Adaptive Implicit Schemes for Hyperbolic Equations

Antoine Regardin – Supervisors: Andrea Thomann, Emmanuel Franck

August 21, 2024

Introduction

Goal: Study a new scheme for discontinuous problems of hyperbolic PDEs, especially in cases of long time simulation and high CFL conditions: the Self-Adaptive Theta scheme.

- Discontinuous hyperbolic problems, for example Riemann problems, are used to modelize many phenomena in physics.
- Standard Euler methods do not behave well when applied to these problems, especially with high CFl conditions.
- The idea is to upgrade the standard Theta scheme into a method that is adapting to discontinuities.

This scheme has been imagined by Pr. Todd Arbogast and Pr. Chieh-Sen Huang, and we based our work on their article from 2021 [1].

Introduction

We will focus on hyperbolic conservation laws:

$$\partial_t u + \partial_x f(u) = 0 \tag{1}$$

And we will study the linear advection, Burgers and RIPA equations with various initial conditions.

Methods

We will present a very short sum of the theory and building of the scheme.
One important point is the Discontinuity-Aware Quadrature, a way to follow and adapt to the displacement of a discontinuity in space and **time**.
Working on a cell of our space-time mesh, we want to find an approximation τ^* of the time coordinate where our perturbation will appear.

$$v(t) = \begin{cases} v^0 + v_L(t), & 0 \leq t < \tau, \\ v^1 + v_R(t), & \tau < t \leq \Delta t. \end{cases} \tag{2}$$

Methods

With $g(t, v(t))$ a smooth function and $\tilde{v} = \frac{1}{\Delta t} \int_0^{\Delta t} v(t)dt$:

$$\int_0^{\Delta t} g(t, v(t))dt \approx \int_0^{\tau^*} g(t, v^0)dt + \int_{\tau^*}^{\Delta t} g(t, v^1)dt \qquad (3.1)$$

With $g(t, v(t)) = v(t)$, we have:

$$\Delta t \tilde{v} = \int_0^{\Delta t} v(t)dt \approx \tau^* v^0 + (\Delta t - \tau^*)v^1 \qquad (3.2)$$

Which gives us, assuming $v^0 \neq v^1$:

$$\tau^* = \frac{v^1 - \tilde{v}}{v^1 - v^0} \Delta t \qquad (3.3)$$

For the case $v^0 = v^1$, we assume $\tau^* = \frac{\Delta t}{2}$.

Methods: introducing the parameter θ

As θ is a balance between implicit and explicit parts, we use our quadrature to adjust it to the propagation of the discontinuity.

$$\theta = 1 - \frac{\tau^*}{\Delta t} = \frac{\tilde{v} - v^0}{v^1 - v^0} \tag{3.4}$$

$$\int_0^{\Delta t} f(v(t))dt \approx (f^0 + \theta(f^1 - f^0))\Delta t \tag{3.5}$$

Methods: Linear advection and Upstream scheme

For the Linear advection equation, our flux f in (1) is $f(u) = au$, a constant. As f is linear, we can use an upstream weighting:

$$\begin{cases} \bar{u}_i^{n+1} = \bar{u}_i - \frac{1}{\Delta x} \int_{t^n}^{t_{n+1}} (\bar{f}_i - \bar{f}_{i-1}) dt, \\ \tilde{\bar{u}}_i^{n+1} = \bar{u}_i - \frac{1}{\Delta x} \int_{t^n}^{t_{n+1}} (\bar{f}_i - \bar{f}_{i-1}) \frac{t^{n+1} - t^n}{\Delta t^{n+1}} dt \end{cases} \tag{4}$$

$$\begin{cases} \bar{u}_i^{n+1} = \bar{u}_i - \frac{\Delta t^{n+1}}{\Delta x} [\bar{f}_i^n + \theta_i^{n+1} (\bar{f}_i^{n+1} - \bar{f}_i^n) - \bar{f}_{i-1}^n - \theta_{i-1}^{n+1} (\bar{f}_{i-1}^{n+1} - \bar{f}_{i-1}^n)], \\ \tilde{\bar{u}}_i^{n+1} = \bar{u}_i - \frac{\Delta t^{n+1}}{\Delta x} [\bar{f}_i^n + (\theta_i^{n+1})^2 (\bar{f}_i^{n+1} - \bar{f}_i^n) - \bar{f}_{i-1}^n - (\theta_{i-1}^{n+1})^2 (\bar{f}_{i-1}^{n+1} - \bar{f}_{i-1}^n)] \end{cases} \tag{5}$$

Methods: Choice of θ

From (3.4), we make this function for the choice of θ_i^n:

$$\theta_i^{n+1} = \begin{cases} \min(\max(\theta_{min}, \frac{w_i^{n+1}}{v_i^{n+1}}), 1) & \text{if } |w_i^{n+1}| > \epsilon. \\ \theta^* & \text{else} \end{cases} \tag{6}$$

Methods: Lax-Friedrichs

What to do with nonlinear fluxes? Lax-Friedrichs weighting.

$$\alpha = \max_u |f'(u)|.$$

We use the following numerical flux:

$$\hat{f}(u^-, u^+) = \frac{1}{2}[f(u^-) + f(u^+) - \alpha(u^+ - u^-)]. \tag{7.1}$$

$$\begin{cases} \bar{u}_i^{n+1} = \bar{u}_i^n - \frac{\Delta t^{n+1}}{2\Delta x_i}[\bar{f}_{i+1}^n + \theta_{i+1}^{n+1}(\bar{f}_{i+1}^{n+1} - \bar{f}_{i+1}^n) - \bar{f}_{i-1}^n - \theta_{i-1}^{n+1}(\bar{f}_{i-1}^{n+1} - \bar{f}_{i-1}^n) \\ \quad - \alpha[\bar{u}_{i+1}^n + \theta_{i+1}^{n+1}(\bar{u}_{i+1}^{n+1} - \bar{u}_{i+1}^n) - 2\bar{u}_i^n - 2\theta_i^{n+1}(\bar{u}_i^{n+1} - \bar{u}_i^n) \\ \quad + \bar{u}_{i-1}^n + \theta_{i-1}^{n+1}(\bar{u}_{i-1}^{n+1} - \bar{u}_{i-1}^n)]], \\ \\ \tilde{\bar{u}}_i^{n+1} = \bar{u}_i^n - \frac{\Delta t^{n+1}}{4\Delta x_i}[\bar{f}_{i+1}^n + (\theta_{i+1}^{n+1})^2(\bar{f}_{i+1}^{n+1} - \bar{f}_{i+1}^n) - \bar{f}_{i-1}^n - (\theta_{i-1}^{n+1})^2(\bar{f}_{i-1}^{n+1} - \bar{f}_{i-1}^n) \\ \quad - \alpha[\bar{u}_{i+1}^n + (\theta_{i+1}^{n+1})^2(\bar{u}_{i+1}^{n+1} - \bar{u}_{i+1}^n) - 2\bar{u}_i^n - 2(\theta_i^{n+1})^2(\bar{u}_i^{n+1} - \bar{u}_i^n) \\ \quad + \bar{u}_{i-1}^n + (\theta_{i-1}^{n+1})^2(\bar{u}_{i-1}^{n+1} - \bar{u}_{i-1}^n)]] \end{cases} \tag{7.2}$$

Methods: Burgers and RIPA

We present the fluxes of the Burgers and RIPA [2] equations:

$$f(u) = \frac{u^2}{2} \tag{Burgers}$$

$$f(W) = \begin{pmatrix} hu \\ hu^2 + g\Theta h^2/2 \\ h\Theta u \end{pmatrix} \tag{RIPA}$$

$W = (h, hu, h\Theta)$

Methods: Computing the θ_i

We write $v_i^{n+1} = \bar{u}_i^{n+1} - \bar{u}_i^n$ and $w_i^{n+1} = \tilde{\bar{u}}_i^{n+1} - \bar{u}_i^n$.
For the upstream-SATh (same for the L-F SATh), we iterate over k the following functions in a Newton method:

$$\begin{cases} v_i^k + \frac{\Delta t}{\Delta x}[\bar{f}_i^{k-1} + \theta_i^{k-1}(\bar{f}_i^k - \bar{f}_i^{k-1}) - \bar{f}_{i-1}^{k-1} - \theta_{i-1}^{k-1}(\bar{f}_{i-1}^k - \bar{f}_{i-1}^{k-1})] = 0, \\ w_i^k + \frac{\Delta t}{\Delta x}[\bar{f}_i^{k-1} + (\theta_i^{k-1})^2(\bar{f}_i^k - \bar{f}_i^{k-1}) - \bar{f}_{i-1}^{k-1} - (\theta_{i-1}^{k-1})^2(\bar{f}_{i-1}^k - \bar{f}_{i-1}^{k-1})] = 0 \end{cases} \tag{8}$$

We use the Newton-Krylov algorithm with the function *scipy.optimize.newton_krylov*

Results: Linear Advection

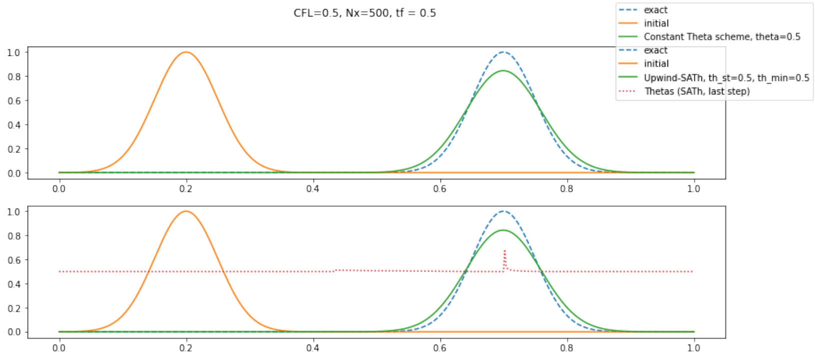

Figure: Smooth bell initial condition, with CFL=0.5 and theta paremeters =0.5, for both constant theta and SATh schemes.

Results: Linear Advection

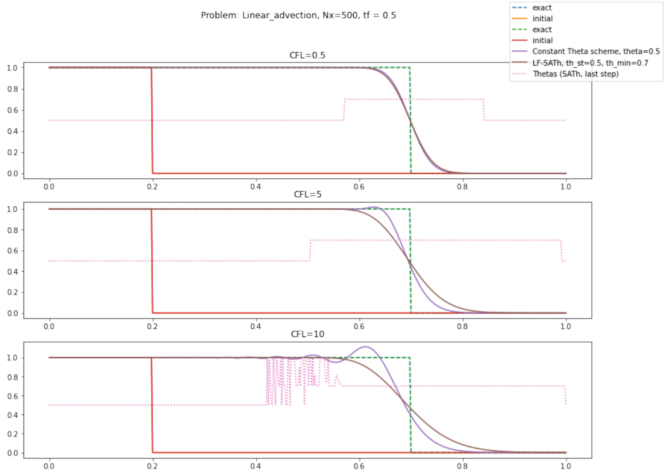

Figure: Riemann problems, various CFL conditions, compared to Crank-Nicholson solution.

Results: Linear Advection

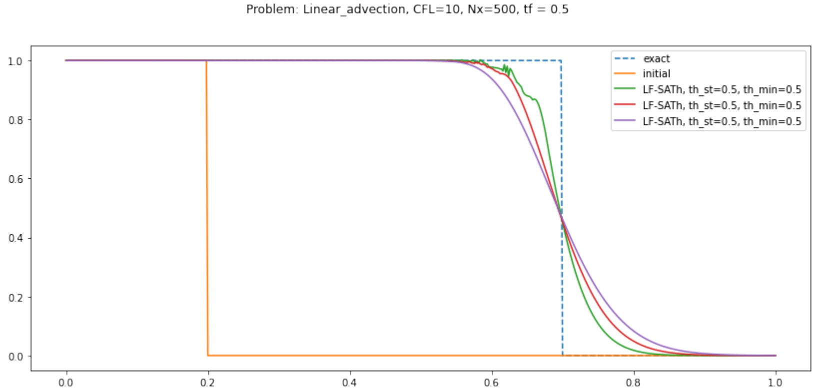

Figure: We notice that when $\theta_{min} = 0.5$ (with $\theta^* = 0.5$), a lot of small oscillations appear. When using higher values, they smear out, but the solutions are more dissipated. Changing the value of ϵ in a range from $1e-3$ to $1e-100$ in (6) does not suppress these oscillations. As this problem is not yet solved, we will plot with $\theta_{min} > 0.5$, but this should be adressed in the future.

Results: Burgers

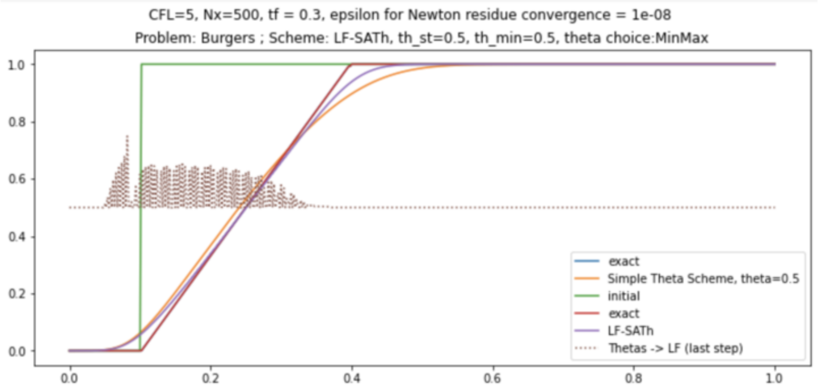

Figure: Simulation for Burgers with one jump initial condition.

Results: Burgers

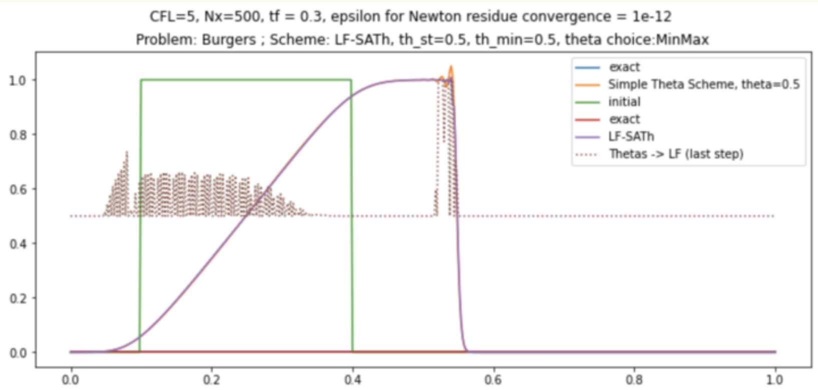

Figure: Simulation for Burgers with the shocks and rarefaction condition.

Results: Burgers

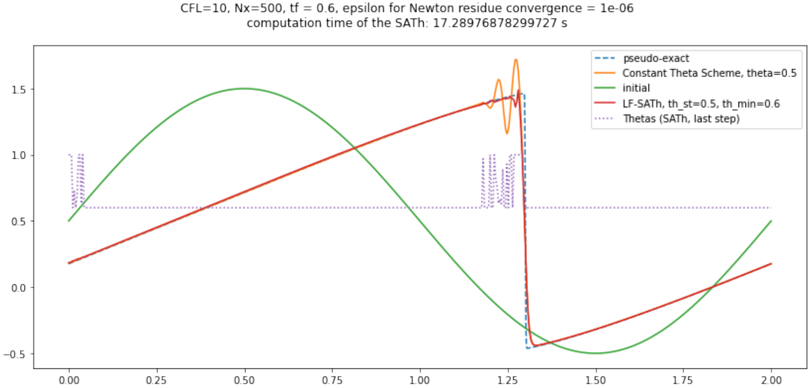

Figure: Simulation for Burgers with the shock creatin condition (with periodical boundary conditions)

Resutls: RIPA model

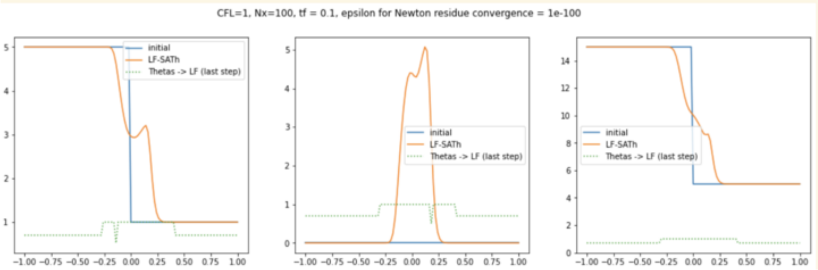

Figure: The first result we had for the RIPA model.

A last experiment

What happens if we change the θ choice function to allow it to go greater than one? We can define $\theta_{max} > 1$:

$$\theta_i^{n+1} = \begin{cases} \min(\max(\theta_{min}, \frac{w_i^{n+1}}{v_i^{n+1}}), \theta_{max}) & \text{if } |w_i^{n+1}| > \epsilon. \\ \theta^* & \text{else} \end{cases} \tag{6bis}$$

If $\theta_i^{n+1} > 1$, that would mean we have $\frac{w_i^{n+1}}{v_i^{n+1}} > 1$, thus $\tilde{u}_i^{n+1} > u_i^{n+1}$. This is still to be studied.

A last experiment

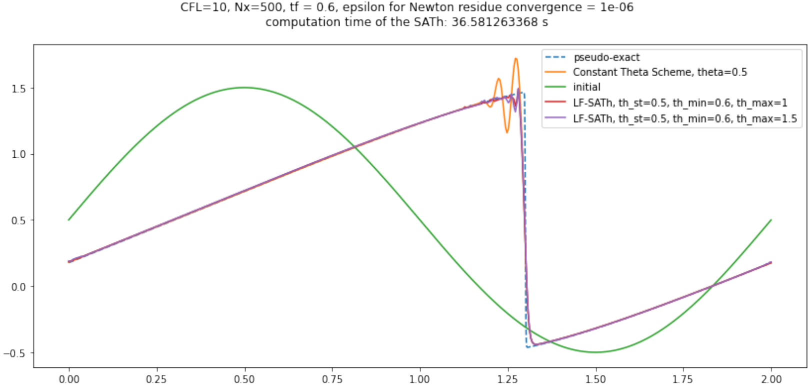

Figure: L2 errors: $\theta_{max} = 1$: 1.20969, $\theta_{max} = 1.5$: 1.20224, inf errors: $\theta_{max} = 1$: 0.80577, errors: $\theta_{max} = 1.5$: 0.78618.

- Our implementation of the scheme needs to be improved. Solutions are not precise and smooth enough.
- Newton-krylov may be not suited (very small oscillations and not compatible with discontinuous behaviour), but still better than the other methods we tried.
- The SATh still seems very promising for handling discontinuities and bad CFL conditions.
- There were (and still are) many issues with the code that have been slowing us and leading to dead ends.

Bibliography

- Arbogast, Presentation support: *Self Adaptive Theta (SATh) Schemes for Solving Hyperbolic Conservation Laws*, 2024
- Arbogast and Huang, *A Self-Adaptive Theta Scheme using discontinuity aware quadrature for solving conservation laws*, 2021.
- Desveaux et al., *Well-balanced Schemes to capture non-explicit steady states: RIPA Model*, 2016.