Goals of this assignment:
1. Solve the given optimal control problem
2. Show that the estimated state $\hat{x}_h$ coincides with the solution of the Kalman filter

Introduction
We wish to estimate the state, at a given time $k = h$, of the following linear system:

$$x_{k+1} = Ax_k + w_k$$

Where each w_k denotes the state disturbance at time $k = \{0,1,2,\dots,h-1\}$.

Assuming that A is invertible the state can be computed by:

$$x_k = A^{-1}x_{k+1} - A^{-1}\overline{w}_{k+1}$$

where $\overline{w}_{k+1} = w_k$. The following output measurements

$$y_k = Cx_k + v_k$$

are available at times $k = \{0,1,2,\dots,h\}$, where v_k denotes the output noise at time k.

The initial state is unknown but an initial estimate, denoted by $\overline{x}_0$, is available.

To obtain the state estimate at time h, denoted by $\bar{x}_h$, the following optimal control problem is solved:

$$J_h(x_h) = \min_{\overline{w}_1,\dots,\overline{w}_h} (x_0 - \bar{x}_0)^T \overline{\Theta}_0^{-1}(x_0 - \bar{x}_0) + (y_0 - Cx_0)^T V^{-1}(y_0 - Cx_0)$$
$$+\textstyle\sum_{k=1}^{h} \overline{w}_k^T W^{-1}\overline{w}_k + (y_k - Cx_k)^T V^{-1}(y_k - Cx_k)$$

For given positive definite weighting matrices W, V and $\overline{\Theta}$. The next step is finding $\bar{x}_h = argmin\, J_h(x_h)$.

Solving optimal control problem
The cost function $J_h(x_h)$ is based on the following elements:
- Deviation from the estimated initial condition: $((x_0 - \bar{x}_0)^T \overline{\Theta}_0^{-1}(x_0 - \bar{x}_0))$
- State disturbances: $(\overline{w}_k^T W^{-1}\overline{w}_k)$
- Output noise: $(((y_k - Cx_k)^T V^{-1}(y_k - Cx_k)) = (v_k^T V^{-1} v_k))$

Substituting $(y_k - Cx_k)$ by v_k in the cost formula results in:

$$J_h(x_h) = \min_{\overline{w}_1,\dots,\overline{w}_h} (x_0 - \bar{x}_0)^T \overline{\Theta}_0^{-1}(x_0 - \bar{x}_0) + v_0^T V^{-1} v_0 + \sum_{k=1}^{h} \overline{w}_k^T W^{-1}\overline{w}_k + v_k^T V^{-1} v_k$$

Now that the optimal cost is known, the optimal policy can be derived.

Estimate coincides with Kalman filter
Now is shown that $\bar{x}_h$ coincides with the solution of the Kalman filter $\hat{x}_{h|h}$. The solution of the Kalman filter, $\hat{x}_{h|h}$, is obtained by iterating for $k = \{0,1,2,\dots,h\}$

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + L_k\left(y_k + C\hat{x}_{k|k-1}\right)$$
$$\hat{x}_{k+1|k} = A\hat{x}_{k|k}$$

With initial condition $\hat{x}_{0|-1}$, where

$$L_k = \Theta_{k|k-1} C^T \left(C\Theta_{k|k-1}C^T + V\right)^{-1}$$
$$\Theta_{k|k} = \Theta_{k|k-1} - \Theta_{k|k-1}C^T\left(C\Theta_{k|k-1}C^T + V\right)^{-1} C\Theta_{k|k-1}$$
$$\Theta_{k+1|k} = A\Theta_{k|k}A^T + W$$

With initial condition $\Theta_{0|-1} = \overline{\Theta}_0$.

For this exercise the DP algorithm going forward in time is considered instead of going backwards and the following initial 'cost-to-come' is used (instead of 'cost-to-go'):

$$J_0(x_0) = (x_0 - \overline{x}_0)^T \overline{\Theta}_0^{-1}(x_0 - \overline{x}_0) + (y_0 - Cx_0)^T V^{-1}(y_0 - Cx_0)$$

The forward DP algorithm and Kalman filter both compute the solution based on the measured output minus the expected output, noted by $(y_k - Cx_k)$.

For the initial condition of the Kalman filter, the state $\hat{x}_{k|k}$ is based on $\Theta_{0|-1}$, C and V. The initial condition of the forward DP algorithm is based on the inverse of $\Theta_{0|-1}$ and V.

Furthermore the forward DP algorithm determines the minimum cost to come to a point, whereas the Kalman filter predicts the error covariance. Both algorithms are running forward in time.

Linking the optimal control problem to the Kalman filter is done using duality. This results in the Riccati equation transformation from (A, B, Q, R) to (A^T, C^T, W, V) with $S = 0$. This results in

$$\bar{P} = A\bar{P}A^T + W - A\bar{P}C^T(C\bar{P}C^T + V)^{-1}C\bar{P}A^T$$

Also does the Riccati recursion for $\Theta_{k|k-1}$ converge to a steady-state value.

Furthermore the estimated state error $\hat{x}_{k|k-1} = x_k - \hat{x}_{k|k-1}$ is resulting in: $y_k - \hat{y}_{k|k-1} = Cx_k + v_k - C\hat{x}_{k|k-1} = C\hat{x}_{k|k-1} + v_k$. This leads to $\hat{x}_{k+1|k} = x_{k+1} - \hat{x}_{k+1|k} = (A - LC)\hat{x}_{k|k-1} + w_k - Lv_k$. Leading to the conclusion that the estimation error is propagating according to a linear system with closed-loop behavior $A - LC$, driven by $w_k - LCv_k$.

In conclusion both the forward DP algorithm and Kalman filter will obtain the same result.