The goal of this assignment is:

- Proving that the relative degree of the open loop transfer function of the LQR compensator is always 1

The following system is considered:

$$\dot{x} = Ax + Bu$$

With $x(t) \in \mathbb{R}^{n_x}$ and $u(t) \in \mathbb{R}$, for every $t \in \mathbb{R}_{\geq 0}$. It is assumed that the pair (A, B) is controllable.

The following Linear Quadratic Regulator (LQR) is considered:

$$u = Kx$$

Where $K = -R^{-1}B^T P$ and P is the unique positive definite solution to

$$P = A^T P + PA + Q - PBR^{-1}B^T P$$

for positive definite weighting matrices Q and R. The open loop transfer function of the LQR loop is denoted as:

$$\frac{n(s)}{d(s)} = -K(sI - A)^{-1}B$$

A schematic block diagram of the considered system is shown in Figure 1.

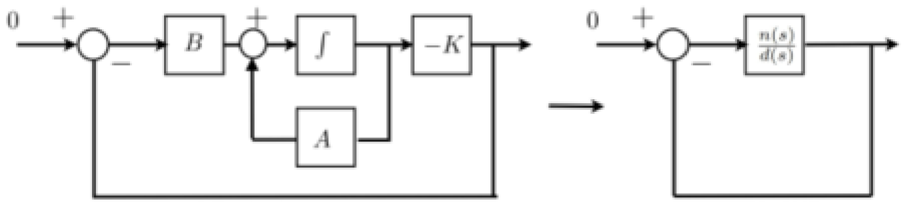

Figure 1 – LQR block diagram

First is proven that the solution of the Nyquist plot is not inside the region $|1 + \phi| < 1$, which indicates the unitary circle around (-1,0). Then is proven that the solution is strictly proper, meaning that the relative degree of the open loop transfer function of the LQR compensator is always 1.

Starting with the continuous time Algebraic Ricatti Equation (ARE):

$$0 = -A^T P - PA - M^T M + PBR^{-1}B^T P$$

where

$$PBR^{-1}B^T P = K^T RK$$

The following is defined:

$$Q = M^T M$$
$$\phi(s) = (sI - A)^{-1}$$

and the ARE is rewritten as:

$$[I - K\phi(-s)B]^T R[I - K\phi(-s)B] = R + [M\phi(-s)B]^T [M\phi(s)B]$$

Now sP is added and subtracted resulting in the following:

$$0 = (-sI - A)^T P + P(sI - A) - M^T M + K^T RK$$

Pre multiplying by $B^T \phi(-s)^T$ and post multiplying by $\phi(s)B$ results in

$$0 = B^T P\phi(s)B + B^T \phi(-s)^T PB - B^T \phi(-s)^T M^T M\phi(s)B + B^T \phi(-s)^T K^T RK\phi(s)B$$

After some rearranging and adding R to both sides of the equation, the FDE is obtained:

$$R - RK\phi(s)B - B^T \phi(-s)^T K^T R + B^T \phi(-s)^T K^T RK\phi(s)B = R + B^T \phi(-s)^T M^T M\phi(s)B$$

Where:

$$R - RK\phi(s)B - B^T\phi(-s)^T K^T R + B^T\phi(-s)^T K^T RK\phi(s)B = [I - K\phi(-s)B]^T R[I - K\phi(-s)B]$$

After substitution this results in:

$$[I - K\phi(-s)B]^T R[I - K\phi(-s)B] = R + B^T\phi(-s)^T M^T M\phi(s)B$$

Dividing by R and replacing s by $j\omega$ results in the following equation:

$$(1 - K\phi(-j\omega)B)(1 - K\phi(j\omega)B) = 1 + \frac{1}{R}(M\phi(-j\omega)B)(M\phi(j\omega)B)$$

This is equivalent to:

$$|1 - K\phi(j\omega)B)|^2 = 1 + \frac{1}{R}|M\phi(j\omega)B|^2$$

From this can be concluded that $|1 - K\phi(j\omega)B)| \geq 1$, meaning that the solution is always outside the region $|1 + \phi| < 1$. The following stability margins can be derived from the Nyquist plot:

- Gain margin between $[-6, \infty)$ dB
- Phase margin of $[-60,60]$ degrees

This indicates that if $\lim_{\omega \to \infty}(K\phi(j\omega)B)$ is computed the imaginary part increases resulting in the gain moving towards zero and the phase to $-90°$, showing that the relative degree of the system is 1.