Homework assignment 3

Optimal control and dynamic programming (4SC000), TU/e, 2018-2019

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Submission: The solutions of Assignment 2 should be submitted via Matlab grader and the solution of Assignment 1, 3, and 4 should be submitted only via canvas.

Grade: The assignment will be graded on a scale 0-10 according to the following distribution

Assignment	1	2	3	4	Total
Max score	2.5	2.5	2.5	2.5	10

For assignment 2 the solution will be considered correct if the chosen tuning matrices meet the specifications (only one test in Matlab grader).
Deadline: January 28th, 23h45, 2019.

1. Relative degree of the LQR open-loop transfer function

Consider a linear system

$$\dot{x} = Ax + Bu,$$

with $x(t) \in \mathbb{R}^{n_x}$ and $u(t) \in \mathbb{R}$, for every $t \in \mathbb{R}_{\geq 0}$. Assume that the pair (A, B) is controllable. Consider also the linear quadratic regulator (LQR):

$$u = Kx,$$

where

$$K = -R^{-1}B^{\mathsf{T}}P,$$

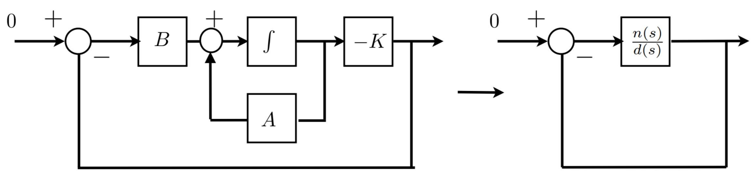
and P is the unique positive definite solution to

$$P = A^{\mathsf{T}}P + PA + Q - PBR^{-1}B^{\mathsf{T}}P,$$

for positive definite weighting matrices Q and R. Let

$$\frac{n(s)}{d(s)} = -K(sI - A)^{-1}B$$

be the open-loop transfer function of the LQR loop.



Assignment 1 Prove that, under the stated assumptions, the relative degree of the open-loop transfer function of the LQR compensator is always 1, i.e., the difference between the degree of the denominator $d(s)$ (number of poles) and the degree of the numerator $n(s)$ (number of zeros) is 1.

2. Guaranteed gain and phase margins with LQG control

Consider a process described by the transfer function

$$t(s) = \frac{s^2 + 4.3s + 1.2}{s^3 - 9s^2 - 12s + 160}$$

which can be written in the standard state-space form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + w(t) \\ y(t) &= Cx(t) + v(t) \end{aligned}$$

where A, B, C, are such that $t(s) = C(sI - A)^{-1}B$ and $w(t)$, $v(t)$ are Gaussian white noise model process disturbances and output noise with $\mathbb{E}[v(t)v(t+\tau)] = V\delta(\tau)$, $\mathbb{E}[w(t)w(t+\tau)] = W\delta(\tau)$. Design an LQG controller, by picking the matrices Q, R of the cost

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \mathbb{E}[x(t)^\intercal Qx(t) + u(t)^\intercal Ru(t)]dt$$

and the matrices W, V, to guarantee the downward and upward gain margins $\text{GM}^- = 0.501$, $\text{GM}^+ = 20$, respectively and the negative and positive phase margins $\text{PM}^- = -40$, $\text{PM}^+ = 40$, respectively. [1]

Assignment 2
Provide the gains meeting the specifications as the output of a Matlab function

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[Q,R,W,V] = gainslqgmargins
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[1] Only the gain margin specifications are tested in cody.

3. Dido's problem

Dido's problem: *'According to a legend about the foundation of Carthage (around 850 B.C.), Dido purchased from a local king the land along the North African coastline that could be enclosed by the hide of an ox. She sliced the hide into very thin strips, tied them together, and was able to enclose a sizable area which became the city of Carthage' [1, Ch.2]*

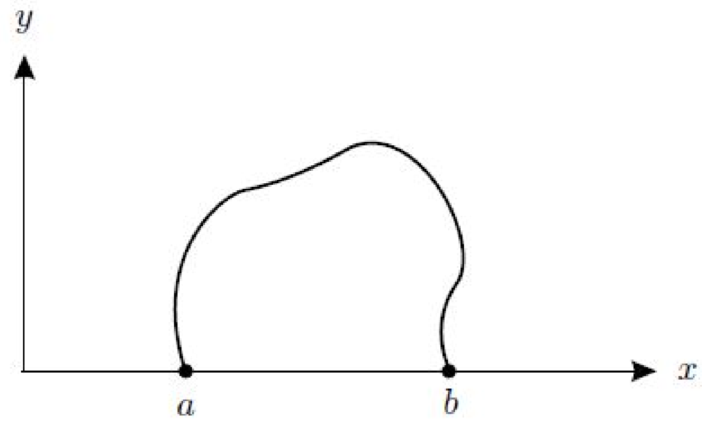

Figure 1: Dido's problem, figure taken from [1, Ch.2]

Let us assume $a = 0$, $b = 3$ are fixed, a length $L = 3.5$, and let us formulate the problem mathematically. We wish to find the curve $y(x)$ of length $L = 3.5$ with fixed end points $y(0) = y(3) = 0$ with maximal area

$$\int_0^3 y(x)dx.$$

The length of the curve is given by $\int_0^3 \sqrt{1 + (y'(x))^2}dx$ and if $y(x)$ is an optimal curve for this problem it is also an optimal curve for the following problem

$$\int_0^3 y(x)dx + \gamma(\int_0^3 \sqrt{1 + (y'(x))^2}dx - L),$$

for some constant γ.

Assignment 3 Find the optimal curve using the Pontryagin's maximum principle and provide the value of γ.

4. Minimum time control of a mass-spring system

Suppose that we wish to drive to rest a mass-spring system

$$\ddot{p}(t) = -p(t) + 2u(t), \quad t \in \mathbb{R}_{\geq 0}$$

in minimum time, i.e., $(p(T), v(T)) = (0, 0)$, where $v(t) := \dot{p}(t)$ and T is the optimal time; the control input is constrained to the set $u(t) \in [-2, 2]$. Suppose that the initial position $p(0)$ and the initial velocity $v(0)$ are given. Then, one can compute the optimal control input $u(t)$, $t \in [0, T]$, where T is the terminal time, which is a function of $(p(0), v(0))$.

Assignment 4
Using Matlab, compute T and plot the optimal u and the corresponding functions p, v for the following initial conditions

(i) $(p(0), v(0)) = (8, 0)$

(ii) $(p(0), v(0)) = (-9, -8)$

(iii) $(p(0), v(0)) = (2, -1)$

References

[1] D. Liberzon, *Calculus of Variations and Optimal Control Theory : A Concise Introduction.* Princeton university press, 2012.