Carleson- Makarov in Pictures

David Pechersky

March 26th

1 Abstract

2 Harmonic Measure

Suppose that $\Omega \subseteq \mathbb{C}$ is open and $z \in \Omega$. Start a two- dimensional Brownian motion at z and run it until it hits $\partial\Omega$ at the time $T_{\partial\Omega}$:

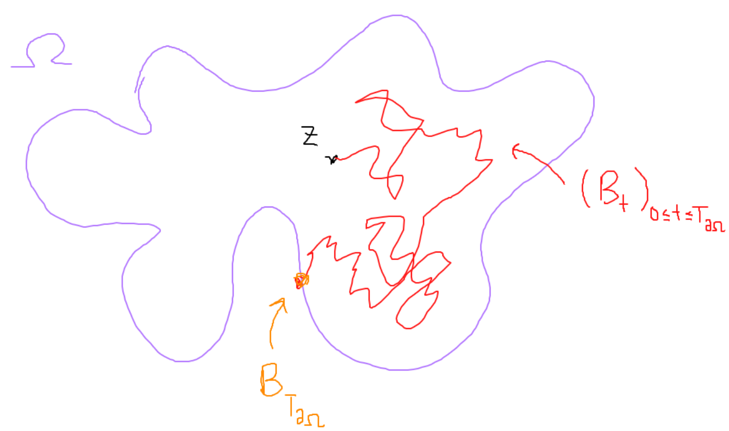

Figure 1

The probability distribution of the point $B_{T_{\partial\Omega}}$ at which our Brownian motion collides with the boundary is known as the harmonic measure in Ω started from z and is denoted by $\omega(z,\cdot,\Omega)$. That is, for any Borel set $E \subseteq \partial\Omega$

$$\omega(z, E, \Omega) = \mathbb{P}^z(B_{T_{\partial\Omega}} \in E)$$

Pick $r > 0$ so that $B(z,r) \subseteq \Omega$. Then a Brownian motion started from z has to exit the disk $B(z,r)$ before hitting $\partial\Omega$. By the strong Markov property we can break our Brownian path into:

1. A Brownian path $(B_t)_{0 \le t \le T_{\partial B(z,r)}}$ starting at z, terminating when it hits $\partial B(z,r)$ at time $T_{\partial B(z,r)}$.

2. An independent Brownian path $(\widetilde{B}_t)_{0 \le t \le T_{\partial\Omega}}$ with initial distribution $\widetilde{B}_0 \overset{d}{=} B_{T_{\partial\Omega}}$.

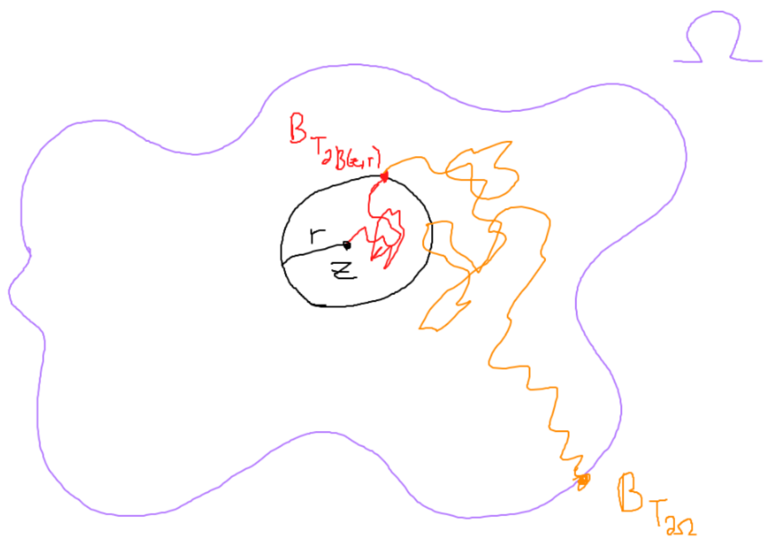

Figure 2

It follows that:

$$\mathbb{P}^z(B_{T_{\partial\Omega}} \in E) = \mathbb{E}[\mathbb{P}^{B_{T_{\partial B(z,r)}}}(\widetilde{B}_{T_{\partial\Omega}} \in E)]$$

Since Brownian motion is rotationally invariant, by symmetry, if we start a Brownian motion from z and wait until it exits $B(z,r)$ the hitting measure of our Brownian motion on $\partial B(z,r) \subseteq \Omega$ is uniform with respect to arclength. Thus,

$$\omega(z, E, \Omega) = \mathbb{E}(\omega(B_{T_{\partial B(z,r)}}, E, \partial\Omega)) = \frac{1}{2\pi}\int_0^{2\pi} \omega(z + re^{i\theta}, E, \Omega)d\theta$$

Since $\omega(z, E, \Omega)$ is a continuous function satisfying the mean value property, $\omega(z, E, \Omega)$ is actually harmonic! (The continuity of $\omega(z, E, \Omega)$ follows from the weak Beurling estimate which we prove in section 4.) Thus we can think of harmonic measure as solving the boundary value problem:

$$\begin{aligned} \Delta\omega(z,E,\Omega) &= 0 \text{ for all } z \in \Omega \\ \omega(z,E,\Omega) &= 1_E(z) \text{ for all } z \in \partial\Omega \end{aligned}$$

This isn't true in any sort of classical sense because the boundary data $1_E(z)$ will not be continuous for general Borel E which means that $\omega(z,E,\Omega)$ cannot extend continuously to the boundary. However, it does suggest to us how one might define harmonic measure without making any reference to Brownian motion.

From the perspective of an analyst, harmonic measure from z, $\omega(z,\cdot,\Omega)$, is the unique measure on $\partial\Omega$ so that for every $f \in C_0(\partial\Omega)$,

$$\int_{\partial\Omega} f(y)\omega(z,dy,\Omega) = u_f(z)$$

where $u_f : \overline{\Omega} \to \mathbb{R}$ is the unique solution to the boundary value problem:

$$\Delta u_f(x) = 0 \text{ for all } x \in \Omega$$

$$u_f(x) = f(x) \text{ for all } x \in \partial\Omega$$

The existence and uniqueness of this measure follows from the Riesz representation theorem. Namely, if X is a locally compact Hausdorff space, let $C_0(X)$ be the closure of $C_c(X)$, the space of continuous functions on X with compact support, with respect to the supremum norm. Then the Riesz representation theorem identifies $(C_0(X))^*$ with $\mathcal{M}(X)$, the space of Radon measures on X. In fact, if we endow $\mathcal{M}(X)$ with the total variation norm, these two spaces are isometrically isomorphic. [Fo99] The map $f \mapsto u_f(z)$ is clearly linear, since our boundary value problem is linear and by the maximum principle for harmonic functions, $|u_f(z)| \leq ||f||_\infty$. This tells us that the map $f \mapsto u_f(z)$ is a bounded linear functional. By the Riesz representation theorem, it corresponds to integrating f against a unique measure on $\partial\Omega$. [1]

To see that the probabilistic and analytic definitions of harmonic measure coincide, apply Ito's lemma to $u_f(B_t)$ to see that it is a martingale, then apply the optional stopping theorem to the martingale $u_f(B_{t\wedge T_{\partial\Omega}})$.

[1] I am lying slightly here. Notice that this definition of the harmonic measure only makes sense provided that you can solve the Dirichlet problem on Ω for arbitrary boundary conditions $f \in C_0(\partial\Omega)$. This is not always possible. In general, you can solve the Dirichlet problem on Ω for arbitrary boundary data $f \in C_0(\partial\Omega)$ iff every point $x \in \partial\Omega$ is "regular." For further details check out chapter 3 of [MP10] or chapter 3 of [GM05].

It can be shown by a fairly routine calculation that the Dirichlet problem in two dimensions is conformally invariant. That is, if $f \in C_0(\partial\Omega)$ for a domain $\Omega \subseteq \mathbb{C}$ and $\phi : \Omega' \to \Omega$ is a conformal map that extends continuously to $\partial\Omega'$ and u solves the boundary value problem:

$$\begin{aligned} \Delta u(z) &= 0 \text{ for all } z \in \Omega \\ u(z) &= f(z) \text{ for all } z \in \partial\Omega \end{aligned}$$

then $\widetilde{u} = u \circ \phi(z)$ solves the boundary value problem:

$$\begin{aligned} \Delta \widetilde{u}(z) &= 0 \text{ for all } z \in \Omega' \\ \widetilde{u}(z) &= f \circ \phi(z) \text{ for all } z \in \partial\Omega' \end{aligned}$$

Since we've defined harmonic measure in terms of a conformally invariant problem, this suggests that harmonic measure itself ought to be conformally invariant. This is indeed the case. Namely, working in the same setting as before, if $E \subseteq \partial\Omega'$ is Borel then:

$$\omega(z, E, \Omega') = \omega(\phi(z), \phi(E), \Omega)$$

This has a probabilistic interpretation! If $(B_t)_{0 \le t \le T_{\partial\Omega'}}$ is a Brownian motion started at $z \in \Omega'$ and stopped when it reaches $\partial\Omega'$ and $\phi : \Omega' \to \Omega$ is conformal then:

$$\left(\phi(B_{\zeta(t)})\right)_{0 \le t \le T_\Omega} \stackrel{d}{=} (\widetilde{B}_t)_{0 \le t \le T_{\partial\Omega}}$$

where $(\widetilde{B}_t)_{0 \le t \le T_{\partial\Omega}}$ is a two dimensional Brownian motion started at $\phi(z)$, run until it hits $\partial\Omega$ at time $T_{\partial\Omega}$ and $\zeta : [0, T_{\partial\Omega}] \to [0, T_{\partial\Omega'}]$ is a nondecreasing homeomorphism. Comparing the quadratic variations of these two processes gives us an explicit formula for $\zeta(t)$:

$$\zeta(t) = \inf\{s \ge 0 : \int_0^s |\phi'(B_s)|^2 ds = t\}$$

The proof of this statement is by stochastic calculus. Namely, applying Ito's lemma to the real and imaginary parts of $\phi(B_t)$ tells us that both components are martingales. By Dubins-Schwarz, it follows that each component is a time- changed one- dimensional Brownian motion. If we look at the explicit formula for this time change for each component, the fact that the components of ϕ satisfy the Cauchy- Riemann equations tells us that the time change is the same for both components. Thus, there is a time change of $\phi(B_t)$ such that the real and imaginary parts are both standard one- dimensional Brownian motions. From here, the only thing we need to check is that the real and imaginary part are independent. One way to do this is to explicitly compute the characteristic function of their joint law using exponential martingales.

Thus, we see that in the plane, the image of a Brownian motion under a conformal map is a time- changed Brownian motion. As per the above formula, unless ϕ is linear, this time change is also random. In particular, if we look at the endpoints of our two Brownian paths we see that:

$$\phi(B_{\zeta(T_{\partial\Omega})}) = \phi(B_{T_{\partial\Omega'}}) \stackrel{d}{=} \widetilde{B}_{T_{\partial\Omega}}$$

This is precisely what it means for harmonic measure to be conformally invariant!

Finally, recall that Harnack's inequality tells us that there exists a constant $C = C(z, w) > 1$ such that for any positive harmonic function u on Ω,

$$C^{-1} \leq \frac{u(z)}{u(w)} \leq C$$

Since $\omega(z, E, \Omega)$ is harmonic in z for any fixed $E \subseteq \Omega$, by Harnack's inequality, the measures $\omega(z, \cdot, \Omega)$ and $\omega(w, \cdot, \Omega)$ are mutually absolutely continuous. For this reason, we often talk about harmonic measure without specifying a particular starting point z.

3 Extremal Length

4 On the Number of Bad Disks

The goal of this section is to provide a sketch of the proof of a technical estimate by Carleson and Makarov which bounds the number of disjoint balls of a given radius that have "large" harmonic measure, in a certain precise sense. [1] To understand why the balls that Carleson and Makarov are looking at have large harmonic measure, one has to know Beurling's estimate:

Theorem (Beurling Estimate) 1. *There exists an absolute constant $C > 0$ such that if $\gamma : [0, 1] \to \overline{B_R(0)}$ is a curve so that $\gamma(0) = 0$, $|\gamma(1)| = R$ and $(B_t)_{t \geq 0}$ is a 2d Brownian motion started at $z \in B_R(0)$ then:*

$$\mathbb{P}^z(B[0, T_{\partial B(0,R)}] \cap \gamma[0, 1] = \emptyset) \leq C\Big(\frac{|z|}{R}\Big)^{1/2}$$

The Beurling estimate gives us an upper bound on the probability that our Brownian motion exits the disk $B(0, R)$ without hitting the curve γ. Furthermore, the exponent $1/2$ here is sharp. In particular, it is known that the probability above is maximized in the case where γ is radial slit pointed away from z:

Proving this estimate with the sharp exponent is quite difficult. Instead we will provide a sketch of the proof of the so- called weak Beurling estimate to give the reader a sense of why such an estimate holds:

Figure 3

Theorem (Weak Beurling Estimate) 2. *There exists absolute constant $C > 0, \beta > 0$ such that if $\gamma : [0,1] \to \overline{B_R(0)}$ is a curve so that $\gamma(0) = 0$, $|\gamma(1)| = R$ and $(B_t)_{t\geq 0}$ is a 2d Brownian motion started at $z \in B_R(0)$ then:*

$$\mathbb{P}^z(B[0, T_{\partial B(0,R)}] \cap \gamma[0,1] = \emptyset) \leq C\Big(\frac{|z|}{R}\Big)^{\beta}$$

Proof. Consider the network of circles $C_0, C_1, ..., C_m$ inside $\overline{B(0,R)}$, where $C_i := \{w : |w| = 2^i|z|\}$. $m = \Big\lfloor \dfrac{\log(R/|z|)}{\log 2} \Big\rfloor$. That is, m is chosen to be as large as possible so that C_m is still inside $\overline{B(0,R)}$. If our Brownian motion hits $\partial B(0,R)$ then at some point it must have hit C_{i+1} before C_{i-1}, having started from some point on C_i earlier. It is a highly believable fact about two- dimensional Brownian motion that if we start a Brownian motion from C_i, it will make a loop around 0 inside the annulus $A_i = \{w : 2^{i-1}|z| < |w| < 2^{i+1}|z|\}$ before hitting ∂A_i with probability $p > 0$. In fact, you actually only need to believe this for the annulus A_1: if this statement is true for A_1, it holds for any annulus A_i by rescaling, since Brownian trajectories are scaling- invariant. Thus:

$$\mathbb{P}^z(B[0, T_{\partial B(0,R)}] \cap \gamma[0,1] = \emptyset) \leq$$

$$\leq \mathbb{P}(\text{Brownian motion started from } C_i \text{ makes loop in } A_i \text{ around 0 before hitting } \partial A_i \text{ for } i = 1, 2, ..., m-1)$$

$$= (1-p)^{m-1} \leq (1-p)^{\lfloor \frac{\log(R/|z|)}{\log 2} \rfloor - 1} \leq \frac{1}{(1-p)^2}(1-p)^{\frac{\log(R/|z|)}{\log 2}} = \frac{1}{(1-p)^2}\Big(\frac{|z|}{R}\Big)^{\frac{\log(1/1-p)}{\log 2}}$$

□

Let $\Omega \subseteq \mathbb{C}$ be a simply connected domain such that $\infty \in \Omega$ (Ω contains a neighbourhood of ∞) normalized so that $\mathrm{diam}(\partial\Omega) = 2$. Define:

$$\omega(E) := \omega(\infty, E \cap \partial\Omega, \Omega)$$

Here $\omega(\infty, E \cap \partial\Omega, \Omega)$ is the harmonic measure from ∞. We can define this as:

$$\omega(\infty, E \cap \partial\Omega, \Omega) := \lim_{|z|\to\infty} \omega(z, E \cap \partial\Omega, \Omega)$$

where the limit on the left hand side always exists. Alternatively, we can use the conformal invariance of Brownian motion and take:

$$\omega(\infty, E \cap \partial\Omega, \Omega) := \omega(0, \psi(E \cap \partial\Omega), \psi(\Omega))$$

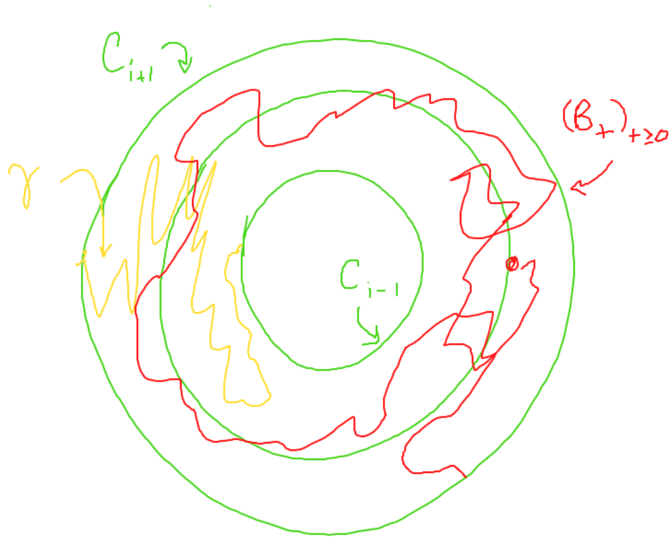

Figure 4: A Brownian motion started from a point on C_i makes a loop around 0 before hitting C_{i+1}. By topology, any such loop must intersect γ.

where $\psi(z) = z^{-1}$. We are using harmonic measure from ∞ in order to be consistent with the original paper by Carleson and Makarov. However, as far as estimating harmonic measure, ∞ is really no different from any other point in the complex plane.

Fix $\varepsilon > 0$ and let $N(\Omega, \varepsilon, \rho)$ be the maximum cardinality of a set of the form:

$$\{\zeta_k \in \partial\Omega : B(\zeta_k, \rho) \text{ are pairwise disjoint and } \omega(B(\zeta_k, \rho)) \geq \rho^{1/2+\varepsilon}\}$$

By the Beurling estimate, $\omega(B(\zeta, \rho)) \leq C\rho^{1/2}$ for some absolute constant $C > 0$ and any $\rho > 0$. Thus, we see that $N(\Omega, \varepsilon, \rho)$ is the largest number of disjoint balls of radius ρ with "large" harmonic measure in the sense that the harmonic measure of each ball is almost the maximum value it could be. Namely, the one given by the Beurling estimate. From here, the theorem of Carleson and Makarov tells us the following:

Theorem 3. *There exist absolute constants $A, K > 0$ so that for any $\varepsilon > 0$, $\rho > 0$ and any domain Ω normalized as we described earlier,*

$$N(\Omega, \varepsilon, \rho) \leq A\rho^{-K\varepsilon}$$

To prove this, fix $\delta > 0$ small and consider the annuli:

$$A_j(\zeta) := \{z : \delta^j < |z - \zeta| < \delta^{j-1}\}$$

Recall that the extremal length of the family of curves connecting the two boundary components of each such annulus is $m := \frac{1}{2\pi} \log 1/\delta$. Let E_j be any subarc of $\Omega \cap \{z : |z - \zeta| = \delta^j\}$. Define:

$$m_j := \min_{\{E_{j-1}, E_j\}} d_{\Omega \cap A_j(\zeta)}(E_j, E_{j-1})$$

Notice that $m_j \geq m$ for all j. This follows almost immediately from the definition of extremal length:

5 References

[Fo99] G. Folland, Real Analysis: Modern Techniques and Their Applications, 2^{nd} edition. *John Wiley Sons Inc., 1999*

[GM05] J. Garnett and D. Marshall, Harmonic Measure. *Cambridge University Press, 2005*

[MP10] P. Morters and Y. Peres, Brownian Motion. *Cambridge University Press, 2010*