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1. Introduction

Lemma 1. hello world

Theorem 2 (Theorem 5.18 pg. 298)**.** Let $A \in M_{n\lambda n}(C)$ be a matrix in which each entry is positive and let λ be an eigenvalue of A such that $|\lambda| = \rho(A)$. Then $\lambda = \rho(A)$ and $\{u\}$ is a basis for E_λ, where $u \in C^n$ is the column vector in which each coordinate equals 1.

Proof. First, note that because A has all postitive values and u has all positive values, then Au has all postive values. Therefore, because $\lambda u = Au$ has all positive values we can conclude $\lambda > 0$ and $\lambda = |\lambda| = \rho(A)$

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Now let v be an eigenvector of A corresponding to λ with coordinates $v_1, v_2, \ldots, v_n$. Then let $v_k = max(|v_1|, |v_2|, \ldots |v_n|)$ and $b = |v_k|$.

Then
$$|\lambda|b = |\lambda||v_k| = |\lambda v_k|$$
But if λ is an eigenvalue of A, then $Av = \lambda v$ and thus $\forall 1 \leq i \leq n \lambda v_i = \sum_{j=1}^{n} A_{ij} v_j$. Thus
$$|\lambda v_k| = |\sum_{j=1}^{n} A_{kj} v_j|$$
By the triangle inequality and then multiplication rules,
$$|\sum_{j=1}^{n} A_{kj} v_j| \leq \sum_{j=1}^{n} |A_{kj} v_j| = \sum_{j=1}^{n} |A_{kj}||v_j|$$
Since we know $b = |v_k| \geq v_i \forall 1 \leq i \leq n$ and similarly $\rho(A) \geq \rho_i(A) \forall 1 \leq i \leq n$, we know
$$\sum_{j=1}^{n} |A_{kj}||v_j| \leq \sum_{j=1}^{n} |A_{kj}|b = b\sum_{j=1}^{n} |A_{kj}| = b\rho_k(A) = \rho_k(A)b \leq \rho(A)b$$
But since we know $|\lambda| = \rho(A)$, we know the three inequalities above are actually equalities.

(1) $|\sum_{j=1}^{n} A_{kj} v_j| = \sum_{j=1}^{n} |A_{kj} v_j|$
(2) $\sum_{j=1}^{n} |A_{kj}||v_j| = \sum_{j=1}^{n} |A_{kj}|b$
(3) $\rho_k(A)b = \rho(A)b$

But now we can use this to show that $\{u\}$ is a basis for E_λ

□

Corollary 2.1 (Corollary 1 pg. 299). Let $A \in M_{n\times n}(C)$ be a matrix in which each entry is positiive and let λ be an eigenvalue of A such that $|\lambda| = \nu(A)$. Then $\lambda = \nu(A)$ and E_λ has dimension 1.

Proof. Consider A^T. We know by execercise 14 of 5.1 in [FIS03] that A and A^T have the same eigenvalues. Let $E_\lambda, E_\lambda{}'$ be the eigenspaces of λ corresponding to A, A^T respectively. Additionally, if $|\lambda| = \nu(A)$, then the row corresponding to $\nu(A)$ is now a column in A^T such that $|\lambda| = \rho(A^T)$. Thus A^T is a matrix in which each entry is positive with an eigenvalue $\lambda = \rho(A^T)$. Thus by Theorem 5.18 [FIS03], the basis of $E_\lambda{}' = \{u\}$, where $u \in C^n$ is the column vector in which each coordinate contains 1 and $\lambda = \rho(A^T)$. Thus $dim(E_\lambda{}') = 1$.

define nu, rho
clarify

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But by exercise 13 of 5.2 of [FIS03], we know $dim(E_\lambda) = dim(E_\lambda{}') = 1$ and $\nu(A) = \rho(A^T) = \lambda$. Thus, we have shown $\lambda = \nu(A)$ and $dim(E_\lambda) = 1$.

□

Corollary 2.2 (Corollary 2 pg. 299). Let $A \in M_{n\times n}(C)$ be a transition matrix in which each entry is positiive and let λ be an eigenvalue of A such that $\lambda \neq 1$. Then $|lambda| < 1$ and the eigenspace corresonding to the eigenvalue 1 has dimension 1.

Proof. Exercise

□

2. Theorem 5.19

3. Theorem 5.20

4. Explanation of Film

References

[FIS03] S.H. Friedberg, A.J. Insel, and L.E. Spence. *Linear Algebra*. Featured Titles for Linear Algebra (Advanced) Series. Pearson Education, 2003.