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1. Introduction

Lemma 1. hello world

Theorem 2 (Theorem 5.18 pg. 298)**.** Let $A \in M_{n\lambda n}(C)$ be a matrix in which each entry is positive and let λ be an eigenvalue of A such that $|\lambda| = \rho(A)$. Then $\lambda = \rho(A)$ and $\{u\}$ is a basis for E_λ, where $u \in C^n$ is the column vector in which each coordinate equals 1.

Proof. First, note that because A has all postitive values and u has all positive values, then Au has all postive values. Therefore, because $\lambda u = Au$ has all positive values we can conclude $\lambda > 0$ and $\lambda = |\lambda| = \rho(A)$

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Now let v be an eigenvector of A corresponding to λ with coordinates $v_1, v_2, \ldots, v_n$. Then let $v_k = max(|v_1|, |v_2|, \ldots |v_n|)$ and $b = |v_k|$.

Then

$$|\lambda|b = |\lambda||v_k| = |\lambda v_k|$$

But if λ is an eigenvalue of A, then $Av = \lambda v$ and thus $\forall 1 \leq i \leq n \lambda v_i = \sum_{j=1}^n A_{ij} v_j$. Thus

$$|\lambda v_k| = |\sum_{j=1}^n A_{kj} v_j|$$

By the triangle inequality and then multiplication rules,

$$|\sum_{j=1}^n A_{kj} v_j| \leq \sum_{j=1}^n |A_{kj} v_j| = \sum_{j=1}^n |A_{kj}||v_j|$$

Since we know $b = |v_k| \geq v_i \forall 1 \leq i \leq n$ and similarly $\rho(A) \geq \rho_i(A) \forall 1 \leq i \leq n$, we know

$$\sum_{j=1}^n |A_{kj}||v_j| \leq \sum_{j=1}^n |A_{kj}|b = b \sum_{j=1}^n |A_{kj}| = b\rho_k(A) = \rho_k(A)b \leq \rho(A)b$$

But since we know $|\lambda| = \rho(A)$, we know the three inequalities above are actually equalities.

(1) $|\sum_{j=1}^n A_{kj} v_j| = \sum_{j=1}^n |A_{kj} v_j|$
(2) $\sum_{j=1}^n |A_{kj}||v_j| = \sum_{j=1}^n |A_{kj}|b$
(3) $\rho_k(A)b = \rho(A)b$

But now we can use this to show that $\{u\}$ is a basis for E_λ

□

Corollary 2.1 (Corollary 1 pg. 299). Let $A \in M_{n\times n}(C)$ be a matrix in which each entry is positiive and let λ be an eigenvalue of A such that $|\lambda| = \nu(A)$. Then $\lambda = \nu(A)$ and E_λ has dimension 1.

Proof. Consider A^T. We know by execercise 14 of 5.1 in [FIS03] that A and A^T have the same eigenvalues. Let $E_\lambda, E_\lambda{}'$ be the eigenspaces of λ corresponding to A, A^T respectively. Additionally, if $|\lambda| = \nu(A)$, then the row corresponding to $\nu(A)$ is now a column in A^T such that $|\lambda| = \rho(A^T)$. Thus A^T is a matrix in which each entry is positive with an eigenvalue $\lambda = \rho(A^T)$. Thus by Theorem 5.18 [FIS03], the basis of $E_\lambda{}' = \{u\}$, where $u \in C^n$ is the column vector in which each coordinate contains 1 and $\lambda = \rho(A^T)$. Thus $dim(E_\lambda{}') = 1$.

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But by exercise 13 of 5.2 of [FIS03], we know $dim(E_\lambda) = dim(E_\lambda{}') = 1$ and $\nu(A) = \rho(A^T) = \lambda$. Thus, we have shown $\lambda = \nu(A)$ and $dim(E_\lambda) = 1$.

□

Corollary 2.2 (Corollary 2 pg. 299). Let $A \in M_{n\times n}(C)$ be a transition matrix in which each entry is positiive and let λ be an eigenvalue of A such that $\lambda \neq 1$. Then $|\lambda| < 1$ and the eigenspace corresonding to the eigenvalue 1 has dimension 1.

Proof. We know by corollary 3 of Theorem 5.16 [FIS03] that if λ is an eigenvalue of a transition matrix, then $|\lambda| \leq 1$. Thus if $|\lambda| \neq 1$, then $|\lambda| < 1$.

If A is a transition matrix, then by Theorem 5.17 [FIS03] we know that 1 is an eigenvalue.

We also know that if A is a transition matrix, then given u as a column vector in which each coordinate equals 1, then $A^T u = u$. But since u is the column vector equal to 1, each u_i in u is equal to the sum along the columns of A^T. But because $u_i = 1 \forall i$, then $\nu_i(A) = 1 \forall i$ and thus $\nu(A) = 1$.

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Thus we have an all-positive-entry matrix with $1 = \lambda = \nu(A)$ and thus by Corollary 1 of Theorem 5.18 , we know $dim(E_\lambda) = 1$.

□

Theorem 3 (Theorem 5.19 pg. 298)**.** Let A be a regular transition matrix and let λ be an eigenvalue of A. Then

(1) $|\lambda| \leq 1$
(2) If $|\lambda| = 1$, then $\lambda = 1$ and $dim(E_\lambda) = 1$.

Proof. We know (1) was proved in Corollary 3 of Theorem 5.16 [FIS03].

Since A is regular, we know by definition $\exists s > 0 \in \mathbb{Z}$ such that A^s has only positive entries Because A is a transition matrix and the entries of A^s are positive, we know the entries of $A^{s+1} = A^s(A)$ are positive Suupose $|\lambda| = 1$, then we know by by Probelem 15b of section 5.1 in [FIS03] that if λ is a eigenvalue of A, then λ^s, λ^{s+1} are eigenvalues of A, A^{s+1} respectively with $|\lambda^s| = |\lambda^{s+1}| = |\lambda| = 1$.

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Because each entry of A^s, A^{s+1} is positive, we know that for any eigenvalues $\lambda*$ of A^s, A^{s+1} such that $\lambda* \neq 1$, then $|\lambda| < 1$. Thus because $|\lambda^s| = |\lambda^{s+1}| = 1$, we can conclude $\lambda^s = \lambda^{s+1} = 1$ and therefore $\lambda = 1$.

Let E_λ and $E_\lambda{}'$ be the eigenspaces of A, A^{s+1} respectively corresponding to $\lambda = 1$. Then $E_\lambda \subseteq E_\lambda{}'$ and because $dim(E_\lambda{}') = 1$, we know $dim(E_\lambda) = 1$.

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□

Corollary 3.1 (Corollary Pg. 300). Let A be a regular transition matrix that is diagonalizble. Then $\lim_{m\to\infty} A^m$ exists.

Proof. We know by Theorem 5.14 [FIS03] that if $A \in M_{n\times n}(C)$ is diagonalizable and has every eigenvalue contained in S, where $S = \{\lambda \in C : |\lambda| < 1 \text{ or } |\lambda| = 1\}$, then $\lim_{m\to\infty} A^m$ exists.

Thus because A is dianolizable by assumption, we just need to show for all eigenvalues λ of A, $\lambda \in S$. But we know by Theorem 5.19 $\forall$ eigenvalues λ that $\lambda = 1$ or $|\lambda| < 1$ and thus $\lambda \in S$. Thus we can conclude that $\lim_{m\to\infty} A^m$ exists.

□

2. Theorem 5.20

3. Explanation of Film

References

[FIS03] S.H. Friedberg, A.J. Insel, and L.E. Spence. *Linear Algebra*. Featured Titles for Linear Algebra (Advanced) Series. Pearson Education, 2003.