$$\gamma = 1.8483$$

Dido's problem

For this problem the following cost function is given:

$$\int_0^3 y(x)dx \tag{1}$$

The rope length is fixed and can be calculated by the following equation, leading to a constraint:

$$Length = \int_0^3 \sqrt{1+\dot{y}(x)^2}dx \tag{2}$$

The system equation is given by:

$$\begin{aligned} \dot{y} =& 0 \cdot y + 1 \cdot u \\ \dot{y} =& u \end{aligned} \tag{3}$$

And the following boundary conditions are given:

$$y(0) = 0, \quad y(3) = 0 \tag{4}$$

Since the constrained rope length is an isoperimetric constraint it is possible to combine the constraint in the cost function and obtain the following combined cost function (Lagrangian) with some constant λ:

$$L = \int_0^3 y(x)dx + \lambda(\int_0^3 \sqrt{1+\dot{y}(x)^2}dx - Length) \tag{5}$$

Note that this is exact the equation given in the assignment, but with λ instead of γ. In this document we will stick to λ.

To obtain the control equation for PMP, the Beltrami Identity is used since the Lagrangian is not explicitly depending on x. This leads to the following equation:

$$\begin{aligned} \dot{y}\frac{\partial L}{\partial \dot{y}} - L =& c_1 \\ \frac{-\lambda \dot{y}^2}{\sqrt{1+\dot{y}^2}} - y + \lambda\sqrt{1+\dot{y}^2} =& c_1 \end{aligned} \tag{6}$$

Which is a separable ODE:

$$\begin{aligned} \dot{y} = \frac{dy}{dx} =& \sqrt{\frac{\lambda^2 - (y+c_1)^2}{(y+c_1)^2}} \\ \int \frac{y+c_1}{\lambda^2 - (y+c_1)^2}dy =& \int 1dx \end{aligned} \tag{7}$$

substituting $y + c_1 = \lambda \cos\theta$ leads to:

$$-\lambda \int \cos\theta d\theta = x + c_2 \tag{8}$$

In conclusion we have the following two relations:

$$\begin{aligned} y + c_1 &= \lambda \cos\theta \\ x + c_2 &= -\lambda \sin\theta \end{aligned} \tag{9}$$

which can be combined to:

$$(y + c_1)^2 + (x + c_2)^2 = \lambda^2 \tag{10}$$

This equation describes a circle, which is the expected result for this problem. In order to determine the three unknowns c_1, c_2 and λ we will use the two given boundary conditions and the given constraint for the length.
First the relation between the length, λ and θ can be defined from the previous conclusions, which leads to:

$$Length = 2|\lambda||\theta| \tag{11}$$

By substituting the boundary conditions (4) into (10) the following two relations are found:

$$\begin{aligned} &for\ y(0) = 0 \\ &\qquad c_1^2 + c_2^2 = \lambda^2 \\ &for\ y(3) = 0 \\ &\qquad c_1^2 + (3 + c_2)^2 = \lambda^2 \\ \\ &Leads\ to: \\ &\qquad c_2 = -1.5 \\ &\qquad c_1^2 + 2.25 = \lambda^2 \end{aligned} \tag{12}$$

Substituting the boundary conditions (4) into (9) leads to the following relation:

$$\begin{aligned} \frac{x + c_2}{y + c_1} &= \frac{-\theta \sin\theta}{\theta \cos\theta} \\ \frac{1.5}{c_1} &= \tan\theta \\ \theta &= \arctan(\frac{1.5}{c_1}) \end{aligned} \tag{13}$$

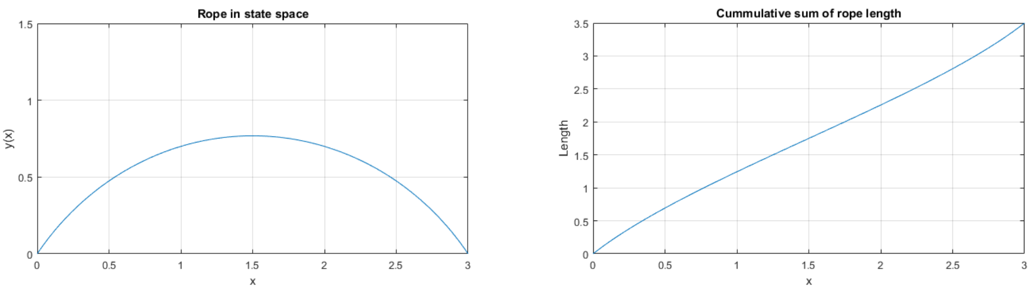
Substituting (12) and (13) into (11) and filling in the given length of 3.5 leads to:

$$\begin{aligned} Length &= 2\sqrt{c_1^2 + 2.25}\arctan(\frac{1.5}{c_1}) \\ 3.5 &= \sqrt{c_1^2 + 2.25}\arctan(\frac{1.5}{c_1}) \\ c_1 &\approx 1.0799 \end{aligned} \tag{14}$$

By substituting c_1 from (14) and c_2 from (12) into the equation for λ from (12) gives the value of λ or γ as used in the assignment:

$$\begin{aligned} \lambda =& \sqrt{c_1^2 + c_2^2} \\ \lambda = \gamma \approx& 1.8483 \end{aligned} \tag{15}$$

The found solution is simulated with MATLAB and the resulting distribution of the rope over the xy space is visualized in figure 1a. Also the cumulative sum of the rope length is shown in figure 1b so it is clear that the length constraint is met.



(a) Rope visualized in the xy space

(b) Cumulative sum of the rope length over x

Figure 1: Simulated solution of the Dido's problem