Technische Universiteit Eindhoven
University of Technology
Department of Mechanical Engineering

Advanced motion control (4CM60)

Exercise set 1

Student name:	**Student ID:**
Tom van de Laar	1265938
Job Meijer	1268155

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1 Exercise 1.1

In this exercise the spectral radius and five matrix norms listed below of the following matrices are determined:

$$G_1 = \begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix} \quad G_2 = \begin{bmatrix} 0 & 100 \\ 0 & 0 \end{bmatrix}$$

The spectral radius is determined by taking the largest eigenvalue of the matrix, resulting in: $\lambda_{max}(G1) = 7.27$, $\lambda_{max}(G2) = 0$

The five matrix norms that are determined:
1. Frobenius matrix norm: $||G||_F : \sqrt{\Sigma_{i,j} = |a_{i,j}|^2} \longrightarrow ||G_1||_F = 7.35, ||G_2||_F = 100$
2. Sum matrix norm: $||G||_{sum} : \Sigma_{i,j} = |a_{i,j}| \longrightarrow ||G_1||_{sum} = 14, ||G_2||_{sum} = 100$
3. Maximum colmn sum: $||G||_{i1} : max_j(\Sigma_i|a_{ij}|) \longrightarrow ||G_1||_{i1} = 8, ||G_2||_{i1} = 100$
4. Maximum row sum: $||G||_{i\infty} : max_i(\Sigma_j|a_{ij}|) \longrightarrow ||G_1||_{i\infty} = 9, ||G_2||_{i\infty} = 100$
5. Maximum singular value: $||G||_{i2} = \overline{\sigma}(G) \longrightarrow \overline{\sigma}(G_1) = 7.35, \overline{\sigma}(G_2) = 100$

2 Exercise 1.2

There are six different pairing possibilities with the given 3x3 plant G. The RGA number indicates which pairing has the least coupling.Therefore, each pairing has a RGA number which can be calculated by subtrackting 1 from the selected pairing. For diagonal pairing equation 2.1 is used.

$$G = \begin{bmatrix} 16.8 & 30.5 & 4.30 \\ -16.7 & 31.0 & -1.41 \\ 1.27 & 54.1 & 5.40 \end{bmatrix} \quad \Lambda(G) = \begin{bmatrix} 1.50 & 0.99 & -1.48 \\ -0.41 & 0.97 & 0.45 \\ -0.08 & -0.95 & 2.03 \end{bmatrix}$$

$$||\Lambda(G) - I||_{sum} \tag{2.1}$$

The pairing with the least coupling is determined using the RGA numbers. The smallest RGA number is the "best" pairing. The diagonal pairing is the "best" pairing following from table 2.1.

Pairing	u1	u2	u3	RGA number
1	y1	y2	y3	**5.92**
2	y1	y3	y2	8.96
3	y2	y1	y3	7.89
4	y2	y3	y1	11.86
5	y3	y1	y2	8.99
6	y3	y2	y1	9.93

Table 2.1: Table RGA numbers of the pairings

3 Exercise 1.3

A lower triangular $m \times m$ matrix A is considered with $a_{ii} = -1$, $a_{ij} = 1$ for all $i > j$, and $a_{ij} = 0$ for all $i < j$.

a) What is det(A)?
The determinant of A is -1 when m is an uneven number and 1 when m is an even number.

b) What are the eigenvalues of A?
The eigenvalues of A are -1 with an multiplicity of m.

c) What is the RGA of A?
The RGA of A is an identity matrix with size m.

d) Now $m = 4$, finding an E with the smallest value of $\overline{\sigma}(E)$ such that $A + E$ is singular
The smallest value of $\overline{\sigma}(E) = 1$ while making $A + E$ singular is obtained with:

Computing the singular value decomposition (SVD) of A, resulting in $[U, \Sigma, V] = svd(A)$. Next the lowest singular value of A ($\underline{\sigma}(A)$) is determined from the diagonal matrix Σ. Also the corresponding input and output vectors of $\underline{\sigma}(A)$ are determined by $\underline{u}$ and $\underline{v}$. Now E can be computed by:

$$E = -\underline{u} \cdot \underline{\sigma}(A) \cdot \underline{v}^T = \begin{bmatrix} 0.0244 & 0.0368 & 0.0679 & 0.1355 \\ 0.0124 & 0.0187 & 0.0345 & 0.0679 \\ 0.0067 & 0.0101 & 0.0187 & 0.0368 \\ 0.0045 & 0.0067 & 0.0124 & 0.0244 \end{bmatrix}$$

This results in the maximum singular value of E being $\overline{\sigma}(E) = 0.1826$ and $det(A + E) = 0$ and therefore the system is singular.

4 Exercise 1.4

Now the system G is considered:

$$G = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

a) Computing the RGA results and interpretation:
The matrix G is singular therefore, a regular inverse cannot be used. Furthermore, the MoorePenrose inverse is used to approximate the inverse, in MATLAB (pinv). Eventually the equation 4.1 is used to compute the RGA values. The resulting values are evenly distributed therefore, the system is evenly coupled.

$$\Lambda(G) = G \times (G^{-1})^T \tag{4.1}$$

$$\Lambda(G) = \begin{bmatrix} 0.25 & 0.25 \\ 0.25 & 0.25 \end{bmatrix}$$

b) What happens when disturbance δ is added to the first element (1,1) of G?
If a disturbance is added the matrix G becomes non-singular again. Furthermore, if δ is increased the RGA converges to identity, see 4.1. Therefore, if δ goes to ∞ the RGA becomes identity (I). Which, implies that the system is perfectly decoupled. The disturbance only acts on one transfer function and is significantly higher than the transfer function itself.

δ	$\Delta(G)$
0	$\begin{bmatrix} 0.25 & 0.25 \\ 0.25 & 0.25 \end{bmatrix}$
1	$\begin{bmatrix} 2.00 & -1.00 \\ -1.00 & 2.00 \end{bmatrix}$
3	$\begin{bmatrix} 1.33 & -0.33 \\ -0.33 & 1.33 \end{bmatrix}$
10	$\begin{bmatrix} 1.10 & -0.10 \\ -0.10 & 1.10 \end{bmatrix}$
100	$\begin{bmatrix} 1.01 & -0.01 \\ -0.01 & 1.01 \end{bmatrix}$

Table 4.1: Disturbance values with corresponding RGA values

5 Exercise 1.5

Now the systems $G_a s$ and $G_b(s)$ are considered:

$$G_a(s) = \begin{bmatrix} G_1(s) & G_2(s) \\ G_1(s) & G_2(s) \end{bmatrix} \quad G_b(s) = \begin{bmatrix} G_1(s) & G_2(s) \\ G_2(s) & G_1(s) \end{bmatrix}$$

Where $G_1(s)$ and $G_2(s)$ are scalar.

The eigenvalue decompositions for both sytems $G_a(s)$ and $G_b(s)$ are:

$$det(G_a(s) - \lambda I) = 0 \longrightarrow \quad \begin{bmatrix} G_1(s) - \lambda & G_2(s) \\ G_1(s) & G_2(s) - \lambda \end{bmatrix} \quad \lambda^2 - G_1(s)\lambda - G_2(s)\lambda = 0$$

Resulting in $\lambda_1 = 0$ and $\lambda_2 = G_1(s) + G_2(s)$.

$$det(G_b(s) - \lambda I) = 0 \longrightarrow \quad \begin{bmatrix} G_1(s) - \lambda & G_2(s) \\ G_2(s) & G_1(s) - \lambda \end{bmatrix} \quad \lambda^2 - 2G_1(s))\lambda + G_1(s)^2 - G_2(s)^2 = 0$$

Resulting in $\lambda_{1,2} = G_1(s) \pm G_2(s)$.

This results in the eigenvalue matrices Λ_a and Λ_b and the eigenvector matrices V_a and V_b:

$$\Lambda_a = \begin{bmatrix} 0 & 0 \\ 0 & G_1(s) + G_2(s) \end{bmatrix} \quad \Lambda_b = \begin{bmatrix} G_1(s) - G_2(s) & 0 \\ 0 & G_1(s) + G_2(s) \end{bmatrix}$$

$$V_a = \begin{bmatrix} -G_2(s)/G_1(s) & 1 \\ 1 & 1 \end{bmatrix} \quad V_b = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

Also must be noted that $G_a(s)V_a = V_a\Lambda_a$ and $G_b(s)V_b = V_b\Lambda_b$. Now the systems $G_a(s)$ and $G_b(s)$ are decoupled by using the eigenvalue decompositions, where Λ_a and Λ_b are the corresponding decoupled systems. The particular property of G that allows for this kind of decoupling is orthogonality.

It cannot be guaranteed that the system is decoupled for a larger frequency range ω because the matrices $\Lambda_{a,b}$ might not be diagonal anymore if the values for $G_1(s)$ and $G_2(s)$ change. Furthermore, the eigenvector only decouples the system at the corresponding eigenvalues.