Existence of parabolic minimizers to the total variation flow on metric measure spaces

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Abstract

We give an existence proof for variational solutions u associated to the total variation flow. Here, the functions being considered are defined on a metric measure space $(\mathcal{X}, d, \mu)$. For such parabolic minimizers that coincide with a time-independent Cauchy-Dirichlet datum u_0 on the parabolic boundary of a space-time-cylinder $\Omega \times (0,T)$ with $\Omega \subset \mathcal{X}$ an open set and $T > 0$, we prove existence in the weak parabolic function space $L^1_w(0,T;\mathrm{BV}(\Omega))$. In this paper, we generalize results from a previous work by Bögelein, Duzaar and Marcellini and argue completely on a variational level. This is a join project with Vito Buffa and Michael Collins, from Friedrich-Alexander-Universität Erlangen-Nürnberg.

1 Motivation

Our aim is to show existence for parabolic minimizers to the total variation flow on metric measure spaces. More precisely, we consider minimizers of integral functionals that are related to scalar functions $u : \Omega \times (0,T) \to \mathbb{R}$ which satisfy the inequality

$$\iint u\partial_t\phi \mathrm{d}\mu\mathrm{d}t + \int \|Du(t)\|\mathrm{d}t \leq \int \|D(u+\varphi)(t)\|\mathrm{d}\mu, \tag{1}$$

where $\|Du\|$ denotes the total variation of u. Here, $\Omega \subset \mathcal{X}$ is a bounded domain, where $(\mathcal{X}, d, \mu)$ is a metric measure space with a metric d and a measure μ.

The Total Variation Flow (TVF) does not have any regularizing effects. Therefore, it is natural to expect the existence of solutions in the class of functions of bounded variation

(BV functions). As already mentioned, instead of the classical Euclidean setting, we intend to work in the general setting of metric measure spaces. During the past two decades, a theory of Sobolev and BV functions has been developed in this abstract setting. A central motivation for developing such a theory has been the desire to unify the assumptions and methods employed in various specific spaces, such as weighted Euclidean spaces, Riemannian manifolds, Heisenberg groups, graphs, etc.

In the setting of a metric measure space, the classical calculus known from the Euclidean space $\mathbb{R}^n$ is no longer available and instead of distributional derivatives, the space BV of functions with bounded variation has to be introduced by a relaxation approach that makes use of the notion of upper gradients. An alternative approach to BV via derivations is also presented. BV functions are a somewhat more general class than Sobolev functions, in the sense that they may have discontinuities and even "jumps", but are nonetheless differentiable in a very weak sense. The class has many applications, for example, as generalized solutions to partial differential equations with linear growth conditions, which often arise in the calculus of variations, physics, mechanics and image processing.

We deal with parabolic minimizers on parabolic cylinders $\Omega_T := \Omega \times (0, T)$ with $\Omega \subset \mathcal{X}$ bounded and open and $T > 0$. $\mathcal{X}$ denotes a metric measure space that fulfills a doubling property with respect to the metric d and the measure μ and supports a suitable Poincaré inequality. We generalize results which have recently been proven by Bögelein, Duzaar and Marcellini, while we restrict ourselves to the simplest case where the functional in question depends only on the total variation itself.

2 Basic Definitions and Examples

2.1 Notations

Let $(\mathcal{X}, d, \mu)$ be a separable, connected metric measure space, i.e. $(\mathcal{X}, d)$ is a complete, separable and connected metric space endowed with a Borel measure μ on $\mathcal{X}$. The measure μ is assumed to fulfill a *doubling property*, i.e. there exists a constant $c \geq 1$, such that

$$0 < \mu\left(B_{2r}(x)\right) \leq c \cdot \mu\left(B_r(x)\right) < \infty \tag{2}$$

for all radii $r > 0$ and centres $x \in \mathcal{X}$. Here $B_r(x) \coloneqq \{y \in \mathcal{X} : d(x, y) < r\}$ denotes the open ball with radius r and centre x with respect to the metric d. The *doubling constant* is defined as

$$c_d \coloneqq \inf\{c \geq 1 : (\mathbf{??}) \text{ holds true}\}. \tag{3}$$

A complete metric measure space that fulfills the doubling property is proper, meaning that all closed and bounded subsets are compact.

Following the concept of Heinonen and Koskela, we call a Borel function $g : \mathcal{X} \to [0, \infty]$ an *upper gradient* for an extended real-valued function $u : \mathcal{X} \to [-\infty, \infty]$ if for all $x, y \in \mathcal{X}$ and all rectifiable curves $\gamma : [0, L_\gamma] \to \mathcal{X}$ with $\gamma(0) = x, \gamma(L_\gamma) = y$ there holds

$$|u(x) - u(y)| \leq \int_\gamma g \, \mathrm{d}s. \tag{4}$$

Moreover, if a non-negative and measurable function g fulfills (**??**) for p-almost every curve as before, meaning that the family of curves for which (**??**) fails has p-modulus zero, then g is called *p-weak upper gradient*.
For $1 \leq p < \infty$ and a fixed open subset $\Omega \subset \mathcal{X}$ we define the vector space

$$\tilde{\mathbb{N}}^{1,p}(\Omega) := \{u \in L^p(\Omega) : \exists \; p\text{-weak upper gradient } g \in L^p(\Omega) \text{ of } u\}.$$

$L^p(\Omega)$ denotes the usual Lebesgue space. The space $\tilde{\mathbb{N}}^{1,p}(\Omega)$ is endowed with the semi-norm

$$\|u\|_{\tilde{\mathbb{N}}^{1,p}(\Omega)} := \|u\|_{L^p(\Omega)} + \|g_u\|_{L^p(\Omega)}, \tag{5}$$

where g_u denotes the *minimal p-weak upper gradient* of u, i.e. $\|g_u\|_{L^p(\Omega)} = \inf \|g\|_{L^p(\Omega)}$, with the infimum being taken over all p-weak upper gradients of u. Introducing the equivalence relation

$$u \sim v \Longleftrightarrow \|u - v\|_{\tilde{\mathbb{N}}^{1,p}(\Omega)} = 0,$$

we define the *Newtonian space* $\mathbb{N}^{1,p}(\Omega)$ as the quotient space

$$\mathbb{N}^{1,p}(\Omega) := \tilde{\mathbb{N}}^{1,p}(\Omega)/\sim,$$

which we endow with the quotient norm $\| \cdot \|_{\mathcal{N}^{1,p}(\Omega)}$ defined as in (**??**). Since this definition clearly depends on the metric d and the measure μ, we abuse the notation $\mathbb{N}^{1,p}(\Omega)$ as an abbrevation for $\mathbb{N}^{1,p}(\Omega, d, \mu)$.
In addition to the doubling property, we demand that the metric measure space $(\mathcal{X}, d, \mu)$ supports a weak $(1,1)$-Poincaré inequality, in the sense that there exist a constant $c_P > 0$ and a dilatation factor $\tau \geq 1$ such that for all open balls $B_\varrho(x_0) \subset B_{\tau\varrho}(x_0) \subset \mathcal{X}$, for all L^1-functions u on $\mathcal{X}$ and all upper gradients $\tilde{g}_u$ of u there holds

$$⨍_{B_\varrho(x_0)} |u - u_{\varrho,x_0}| \, \mathrm{d}\mu \leq c_P \, \varrho ⨍_{B_{\tau\varrho}(x_0)} \tilde{g}_u^p \, \mathrm{d}\mu, \tag{6}$$

where the symbol

$$u_{\varrho,x_0} := ⨍_{B_\varrho(x_o)} u \; \mathrm{d}\mu := \frac{1}{\mu(B_\varrho(x_0))} \int_{B_\varrho(x_o)} u \; \mathrm{d}\mu$$

denotes the mean value integral of the function u on the ball $B_\varrho(x_0)$ with respect to the measure μ.

Now, we recall the definition and some basic properties of functions of bounded variation. For $u \in L^1_{\mathrm{loc}}(\mathcal{X})$, we define the total variation of u on $\mathcal{X}$ to be

$$\|Du\|(\mathcal{X}) \coloneqq \inf\left\{\liminf_{i\to\infty}\int_{\mathcal{X}} g_{u_i}\,\mathrm{d}\mu : u_i \in \mathrm{Lip}_{\mathrm{loc}}(\mathcal{X}),\ u_i \to u \text{ in } L^1_{\mathrm{loc}}(\mathcal{X})\right\},$$

where each g_{u_i} is the minimal 1-weak upper gradient of u_i. We say that a function $u \in L^1(\mathcal{X})$ is of *bounded variation*, by notation $u \in \mathrm{BV}(\mathcal{X})$, if $\|Du\|(\mathcal{X}) < \infty$. By replacing $\mathcal{X}$ with an open set $\Omega \subset \mathcal{X}$ in the definition of the total variation, we can define $\|Du\|(\Omega)$. The norm in BV is given by

$$\|u\|_{\mathrm{BV}(\Omega)} \coloneqq \|u\|_{L^1(\Omega)} + \|Du\|(\Omega).$$

It is known that for $u \in \mathrm{BV}(\mathcal{X})$, the total variation $\|Du\|$ is the restriction to the class of open sets of a finite Radon measure defined on the class of all subsets of $\mathcal{X}$. This outer measure is obtained from the map $\Omega \mapsto \|Du\|(\Omega)$ on open sets $\Omega \subset \mathcal{X}$ via the standard Carathéodory construction. Thus, for an arbitrary set $A \subset \mathcal{X}$ one can define

$$\|Du\|(A) \coloneqq \inf\left\{\|Du\|(\Omega) : \Omega \text{ open}, A \subset \Omega\right\}.$$

2.2 Parabolic function spaces

For a Banach space B and $T > 0$, the space

$$C^0([0,T];B)$$

consists of all continuous functions $u : [0,T] \to B$ satisfying

$$\|u\|_{C^0([0,T];B)} \coloneqq \max_{0\le t\le T}\|u(t)\|_B < \infty.$$

Naturally, for $\alpha \in (0,1]$, the space

$$C^{0,\alpha}([0,T];B)$$

consists of those functions $u \in C^0([0,T];B)$, for which additionally

$$\sup_{s,t\in[0,T]}\frac{\|u(s)-u(t)\|_B}{|s-t|^\alpha} < \infty$$

holds true.

In the Euclidean case, it can be shown via integration by parts that the space BV can be written as the dual space of a separable Banach space. Since this tool is not available in the metric setting (at least not in the sense as it is understood in the Euclidean case), a different approach has to be taken.

2.2.1 The space BV via derivations

Let $\mathrm{Lip}_{\mathrm{bs}}(\mathcal{X})$ denote the space of Lipschitz functions on $\mathcal{X}$ with bounded support and $L^0(\mathcal{X})$ the space of measurable functions. By a (Lipschitz) *derivation* we denote a linear map $\mathfrak{d} : \mathrm{Lip}_{\mathrm{bs}}(\mathcal{X}) \to L^0(\mathcal{X})$ such that the Leibniz rule

$$\mathfrak{d}(fg) = f\mathfrak{d}(g) + g\mathfrak{d}(f)$$

holds true for all $f, g \in \mathrm{Lip}_{\mathrm{bs}}(\mathcal{X})$ and for which there exists a function $h \in L^0(\mathcal{X})$ such that for μ-a.e. (almost every) $x \in \mathcal{X}$ and all $f \in \mathrm{Lip}_{\mathrm{bs}}(\mathcal{X})$ there holds

$$|\mathfrak{d}(f)|(x) \le h(x) \cdot \mathrm{Lip}_a(f)(x), \tag{7}$$

where $\mathrm{Lip}_a(f)(x)$ denotes the asymptotic Lipschitz constant of f at x. The set of all such derivations will be denoted by $\mathrm{Der}(\mathcal{X})$. The smallest function h satisfying (**??**) will by denoted by $|\mathfrak{d}|$ and we are going to write $\mathfrak{d} \in L^p$ when we mean to say $|\mathfrak{d}| \in L^p$.

For given $\mathfrak{d} \in \mathrm{Der}(\mathcal{X})$ with $\mathfrak{d} \in L^1_{\mathrm{loc}}(\mathcal{X})$ we define the *divergence operator* $\nabla \cdot (\mathfrak{d}) : \mathrm{Lip}_{\mathrm{bs}}(\mathcal{X}) \to \mathbb{R}$ as

$$f \mapsto -\int_{\mathcal{X}} \mathfrak{d}(f)\,\mathrm{d}\mu.$$

We say that $\nabla \cdot (\mathfrak{d}) \in L^p(\mathcal{X})$ if this operator admits an integral representation via a unique L^p-function h, i.e.

$$\int_{\mathcal{X}} \mathfrak{d}(f)\,\mathrm{d}\mu = -\int_{\mathcal{X}} hf\,\mathrm{d}\mu.$$

For all $p, q \in [1, \infty]$ we shall set

$$\mathrm{Der}^p(\mathcal{X}) \coloneqq \{\mathfrak{d} \in \mathrm{Der}(\mathcal{X}) : \mathfrak{d} \in L^p(\mathcal{X})\}$$

and

$$\mathrm{Der}^{p,q}(\mathcal{X}) \coloneqq \{\mathfrak{d} \in \mathrm{Der}(\mathcal{X}) : \mathfrak{d} \in L^p(\mathcal{X}),\ \nabla \cdot (\mathfrak{d}) \in L^q(\mathcal{X})\}.$$

When $p = \infty = q$ we will write $\mathrm{Der}_b(\mathcal{X})$ instead of $\mathrm{Der}^{\infty,\infty}(\mathcal{X})$. The domain of the divergence is characterized as

$$D(\nabla\cdot) \coloneqq \{\mathfrak{d} \in \mathrm{Der}(\mathcal{X}) : |\mathfrak{d}|, \nabla \cdot (\mathfrak{d}) \in L^1_{L^1_{loc}}(\mathcal{X})\}.$$

For $u \in L^1(\mathcal{X})$ we say that u is of *bounded variation (in the sense of derivations)* in $\mathcal{X}$, denoted $u \in \mathrm{BV}_{\mathfrak{d}}(\mathcal{X})$, if there is a linear and continuous map $L_u : \mathrm{Der}_b(\mathcal{X}) \to \mathbf{M}(\mathcal{X})$ such that

$$\int_{\mathcal{X}} \mathrm{d}L_u(\mathfrak{d}) = -\int_{\mathcal{X}} u\nabla \cdot (\mathfrak{d})\,\mathrm{d}\mu \tag{8}$$

for all $\mathfrak{d} \in \mathrm{Der}_b(\mathcal{X})$ and satisfying $L_u(h\mathfrak{d}) = hL_u(\mathfrak{d})$ for any bounded $h \in \mathrm{Lip}(\mathcal{X})$, where $\mathbf{M}(\mathcal{X})$ denotes the space of finite signed Radon measures on $\mathcal{X}$.

The characterization of BV in the sense of derivations is well-posed. If we take any two maps L_u, $\tilde{L}_u$ as in (**??**), the Lipschitz-linearity of derivations ensures that $L_u(\mathfrak{d}) = \tilde{L}_u(\mathfrak{d})$ μ-a.e. for all $\mathfrak{d} \in \mathrm{Der}_b(\mathcal{X})$. The common value will be then denoted by $Du(\mathfrak{d})$.

We know that for $u \in \mathrm{BV}_{\mathfrak{d}}(\mathcal{X})$ there exists a non-negative, finite Radon measure $\nu \in \mathbf{M}(\mathcal{X})$ such that for every Borel set $B \subset \mathcal{X}$ one has

$$\int_B \mathrm{d}Du(\mathfrak{d}) \leq \int_B |\mathfrak{d}|^* \,\mathrm{d}\nu \tag{9}$$

for all $\mathfrak{d} \in \mathrm{Der}_b(\mathcal{X})$, where $|\mathfrak{d}|^*$ denotes the upper-semicontinuous envelope of $|\mathfrak{d}|$. The least measure ν satisfying (**??**) will be denoted by $\|Du\|_{\mathfrak{d}}$, the total variation of u (in the sense of derivations). Moreover, we have

$$\|Du\|_{\mathfrak{d}}(\mathcal{X}) = \sup\{|Du(\mathfrak{d})(\mathcal{X})| : \mathfrak{d} \in \mathrm{Der}_b(\mathcal{X}), |\mathfrak{d}| \leq 1\}.$$

Finally, the classical representation formula for $\|Du\|_{\mathfrak{d}}$ holds, in the sense that if $\Omega \subset \mathcal{X}$ is any open set, then

$$\|Du\|_{\mathfrak{d}}(\Omega) = \sup\left\{\int_\Omega u\nabla\cdot(\mathfrak{d})\,\mathrm{d}\mu : \mathfrak{d} \in \mathrm{Der}_b(\mathcal{X}), \mathrm{supp}(\mathfrak{d}) \Subset \Omega, |\mathfrak{d}| \leq 1\right\}. \tag{10}$$

We obtain that if $(\mathcal{X}, d, \mu)$ is a complete and separable metric measure space endowed with a locally finite measure μ (as in the case of this paper), then

$$\mathrm{BV}(\mathcal{X}) = \mathrm{BV}_{\mathfrak{d}}(\mathcal{X})$$

and in particular, the respective notions of the total variation coincide. Therefore, from now on, we are only going to write $\mathrm{BV}(\mathcal{X})$ and $\|Du\|$ without making any further distinction.

2.2.2 Weak parabolic function spaces

For $T > 0$ and an open subset $\Omega \subset \mathcal{X}$ we write Ω_T for the space-time cylinder $\Omega \times (0, T)$. For the concept of variational solutions we are going to make use of the space

$$L^1_w(0, T; \mathrm{BV}(\Omega)),$$

where the suffix w stands for 'weak'. This space consists of those $v \in L^1(\Omega_T)$, such that there holds:

- $v(\cdot, t) \in \mathrm{BV}(\Omega)$ for a.e. $t \in (0, T)$,

- $\displaystyle\int_0^T \|Dv(t)\|(\Omega)\mathrm{d}t < \infty.$
- The mapping $t \mapsto v(\cdot, t)$ is *weakly measurable*, i.e. the mapping

$$(0,T) \ni t \longmapsto \int_\Omega v(t)\nabla \cdot (\mathfrak{d})\,\mathrm{d}\mu \tag{11}$$

is measurable for all $\mathfrak{d} \in \mathrm{Der}_b(\Omega)$ with $\mathrm{supp}(\mathfrak{d}) \Subset \Omega$.

Remark 1. *In the case of the gradient flow, i.e. a functional with p-growth for $p > 1$, the parabolic function spaces considered are usually $L^p(0,T;\mathbb{N}^{1,p}(\Omega))$, which consist of mappings $v : (0,T) \to \mathbb{N}^{1,p}(\Omega)$ that are strongly measurable in the sense of Bochner. In the case at hand, that is $p = 1$, one would consider the Bochner space $L^1(0,T;\mathrm{BV}(\Omega))$. But the strong measurability in the sense of Bochner is too restrictive, since many simple examples - like the space-time cone $u(t) = \mathbb{1}_{B_t(x_0)}$ - are not strongly measurable in the sense of Bochner, since their image is not separable in $\mathrm{BV}(\Omega)$. Therefore, the strong measurability condition is replaced with a weaker one.*
Note that the weak measurability of a function in $L^1_w(0,T;\mathrm{BV}(\Omega))$ is not to be confused with the weak measurability of a Banach space-valued function in the sense of Pettis' theorem.

Remark 2. *In the Euclidean case, i.e. $\mathcal{X} = \mathbb{R}^n$ for some $n \in \mathbb{N}$, the notion of weak measurability as in (**??**) is usually understood in the sense that the pairing*

$$(0,T) \ni t \mapsto \langle Dv(t), \varphi\rangle = -\int_\Omega v(t)\nabla \cdot (\varphi)\,\mathrm{d}x$$

is measurable for any $\varphi \in C^1_0(\Omega;\mathbb{R}^n)$.

Indeed, the approach by derivations as introduced before yields this classical notion of weak measurability. To understand this, define for any $\varphi \in C^1_0(\Omega;\mathbb{R}^n)$ the mapping

$$\mathfrak{d}_\varphi : \mathrm{Lip}_{bs}(\Omega) \ni f \mapsto \langle \varphi, Df\rangle.$$

*By Rademacher's theorem, the gradient Df is defined almost everywhere on Ω for a Lipschitz function f. It is easy to check that $\mathfrak{d}_\varphi$ fulfills the Leibniz rule and the property (**??**) with $g(x) = |\varphi(x)|$ almost everywhere. By integration by parts, we find that for the divergence operator of $\mathfrak{d}_\varphi$ there holds*

$$\nabla \cdot (\mathfrak{d}_\varphi) : f \mapsto -\int_\Omega \langle \varphi, Df\rangle\,\mathrm{d}x = \int_\Omega \nabla \cdot (\varphi) f\,\mathrm{d}x.$$

*Hence, the divergence of $\mathfrak{d}_\varphi$ is represented by $\nabla \cdot (\varphi)$. Thus, the weak measurabilty in the sense of (**??**) yields the measurabilty of the mapping*

$$(0,T) \ni t \mapsto \int_\Omega v(t)\nabla \cdot (\varphi)\,\mathrm{d}x.$$

In view of (**??**), the mapping $[0,T] \ni t \mapsto \|Dv(t)\|(\Omega)$ is measurable for $v \in L^1_w(0,T;\mathrm{BV}(\Omega))$.

Furthermore, the limit of a sequence of functions in $L^1_w(0,T;\mathrm{BV}(\Omega))$ with uniformly bounded total variation is again a $L^1_w(0,T;\mathrm{BV}(\Omega))$-function:

2.3 Variational solutions

In the Euclidean case, i.e. $\mathcal{X} = \mathbb{R}^n$, one might consider the Cauchy-Dirichlet problem

$$\begin{cases} \partial_t u - \nabla \cdot \left(\dfrac{Du}{|Du|} \right) = 0 & \text{in } \Omega_T, \\ u = u_0 & \text{on } \partial_{\mathrm{par}} \Omega_T, \end{cases} \tag{12}$$

where $\partial_{\mathrm{par}}\Omega_T := (\overline{\Omega} \times \{0\}) \cup (\partial\Omega \times (0,T))$ denotes the *parabolic boundary* of Ω_T and u_0 is some given boundary data.

When trying to define a concept of Cauchy-Dirichlet problems like (**??**) for $u \in L^1_w(0,T;\mathrm{BV}(\Omega))$ on metric measure spaces, one has to overcome several difficulties. We point out that also in our case boundary values of BV-functions are a delicate to manage, since the trace operator is not continuous with respect to the weak*-convergence in $\mathrm{BV}(\Omega)$ and the pairing in (**??**). A suitable strategy to treat this issue is to consider a slightly larger domain Ω^* that compactly countains the bounded open set Ω and to assume that the datum u_0 is defined on Ω^*. The boundary condition $u = u_0$ on the lateral boundary $\partial\Omega \times (0,T)$ could then be interpreted by requiring that $u(\cdot,t) = u_0$ a.e. on $\Omega^* \setminus \Omega$ for all $t \in (0,T)$. Thus said, from now on boundary values shall be understood in the following sense:

given $u_0 \in \mathrm{BV}(\Omega^*)$, a function u belongs to $\mathrm{BV}_{u_0}(\Omega)$
if and only if $u \in \mathrm{BV}(\Omega^*)$ and $u = u_0$ a.e. on $\Omega^* \setminus \Omega$.

The condition on the lateral boundary has to be read in the sense that there holds $u(\cdot,t) \in \mathrm{BV}_{u_0}(\Omega)$ for a.e. $t \in (0,T)$.

On the other hand, we do not have the possibility to explain derivatives such as in (**??**). Therefore we cannot consider Cauchy-Dirichlet problems like this. However, by an idea of Lichnewsky and Temam, one can define the concept of *variational solutions.* Since this concept for solutions to a Cauchy-Dirichlet problem is described purely on a variational level, it can be extended to the concept of metric measure spaces.

To be precise, we assume Ω to be open and bounded, Ω^* open and bounded with $\Omega \Subset \Omega^*$ and

$$u_0 \in L^2(\Omega^*) \cap \mathrm{BV}(\Omega^*). \tag{13}$$

Where it makes sense, we are going to abbreviate $v(t) := v(\cdot, t)$.

Definition 3. Assume that the Cauchy-Dirichlet datum u_0 fulfills (**??**). A map $u : \Omega_T^* \to \mathbb{R}$, $T \in (0, \infty)$ in the class

$$L^1_w\left(0, T; \mathrm{BV}_{u_0}(\Omega)\right) \cap C^0\left([0, T]; L^2(\Omega^*)\right)$$

will be referred to as a *variational solution* on Ω_T to the Cauchy-Dirichlet problem for the total variation flow if and only if the variational inequality

$$\begin{aligned}\int_0^T \|Du(t)\|(\Omega^*)\,\mathrm{d}t \le \int_0^T &\left[\int_{\Omega^*} \partial_t v(v-u)\,\mathrm{d}\mu + \|Dv(t)\|(\Omega^*)\right]\mathrm{d}t \\ &- \frac{1}{2}\|(v-u)(T)\|^2_{L^2(\Omega^*)} + \frac{1}{2}\|v(0) - u_0\|^2_{L^2(\Omega^*)}\end{aligned} \tag{14}$$

holds true for any $v \in L^1_w\left(0, T; \mathrm{BV}_{u_0}(\Omega)\right)$ with $\partial_t v \in L^2(\Omega_T^*)$ and $v(0) \in L^2(\Omega^*)$. A map $u : \Omega_\infty^* \to \mathbb{R}$ is termed a *global variational solution* if

$$u \in L^1_w\left(0, T; \mathrm{BV}_{u_0}(\Omega)\right) \cap C^0\left([0, T]; L^2(\Omega^*)\right) \text{ for any } T > 0$$

and u is a variational solution on Ω_T for any $T \in (0, \infty)$.

Note that the time indepenent extension $v(\cdot, t) := u_0$ is an admissible comparison map in (**??**). Therefore, we have that

$$\int_0^T \|Du(t)\|(\Omega^*)\,\mathrm{d}t < \infty$$

for any variational solution u.

3 Results and proofs

Our main results concern the existence, uniqueness and regularity of variational solutions as follows:

Theorem 4. *Suppose that the Cauchy-Dirichlet datum u_0 fulfills the requirements of (**??**). Then, there exists a unique global variation solution in the sense of Definition **??**.*

Theorem 5. *Suppose that the Cauchy-Dirichlet datum u_0 fulfills the requirements of (**??**). Then, any variational solution in the sense of Definition **??** on Ω_T with $T \in (0, \infty]$ satisfies*

$$\partial_t u \in L^2(\Omega^*) \text{ and } u \in C^{0,\frac{1}{2}}\left([0, \tau]; L^2(\Omega^*)\right) \text{ for all } \tau \in \mathbb{R} \cap (0, T].$$

Furthermore, for the time derivative $\partial_t u$ there holds the quantitative bound

$$\int_0^T \int_{\Omega^*} |\partial_t u|^2 \, \mathrm{d}\mu \, \mathrm{d}t \leq \|Du_0\|(\Omega^*).$$

Finally, for any $t_1, t_2 \in \mathbb{R}$ with $0 \leq t_1 < t_2 \leq T$ one has the energy estimate

$$\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \|Du(t)\|(\Omega^*) \, \mathrm{d}t \leq \|Du_0\|(\Omega^*). \tag{15}$$

4 Brief Summary of Talk

1. Motivation. Why the study in metric measure spaces?
2. Basic Definitions for analysis in the general metric setting.
3. Parabolic function spaces. Understanding the function space we are in.
4. Proof of existence of a variational solution for the total variation flow.