AST1420 "Galactic Structure and Dynamics" Problem Set 2

Due on Oct. 27 at the start of class

Most of the exercises in this problem set must be solved on a computer and the best way to hand in the problem set is as an `jupyter notebook`. Rather than sending me the notebook, you can upload it to `GitHub`, which will automatically render the notebook. Rather than starting a repository for a single notebook, you can upload your notebook as a `gist`, which are version-controlled snippets of code that can optionally be made private.

If you want to upload your notebook as a gist from the command-line, you can use the package at this http URL and use it as follows. Log into your `GitHub` account:

```
gist --login
```

and then upload your notebook `AST1420_2017_PS2_YOURNAME.ipynb` as

```
gist -p AST1420_2017_PS2_YOURNAME.ipynb
```

(the `-p` option will make the gist private). If you want to make further changes, you can clone your gist in a separate directory and use it as you would any other git repository. *Please re-run the entire notebook (with `Cell > Run All`) after re-starting the notebook kernel before uploading it*; this will make sure that the input and output are fully consistent.

If you are unfamiliar with notebooks, you can also hand in a traditional write-up, but you also need to send in well-commented code for how you solved the problems. Thus, notebooks are strongly preferred :-)

Problem 1: (10 points) The total dark-matter halo mass of Dragonfly 44. In class, we discussed the ultra-diffuse galaxy Dragonfly 44, which has a total mass of $M(< 4.6\,\mathrm{kpc}) = 0.72 \pm 0.22 \times 10^{10}\,M_\odot$. Here we will investigate what this implies for the total dark-matter halo mass of Dragonfly 44.

(a) An alternative parameterization of an NFW halo is through its *virial mass* M_{vir} and its concentration c. The virial mass is the mass enclosed within the *virial radius* r_{vir}, which is defined as the radius within which the mean density of the NFW halo is some multiple Δ_h of the mean-matter density $\bar{\rho} = \rho_{\mathrm{crit}}\,\Omega_m$ in the Universe (where ρ_{crit} is the critical density and Ω_m is the matter density parameter). For the purposes of this exercise, use $H = 70\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$, $\Omega_m = 0.3$, and $\Delta_h = 200$. Setting $c = 12$, compute the virial radius (in kpc) for a halo with $M_{\mathrm{vir}} = 10^{12}\,M_\odot$.

(b) Plot the enclosed mass curves for $M_{\mathrm{vir}} = 10^{10}, 10^{11}, 10^{12}, 10^{13}\,M_\odot$ assuming $c = 12$ over the range $1\,\mathrm{kpc} \leq r \leq 100\,\mathrm{kpc}$.

(c) Now plot the same curves, but also add for each curve the range spanned by reducing the concentration to $c = 9$ and increasing it to $c = 16$ (this is roughly the scatter in the concentration found in numerical simulations; we are ignoring the minor dependence of concentration on mass here). Also add the observed enclosed-mass data point for Dragonfly 44. You should end up with something that looks similar to Figure 5 in van Dokkum et al. (2016; ApJL, 828, 6). Comment on how strongly the observational point constrains the virial mass.

Problem 2: (10 points) The Kuzmin disk. We discussed the Kuzmin disk as a simple example of a razor-thin disk potential. We explore it and orbits in it a little further in this problem.

(a) Draw contours of the Kuzmin potential with $a = 2$ in the (x, z) plane *without explicitly evaluating the potential* (i.e., do not just evaluate the potential and contour this).

(b) Prolate spheroidal coordinates are defined by $R = \Delta\,\sinh u\,\sin v\,,\ z = \Delta\,\cosh u\,\cos v\,,$ where Δ is a constant. Show that

$$R^2 + (|z| + \Delta)^2 = \Delta^2\,(\cosh u + |\cos v|)^2\,. \tag{1}$$

(c) Use Equation (1) to express the Kuzmin potential as (choose Δ wisely)

$$\Phi(u, v) = -\frac{G\,M}{a}\,\frac{\cosh u - |\cos v|}{\sinh^2 u + \sin^2 v}\,. \tag{2}$$

(d) Write down the Lagrangian $\mathcal{L}(u, v, \dot{u}, \dot{v})$ and find the relation between the momenta (p_u, p_v) and $(p_R, p_z) = (\dot{R}, \dot{z})$.

(e) Write down the Hamiltonian $H(u, v, p_u, p_z)$. What do you notice about this Hamiltonian? Show that the equation expressing that the Hamiltonian is equal to the conserved energy, $H = E$, can be written as

$$2\Delta^2\,\left[E\,\sinh^2 u + \frac{GM}{a}\,\cosh u\right] - p_u^2 - \frac{L_z^2}{\sinh^2 u} = \frac{L_z^2}{\sin^2 v} + p_v^2 - 2\Delta^2\,\left[E\,\sin^2 v - \frac{GM}{a}|\cos v|\right]\,. \tag{3}$$

Because the left-hand side only depends on u and the right-hand side only depends on v, this means that both sides must equal a constant, which we define to be $2\,\Delta^2\,I_3$. This I_3 is a third integral of the motion in addition to (E, L_z) and the Kuzmin disk therefore has three explicit integrals of the motion.

(f) Setup a Kuzmin potential in `galpy` with $GM = 1.8$ and $a = 0.7$. Integrate an orbit with initial conditions $(R, v_R, v_T, z, v_z, \phi) = (1, 0.2, 0.9, 0, 0.1, 0)$ in this potential from $t = 0$ to $t = 20$, obtaining 10,001 points along the orbit (because of the force discontinuity at $z = 0$, the default orbit integration in `galpy` does not conserve energy well enough for this orbit, use the parameter `dt=0.000001` in `Orbit.integrate` to make sure that a small enough step size is used for the orbit integration). Plot the following

- The orbit in (R, z)
- The orbit in (u, v)

- The orbit in (u, p_u)
- The orbit in (v, p_v)
- I_3 vs. time using the left-hand side of Equation (3) and overplot I_3 vs. time using the right-hand side of the same equation.

Comment on what you see in these plots.