REGULAR TRANSITION MATRICES

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1. Introduction

2. Theorems

Theorem 1 (Theorem 5.18 pg. 298)**.** Let $A \in M_{n\lambda n}(C)$ be a matrix in which each entry is positive and let λ be an eigenvalue of A such that $|\lambda| = \rho(A)$. Then $\lambda = \rho(A)$ and $\{u\}$ is a basis for E_λ, where $u \in C^n$ is the column vector in which each coordinate equals 1.

Proof. First, note that because A has all postitive values and u has all positive values, then Au has all postive values. Therefore, because $\lambda u = Au$ has all positive values we can conclude $\lambda > 0$ and $\lambda = |\lambda| = \rho(A)$

Now let v be an eigenvector of A corresponding to λ with coordinates $v_1, v_2, \ldots, v_n$. Then let $v_k = max(|v_1|, |v_2|, \ldots |v_n|)$ and $b = |v_k|$.

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Then

$$|\lambda| b = |\lambda||v_k| = |\lambda v_k|$$

But if λ is an eigenvalue of A, then $Av = \lambda v$ and thus $\forall 1 \leq i \leq n \lambda v_i = \sum_{j=1}^{n} A_{ij} v_j$. Thus

$$|\lambda v_k| = |\sum_{j=1}^{n} A_{kj} v_j|$$

By the triangle inequality and then multiplication rules,

$$|\sum_{j=1}^{n} A_{kj} v_j| \leq \sum_{j=1}^{n} |A_{kj} v_j| = \sum_{j=1}^{n} |A_{kj}||v_j|$$

Since we know $b = |v_k| \geq v_i \forall 1 \leq i \leq n$ and similarly $\rho(A) \geq \rho_i(A) \forall 1 \leq i \leq n$, we know

$$\sum_{j=1}^{n} |A_{kj}||v_j| \leq \sum_{j=1}^{n} |A_{kj}| b = b \sum_{j=1}^{n} |A_{kj}| = b\rho_k(A) = \rho_k(A) b \leq \rho(A) b$$

But since we know $|\lambda| = \rho(A)$, we know the three inequalities above are actually equalities.

(1) $|\sum_{j=1}^{n} A_{kj} v_j| = \sum_{j=1}^{n} |A_{kj} v_j|$
(2) $\sum_{j=1}^{n} |A_{kj}||v_j| = \sum_{j=1}^{n} |A_{kj}| b$
(3) $\rho_k(A) b = \rho(A) b$

But now we can use this to show that $\{u\}$ is a basis for E_λ

□

Corollary 1.1 (Corollary 1 pg. 299). Let $A \in M_{n \times n}(C)$ be a matrix in which each entry is positiive and let λ be an eigenvalue of A such that $|\lambda| = \nu(A)$. Then $\lambda = \nu(A)$ and E_λ has dimension 1.

Proof. Consider A^T. We know by execercise 14 of 5.1 in [FIS03] that A and A^T have the same eigenvalues. Let $E_\lambda, E_\lambda{}'$ be the eigenspaces of λ corresponding to A, A^T respectively. Additionally, if $|\lambda| = \nu(A)$, then the row corresponding to $\nu(A)$ is now a column in A^T such that $|\lambda| = \rho(A^T)$. Thus A^T is a matrix in which each entry is positive with an eigenvalue $\lambda = \rho(A^T)$. Thus by Theorem 5.18 [FIS03], the basis of $E_\lambda{}' = \{u\}$, where $u \in C^n$ is the column vector in which each coordinate contains 1 and $\lambda = \rho(A^T)$. Thus $dim(E_\lambda{}') = 1$.

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But by exercise 13 of 5.2 of [FIS03], we know $dim(E_\lambda) = dim(E_\lambda{}') = 1$ and $\nu(A) = \rho(A^T) = \lambda$. Thus, we have shown $\lambda = \nu(A)$ and $dim(E_\lambda) = 1$.

$\square$

Corollary 1.2 (Corollary 2 pg. 299). Let $A \in M_{n\times n}(C)$ be a transition matrix in which each entry is positiive and let λ be an eigenvalue of A such that $\lambda \neq 1$. Then $|\lambda| < 1$ and the eigenspace corresonding to the eigenvalue 1 has dimension 1.

Proof. We know by corollary 3 of Theorem 5.16 [FIS03] that if λ is an eigenvalue of a transition matrix, then $|\lambda| \leq 1$. Thus if $|\lambda| \neq 1$, then $|\lambda| < 1$.

If A is a transition matrix, then by Theorem 5.17 [FIS03] we know that 1 is an eigenvalue.

We also know that if A is a transition matrix, then given u as a column vector in which each coordinate equals 1, then $A^T u = u$. But since u is the column vector equal to 1, each u_i in u is equal to the sum along the columns of A^T. But because $u_i = 1 \forall i$, then $\nu_i(A) = 1 \forall i$ and thus $\nu(A) = 1$.

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Thus we have an all-positive-entry matrix with $1 = \lambda = \nu(A)$ and thus by Corollary 1 of Theorem 5.18 , we know $dim(E_\lambda) = 1$.

$\square$

Theorem 2 (Theorem 5.19 pg. 298)**.** Let A be a regular transition matrix and let λ be an eigenvalue of A. Then

(1) $|\lambda| \leq 1$
(2) If $|\lambda| = 1$, then $\lambda = 1$ and $dim(E_\lambda) = 1$.

Proof. We know (1) was proved in Corollary 3 of Theorem 5.16 [FIS03].

Since A is regular, we know by definition $\exists s > 0 \in \mathbb{Z}$ such that A^s has only positive entries Because A is a transition matrix and the entries of A^s are positive, we know the entries of $A^{s+1} = A^s(A)$ are positive Suupose $|\lambda| = 1$, then we know by by Probelem 15b of section 5.1 in [FIS03] that if λ is a eigenvalue of A, then λ^s, λ^{s+1} are eigenvalues of A, A^{s+1} respectively with $|\lambda^s| = |\lambda^{s+1}| = |\lambda| = 1$.

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Because each entry of A^s, A^{s+1} is positive, we know that for any eigenvalues $\lambda*$ of A^s, A^{s+1} such that $\lambda* \neq 1$, then $|\lambda| < 1$. Thus because $|\lambda^s| = |\lambda^{s+1}| = 1$, we can conclude $\lambda^s = \lambda^{s+1} = 1$ and therefore $\lambda = 1$.

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Let E_λ and $E_\lambda{}'$ be the eigenspaces of A, A^{s+1} respectively corresponding to $\lambda = 1$. Then $E_\lambda \subseteq E_\lambda{}'$ and because $dim(E_\lambda{}') = 1$, we know $dim(E_\lambda) = 1$.

$\square$

Corollary 2.1 (Corollary Pg. 300). Let A be a regular transition matrix that is diagonalizble. Then $\lim_{m\to\infty} A^m$ exists.

Proof. We know by Theorem 5.14 [FIS03] that if $A \in M_{n\times n}(C)$ is diagonalizable and has every eigenvalue contained in S, where $S = \{\lambda \in C : |\lambda| < 1 \text{ or } |\lambda| = 1\}$, then $\lim_{m\to\infty} A^m$ exists.

Thus because A is dianolizable by assumption, we just need to show for all eigenvalues λ of A, $\lambda \in S$. But we know by Theorem 5.19 $\forall$ eigenvalues λ that $\lambda = 1$ or $|\lambda| < 1$ and thus $\lambda \in S$. Thus we can conclude that $\lim_{m\to\infty} A^m$ exists.

$\square$

Theorem 3 (Theorem 5.20)**.** Let A be an $n \times n$ regular transition matrix. Then:

(1) The multiplicity of 1 as an eigenvalue of A is 1.
(2) $\lim_{m\to\infty} A^m$ exists.
(3) $L = \lim_{m\to\infty} A^m$ is a transition matrix.
(4) $LA = AL = L$.
(5) The columns of L are identical and equal to the unique probability vector v that is equal to the eigenvalue 1.
(6) For any probability vector w, $\lim_{m\to\infty} A^m w = v$.

Proof. (1) See exercise 20 of section 7.2
(2) 1, 5.13, 5.19

(3) By theorem 5.15 [FIS03], we know L is a transition matrix if $L^t u = u$, where u is the column vector where every entry is equal to 1 Due to the equivalence, we can know L is a transition matrix is $u^t L = u^t$. But

$$u^t L = u^t \lim_{m\to\infty} A^m = \lim_{m\to\infty} u^t A^m$$

But A^m is a transition matrix which means $\forall m u^t A^m = u^t$ and thus

$$\lim_{m\to\infty} u^t A^m = \lim_{m\to\infty} u^t = u^t$$

Thus L is a transition matrix.

(4) By Theorem 5.12 [FIS03], we know $AL = \lim_{m\to\infty} AA^m = \lim_{m\to\infty} A^{m+1} = \lim_{m\to\infty} A^m A = LA$.
But $\lim_{m\to\infty} A^{m+1} = \lim_{m\to\infty} A^m = L$.
Thus $LA = AL = L$.

(5) We know $AL = L$ by (4). Let L_i be the ith column vector of L. Then because $AL = L$, we know $AL_i = L_i \forall 1 \leq i \leq n$. Thus L_i is an eigenvector of A corresponding to eigenvalue $\lambda = 1$.

Additionally, because L is a transition matrix by (3), we know $L^t u = u$ and thus $\forall 1 \leq i \leq n L_i^t u = u$. Thus $\forall 1 \leq i \leq n L_i$ is a probability vector.

But then using (1), we know that the multiplicity of 1 as an eigenvalue of A is 1. Thus all the columns of L have to be a scalar multiple of the same vector. But because every column is a probability vector, they have to be the same in order to satisify $\forall 1 \leq i \leq n L_i^t u = u$. need to prove

Thus each column of L is equal to the the unique probability vector v corresponding to eigenvalue 1.

(6) Let w be any probability vector and $y = \lim_{m\to\infty} A^m w = Lw$. We want to show $y = v$. , If $y = Lw$, then by the corollary to Theorem 5.15 [FIS03], we know y is a probability vector.

Additionally,

$$Ay = A(Lw) = (AL)w = Lw$$

by part (4). But $Lw = y$ and thus $Ay = y$.

Therefore y is an probability vector and eigenvector of A corresponding to $\lambda = 1$ But v is the unique probability vector and eigenvector of A corresponding to $\lambda = 1$ and thus $y = v$.

Thus $\lim_{m\to\infty} A^m w = v$.

$\square$

3. Applications

References

[FIS03] S.H. Friedberg, A.J. Insel, and L.E. Spence. *Linear Algebra*. Featured Titles for Linear Algebra (Advanced) Series. Pearson Education, 2003.