigma) \ \partial_\xi \overline{\hat{f}}(\xi-\sigma-\eta) \Big\} .\end{aligned}$$

Since the three terms in the curly braces are so similar, we'll only show how to bound the first term

$$I(t,\xi) \doteq \int_1^t \mathrm{d}s \int \mathrm{d}\eta \int \mathrm{d}\sigma \ e^{is\eta\sigma} \ \partial_\xi \hat{f}(\xi-\eta) \ \hat{f}(\xi-\sigma) \ \overline{\hat{f}}(\xi-\sigma-\eta)$$

in detail.

Our first step in bounding $\|I(t,\xi)\|_{L^2_\xi}$ is to immediately undo the change of variables that just saved our lives. This has the advantage of allowing us to "un-convolute", in the same vein of the first stage of the proof. Of course, the presence of the ∂_ξ means we have to go through a few more details explicitly, but the main idea is the same.

$$
\begin{aligned}
I(t,\xi) &= \int_1^t \mathrm{d}s \int \mathrm{d}\eta \int \mathrm{d}\sigma \; e^{is\eta\sigma} \; \partial_\xi \hat{f}(\xi-\eta) \; \hat{f}(\xi-\sigma) \; \overline{\hat{f}}(\xi-\sigma-\eta) \\
&= -\int_1^t \mathrm{d}s \int \mathrm{d}\eta \int \mathrm{d}\sigma \; e^{is\eta(\xi-\sigma)} \; \partial_\xi \hat{f}(\xi-\eta) \; \hat{f}(\sigma) \; \overline{\hat{f}}(\sigma-\eta) \\
&= -\int_1^t \mathrm{d}s \int \mathrm{d}\eta \int \mathrm{d}\sigma \; e^{\frac{is}{2}\left(\xi^2-(\xi-\eta)^2-\sigma^2+(\eta-\sigma)^2\right)} \; \partial_\xi \hat{f}(\xi-\eta) \; \hat{f}(\sigma) \; \widehat{\overline{f}}(\eta-\sigma) \\
&= -\int_1^t \mathrm{d}s \; e^{\frac{is\xi^2}{2}} \left[e^{\frac{-is\diamondsuit^2}{2}} \widehat{(xf)}(\diamondsuit) * e^{\frac{-is\diamondsuit^2}{2}} \hat{f}(\diamondsuit) * e^{\frac{is\diamondsuit^2}{2}} \overline{\hat{f}}(\diamondsuit) \right]
\end{aligned}
$$

(we have let $\diamondsuit$ stand for the dummy variable over which we perform the convolution). Now, we can easily estimate the L^2_ξ-norm of $I(t,\xi)$ in terms of spatial norms of u:

$$
\begin{aligned}
\|I(t,\xi)\|_{L^2_\xi} &= \left\| \int_1^t \mathrm{d}s \; e^{\frac{is\xi^2}{2}} \left[e^{\frac{-is\diamondsuit^2}{2}} \widehat{(xf)}(\diamondsuit) * e^{\frac{-is\diamondsuit^2}{2}} \hat{f}(\diamondsuit) * e^{\frac{is\diamondsuit^2}{2}} \overline{\hat{f}}(\diamondsuit) \right] \right\|_{L^2_\xi} \\
&\simeq \left\| \int_1^t \mathrm{d}s \; e^{\frac{-is}{2}\partial_x^2} \left[\left(e^{\frac{is}{2}\partial_x^2}(xf) \right) \; \left(e^{\frac{is}{2}\partial_x^2} f \right) \left(e^{-\frac{is}{2}\partial_x^2} \overline{f} \right) \right] \right\|_{L^2_x} \\
&= \left\| \int_1^t \mathrm{d}s \; e^{\frac{-is}{2}\partial_x^2} \left[\left(e^{\frac{is}{2}\partial_x^2}(xf(s,x)) \right) \; |u(s,x)|^2 \right] \right\|_{L^2_x} \\
&\lesssim \int_1^t \mathrm{d}s \; \left\| e^{\frac{-is}{2}\partial_x^2} \left[\left(e^{\frac{is}{2}\partial_x^2}(xf(s,x)) \right) \; |u(s,x)|^2 \right] \right\|_{L^2_x} .
\end{aligned}
$$

Since the linear Schrödinger flow $e^{\frac{-is}{2}\partial_x^2}$ preserves the L^2_x norm, the above yields

$$
\begin{aligned}
\|I(t,\xi)\|_{L^2_\xi} &\lesssim \int_1^t \mathrm{d}s \; \left\| \left(e^{\frac{is}{2}\partial_x^2}(xf(s,x)) \right) \; |u(s,x)|^2 \right\|_{L^2_x} \\
&\lesssim \int_1^t \mathrm{d}s \; \|u(s,x)\|^2_{L^\infty_x} \; \|xf(s,x)\|_{L^2_x} .
\end{aligned}
$$

Using the definition of the X_T-norm again, we get

$$
\begin{aligned}
\|I(t,\xi)\|_{L^2_\xi} &\lesssim \|u\|^2_{X_T} \int_1^t \mathrm{d}s \; s^{-1} \; \|xf(s,x)\|_{L^2_x} \\
&\lesssim \|u\|^3_{X_T} \int_1^t \mathrm{d}s \; s^{\alpha-1} \\
&\lesssim t^\alpha \|u\|^3_{X_T} .
\end{aligned}
$$

This is clearly enough to conclude that

$$\|x\mathcal{N}[u]\|_{L^2_x} \lesssim t^{\alpha} \|u\|^3_{X_T},$$

so the proof is complete. □

Remark 3.7. *The space-time resonance approach (read: focusing on the profile ODE) inspired us to control weighted norms by taking derivatives in Fourier space. In other approaches to the problem, one needs to resort to alternative techniques to control these norms, including but not limited to the method of vector fields.*

Corollary 3.8. *Let $u(t,x)$ be the local solution constructed in Proposition 3.4. There exists some constant $C > 0$ such that*

$$\sup_{t\in[1,T]} t^{-\alpha}\left[\|u(t,x)\|_{\dot{H}^{1,0}_x} + \|f(t,x)\|_{H^{0,1}_x}\right] \le \epsilon + C\|u\|^3_{X_T}.$$

Proof. This follows immediately from combining (3.8) with lemma 3.6, recalling along the way that Plancherel gives

$$\|u(t,x)\|_{\dot{H}^{1,0}_x} = \|f(t,x)\|_{\dot{H}^{1,0}_x}$$

□

3.3.3 Bootstrapping Linear Decay 2: L^∞_x Estimates

Now, we move on to proving the L^∞_x estimates, which are substantially more complicated. We start by going through a long but mostly straightforward computation.

Lemma 3.9.

$$\int \mathrm{d}\eta \int \mathrm{d}\sigma\ \mathcal{F}^{-1}_{(\eta',\sigma')\to(\eta,\sigma)}\left[\hat{f}(s,\xi-\eta)\ \hat{f}(s,\xi-\sigma)\ \overline{\hat{f}}(s,\xi-\sigma-\eta)\right] = |\hat{f}(s,\xi)|^2\hat{f}(s,\xi)$$

Proof. Call the integral on the left-hand side I. Then, making prodigious use of the convolution theorem, the formula for the Fourier transform of a translation, and the expression

$$\hat{\overline{f}}(\xi) = \overline{\hat{f}}(-\xi),$$

we get

$$
\begin{aligned}
I &= (2\pi)^{-1} \int \mathrm{d}\eta \int \mathrm{d}\sigma \int \mathrm{d}\eta' \int \mathrm{d}\sigma' \; e^{i(\eta\eta'+\sigma\sigma')} \; \hat{f}(s,\xi-\eta') \; \hat{f}(s,\xi-\sigma') \; \overline{\hat{f}}(s,\xi-\sigma'-\eta') \\
&= (2\pi)^{-1/2} \int \mathrm{d}\eta \int \mathrm{d}\sigma \int \mathrm{d}\eta' \; e^{i\eta\eta'} \; \hat{f}(\xi-\eta') \left[\mathcal{F}^{-1}_{\sigma'\to\sigma}\left(\hat{f}(\xi-\sigma') \; \hat{\overline{f}}(-\xi+\eta'+\sigma')\right)\right] \\
&= -(2\pi)^{-1} \int \mathrm{d}\eta \int \mathrm{d}\sigma \int \mathrm{d}\eta' \; e^{i\eta\eta'} \; \hat{f}(\xi-\eta') \left[e^{i\xi\sigma}\hat{\hat{f}}(\sigma) * e^{i(\xi-\eta')\sigma}\left(\hat{\overline{f}}\right)^{\vee}(\sigma)\right] \\
&= -(2\pi)^{-1} \int \mathrm{d}\eta \int \mathrm{d}\eta' \; e^{i\eta\eta'} \; \hat{f}(\xi-\eta') \; \int \mathrm{d}\sigma \int \mathrm{d}\lambda \; e^{i\xi(\sigma-\lambda)}\hat{\hat{f}}(\sigma-\lambda) \; e^{i(\xi-\eta')\lambda} \; \overline{f}(\lambda) \\
&= -(2\pi)^{-1/2} \int \mathrm{d}\eta \int \mathrm{d}\eta' \; e^{i\eta\eta'} \; \hat{f}(\xi-\eta') \; \int \mathrm{d}\lambda \; e^{-i\eta'\lambda} \; \overline{f}(\lambda) \; \mathcal{F}^{-1}_{\sigma\to\xi}\left(\hat{\hat{f}}(\sigma-\lambda)\right) \\
&= -(2\pi)^{-1/2}\hat{f}(\xi) \int \mathrm{d}\eta \int \mathrm{d}\eta' \; e^{i\eta\eta'} \; \hat{f}(\xi-\eta') \; \int \mathrm{d}\lambda \; \overline{f}(\lambda) \; e^{-i\lambda(\eta'-\xi)} \\
&= -(2\pi\hat{f})^{1/2}(\xi) \int \mathrm{d}\eta \; \mathcal{F}^{-1}_{\eta'\to\eta}\left[\hat{f}(\xi-\eta') \; \hat{\overline{f}}(\eta'-\xi)\right] \\
&= \hat{f}(\xi) \int \mathrm{d}\eta \; \left[e^{i\xi\lambda}\hat{\hat{f}}(\lambda) * e^{i\xi\lambda}\overline{f}(\lambda)\right](\eta) \\
&= \hat{f}(\xi) \int \mathrm{d}\eta \int \mathrm{d}\lambda \; e^{i\xi(\eta-\lambda)} \; \hat{\hat{f}}(\eta-\lambda) \; e^{i\xi\lambda} \; \overline{f}(\lambda) \\
&= (2\pi)^{1/2}\hat{f}(\xi)\mathcal{F}^{-1}_{\eta\to\xi}\left(\hat{\hat{f}} * \overline{f}\right) \\
&= \hat{f}^2\check{\overline{f}}(\xi) \\
&= |\hat{f}(\xi)|^2\hat{f}(\xi)
\end{aligned}
$$

$\square$

Using this computation, we can re-write (3.8) in a simpler fashion.

Corollary 3.10. *We may decompose $\hat{f}(t,\xi)$ into the following form:*

$$
\hat{f}(t,\xi) = \hat{f}(1,\xi) + i\lambda \int_1^t \frac{\mathrm{d}s}{s} \; |\hat{f}(s,\xi)|^2 \hat{f}(s,\xi) + \int_1^t \; \mathrm{d}s \; R(s,\xi) \tag{3.13}
$$

where the ***remainder*** *is defined by*

$$
R(s,\xi) = \frac{-i\lambda}{2\pi s} \int \mathrm{d}\eta \int \mathrm{d}\sigma \left(e^{-i\eta\sigma s^{-1}} - 1\right) \; \mathcal{F}^{-1}_{(\eta',\sigma')\to(\eta,\sigma)}\left[\hat{f}(s,\xi-\eta') \; \hat{f}(s,\xi-\sigma') \; \overline{\hat{f}}(s,\xi-\sigma'-\eta')\right] \tag{3.14}
$$

Proof. We start by using (3.11) to write

$$\hat{f}(t,\xi)-\hat{f}(1,\xi)=-\frac{i\lambda}{2\pi}\int_1^t \mathrm{d}s\int \mathrm{d}\eta'\int \mathrm{d}\sigma'\ e^{is\eta'\sigma'}\ \hat{f}(\xi-\eta')\ \hat{f}(\xi-\sigma')\ \overline{\hat{f}}(\xi-\sigma'-\eta')$$

$$=-\frac{i\lambda}{2\pi}\int_1^t \mathrm{d}s\left\langle \hat{f}(\xi-\eta')\ \hat{f}(\xi-\sigma')\ \overline{\hat{f}}(\xi-\sigma'-\eta'),\exp\left(\frac{-is}{2}\begin{pmatrix}\eta'\\ \sigma'\end{pmatrix}^{\mathrm{T}}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}\eta'\\ \sigma'\end{pmatrix}\right)\right\rangle_{L^2_{\eta',\sigma'}}$$

If we use unitarity of the Fourier transform along with the identity

$$\mathcal{F}^{-1}_{(\eta',\sigma')\to(\eta,\sigma)}\left[\exp\left(\frac{-is}{2}\begin{pmatrix}\eta'\\ \sigma'\end{pmatrix}^{\mathrm{T}}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}\eta'\\ \sigma'\end{pmatrix}\right)\right]^{\wedge}=\frac{2\pi}{s}e^{i\eta\sigma s^{-1}}$$

(check the classical formula for the Fourier transform of a Gaussian!) then we can write

$$\hat{f}(t,\xi)-\hat{f}(1,\xi)=-i\lambda\int_1^t \frac{\mathrm{d}s}{s}\int \mathrm{d}\eta\int \mathrm{d}\sigma\ e^{-i\eta\sigma s^{-1}}\ \mathcal{F}^{-1}_{(\eta',\sigma')\to(\eta,\sigma)}\left[\hat{f}(\xi-\eta')\ \hat{f}(\xi-\sigma')\ \overline{\hat{f}}(\xi-\sigma'-\eta')\right]$$

$$=-i\lambda\int_1^t \frac{\mathrm{d}s}{s}\int \mathrm{d}\eta\int \mathrm{d}\sigma\ \mathcal{F}^{-1}_{(\eta',\sigma')\to(\eta,\sigma)}\left[\hat{f}(\xi-\eta')\ \hat{f}(\xi-\sigma')\ \overline{\hat{f}}(\xi-\sigma'-\eta')\right]+\int_1^t \mathrm{d}s\ R(s,\xi)$$

An application of lemma 3.9 completes the proof. $\square$

Now, we focus on estimating the size of the remainder term. Ideally, we want to show that the remainder decays in s faster than

$$\frac{1}{s}\ \left|\hat{f}(s,\xi)\right|^2 f(s,\xi)$$

so by (3.13) we'll have, to "leading order" in t,

$$\partial_t\hat{f}\approx\frac{i\lambda}{2\pi}\left|\hat{f}\right|^2 f,$$

which is the same approximate profile ODE we derived earlier. That is, separating and estimating the remainder term $R(s,\xi)$ is a key step in rigorously justifying many of our heuristics.

Lemma 3.11. *There exists $\delta\in\left(3\alpha,\frac{1}{4}\right)$ such that*

$$|R(s,\xi)|\lesssim s^{-1-\delta+3\alpha}\ \|u\|^3_{X_T}\tag{3.15}$$

Proof. We start by using the definition of $R(s,\xi)$ to write

$$|R(s,\xi)|\lesssim s^{-1}\int \mathrm{d}\eta\int \mathrm{d}\sigma\ \left|e^{-i\eta\sigma s^{-1}}-1\right|\ \left|\mathcal{F}^{-1}_{(\eta',\sigma')\to(\eta,\sigma)}\left[\hat{f}(s,\xi-\eta')\ \hat{f}(s,\xi-\sigma')\ \overline{\hat{f}}(s,\xi-\sigma'-\eta')\right]\right|.$$

By lemma 2.2 we get for any $\delta \in \left(0, \frac{1}{2}\right)$

$$|R(s,\xi)| \lesssim s^{-1-\delta} \int \mathrm{d}\eta \int \mathrm{d}\sigma \ (|\eta||\sigma|)^{\delta} \ \left| \mathcal{F}^{-1}_{(\eta',\sigma')\to(\eta,\sigma)} \left[\hat{f}(s,\xi-\eta') \ \hat{f}(s,\xi-\sigma') \ \overline{\hat{f}}(s,\xi-\sigma'-\eta') \right] \right|. \tag{3.16}$$

Following the computations in lemma 3.9, we can show that

$$\left| \mathcal{F}^{-1}_{(\eta',\sigma')\to(\eta,\sigma)} \left[\hat{f}(s,\xi-\eta') \ \hat{f}(s,\xi-\sigma') \ \overline{\hat{f}}(s,\xi-\sigma'-\eta') \right] \right| \simeq \left| \int \mathrm{d}z \ e^{-iz\xi} \ f(z-\eta) \ f(z-\sigma) \ \overline{f}(z) \right|$$

hence (3.16) becomes

$$|R(s,\xi)| \lesssim s^{-1-\delta} \int \mathrm{d}z \int \mathrm{d}\eta \int \mathrm{d}\sigma \ (|\eta||\sigma|)^{\delta} \ \left| f(z-\eta) \ f(z-\sigma) \ \overline{f}(z) \right|. \tag{3.17}$$

Using the triangle inequality and convexity, we can write for any $z \in \mathbb{R}$

$$\begin{aligned} (|\eta||\sigma|)^{\delta} &\leq \left(|z-\eta|^{\delta} + |z|^{\delta}\right) \left(|z-\sigma|^{\delta} + |z|^{\delta}\right) \\ &= |z-\eta|^{\delta}|z-\sigma|^{\delta} + |z|^{\delta}|z-\sigma|^{\delta} + |z|^{\delta}|z-\eta|^{\delta} + |z|^{2\delta}. \end{aligned}$$

Plugging this into (3.16) we get

$$\begin{aligned} |R(s,\xi)| &\lesssim s^{-1-\delta} \int \mathrm{d}z \int \mathrm{d}\eta \ \mathrm{d}\sigma \ \left(|z-\eta|^{\delta}|z-\sigma|^{\delta} + |z|^{\delta}|z-\sigma|^{\delta} + |z|^{\delta}|z-\eta|^{\delta} + |z|^{2\delta}\right) \ \left| f(z-\eta) \ f(z-\sigma) \ \overline{f}(z) \right| \\ &\leq s^{-1-\delta} \left\{ \int \mathrm{d}z \ |f(z)| \int \mathrm{d}\eta \ |z-\eta|^{\delta} |f(z-\eta)| \int \mathrm{d}\sigma \ |z-\sigma|^{\delta} |f(z-\sigma)| \right. \\ &\quad + \int \mathrm{d}z \ |z|^{\delta} |f(z)| \int \mathrm{d}\eta \ |f(z-\eta)| \int \mathrm{d}\sigma \ |z-\sigma|^{\delta} |f(z-\sigma)| \\ &\quad + \int \mathrm{d}z \ |z|^{\delta} |f(z)| \int \mathrm{d}\eta \ |z-\eta|^{\delta} |f(z-\eta)| \int \mathrm{d}\sigma \ |f(z-\sigma)| \\ &\quad \left. + \int \mathrm{d}z \ |z|^{2\delta} |f(z)| \int \mathrm{d}\eta \ |f(z-\eta)| \int \mathrm{d}\sigma \ |f(z-\sigma)| \right\} \\ &\leq s^{-1-\delta} \left\{ 3 \left\| \langle x \rangle^{\delta} f(s,x) \right\|^2_{L^1_x} \|f(s,x)\|_{L^1_x} + \left\| \langle x \rangle^{2\delta} f(s,x) \right\|_{L^1_x} \|f(s,x)\|^2_{L^1_x} \right\}. \end{aligned}$$

If we apply Cauchy-Schwarz, use that $\langle x \rangle^{-\gamma} \in L^2_x$ for $\gamma > 1/2$, and demand $\delta < 1/4$ we know

$$\|f(s,x)\|_{L^1_x}, \ \left\| \langle x \rangle^{\delta} f(s,x) \right\|_{L^1_x} \ \left\| \langle x \rangle^{2\delta} f(s,x) \right\|_{L^1_x} \lesssim \|f(s,x)\|_{H^{0,1}_x} \leq s^{\alpha} \|u\|_{X_T}$$

(though the individual constants in the bounds are different). Therefore,

$$|R(s,\xi)| \lesssim s^{-1-\delta+3\alpha} \|u\|^3_{X_T}$$

This completes the proof, if we pick $\delta \in \left(3\alpha, \frac{1}{4}\right)$ to ensure the factor of s on the right-hand side decays faster than s^{-1}. $\square$

With the error estimate at hand, we can finish proving the L^∞_x estimates we need. Along the way, we'll rigorously discuss the profile ODE that gave us so much guidance earlier.

Corollary 3.12. *There exists a constant $C > 0$ such that*

$$\sup_{t\in[1,T]} t^{1/2} \|u\|_{L^\infty_x} \leq \epsilon + C \|u\|^3_{X_T}$$

Proof. Before starting the proof in earnest, we'll need to use a neat trick to simplify (3.13) even more. We differentiate (3.13) with respect to t in order to find

$$\partial_t \hat{f}(t,\xi) = \frac{i\lambda}{t} \left|\hat{f}(t,\xi)\right|^2 \hat{f}(t,\xi) + R(t,\xi).$$

(compare with (3.6)). If we pretend $\left|\hat{f}(t,\xi)\right|^2$ does not depend on $\hat{f}$ (similar to what we often did when discussing the physics of NLS), then we can simplify the above ODE by introducing an *integrating factor.* Let

$$B(t,\xi) \doteq \lambda \int_1^t \frac{\mathrm{d}s}{s} \left|\hat{f}(s,\xi)\right|^2.$$

Then, it's trivial to show

$$\partial_t \left(\hat{f}(t,\xi) e^{-iB(t,\xi)}\right) = R(t,\xi) e^{-iB(t,\xi)}.$$

(with a bit of prescience, this method also amounts to treating B like a modified frequency). We then integrate the new ODE to obtain

$$\hat{w}(t,\xi) \doteq \hat{f} e^{-iB} = \hat{f}(1,\xi) + \int_1^t \mathrm{d}s\ R(s,\xi)\ e^{-iB(s,\xi)} \tag{3.18}$$

$\hat{w}$ has the same modulus as $\hat{f}$, so for many estimates of interest it suffices to look at the simpler equation (3.18) instead of (3.13).

Now that we have a more convenient starting point, we can get into the meat of the proof. By lemma 2.3 there is some $\kappa \ll 1$ such that

$$\|u(t,x)\|_{L^\infty_x} \leq t^{-1/2} \left\|\hat{f}(t,\xi)\right\|_{L^\infty_\xi} + t^{-1/2-\kappa} \|f(t,x)\|_{H^{0,1}_x}.$$

So, we need to bound $\left\|\hat{f}(t,\xi)\right\|_{L^\infty_\xi}$ and $\|f(t,x)\|_{H^{0,1}_x}$. Using (3.18) and lemma 3.11, we have

$$\left\|\hat{f}\right\|_{L^\infty_\xi} \leq \left\|\hat{f}(1,\xi)\right\|_{L^\infty_\xi} + \|u\|^3_{X_T} \int_1^t \mathrm{d}s\ s^{-1-\delta+3\alpha} \tag{3.19}$$

Using boundedness of the Fourier transform on $L^1_x \to L^\infty_\xi$, we have that

$$\left\|\hat{f}(s,\xi)\right\|_{L^\infty_\xi} \leq \|f\|_{L^1_x}.$$

We can then use Cauchy-Schwarz along with $\langle x\rangle^{-1} \in L_x^2$ to discover

$$\left\|\hat{f}(s,\xi)\right\|_{L_\xi^\infty} \lesssim \|f(s,x)\|_{H_x^{0,1}}.$$

Combining the above with (3.19), we find

$$\begin{aligned}\left\|\hat{f}\right\|_{L_\xi^\infty} &\lesssim \|f(1,x)\|_{H_x^{0,1}} + \|u\|_{X_T}^3 \int_1^t \mathrm{d}s\ s^{-1-\delta+3\alpha} \\ &\lesssim \epsilon + t^{-\delta+3\alpha}\|u\|_{X_T}^3.\end{aligned}$$

In turn we find

$$\|u(t,x)\|_{L_x^\infty} \lesssim t^{-1/2}\left(\epsilon + t^{-\delta+3\alpha}\|u\|_{X_T}^3 + t^{-\kappa}\|f(t,x)\|_{H_x^{0,1}}\right).$$

At this point, we may use corollary 3.8 to get

$$\|u(t,x)\|_{L_x^\infty} \lesssim t^{-1/2}\left(\epsilon + t^{-\delta+3\alpha}\|u\|_{X_T}^3 + t^{\alpha-\kappa}\left(\epsilon + C\|u\|_{X_T}^3\right)\right).$$

Thus, after demanding $\alpha < \kappa$ and possibly shrinking ϵ, the proof is complete. □

3.3.4 Bootstrapping Linear Decay 3: Conclusion

Now, we are ready to establish our main technical ingredient for establishing bootstrap estimates:

Proposition 3.13. *If $u(t,x)$ is the local solution to NLS constructed in proposition 3.4, then we have*

$$\|u\|_{X_T} \leq \epsilon + C\|u\|_{X_T}^3.$$

Proof. This follows easily from combining corollaries 3.8 and 3.12, and L_x^2-norm conservation. □

At long last, we now prove global existence and the dispersive estimate in one fell swoop. I emphasize that the smallness of ϵ is determined in the course of this particular proof.

Theorem 3.14 (Global Existence & Linear Decay)**.** *If ϵ is sufficiently small, the local solution constructed in proposition 3.4 can be extended to a global solution*

$$u(t,x) \in C\left(\mathbb{R}; H_x^{1,0} \cap H_x^{0,1}\right)$$

Also, for all $T > 0$, we have

$$\|u\|_{X_T} \lesssim \epsilon. \tag{3.20}$$

In particular, $u(t,x)$ satisfies the same decay bound as solutions to the linear Schrödinger equation:

$$\|u(t,x)\|_{L_x^\infty} \lesssim \epsilon\langle t\rangle^{-1/2} \quad \forall\ t > 0. \tag{3.21}$$

Proof. Note that (3.20) immediately implies (3.21). By our local existence result in proposition 3.4, once ϵ is fixed there is a constant $C_0 \geq 2$ and a fixed time $T > 1$ such that a unique solution $u(t,x)$ exists and satisfies

$$\|u\|_{X_T} \leq C_0\epsilon. \tag{3.22}$$

We can then treat $u(T)$ as initial data and apply proposition 3.4 to find a larger existence time $T' > T$. Since $\tau \mapsto \|u\|_{X_\tau}$ is continuous, we can without loss of generality shrink T' so that

$$\|u\|_{X_{T'}} \leq 2C_0\epsilon. \tag{3.23}$$

Then, if we apply proposition 3.13, we find

$$\|u\|_{X_{T'}} \leq \epsilon + C\,\|u\|^3_{X_{T'}} \leq \epsilon + 8CC_0^3\epsilon^3. \tag{3.24}$$

If ϵ satisfies

$$\epsilon \leq \left(8CC_0^3\right)^{-1/2}$$

then (3.24) and $C_0 \geq 2$ yields

$$\|u\|_{X_{T'}} \leq C_0\epsilon,$$

which is stronger than (3.23). Thus, by iterating this argument, our solution can be extended to exist for all time and we can actually take T as large as we like in (3.22). We then finish the proof by noting that

$$u(t,x) \in C\left(\mathbb{R}; H_x^{1,0} \cap H_x^{0,1}\right)$$

follows from the definition of the X_T norm. □

3.3.5 Existence of Asymptotic Profile

To get L^∞_x-estimates, we used the factorization

$$\hat{w}(t,\xi) = \hat{f}(t,\xi)e^{-iB(t,\xi)}$$

inspired by the method of integrating factors from basic ODE theory. In light of our heuristic computation from earlier, we expect

$$\hat{f}(t,\xi) \approx e^{i\lambda|F(\xi)|^2\log t}F(\xi), \quad t \gg 1,$$

so since

$$B(t,\xi) = \lambda\int_1^t \frac{\mathrm{d}s}{s}\ \left|\hat{f}(s,\xi)\right|^2 \approx \lambda|F(\xi)|^2\log t$$

if $\hat{f}(s,\xi)$ varies slowly in s, we conjecture

$$\lim_{t\to\infty}\hat{w}(t,\xi) = F(\xi).$$

To prove this claim rigorously, we first need to show that $\hat{w}(t,\xi)$ actually has a limit.

Proposition 3.15. *There exists a unique $F(\xi) \in L^\infty_\xi$ such that*

$$\lim_{t\to\infty} \hat{w}(t,\xi) = F(\xi).$$

Also, $\hat{w}(t,\xi) \in L^\infty_t L^\infty_\xi$.

Proof. It suffices to show that $\hat{w}(t,\xi)$ is Cauchy in time: since $\hat{\omega}$ is necessarily bounded in ξ, the limiting function $F(\xi)$ must be bounded as well. We start using the ODE for $\hat{\omega}$ in the form (3.18) to write for any $t < t' \in \mathbb{R}$

$$|\hat{w}(t,\xi) - \hat{w}(t',\xi)| \lesssim \int_t^{t'} \mathrm{d}s \ |R(s,\xi)|.$$

The error estimate in lemma 3.11 and the global well-posedness result from theorem 3.14 together give

$$|\hat{w}(t,\xi) - \hat{w}(t',\xi)| \lesssim_\epsilon \left| \left[s^{-\delta+3\alpha} \right]_t^{t'} \right| \quad (\text{uniformly in } \xi). \tag{3.25}$$

Since $-\delta+3\alpha < 0$, we can take $t \to \infty$ to find that $\hat{w}(t,\xi)$ is Cauchy in time, so the existence of $F(\xi)$ is established. Using (3.25), it is trivial to show $|\hat{w}(t,\xi)|$ is bounded in time. □

We now estimate the error in treating $F(\xi)$ like the asymptotic profile from our heuristic computation, and in doing so prove the second part of the main theorem.

Corollary 3.16. *Let $F(\xi)$ be defined as in proposition 3.15. Then, there is some $\rho > 0$ such that the global solution $u(t,x)$ to NLS with sufficiently small initial data satisfies*

$$\left\| \hat{f}(\xi) \exp\left(-i\lambda \int_1^t |\hat{u}(s,\xi)|^2 \frac{\mathrm{d}s}{s} \right) - F(\xi) \right\|_{L^\infty_\xi} \lesssim t^{-\rho}.$$

Proof. Take $t' \to \infty$ in (3.25), then take the sup over $\xi \in \mathbb{R}$ and choose $\rho = \delta - 3\alpha$. □

3.3.6 Proof of Leading-Order Asymptotics

Finally, we turn to establishing the leading-order asymptotic expansion (3.10). For the reader's convenience, we rewrite this equation here:

$$u(t,x) = (it)^{-1/2} F(x/t) \exp\left(\frac{i|x|^2}{2t} + i\,|F(x/t)|^2 \log t + i\phi(x/t) \right) + \mathcal{O}\left(t^{-1/2-\beta}\right). \tag{3.10}$$

Lemma 3.17. *Define*

$$\psi(t,\xi) = \lambda \int_1^t \frac{\mathrm{d}s}{s} \ \left(|\hat{w}(s)|^2 - |\hat{w}(t)|^2 \right) \tag{3.26}$$

Then, there exists a unique real-valued function $\phi(\xi)$ and some $\nu > 0$ such that

$$\|\psi(t,\xi) - \phi(\xi)\|_{L^\infty_\xi} \lesssim t^{-\nu}.$$

Proof. For any $0 < t' < t$, we have

$$\psi(t,\xi) - \psi(t',\xi) = -\lambda\left(|\hat{w}(t)|^2 \log t - |\hat{w}(t')|^2 \log t'\right) + \lambda \int_{t'}^{t} \frac{\mathrm{d}s}{s}\ |\hat{w}(s)|^2$$
$$= -\lambda\left(|\hat{w}(t)|^2 - |\hat{w}(t')|^2\right)\log t' + \lambda \int_{t'}^{t} \frac{\mathrm{d}s}{s}\ \left(|\hat{w}(s)|^2 - |\hat{w}(t)|^2\right).$$

So, we need to estimate

$$\left||\hat{w}(t)|^2 - |\hat{w}(t')|^2\right|.$$

Since $\hat{w}(t,\xi) \in L_t^\infty$, we can apply the mean value theorem and (3.25) to get

$$\left||\hat{w}(t)|^2 - |\hat{w}(t')|^2\right| \lesssim \left|\left[s^{-\rho}\right]_t^{t'}\right|.$$

Thus

$$|\psi(t,\xi) - \psi(t',\xi)| \lesssim \left|\left[s^{-\rho}\right]_t^{t'}\right| \log t' + \int_{t'}^{t} \mathrm{d}s\ s^{-1}\ |s^{-\rho} - t^{-\rho}|.$$

Since $\log t'$ is asymptotically negligible compared to any positive power of t', we can set $\nu = \rho/2$ so the above implies

$$|\psi(t,\xi) - \psi(t',\xi)| \lesssim t^{-\nu} + (t')^{-\nu},$$

which is clearly enough to establish the claim. □

Now, we can use the above result to write logarithmic frequency correction established in the previous subsection in a more friendly way.

Lemma 3.18. *For some $\nu > 0$, the function $\phi(\xi)$ constructed in lemma 3.17 satisfies*

$$\left\|\lambda \int_1^t |\hat{u}(s,\xi)|^2\ \frac{\mathrm{d}s}{s} - \lambda |F(\xi)|^2 \log t - \phi(\xi)\right\|_{L_\xi^\infty} \lesssim t^{-\nu}.$$

Proof. By simple arithmetic,

$$\lambda \int_1^t \frac{\mathrm{d}s}{s}\ |\hat{w}(s,\xi)|^2 = \lambda|F(\xi)|^2 \log t - \lambda |F(\xi)|^2 \log t + \phi(\xi) - \phi(\xi) - \psi(t,\xi) + \lambda |\hat{w}(t,\xi)|^2 \log t$$
$$= \left[\lambda |F(\xi)|^2 \log t + \phi(\xi)\right] + \left\{\left[\psi(t,\xi) - \phi(\xi)\right] + \lambda \log t \left[|\hat{w}(t,\xi)|^2 - |F(\xi)|^2\right]\right\}.$$

Since the term in curly braces vanishes at infinity at least as fast as $t^{-\nu}$, the claim follows upon recalling that $|\hat{u}| = |\hat{w}|$. □

Proposition 3.19. *The leading-order asymptotic estimate* (3.10) *holds.*

Proof. From our linear dispersive estimates, we know there is some small α so that

$$\|f(t,x)\|_{H_x^{0,1}} \lesssim_\epsilon t^\alpha,$$

so by lemma 2.3 we can write (possibly after shrinking α again and re-defining γ)

$$u(t,x) = (it)^{-1/2}\, e^{\frac{ix^2}{2t}} \hat{f}\left(t, \frac{x}{t}\right) + \mathcal{O}\left(t^{-1/2-\gamma}\right). \tag{3.27}$$

Then, we use corollary 3.16 to say

$$\begin{aligned}
\left\|\hat{f}(t,\xi) - F(\xi) e^{i\lambda|F(\xi)|^2 \log t + i\phi(\xi)}\right\|_{L_\xi^\infty} &= \left\|\hat{f}(t,\xi) e^{-i\lambda|F(\xi)|^2 \log t - i\phi(\xi)} - F(\xi)\right\|_{L_\xi^\infty} \\
&\le \left\|\hat{f}(t,\xi) e^{-i\lambda|F(\xi)|^2 \log t - i\phi(\xi)} - \hat{f}(t,\xi) e^{-i\lambda \int_1^t |\hat{u}(s,\xi)|^2 \ \frac{\mathrm{d}s}{s}}\right\|_{L_\xi^\infty} \\
&\quad + \left\|\hat{f}(t,\xi) e^{-i\lambda \int_1^t |\hat{u}(s,\xi)|^2 \ \frac{\mathrm{d}s}{s}} - F(\xi)\right\|_{L_\xi^\infty} \\
&\lesssim \left\|\hat{f}(t,\xi)\right\|_{L_\xi^\infty} \left\|\exp\left(-i\lambda|F(\xi)|^2 \log t - i\phi(\xi) + i\lambda \int_1^t |\hat{u}(s,\xi)|^2 \ \frac{\mathrm{d}s}{s}\right) - 1\right\|_{L_\xi^\infty} + t^{-\rho}.
\end{aligned}$$

Combining the above with

$$\left\|\hat{f}(t,\xi)\right\|_{L_\xi^\infty} \lesssim_\epsilon t^\alpha$$

(a simple consequence of $\|u\|_{X_T} \lesssim \epsilon$) as well as lemmas 2.2 and 3.18, we know that for any $\kappa \in (0, 1/2)$

$$\begin{aligned}
\left\|\hat{f}(t,\xi) - F(\xi) e^{i\lambda|F(\xi)|^2 \log t + i\phi(\xi)}\right\|_{L_\xi^\infty} &\lesssim_\epsilon t^\alpha \left\|\lambda \int_1^t |\hat{u}(s,\xi)|^2 \ \frac{\mathrm{d}s}{s} - \lambda|F(\xi)|^2 \log t - \phi(\xi)\right\|_{L_\xi^\infty}^\kappa + t^{-\rho} \\
&\le t^{\alpha - \kappa\nu} + t^{-\rho}.
\end{aligned}$$

Plugging the above into (3.27) gives

$$\begin{aligned}
u(t,x) &= (it)^{-1/2}\, e^{\frac{ix^2}{2t}} \left\{F\left(\frac{x}{t}\right) e^{i\lambda\left|F\left(\frac{x}{t}\right)\right|^2 \log t + i\phi\left(\frac{x}{t}\right)} + \mathcal{O}\left(t^{\alpha-\kappa\nu} + t^{-\rho}\right)\right\} + \mathcal{O}\left(t^{-1/2-\gamma}\right) \\
&= (it)^{-1/2}\, F\left(\frac{x}{t}\right) \exp\left(\frac{ix^2}{2t} + i\lambda \left|F\left(\frac{x}{t}\right)\right|^2 \log t + i\phi\left(\frac{x}{t}\right)\right) + \mathcal{O}\left(t^{-1/2-\beta}\right)
\end{aligned}$$

after (possibly) shrinking α one last time and defining β appropriately. □

Notice that we can without loss of generality take $F(\xi)$ to be real-valued and positive by changing $\phi(\xi)$.

4 Summary & Main Takeaways

- NLS is a physically important nonlinear dispersive PDE, and we have provided a useful leading-order asymptotic description of its small solutions. In particular, we have found that on long time scales solutions to NLS *do not* converge to solutions of the linearized PDE.
- There are many different roads to rigorously proving our main theorem, but following Kato and Pusateri [10] we used the method of space-time resonances, which provides an elegant framework based on the ODE obeyed by our solution's profile.
- The space-time resonance approach allows us to estimate weighted norms by taking derivatives in Fourier space!
- Once one has weighted estimates, it's very easy to get quality asymptotics using the general procedure laid out by Hayashi and Naumkin [7]

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