Statistical Distributions on the Sphere and their Applications to Light Polarization

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Laboratory Overview

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[1]

- research center in Strasbourg
- engineering science, computer science, and imaging
- collaborative effort with the University of Strasbourg, CNRS, ENGEES, and INSA Strasbourg
- 650 members
-
    - the Imaging, Robotics, Remote Sensing, and Biomedical department (D-IRTS)
    - the Remote Sensing, Radiometry, and Optical Imaging team (TRIO)

Objectives

- **Polarization Fundamentals :** Study light polarization concepts and relevant statistical distributions.
- **Statistical Modeling :** Implement and integrate spherical distributions in statistical software.
- **Data Analysis :** Analyze polarization data using statistical models and visualize on the Poincaré sphere.
- **Practical Applications :** Apply these methods in real-world scenarios like image analysis and material characterization.

Introduction to Polarization

What is polarization ?

An electric field propagating along the z axis is described by the vector : $E = (E_x \exp(i\omega t), E_y \exp(i\omega t), 0)^T$.

- In unpolarized light, these vibrations occur in all directions perpendicular to the direction of travel.
- Polarization occurs when these vibrations are restricted to a specific direction.
- Polarization is used in photography, LCD screens, and in scientific measurements to analyze light properties and materials.

Introduction to Polarization

What is polarization ?

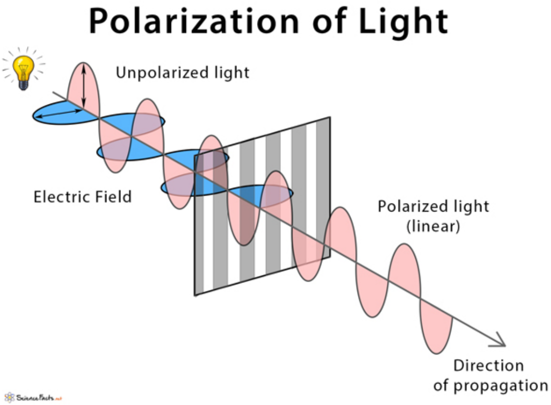

Figure: Diagram illustrating the concept of light polarization.[2]

Introduction to Polarization

Stokes vectors

[7] Over time, the total polarization state describes an ellipse.

- λ is the *azimuth*
- ϵ is defined as $\tan(\epsilon) = \frac{B}{A}$
- The *Stokes vectors* $S = (I, Q, U, V)^T$ describes the polarization state :

$$\begin{cases} I &= A^2 + B^2 \\ Q &= (A^2 - B^2)\cos(2\lambda) \\ U &= (A^2 - B^2)\sin(2\lambda) \\ V &= 2ABh, \ h = \mathrm{sgn}(V) \end{cases}$$

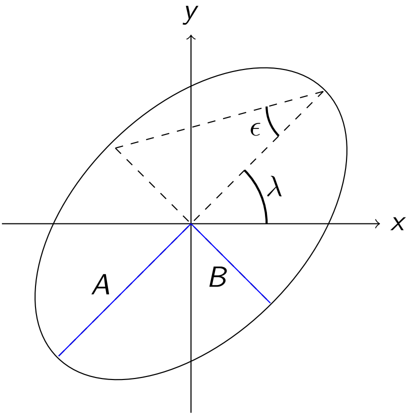

Figure: Definition of the azimuth λ, ϵ and the semi major A and minor B axis of the ellipse.

The degree of polarization (DOP) is define :

$$\boxed{DOP(S) = \frac{\sqrt{Q^2+U^2+V^2}}{I} \leq 1}, \; S = (I, Q, U, V)^T.$$

Remark

If $DOP(S) = 1$, the light is totally polarized.

Remark

If we normalize S by I : $S_{norm} = \left(1, \frac{Q}{I}, \frac{U}{I}, \frac{V}{I}\right)^T$, the expression of the DOP is the norm of the vector of the last three coordinates of S.

Introduction to Polarization

The Poincaré sphere

$$S_{norm} = \left(1, \frac{Q}{I}, \frac{U}{I}, \frac{V}{I}\right)^T, \; DOP(S) = \frac{\sqrt{Q^2+U^2+V^2}}{I}.$$

- $DOP = 1$, totally polarized. (outline on the sphere)
- $0 \leq DOP < 1$, partially polarized. (inside the sphere)

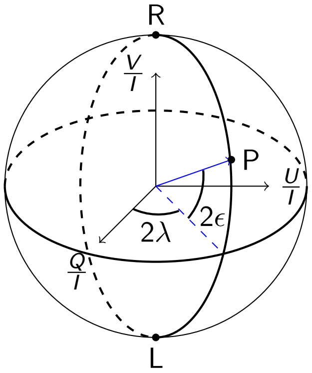

Figure: Poincaré sphere with the definition of the direction vectors, 2λ and 2ϵ for a random point P on the sphere. [5]

Directional statistics

von Mises-Fisher distribution, on 2-Sphere (Theory)

Density function : $\boxed{\begin{aligned} f_3(x;\mu,\kappa) &= C_3(\kappa)\exp(\kappa\mu^T x) \\ &= \tfrac{\kappa}{4\pi\sinh(\kappa)}\exp(\kappa\mu^T x) \end{aligned}}$ [6]

$$\begin{cases} \kappa & \geq & 0\text{, the concentration parameter} \\ \|\mu\| & = & 1\text{, the mean direction} \\ C_3(\kappa) & , & \text{the normalization constant} \end{cases}$$

Directional statistics

von Mises-Fisher distribution, on 2-Sphere (Theory)

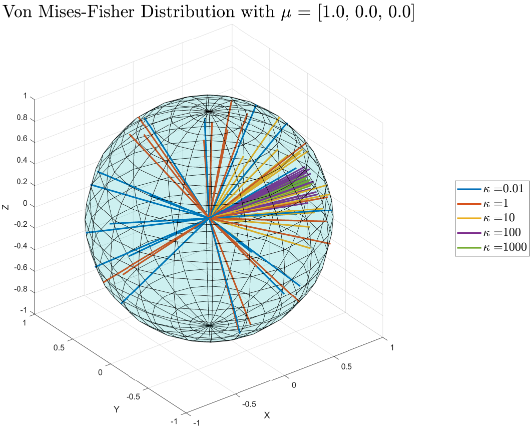

Figure: The Poincare Sphere with vectors generate by the VMF distribution for $\mu = (1; 0; 0)$, $\kappa \in \{0.01, 1, 10, 100, 1000\}$ and $n = 20$.

Directional statistics

von Mises-Fisher distribution, on 2-Sphere (Link with the DOP)

Reminder : If we normalize S by I : $S_{norm} = \left(1, \frac{Q}{I}, \frac{U}{I}, \frac{V}{I}\right)^T$, the expression of the DOP is the norm of the vector of the last three coordinates of S.

The expression of the degree of polarization as a function of κ :

$$\boxed{DOP(\kappa) = \frac{1}{\tanh(\kappa)} - \frac{1}{\kappa}} \tag{1}$$

, for vectors following VMF distribution.

Directional statistics

von Mises-Fisher distribution, on 2-Sphere (Link with the DOP)

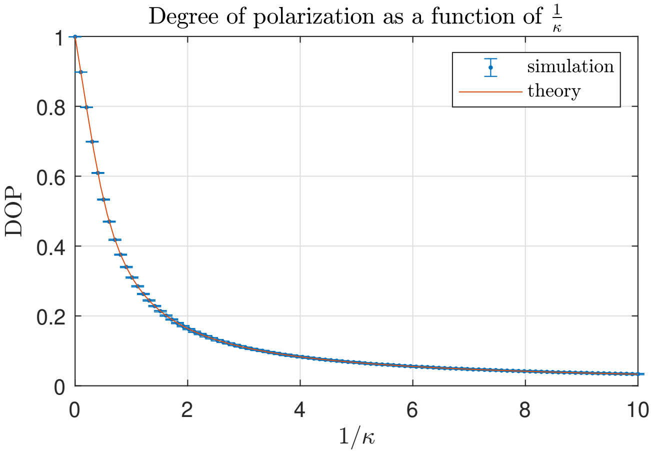

Figure: The DOP as a function of $\frac{1}{\kappa}$ for $\mu = (1;0;0)$ and $\kappa \in [0.1;1000]$ and $(n,t,s) = (100{,}100{,}100000)$.

Directional statistics

von Mises-Fisher distribution, on 2-Sphere (Angular error)

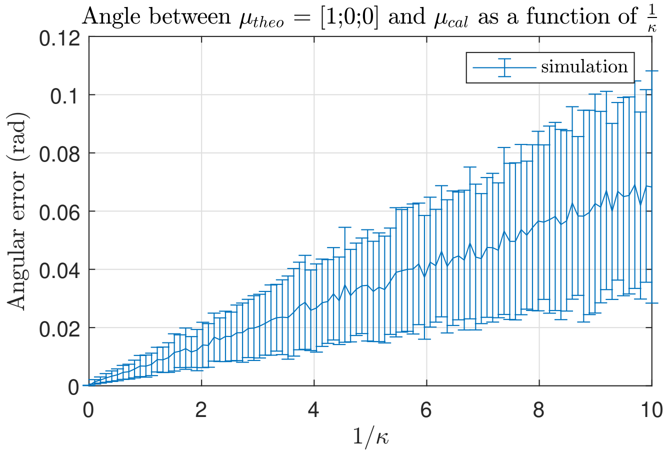

Figure: The angular error as a function of $\frac{1}{\kappa}$, for $\mu = (1; 0; 0)$ and $\kappa \in [0.1; 1000]$ and $(n, t, s) = (100, 100, 100000)$.

Mueller imager and Segmentation

Mueller imager

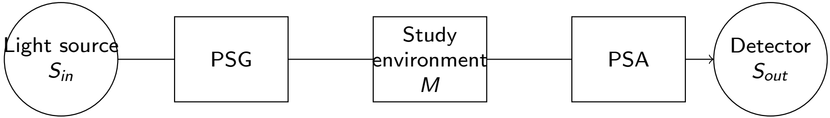

Figure: Diagram of a Mueller polarimetric imaging system.
PSG = Polarization State Generator and PSA = Polarization State Analyzer. [3]

$$\begin{cases} S_{out} & = & M.S_{in},\ M \text{ Mueller matrix} \\ I_{in} & = & A.S_{in},\ A \text{ the vertices of a regular tetrahedron} \\ I_{in,meas} & = & I_{in} + \epsilon,\ \epsilon \text{ the noise in each pixel} \end{cases}$$

So : $S_{in,meas} = A^{-1}.I_{in,meas}$

Mueller imager and Segmentation

`Stokes_segmentation.m`

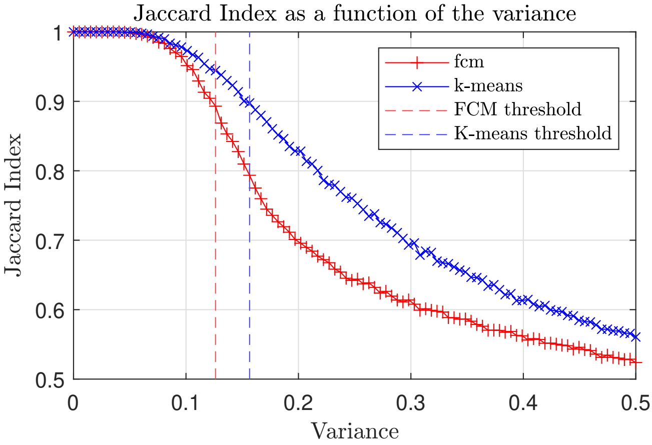

Figure: The Jaccard Index as a function of the variance for vectors $(S_{z_0}, S_{z_1}, S_{z_2}) = ((1,1,0,0)^T, (1,0,1,0)^T, (1,0,0,1)^T)$ with thresholds for which the Jaccard index is less than 0.9.

Mueller imager and Segmentation

Stokes_segmentation.m

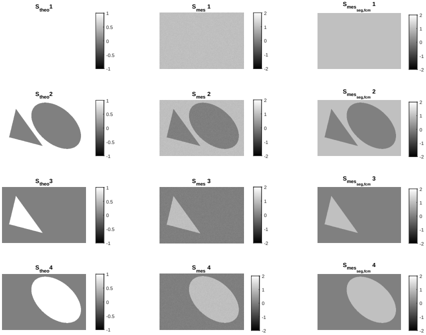

Figure: The coordinate of each pixel theoretical, measured (with noise) and after application of the fcm segmentation for vectors $(S_{z_0}, S_{z_1}, S_{z_2}) = ((1,1,0,0)^T, (1,0,1,0)^T, (1,0,0,1)^T)$ and a variance of 0.01.

Mueller imager and Segmentation

`Stokes_segmentation.m`

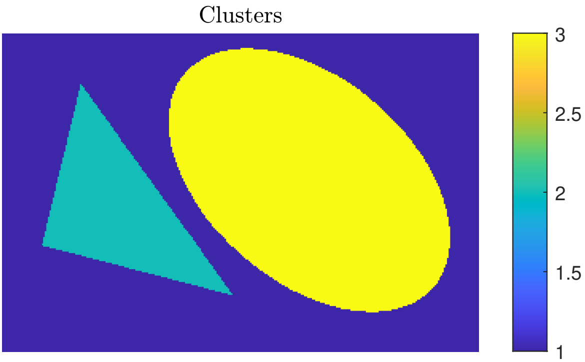

Figure: The clusters selects for each pixel by the method fcm for vectors $(S_{z_0}, S_{z_1}, S_{z_2}) = ((1,1,0,0)^T, (1,0,1,0)^T, (1,0,0,1)^T)$ and a variance of 0.01.

Mueller imager and Segmentation

Stokes_segmentation.m

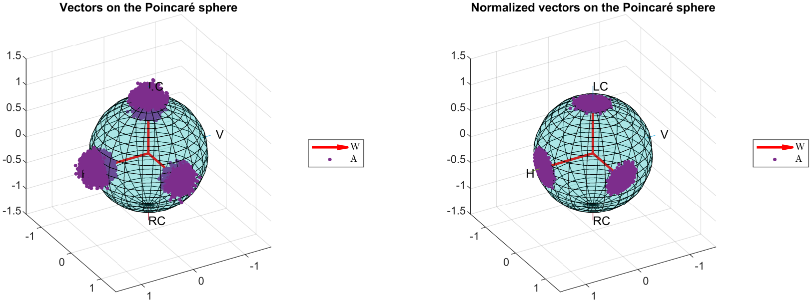

Figure: The clusters selects for each pixel by the method fcm for vectors $(S_{z_0}, S_{z_1}, S_{z_2}) = ((1,1,0,0)^T, (1,0,1,0)^T, (1,0,0,1)^T)$ and a variance of 0.01.

Mueller imager and Segmentation

`Stokes_segmentation.m` (Link with the DOP)

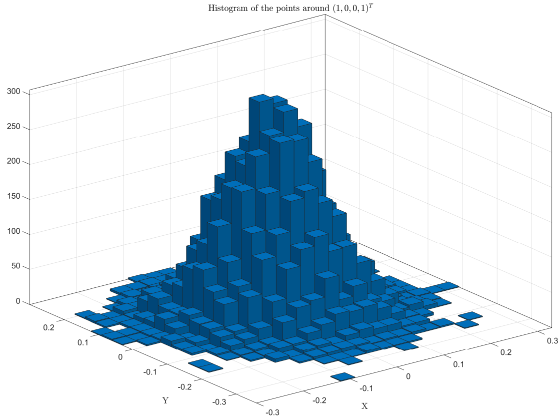

Figure: The histogram of the points around $(1,0,0,1)^T$ for vectors $(S_{z_0}, S_{z_1}, S_{z_2}) = ((1,1,0,0)^T, (1,0,1,0)^T, (1,0,0,1)^T)$ and a variance of 0.01.

Mueller imager and Segmentation

`Stokes_segmentation.m` (Link with the DOP)

On Matlab by doing :

1. `[XX,YY] = meshgrid((Yedges(1:end-1)+Yedges(2:end))/2, (Xedges(1:end-1)+Xedges(2:end))/2);`
   Calculating the centers of each bin consecutive for x and y axis.
2. `M = N/sum(N(:));`
   Transforms frequency probabilities by normalizing frequencies by the total number of frequencies.
3. Using curve fitting, for $(x, y, z) = (XX, YY, M)$ and an expression model curve : $\frac{0.02*0.02*k}{4*\pi*\sinh(k)} \exp(k * \sqrt{1 - x^2 - y^2})$.
   0.02 is the step in x and y.

We found a R-square value of 0.97 and $k = 160$.

Conclusion

- Found the formula for the Degree of Polarization (DOP) for vectors following a Von Mises-Fisher (VMF) distribution.
- Applied fcm segmentation to noisy images, normalized data, and confirmed the VMF distribution with a high correlation coefficient.

Next Steps

- Plan to analyze real polarized images, focusing on the relationship between variance and angles of Stokes vectors for better segmentation.
- Aim to improve segmentation accuracy by addressing outliers on the Poincaré sphere, enhancing data reliability.

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