Heat diffusion

Project for the course on Model Reduction (5MB10)

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Introduction

In this project we consider how a temperature distribution on a locally heated surface evolves over time. We find this theme in many technical applications ranging from fused deposit modeling in 3D printing, the application of thermal loads on wafers and reflective materials in EUV optical lithography, thermal studies of composite materials and medical applications such as hyperthermia treatments.

To only expand on the latter, hyperthermia treatments are among the successful medical techniques to reduce the growth of tumor cells in body tissue and to enhance the effectiveness of other therapies such as radiation or chemotherapeutic medication. It is a therapy in which tissue is exposed to temperatures that are higher than the nominal body temperature for a prolonged period of time. Although there is no general consensus as to what is the most effective and safest temperature elevation in hyperthermia treatments, typical exposure temperatures range between 39.5 and 41.5 degrees Celsius applied at a very localized area of tissue during an exposure time between 5 and 30 minutes.

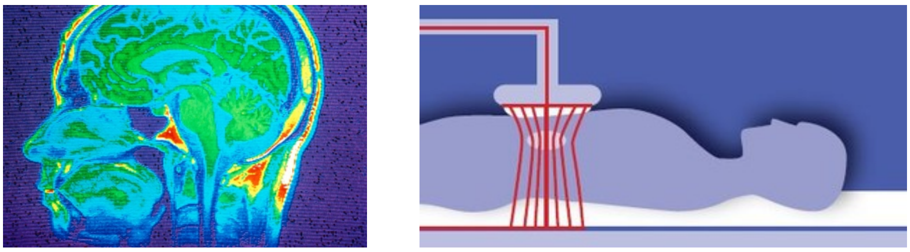

Figure 1: Hyperthermia treatment

The heat that is necessary for such a treatment is supplied by various techniques, including electromagnetic waves, infrared lighting, ultrasound waves, radiofrequency ablation or by direct contact with the tissue through the use of probes for the infusion of warmed liquids.

In local hyperthermia treatments a very small area of tissue is heated with the purpose to reduce the growth of tumor cells while not damaging any neighbouring cells. In regional hyperthermia treatments, larger parts of the body, such as an entire organ or limb are being heated.

Since external heating may cause surface burns, tissue damage or serious injuries, it is particularly challenging to control the heat supply in a careful and optimal manner so that the right temperature is reached in a sustained time frame and in the correct part of the patient's body. Typically, side effects and injuries are reduced if the heated area is as small as possible, the treatment time is as short as possible and knowledge of the thermal properties of tissue is as accurate as possible. Various departments at TU/e are doing active research on this challenge.

The model

In this project we consider a model that describes the temperature evolution on a surface. We consider a rectangular area of dimension $L_x \times L_y$ and aim to describe the temperature variation at the surface as a function of time. See Figure 2.

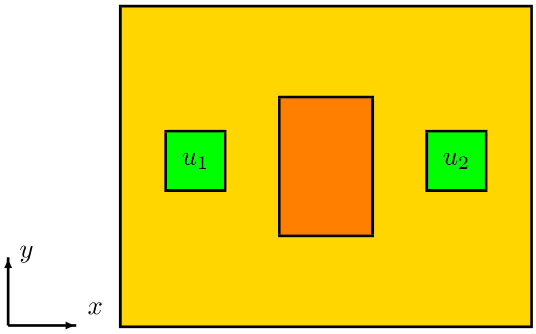

Figure 2: Configuration of surface with heat flux locations

The temperature at surface position (x, y) and at time t is given by $T_{\text{amb}} + T(x, y, t)$, where T_{amb} is the ambient temperature which is assumed to be constant. The model features two time-dependent heat fluxes $u_i(t)$, $i = 1, 2$, that are applied at disjoint areas centered at the points (X_i, Y_i) on the surface. Taking the locations of the sources into account, this means that the total heat flux that is applied at an arbitrary position (x, y) on the surface at time t is given by

$$u(x, y, t) = \begin{cases} u_1(t) & \text{if } |x - X_1| \leq \frac{W}{2}, \quad |y - Y_1| \leq \frac{W}{2} \\ u_2(t) & \text{if } |x - X_2| \leq \frac{W}{2}, \quad |y - Y_2| \leq \frac{W}{2} \\ 0 & \text{elsewhere} \end{cases}. \tag{1}$$

Hence, heat is supplied at two square-shaped areas of width W at midpoint positions (X_i, Y_i). The surface is assumed to be a cross section of possibly different tissue materials that are indicated as the yellow and the orange area in Figure 2. Both of these areas are assumed to have a uniform material density ρ [kg/m^3] and a uniform heat capacity c [J/kg K]. The orange area is rectangular, centered in the middle and has dimension $\ell_x \times \ell_y$ with $\ell_x < L_x$ and $\ell_y < L_y$.

The temperature variation on the surface is described by the parabolic partial differential equation

$$\rho c \frac{\partial T}{\partial t} = \nabla \cdot (K \nabla T) + u \tag{2}$$

where ρ and c are material density and heat capacity. In general, K is a positive definite matrix of dimension 2×2 representing the thermal conductivity, the input u is defined in (1) and ∇ denotes the usual gradient operator.

In Euclidean surface coordinates with $x \in [0, L_x]$ and $y \in [0, L_y]$, the model (2) assumes the form

$$\rho(x, y)c(x, y)\frac{\partial T}{\partial t}(x, y, t) = \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \end{bmatrix} K(x, y) \begin{bmatrix} \frac{\partial T(x,y)}{\partial x} \\ \frac{\partial T(x,y)}{\partial y} \end{bmatrix} + u(x, y, t) \tag{3}$$

where $\rho(x, y)$, $c(x, y)$ are the location dependent density and heat capacity, $K(x, y) \succ 0$ is the location dependent thermal conductivity matrix.

Some terminology:

- The model (3) is called *homogeneous* if ρ, c and K do not depend on (x, y).
- The model is called *isotropic* if $K = \kappa I$ with I the 2×2 identity matrix. In that case, also κ is called the thermal conductivity and has [W/m K] as its unit.

The surface is assumed to be insulated at its boundaries in the sense that

$$\frac{\partial T}{\partial x}(0, y, t) = 0; \qquad \frac{\partial T}{\partial x}(L_x, y, t) = 0 \tag{4}$$

$$\frac{\partial T}{\partial y}(x, 0, t) = 0; \qquad \frac{\partial T}{\partial y}(x, L_y, t) = 0 \tag{5}$$

for all $t \geq 0$ and all coordinates $0 \leq x \leq L_x$ and $0 \leq y \leq L_y$.

Purpose

We will be interested in deriving a fast and simple model that accurately describes the temperature evolution of this system. In particular, we will be interested in the diffusion process that takes place as a result of individual heat loads u_1 and u_2 applied at the surface.