Calculus of Variation
An Introduction To Isoperimetric Problems

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Abstract

Isoperimetric problems in Calculus of Variation can be loosely translated into "optimisation of a functional under the constraint of an integral". The classical Dido's Problem is an example of this. This article aims to explore some introductory theories of isoperimetric problems using Lagrange multipliers. Even though the motivation for exploring isoperimetric problem originated from lower dimensions, these results and techniques can be readily generalised to higher dimensions and multiple constraints.

Part I

Lagrange Multipliers

The aim of this part is to understand the theory and application of Lagrange multipliers and later, we will use it to prove theorems in Part II.

In this part we begin with the Lagrange multipliers method, and then justify its formulations. We will finish the part with some important assumptions and some generalisations.

1 Single Constraint Lagrange Multipliers

The idea of Lagrange multipliers is simple: we want to optimise a function, $f(x, y)$, under the constraint of another function $g(x, y) = 0$. Lagrange multipliers is a powerful method that can be used to solve this type of "constrained optimisation" problems.

We will firstly look at the the theorem of Lagrange multipliers and then justify it.

Theorem 1 (Method of Lagrange Multipliers) *To optimise for $f(x, y)$ subject to the constraint $g(x, y) = 0$ (assuming that such extrema exist and $\nabla g \neq \mathbf{0}$):*

(a) ***Simultaneously*** *solve for all values of x, y and λ in the following equations:*

$$g(x, y) = 0 \tag{1.1}$$

$$\nabla f(x, y) = \lambda \nabla g(x, y) \tag{1.2}$$

where λ is a non-zero constant, it is called the "Lagrange multiplier".

(b) Evaluate all points in in part (a) to see if these are minimum or maximum points (possibly using Hessian matrix).

We can think of Lagrange multipliers in the following way: if a "constrained critical point" does exist and $\nabla g(x, y) \neq 0$, then the constrained critical point must simultaneously satisfy two criteria: (1.1) and (1.2).

Being a constraint, $g(x, y) = 0$ must always be satisfied, hence, the first criterion is justified. We will justify (1.2) by thinking about the contour plots of $f(x, y)$ and $g(x, y) = 0$. The contour plot of $f(x, y)$ and $g(x, y) = 0$ can look something like Figure 1.

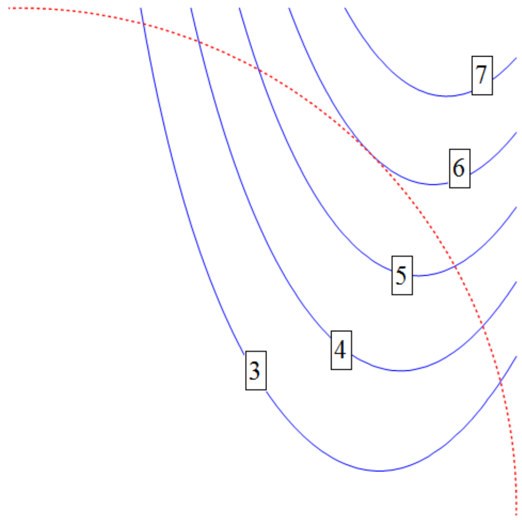


Figure 1: The red line is the contour line of $g(x, y) = 0$ and the blue lines are the contour plot of $f(x, y) = c$, where $c = 3, 4, 5, 6, 7$

We want to optimise $f(x, y)$ under the constraint of $g(x, y) = 0$. Geometrically on the contour plot, this is equivalent to finding the largest (or smallest) value of c such that $f(x, y) = c$ still intersects with $g(x, y) = 0$. And such intersection is our desired "constrained critical point".

From our diagram, it appears such constrained critical point occur when the two contour lines intersect **tangentially**, i.e. when $c = 6$. This can be justified in the following way: imagine moving along the contour line of $g(x, y) = 0$ (this movement is necessary because we must satisfy the constraint at all times); if the two contour lines are tangential to each other, then the value of $f(x, y)$ does not change locally as we move. Hence, the point where the two contour lines intersect tangentially correspond to a constrained critical point of $f(x, y)$ with reference to $g(x, y) = 0$.

If the contour lines of $f(x, y)$ and $g(x, y)$ intersect tangentially at a constrained critical point, this is equivalent to saying that these two contour lines has parallel tangent vectors at this point. Which means their normal vectors must be parallel, or **linearly dependent**. Since the normal vectors for $f(x, y)$ and $g(x, y) = 0$ are $\nabla f(x, y)$ and $\nabla g(x, y)$ respectively, this linear dependence can be expressed as $\nabla f(x, y) = \lambda \nabla g(x, y)$, where λ is a non-zero constant. And hence, we justified the second criterion of Lagrange multiplier method, which is equation (1.2).

Remarks:

Some underlying assumptions are:

1. Such extrema do exist.

2. $f(x, y)$ and $g(x, y)$ are smooth functions. If this were not true, then our geometric justification of for (1.2) might become flawed, due to possible discontinuity.

3. $\nabla g(x, y) \neq \mathbf{0}$, otherwise we will be dealing with an abnormal case. A abnormal case is when Lagrange multipliers method gives inconclusive results. For example:

If $\nabla g(x, y) = \mathbf{0}$ does occur, then, there could potentially be no solutions for λ (See Example 4.1.3. in [1]). Or equally troubling, the system can yield infinitely many solutions for λ (See Example 4.1.4 in [1]). In both cases, we will have to examine the contour plots of functions to determine whether our solutions are valid or not. There are many examples for abnormal cases, however, these are not discussed in this article.

1.1 An example of Lagrange Multipliers of Single Constraint

Example 1 *Optimise the function $f(x, y) = 3x^2 + 2y^2$, subject to the constraint of $g(x, y) = x^2 + y^2 - 1 = 0$.*

Solution: *Firstly calculate the gradients: $\nabla f(x, y) = (6x, 4y)$ and $\nabla g(x, y) = (2x, 2y)$. Equating the partials and include the constraint $g(x, y) = 0$, we have:*

$$\left.\begin{aligned} 6x &= \lambda(2x) \\ 4y &= \lambda(2y) \\ x^2 + y^2 - 1 &= 0 \end{aligned}\right\} \quad \textit{Solve these simultaneously:}$$

This system yield four solutions. When $\lambda = 2$, we get $(0, -1)$ and $(0, 1)$, which gives $f(0, -1) = f(0, 1) = 2$, i.e. minimum points. When $\lambda = 3$, we get $(-1, 0)$ and $(1, 0)$, which gives $f(-1, 0) = f(1, 0) = 3$, i.e. maximum points.

Also quickly check that $\nabla g \neq \mathbf{0}$ at these points, so the abnormal case is avoided. It is worth noting that at different points, the Lagrange multiplier λ could be different.

2 Multiple Constraints

Multiple constraints is where Lagrange multiplier method really shines as a powerful optimisation method. We will firstly generalise our Lagrange multipliers to $\mathbb{R}^N$ and later, multiple constraints.

2.1 Higher Dimensions

Even though our arguments for Theorem 1 only involved looking at 2-dimensional contour plots, the Lagrange multipliers can be readily extend to higher finite dimensions. If we define $\boldsymbol{x} = (x_1, x_2, \ldots, x_N)$, Theorem 1 can be modified into:

Theorem 2 *To optimise $f(\boldsymbol{x})$ under the constraint of $g(\boldsymbol{x}) = 0$, assuming existence of extrema and $\nabla g(\boldsymbol{x}) \neq \mathbf{0}$:*

(a) Solve $\boldsymbol{x}$ and λ simultaneously in

$$g(\boldsymbol{x}) = 0 \tag{2.1}$$

$$\nabla f(\boldsymbol{x}) = \lambda \nabla g(\boldsymbol{x}) \tag{2.2}$$

(b) Evaluate the values for points in part(a) to determine if they are maximum or minimum.

Remarks: $\nabla f(\boldsymbol{x})$ and $\nabla g(\boldsymbol{x})$ are made up of N partials each, hence, overall in Theorem 2, we are solving $N + 1$ equations simultaneously.

2.2 Lagrange Multipliers of Multiple Constraints

In multiple constraints, we will have M constraints, and each of these constraints is a function of the form $g_k(\boldsymbol{x}) = 0$ where $k = 0, 1 \ldots M$. Again, we will assume the existence of the extrema.

Theorem 3 (Lagrange Multipliers for Multiple Constraints) *To optimise the function $f(\boldsymbol{x})$, under constraints $g_k(\boldsymbol{x}) = 0$ where $k = 0, 1 \ldots M$:*

(a) Solve $\boldsymbol{x}$ and λ_k's simultaneously in:

$$g_k(\boldsymbol{x}) = 0 \tag{2.3}$$

$$\nabla f(\boldsymbol{x}) = \sum_{k=1}^{M} \lambda_k \nabla g_k(\boldsymbol{x}) \tag{2.4}$$

where λ_k are constants [1]

(b) Evaluate the value of points in part(a) to determine if they are maximum or minimum.

Again, the first criterion (2.3) is simply the constraints we have. The second criterion can be viewed as a generalisation of our argument of linear dependence in (1.2). A more technical and formal way to justify this is to say the tangent space of f at the constrained critical point $\boldsymbol{a}$ is contained in the tangent space defined by the constraints $g_k(\boldsymbol{x})$ at this point $\boldsymbol{a}$. Geometrically, this means $\nabla f(\boldsymbol{x})$ lies in the normal space spanned by the vectors $\nabla g_k(\boldsymbol{x})$. [5] provides a good alternative explanation on multiple constraint Lagrange multipliers.

The computation of multiple constraint problems is very similar to the single constraint case, that is, solving a system of equations simultaneously. Numerical solutions might be used, since (2.3) and (2.4) implies there are, in total $N + M$ equations.

Part II
Isoperimetric Problems

The word "iso-perimetric" in the study of Calculus of Variation can be loosely translated into optimisation of a function under the constraint of an integral. The oldest problem in Calculus of Variation, the Dido's problem is a good example of isoperimetric problem, and it offers some deep mathematical insights into our intuitions about simple geometry.

In this part, we will firstly define the isoperimetric problem and quickly revise some ideas in unconstrained Euler-Lagrange equation. With these necessary preparations, we can derive the so called "**Euler-Lagrange equation for the isoperimetric problem**". Then, we will consolidate these ideas by exploring Dido's problem in detail. Finally, we will end with some generalisations of isoperimetric problems, namely higher order derivatives and multiple constraints.

[1] This means ∇f is a linear combination of ∇g_k. To preserve linear dependence, not all λ_k are equal to 0 simultaneously.

3 Introduction to Isoperimetric Problems

3.1 Defining the Isoperimetric Problem

Previously from [2] we know given a functional of the form:

$$J(y) = \int_{x_0}^{x_1} f(x, y, y') \; dx \tag{3.1}$$

where y satisfies the boundary conditions:

$$y(x_0) = y_0 \qquad y(x_1) = y_1 \tag{3.2}$$

we can simply apply the Euler-Lagrange equation to find the extremal of this functional.

This time, we impose something called the " **isoperimetric constraint**" onto (3.1), which takes the form of an integral that always evaluates to a constant, L:

$$I(y) = \int_{x_0}^{x_1} g(x, y, y') \; dx = L \tag{3.3}$$

Hence, isoperimetric problem can be defined more precisely as:

Definition 1 (Isoperimetric Problem) *An isoperimetric problem seeks to optimise a functional of the form* (3.1) *under the condition that the isoperimetric constraint of* (3.3) *must be satisfied.*

3.2 Revision of the Fundamental Lemma

If there is no constraint, the first step in deriving the first order Euler-Lagrange equation is to construct a perturbation of the form $\hat{y} = y + \epsilon\eta$; where function y is the extremal for the functional without the isoperimetric constraint, and $\epsilon\eta$ is an arbitrary function dependent on x that satisfy the boundary condition. After calculating partial deriveatives, the final step of the proof was to apply the Fundamental Lemma of Calculus of Variation[2]:

Lemma 1 (The Fundamental Lemma of Calculus of Variation) *Let $h(x)$ be a smooth function on the interval $[x_0, x_1]$. Suppose that for any arbitrary smooth function $\eta(x)$ with $\eta(x_0) = \eta(x_1) = 0$ we have*

$$\int_{x_0}^{x_1} h(x)\eta(x) \; dx = 0 \tag{3.4}$$

Then $h(x)$ is identically zero on $[x_0, x_1]$.

3.3 Rigid Extremal

An important assumption we will be making throughout our discussion of isoperimetric problems is that our constrained extremal y is not an extremal of the isoperimetric constraint, (3.3). Because if that is the case, we will run into a case of rigid extremal, which means the perturbation, $\hat{y}$, will not have any variations.

[2] See [2] for details of this formulation and proof

A simple example of rigid extremal is to consider the isoperimetric constraint of:

$$I(y) = \int_{-1}^{1} \sqrt{1+y'^2}\ dx = 2 \tag{3.5}$$

This is the arclength of a function between $x = -1$ and $x = 1$. Since $L = 2$, which is already the value of the geodesic on a plane. This problem becomes uninteresting because there is no other choice for $\hat{y}$ (or y) except a straight line segment between $x = -1$ and $x = 1$.

This rigid extremal case will be avoided where possible. Before our calculations, we should always check that the rigid extremal case does not exist.

4 Deriving Euler-Lagrange equation for Isoperimetric Problems

After preparations made in section 3 , we are now ready to derive the Euler-Lagrange equation with the complication of the isoperimetric constraint.

In deriving this equation, we will need a perturbation $\hat{y}$ to satisfy (3.3), and later, we will try to apply Lemma 1 to get the Euler-Lagrange equation. However, we will run into a hurdle, because while the lemma relies on the arbitrary nature of $\epsilon\eta$, but not all $\epsilon\eta$ can satisfy the isoperimetric constraint. This means Lemma 1 cannot be applied directly for the formulation of $\hat{y} = y + \epsilon\eta$.

To overcome this hurdle, we rewrite our perturbation as:

$$\hat{y} = y + \epsilon_1\eta_1 + \epsilon_2\eta_2 \tag{4.1}$$

The addition of $\epsilon_2\eta_2$ acts as a "correction term", which makes sure the entire $\hat{y} = y + \epsilon_1\eta_1 + \epsilon_2\eta_2$ will satisfy the isoperimetric constraint, regardless of what $\epsilon_1\eta_1$ is. And in doing so, the arbitrary nature of $\epsilon_1\eta_1$ is retained, and the Fundamental Lemma can be applied later.

So now we can write down $J(\hat{y})$. An equivalent expression is to regard it as a function $\Theta(\epsilon_1, \epsilon_2)$, taking ϵ_1 and ϵ_2 as parameters, hence:

$$\begin{aligned} J(\hat{y}) &= \int_{x_0}^{x_1} f(x, \hat{y}, \hat{y}')\ dx \\ &= \int_{x_0}^{x_1} f(x, y + \epsilon_1\eta_1 + \epsilon_2\eta_2, y + \epsilon_1\eta_1' + \epsilon_2\eta_2')\ dx \\ &= \Theta(\epsilon_1, \epsilon_2) \end{aligned} \tag{4.2}$$

Similarly for $I(\hat{y}) = L$:

$$I(\hat{y}) = \int_{x_0}^{x_1} g(x, \hat{y}, \hat{y}')\ dx = \Phi(\epsilon_1, \epsilon_2) = L \tag{4.3}$$

So our problem of optimising functionals under constraint is reduced into a simpler problem of optimising the function $\Theta(\epsilon_1, \epsilon_2)$ under the constraint of $\Phi(\epsilon_1, \epsilon_2) - L = 0$.

Using Lagrange multipliers in (1.2):

$$\begin{aligned}\nabla\Theta(\epsilon_1,\epsilon_2) &= \lambda\nabla(\Phi(\epsilon_1,\epsilon_2 - L)) \\ &= \lambda\nabla\Phi(\epsilon_1,\epsilon_2) \\ 0 &= \nabla\Theta(\epsilon_1,\epsilon_2) - \nabla\Phi(\epsilon_1,\epsilon_2) \end{aligned} \tag{4.4}$$

At $\boldsymbol{\epsilon} = (\epsilon_1, \epsilon_2) = (0,0) = \mathbf{0}$, we have $\hat{y} = y$, which is an extremal. So we will now evaluate the partial derivatives:

$$\left.\frac{\partial}{\partial\epsilon_1}\Theta(\epsilon_1,\epsilon_2) - \lambda\frac{\partial}{\partial\epsilon_1}\Phi(\epsilon_1,\epsilon_2)\right|_{\boldsymbol{\epsilon}=\mathbf{0}} = 0 \tag{4.5}$$

$$\left.\frac{\partial}{\partial\epsilon_2}\Theta(\epsilon_1,\epsilon_2) - \lambda\frac{\partial}{\partial\epsilon_2}\Phi(\epsilon_1,\epsilon_2)\right|_{\boldsymbol{\epsilon}=\mathbf{0}} = 0 \tag{4.6}$$

here:

$$\begin{aligned}\left.\frac{\partial}{\partial\epsilon_1}\Theta(\epsilon_1,\epsilon_2)\right|_{\boldsymbol{\epsilon}=\mathbf{0}} &= \left.\frac{\partial}{\partial\epsilon_1}\int_{x_0}^{x_1} f(x, y+\epsilon_1\eta_1+\epsilon_2\eta_2, y+\epsilon_1\eta_1'+\epsilon_2\eta_2')\, dx\right|_{\boldsymbol{\epsilon}=\mathbf{0}} \\ &= \left.\int_{x_0}^{x_1} \frac{\partial}{\partial\epsilon_1} f(x, y+\epsilon_1\eta_1+\epsilon_2\eta_2, y+\epsilon_1\eta_1'+\epsilon_2\eta_2')\, dx\right|_{\boldsymbol{\epsilon}=\mathbf{0}} \\ &= \int_{x_0}^{x_1} \left(\eta_1\frac{\partial f}{\partial y} + \eta_1'\frac{\partial f}{\partial y'}\right)\, dx\end{aligned}$$

integrating by parts on the second term yields:

$$\left.\frac{\partial}{\partial\epsilon_1}\Theta(\epsilon_1,\epsilon_2)\right|_{\boldsymbol{\epsilon}=\mathbf{0}} = \int_{x_0}^{x_1} \eta_1\left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'}\right)\, dx$$

similarly:

$$\left.\frac{\partial}{\partial\epsilon_1}\Phi(\epsilon_1,\epsilon_2)\right|_{\boldsymbol{\epsilon}=\mathbf{0}} = \int_{x_0}^{x_1} \eta_1\left(\frac{\partial g}{\partial y} - \frac{d}{dx}\frac{\partial g}{\partial y'}\right)\, dx$$

Substitute these two equations back to (4.5) gives:

$$\int_{x_0}^{x_1} \eta_1 \underbrace{\left\{\left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'}\right) - \lambda\left(\frac{\partial g}{\partial y} - \frac{d}{dx}\frac{\partial g}{\partial y'}\right)\right\}}_{=0 \text{ by Lemma 1}}\, dx$$

Because of our previous set up, η_1 is now an arbitrary function, so we can apply the Lemma 1 and everything inside the brace will be equal to 0, which can be succinctly be written as:

$$\frac{d}{dx}\frac{\partial F}{\partial y'} - \frac{\partial F}{\partial y} = 0 \tag{4.7}$$

where:

$$F = f - \lambda g \tag{4.8}$$

If we use the same technique upon (4.6), we will get the same equations, and so we arrive at a theorem:

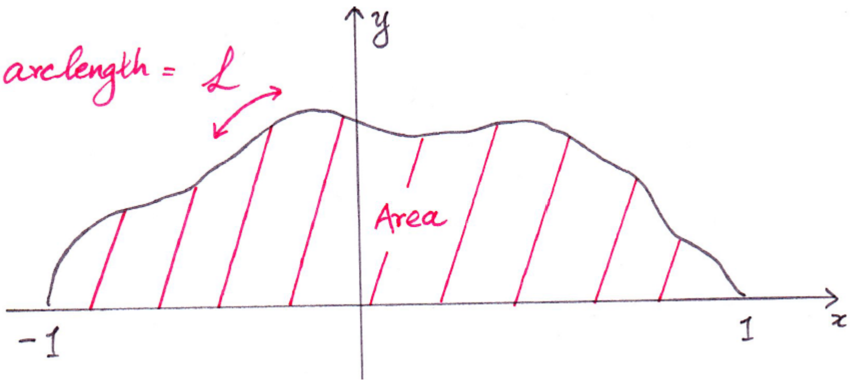
Theorem 4 (Euler-Lagrange equation for isoperimetric problems) *Suppose that J has an extremal at $y \in C^2[x_0, x_1]$, subjected to the boundary condition (3.2) and the isoperimetric constraint of (3.3). Suppose further that y is not a rigid extremal for $I(y)$. Then there exists a constant λ such that y satisfies (4.7) and (4.8).*

Armed with this theorem, we are now ready to tackle the Dido's problem.

4.1 Dido's problem

Question 1 (The Classical Dido's Problem) *Given a rope of fixed length and a line segment(the x-axis), what shape should the rope take so that the area enclosed by the rope and the line segment is maximised?*

Construct a simple sketch, the actual dimensions of the area or length can be rescaled to be between the interval $[-1, 1]$:



Since the area we want to maximise is simply the area under the curve, which can be simply be expressed as the functional:

$$J(y) = \int_{-1}^{1} y \; dx \tag{4.9}$$

Also, since the rope is of fixed length, the isoperimetric constraint is just the arclength of the curve over the interval $[-1, 1]$:

$$I(y) = \int_{-1}^{1} \sqrt{1 + y'^2} \; dx = L \tag{4.10}$$

We will avoid the rigid extremal by setting $L > 2$. This is a necessary assumption, because if $L = 2$, we will simply get a straight line (geodesic on a plane) i.e. without any variation on the function. Furthermore, if $L < 2$, there isn't any function that could satisfy the constraint.

Applying Theorem 4, we can write (4.8) as:

$$\begin{aligned} F &= f - \lambda g \\ &= y - \lambda \sqrt{1 + y'^2} \end{aligned}$$

Here is where we can apply the Euler-Lagrange equation onto F, but since there is no explicit x dependence in F, we can apply the Beltrami Identity [3] [5], which takes the form of:

$$y'\frac{\partial F}{\partial y'} - F = c_1$$
$$\frac{-\lambda y'^2}{\sqrt{1+y'^2}} - y + \lambda\sqrt{1+y'^2} = c_1$$

which simplifies to a separable ODE:

$$y' = \frac{dy}{dx} = \sqrt{\frac{\lambda^2 - (y+c_1)^2}{(y+c_1)^2}}$$
$$\int \frac{y+c_1}{\sqrt{\lambda^2-(y+c_1)^2}}\, dy = \int\, dx$$

If we make a substitution of $y + c_1 = \lambda\cos\theta$, the expression simplifies to:

$$-\lambda \int \cos\theta\ d\theta = x + c_2$$

And so we have:

$$x + c_2 = -\lambda\sin\theta \tag{4.11}$$
$$y + c_1 = \lambda\cos\theta \tag{4.12}$$

We can combine the above to get:

$$(x+c_2)^2 + (y+c_1)^2 = \lambda^2 \tag{4.13}$$

We can tell immediately that this is a circle. However, if we are given the actual value of L, we need to describe the radius and center of this circle precisely. So, we will now solve some constants and explore their relationship with the prescribed constant L. If we substitute (4.11) and (4.12) back to (4.10), we have:

$$L = 2|\lambda||\theta| \tag{4.14}$$

Substituting the boundary conditions:

$$y(-1) = y(1) = 0 \tag{4.15}$$

into (4.13), we get:

$$c_2 = 0 \tag{4.16}$$
$$1 + c_1^2 = \lambda^2 \tag{4.17}$$

[3] Beltrami Identity is a direct result of Euler Lagrange equation when the function has no explicit x dependence

Substituting the same boundary conditions into (4.11) and (4.12) gives:

$$\frac{-1+0}{0+c_1} = \frac{-\lambda \sin\theta}{\lambda \cos\theta}$$

$$\tan\theta = \frac{1}{c_1}$$

$$\theta = \arctan(\frac{1}{c_1}) \tag{4.18}$$

Finally, substitute (4.16), (4.17), (4.18) into equations (4.14) and (4.13) we finally arrive at:

$$L = 2\sqrt{1+c_1^2}\arctan(\frac{1}{c_1}) \tag{4.19}$$

$$x^2 + (y+c_1)^2 = 1 + c_1^2 \tag{4.20}$$

This is a circle centred at $(0, -c_1)$, with radius $1 + c_1^2$. So given a value for L, we can solve (4.19) numerically and thus determine the position and radius of the circle precisely. In conclusion, if we want to maximise [4] the enclosed area between a fixed length of rope and a line segment, then the rope is an arc of a circle described by (4.19) and (4.20).

Notice that as $c_1 \to 0$, we will get the semi-unit circle, with $L \to \pi$. And using the reflection symmetry of semi-unit circle, we can almost claim an intuitive result: given a fixed length of rope, the maximum area it can enclose is a *circle*. In fact, this claim is true, and it can be capture by the "*isoperimetric inequality*", which states: $4\pi A \leq L^2$, where L is the perimeter of a 2D curve and A is the enclosed area. However, this result is beyond the scope of this introductory article. [3]

Another good example of isoperimetric problem is the catenary problem with length constraint. Its solution bears close resemblances with the Dido's problem. See Example 4.2.1 in [1].

5 Some Generalisations

In this section, we will briefly sketch some generalisations of Theorem 4.

5.1 Connection Between Lagrange Multipliers and Isoperimetric Problems

The Lagrange multiplier λ is more than just a physical constant; it has a more deep and meaningful relation with the isoperimetric problem. The functional $J(y)$ can be written as:

$$J(y) = \int_{x_0}^{x_1} \left\{ f(x,y,y') + \lambda \left(\frac{L}{x_1 - x_0} - g(x,y,y') \right) \right\} \, dx$$

As we have seen in Dido's problem, the solution to the Euler-Lagrange equation involved the boundary conditions and the prescribed constant L. Thus, the Lagrange

[4] To be completely rigorous, the real claim is that circle is a *stationary extremal*. A proof as to if this is really the maximum of the functional involves the *second variation* in calculus of variation

multipliers λ depends on x_0, x_1, y_0, y_0 and L. If the boundary is fixed, then $J(y)$ can be considered as a function of the parameter L, so we can take the partial derivative:

$$\begin{aligned}
\frac{\partial J}{\partial L} &= \int_{x_0}^{x_1} \frac{\partial}{\partial L} \left\{ (f(x,y,y') - \lambda g(x,y,y')) + \frac{\lambda L}{x_1 - x_0} \right\} dx \\
&= \int_{x_0}^{x_1} \left\{ \left(\frac{\partial F}{\partial y}\frac{\partial y}{\partial L} + \frac{\partial F}{\partial y'}\frac{\partial y'}{\partial L} \right) + \frac{\partial \lambda}{\partial L} \left(\frac{L}{x_1 - x_0} - g(x,y,y') \right) + \frac{\lambda}{x_1 - x_0} \right\} dx \\
&= \int_{x_0}^{x_1} \left(\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} \right) \frac{\partial y}{\partial L} \, dx + \frac{\partial \lambda}{\partial L} \left(L - \int_{x_0}^{x_1} g(x,y,y') \, dx \right) + \lambda \\
&= \int_{x_0}^{x_1} \underbrace{\left(\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} \right)}_{=0} \frac{\partial y}{\partial L} \, dx + \frac{\partial \lambda}{\partial L} \underbrace{\left(L - \int_{x_0}^{x_1} g(x,y,y') \, dx \right)}_{=0} + \lambda
\end{aligned}$$

The first bracket equals to 0, since y is an extremal for the functional, and thus satisfies the Euler Lagrange equation. The second bracket also goes to 0, since this the isoperimetric constraint that must be satisfied. And thus

$$\frac{\partial J}{\partial L} = L \tag{5.1}$$

Thus, the Lagrange multipliers has a special meaning: it correspond to the the rate of change of the extremum $J(y)$ with respect to the isoperimetric parameter L.

5.2 Higher Order Derivatives

So far, we only considered functionals involving up to the first derivative in equation (3.1) and (3.3). This can be extended to higher derivatives. For instance, consider:

$$\begin{aligned}
J(y) &= \int_{x_0}^{x_1} f(x,y,y',y'') \, dx \\
I(y) &= \int_{x_0}^{x_1} g(x,y,y',y'') \, dx = L
\end{aligned}$$

The same techniques for deriving (4.7) can be used to show that any smooth extremal for J subjected to the isoperimetric constraint $I(y) = L$ and boundary conditions (3.2), must satisfy the Euler-Lagrange equation:

$$\frac{d^2}{dx^2}\frac{\partial F}{\partial y''} - \frac{d}{dx}\frac{\partial F}{\partial y'} + \frac{\partial F}{\partial y} = 0 \tag{5.2}$$

Where $F = f - \lambda g$. The existence of λ is assured if y is not an extremal of the isoperimetric constraint. If even higher order derivatives are involved, then it is possible to use a higher order Euler-Lagrange equation. See [1] and [4] Euler-Lagrange equation for higher order derivatives.

5.3 Multiple Isoperimetric Constraints

Suppose that we have M constraints on the functional (3.1), and each of these constraints $I_1, I_2, \dots I_M$ are in the form of:

$$I_k(y) = \int_{x_0}^{x_1} g_k(x, y, y') \, dx = L_k \tag{5.3}$$

for $k = 1, 2, \dots M$. The boundary condition of (3.2) is preserved. We will discuss the special case of two isoperimetric constraints.

Theorem 5 *Suppose that J has an extremal at y, subjected to the boundary condition* (3.2) *and the isoperimetric constraint of* (5.4)*. Suppose further that y is not a rigid extremal for $I_k(y)$. Then the Euler-Lagrange equation*

$$\frac{d}{dx}\frac{\partial F}{\partial y'} - \frac{\partial F}{\partial y} = 0 \tag{5.4}$$

is satisfied with: $F = f - \lambda_1 g_1 - \lambda_2 g_2$

The proof of this theorem is very similar the proof given in Section 4. Here, we will only outline some differences:

1. We must construct a perturbation of $\hat{y} = y + \epsilon_1\eta_1 + \epsilon_2\eta_2 + \epsilon_3\eta_3$, so that both η_1 and η_2 can be arbitrary functions, and doing so means Lemma 1 can be applied.

2. Analogous to (4.2) and (4.3), we will have $\Theta(\epsilon_1, \epsilon_2, \epsilon_3)$, $\Phi_1(\epsilon_1, \epsilon_2, \epsilon_3)$ and $\Phi_2(\epsilon_1, \epsilon_2, \epsilon_3)$. And using equation (2.4), i.e. the multiple constraints version of Lagrange multipliers, (4.4) becomes

$$\nabla(\Theta - \lambda_1\Phi_1 - \lambda_2\Phi_2)\big|_{\boldsymbol{\epsilon}=\mathbf{0}} = \mathbf{0} \tag{5.5}$$

The details of this proof can be found in Chapter 4 of [1]. It is worth noting that the definition of the function F can be extended if more isoperimetric constraints are imposed.

Concluding Remarks

The Lagrange multipliers method is a well-known and powerful method for optimisation under constraints. In this article, we used the Lagrange multipliers to analyse the isoperimetric problem, in particular, we derived the Euler-Lagrange equation for isoperimetric problem.

The isoperimetric problem is an interesting topic within the Calculus of Variations. In particular, the Dido's problem provides a relatively rigorous justification for our simple geometric intuition about maximisation of area. The techniques used here are also applicable to more abstract and difficult problems in higher orders and multiple isoperimetric constraints. Some extensions on our isoperimetric problem include eigenvalue problems, holonomic and non-holonomic constraints in Calculus of Variations.

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