AST 1430

Cosmology

Week Index	Dates	Topics
Week 1	Jan 11, 13	logistics, introduction, basic observations
Week 2	Jan 18, 20	Basic GR, RW metric, Distances, coordinates, Friedmann equations, cosmological models
Week 3	Jan 25, 27	Consistency with observations, Early Hot Universe, BBN
Week 4	Feb 1, 3	Inflation, Perturbations & Structure pre-recombination
Week 5	Feb 8, 10	CMB: basics, polarization, secondaries
Week 6	Feb 15 (2h)	Early-Universe Presentations
		Reading Week – No Class
Week 7	Mar 1, 3	Post-recombination growth of structure, formation of dark matter halos, halo mass function
Week 8	Mar 8, 10	The relation between dark matter halos and galaxies
Week 9	Mar 15, 17	Probing the cosmic density field / clustering
Week 10	Mar 22, 24	Late Universe Presentations; lcosmological observations: BAO, supernovae, weak lensing, etc.
Week 11	Mar 29, 31	H0 controversy: how fast exactly is the Universe expanding today?
Week 12	Apr 5	Review

Keith

Jo

asst1

asst2

asst3

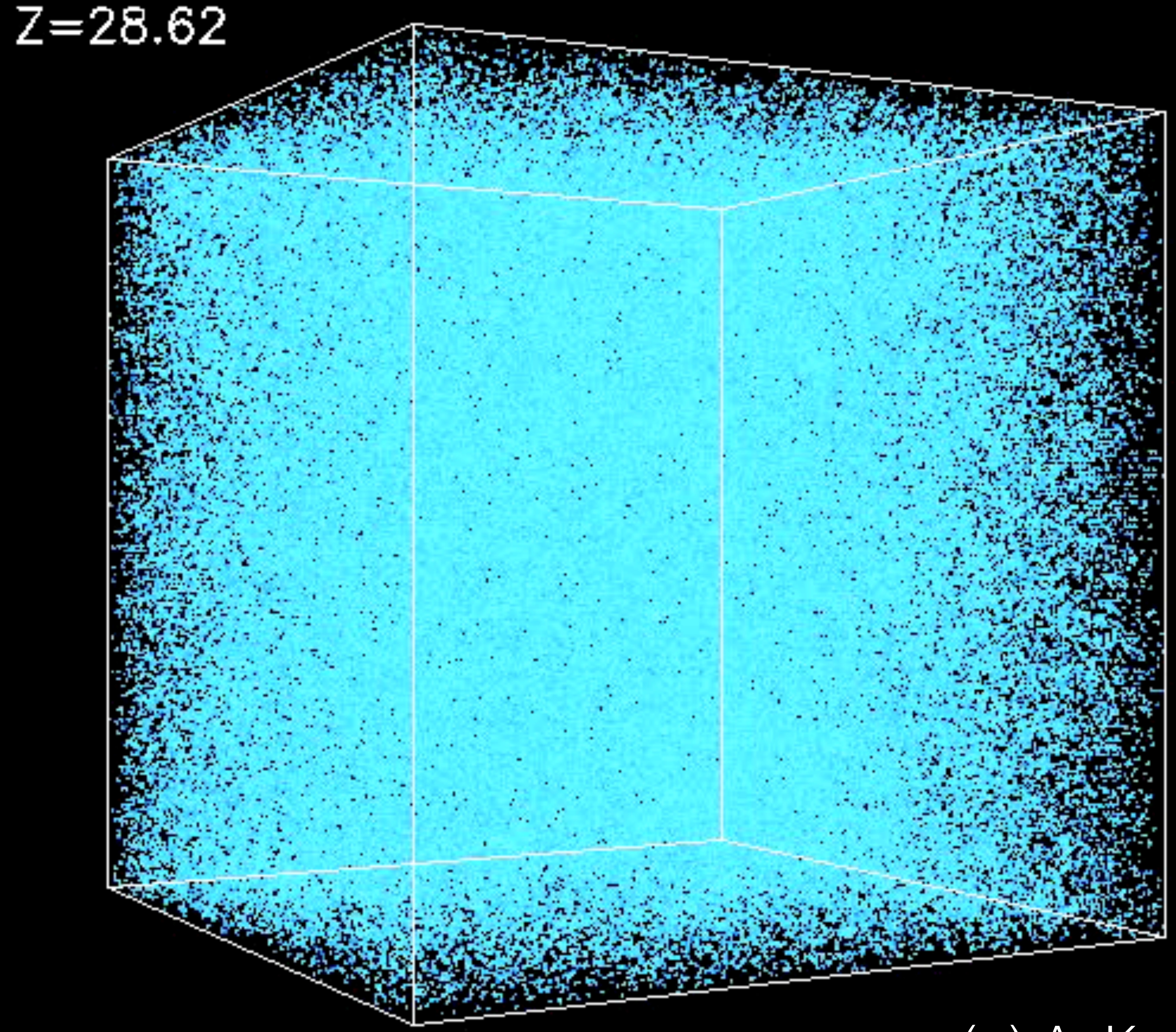
Late-time cosmology

https://www.illustris-project.org/movies/illustris_movie_cube_sub_frame.mp4

The growth of structure

Post-recombination

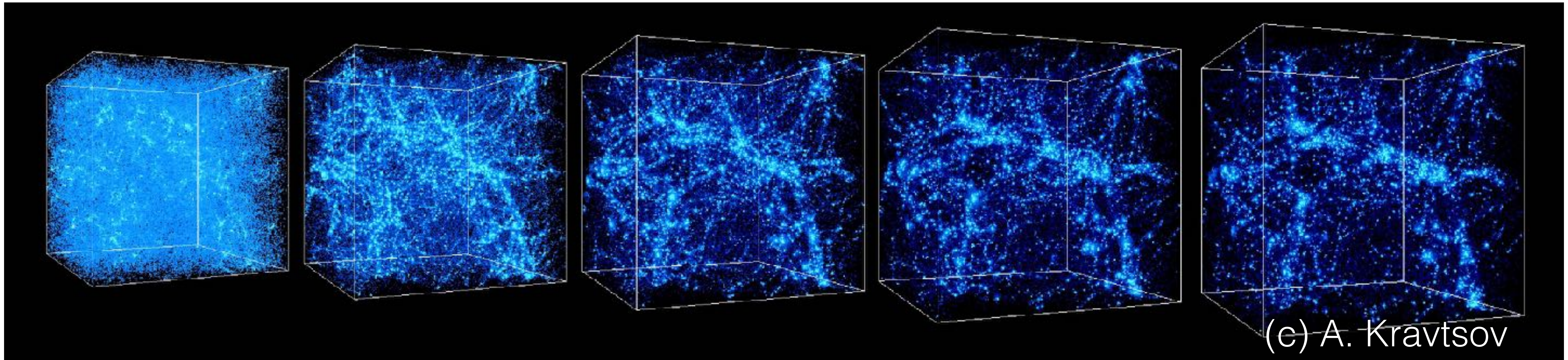
Evolution of small DM perturbations

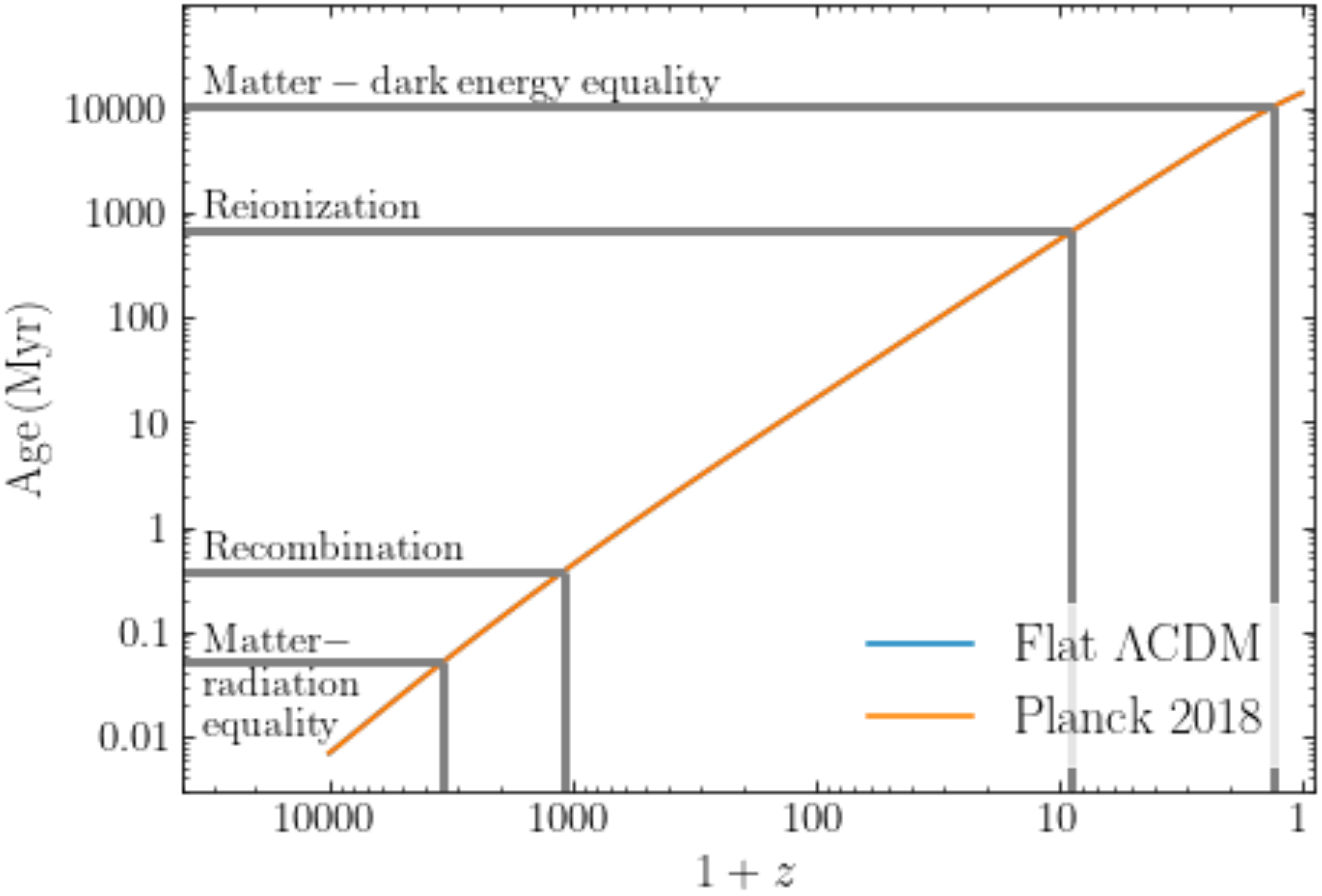


(c) A. Kravtsov

Evolution of small DM perturbations

- All components of the Universe have initial conditions set at very early times —> initial pattern of under- and over-densities δ with respect to the mean density
- (We think these are set through quantum fluctuations arising during inflation)
- (We think the initial distribution of δ is Gaussian and adiabatic, i.e., all different components are *in phase*)
- These over densities evolve over time, due to gravity and other physical processes
- DM evolution is in many ways the simplest, because no physics other than gravity
- (of course, gravity couples DM to all other energy components, so detailed evolution depends on all physics)





Matter – dark energy equality
Reionization
Recombination
Matter–
radiation
equality
Flat ΛCDM
Planck 2018
Age (Myr)
10000
1000
100
10
1
0.1
0.01
10000
1000
100
10
1
1 + z

Evolution of DM perturbations

- DM in the Universe is described by a distribution function f(**q**,**p**,t): number density of DM particles as a function of position, momentum, and time

- DM is collisionless in simplest model

- On scales small compared to the scale of the Universe, its evolution is therefore given by the collisionless Boltzmann equation (CBE; essentially a continuity equation for the phase-space density)

$$\frac{\partial f(\mathbf{q},\mathbf{p},t)}{\partial t} + \frac{\partial H}{\partial \mathbf{p}}\,\frac{\partial f(\mathbf{q},\mathbf{p},t)}{\partial \mathbf{q}} - \frac{\partial H}{\partial \mathbf{q}}\,\frac{\partial f(\mathbf{q},\mathbf{p},t)}{\partial \mathbf{p}} = 0\,,$$

- We can derive a good form for the CBE in this case, by starting from the Lagrangian in *proper* coordinates

$$\mathcal{L}(\mathbf{r},\dot{\mathbf{r}},t) = \frac{m|\dot{\mathbf{r}}^2|}{2} - m\Psi(\mathbf{r},t)$$

Evolution of DM perturbations in co-moving coordinates

- Universe is expanding, which causes displacements and velocities to grow —> makes sense to remove this effect

- Co-moving coordinates: separate expansion from positions and velocities relative to expansion

  - **r**(t) = a(t) **x** where a(t) is the *scale factor*

- Lagrangian in terms of co-moving **x**

$$\mathcal{L}(\mathbf{x}, \dot{\mathbf{x}}, t) = \frac{m}{2}\,(\dot{a}\,\mathbf{x} + a\,\dot{\mathbf{x}})^2 - m\Psi(\mathbf{x}, t)$$

Hamiltonian for the DM

- Following Lagrangian formalism, can compute generalized momentum

$$\mathbf{p}' = \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}} = m\,a\;(\dot{a}\,\mathbf{x} + a\,\dot{\mathbf{x}})$$

- and the Hamiltonian

$$H(\mathbf{x}', \mathbf{p}', t) \;=\; \frac{m\,a^2}{2}\,\dot{\mathbf{x}}'^2 - \frac{m}{2}\,\dot{a}^2\,\mathbf{x}'^2 + m\Psi(\mathbf{x}', t)$$

- To further simplify, we perform a canonical transformation of phase-space (**x**’,**p**’) —> (**x**,**p**) generated by F_3(**p**’,**x**)

$$F_3(\mathbf{p}', \mathbf{x}) = -\mathbf{p}' \cdot\; \mathbf{x} + m\,a\,\dot{a}\,|\mathbf{x}|^2/2$$

- This gives the final Hamiltonian and potential

$$H(\mathbf{x}, \mathbf{p}, t) = \frac{1}{2\,m\,a^2}\,\mathbf{p}^2 + m\Phi(\mathbf{x}, t) \qquad \nabla^2\Phi(\mathbf{x}, t) = 4\pi\,G\overline{\rho}(t)\,a(t)^2\,\delta(\mathbf{x}, t)$$

CBE for the DM

- The collisionless Boltzmann equation then becomes (starting from the general form):

$$\frac{\partial f}{\partial t} + \frac{1}{m\,a^2}\,\mathbf{p}\,\frac{\partial f}{\partial \mathbf{x}} - m\,\frac{\partial \Phi}{\partial \mathbf{x}}\,\frac{\partial f}{\partial \mathbf{p}} = 0$$

- Integrating over momentum (~Jeans equation) —> continuity equation

$$\frac{\partial \delta}{\partial t} + a^{-1}\,\nabla\cdot\langle\mathbf{v}\rangle = 0$$

- Multiply by v_i and integrate over momentum

$$\frac{\partial}{\partial t}([1+\delta]\ \langle v_i\rangle) + \frac{\dot{a}}{a}\,(1+\delta)\ \langle v_i\rangle = -\frac{(1+\delta)}{a}\,\frac{\partial \Phi}{\partial x_i} - a^{-1}\sum_j \frac{\partial}{\partial x_j}\big([1+\delta]\,\langle v_i\,v_j\rangle\big)$$

- Re-arrange

$$\frac{\partial\langle\mathbf{v}\rangle}{\partial t} + \frac{\dot{a}}{a}\,\langle\mathbf{v}\rangle + a^{-1}\,(\langle\mathbf{v}\rangle\cdot\nabla)\langle\mathbf{v}\rangle = -a^{-1}\,\nabla\Phi - \frac{1}{a\,(1+\delta)}\,\nabla\cdot\big([1+\delta]\ \sigma^2\big)$$

- Dark matter is cold before halo formation —> sigma is small and can be ignored

$$\frac{\partial\langle\mathbf{v}\rangle}{\partial t} + \frac{\dot{a}}{a}\,\langle\mathbf{v}\rangle + a^{-1}\,(\langle\mathbf{v}\rangle\cdot\nabla)\langle\mathbf{v}\rangle = -a^{-1}\,\nabla\Phi$$

Equations for δ << 1

- Keeping only first-order terms in δ or v [which is O(δ)]

$$\frac{\partial \delta}{\partial t} + a^{-1} \nabla \cdot \langle \mathbf{v} \rangle = 0\,,$$
$$\frac{\partial \langle \mathbf{v} \rangle}{\partial t} + \frac{\dot{a}}{a} \langle \mathbf{v} \rangle = -a^{-1} \nabla \Phi$$

- Can combine to a single equation

$$\frac{\partial^2 \delta}{\partial t^2} + 2\,\frac{\dot{a}}{a}\,\frac{\partial \delta}{\partial t} = 4\pi\, G \bar{\rho}\, \delta\,.$$

- These equations are just like those of a classical fluid with zero pressure

Evolution of baryonic overdensities

- Baryons are not collisionless, because in gaseous form —> mean-free path << scale of the Universe —> classical fluid (assume no EM coupling to photons here, z < 1000)

- Equations: continuity, Euler, and Poisson

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \, \mathbf{u} = 0 \, , \qquad \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \, \mathbf{u} = -\frac{\nabla P}{\rho} - \nabla \Psi \, ,$$

- Convert to co-moving coordinates and take limit of small δ, eventually end up with

$$\frac{\partial^2 \delta}{\partial t^2} + 2 \, \frac{\dot{a}}{a} \, \frac{\partial \delta}{\partial t} = 4\pi \, G \, \overline{\rho} \, \delta + \frac{c_s^2}{a^2} \, \nabla^2 \delta$$

Co-evolution of DM and baryons

- Gravity is the only coupling, so equations become

$$\frac{\partial^2 \delta_c}{\partial t^2} + 2\,\frac{\dot{a}}{a}\,\frac{\partial \delta_c}{\partial t} = 4\pi\, G\,\overline{\rho}\,\left(\delta_c + \delta_b\right)\,,$$
$$\frac{\partial^2 \delta_b}{\partial t^2} + 2\,\frac{\dot{a}}{a}\,\frac{\partial \delta_b}{\partial t} = 4\pi\, G\,\overline{\rho}\,\left(\delta_c + \delta_b\right) + \frac{c_s^2}{a^2}\,\nabla^2 \delta_b$$

- On large scales, pressure is small —> DM and baryons behave the same

Fourier solution

- Equations are linear partial differential equations —> Fourier transform them to obtain ordinary differential equations

$$\frac{\partial^2 \delta_{\mathbf{k}}}{\partial t^2} + 2\,\frac{\dot{a}}{a}\,\frac{\partial \delta_{\mathbf{k}}}{\partial t} = 4\pi\, G\,\overline{\rho}\,\delta_{\mathbf{k}} - \frac{k^2\,c_s^2}{a^2}\,\delta_{\mathbf{k}}$$

$$\delta_{\mathbf{k}}(t) = \frac{1}{V}\int_V \mathrm{d}\mathbf{x}\,\delta(\mathbf{x},t)\,\exp(-i\mathbf{k}\cdot\mathbf{x})$$

- Or in terms of the Jeans length

$$\frac{\partial^2 \delta_{\mathbf{k}}}{\partial t^2} + 2\,\frac{\dot{a}}{a}\,\frac{\partial \delta_{\mathbf{k}}}{\partial t} = 4\pi\, G\,\overline{\rho}\left(1 - \frac{k^2}{k_J^2}\right)\delta_{\mathbf{k}}$$

$$k_J^2 = \frac{4\pi\, G\,\overline{\rho}\,a^2}{c_s^2} \qquad \lambda_J = \frac{2\pi a}{k_J} = c_s\sqrt{\frac{\pi}{G\,\overline{\rho}}}\,.$$

Fourier solution

- CDM: no scale dependence anywhere in equation —> over densities grow the same on all scales
- Baryons:
  - at $k < k_J$ (large scales), no scale dependence, so same growth as CDM
  - on small scales, growth suppressed due to pressure

Solutions for CDM

- CDM equation has two solutions: one growing, one decaying —> growing 'mode' is the important one on large timescales

- Can show relation between solutions:

$$\delta_+ \frac{\partial \delta_-}{\partial t} - \frac{\partial \delta_+}{\partial t} \delta_- \propto a^{-2}$$

- One solution given by

$$\delta_-(t) \propto H.$$

- Growing mode

$$\delta_+(t) \propto H(t) \int_0^t \frac{\mathrm{d}t'}{\dot{a'}^2(t')},$$

Einstein—de Sitter model

- Background evolution:

$$a(t) = a_0 \left(\frac{t}{t_0} \right)^{\frac{2}{3}}$$
$$H(t) = \frac{2}{3}\, t^{-1} \,.$$

- Growing mode: $\delta_+(t) \propto t^{2/3} \propto a(t)$

Dark energy only Universe

- Hubble function is constant —> ‘decaying’ mode is constant
- Growing mode turns out to be decaying as $1/a^2$
- In dark-energy dominated Universe, over densities therefore do not grow

Our Universe

- Approximate as: matter domination early —> dark-energy domination later, always flat
- During matter domination, over densities grow as a
- As z ~ few, growth slows down
- Now transitioning to DE domination —> growth of perturbations will end, structure formation will cease (peculiar velocities also decay, so accretion stops as well)

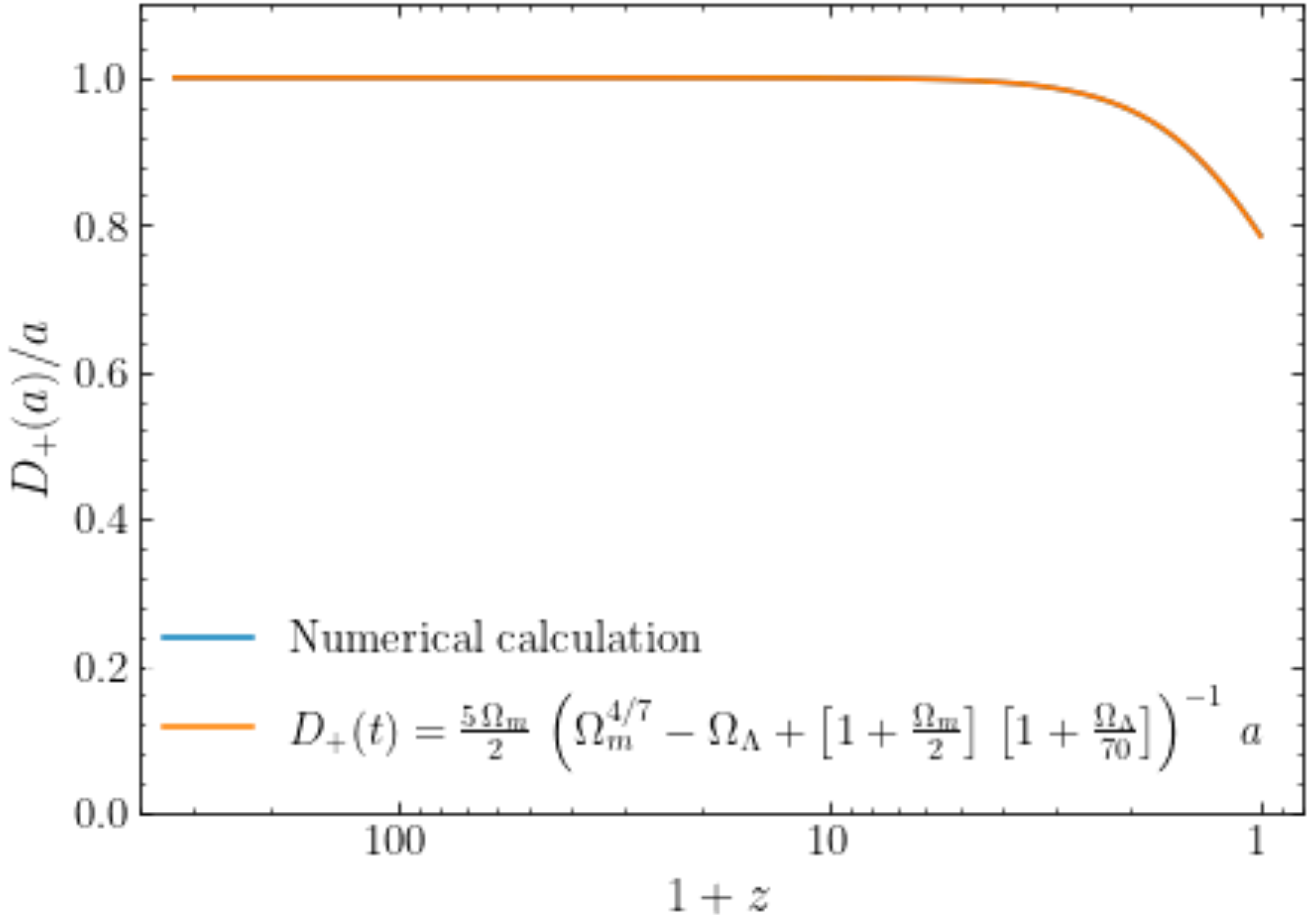
The growth factor

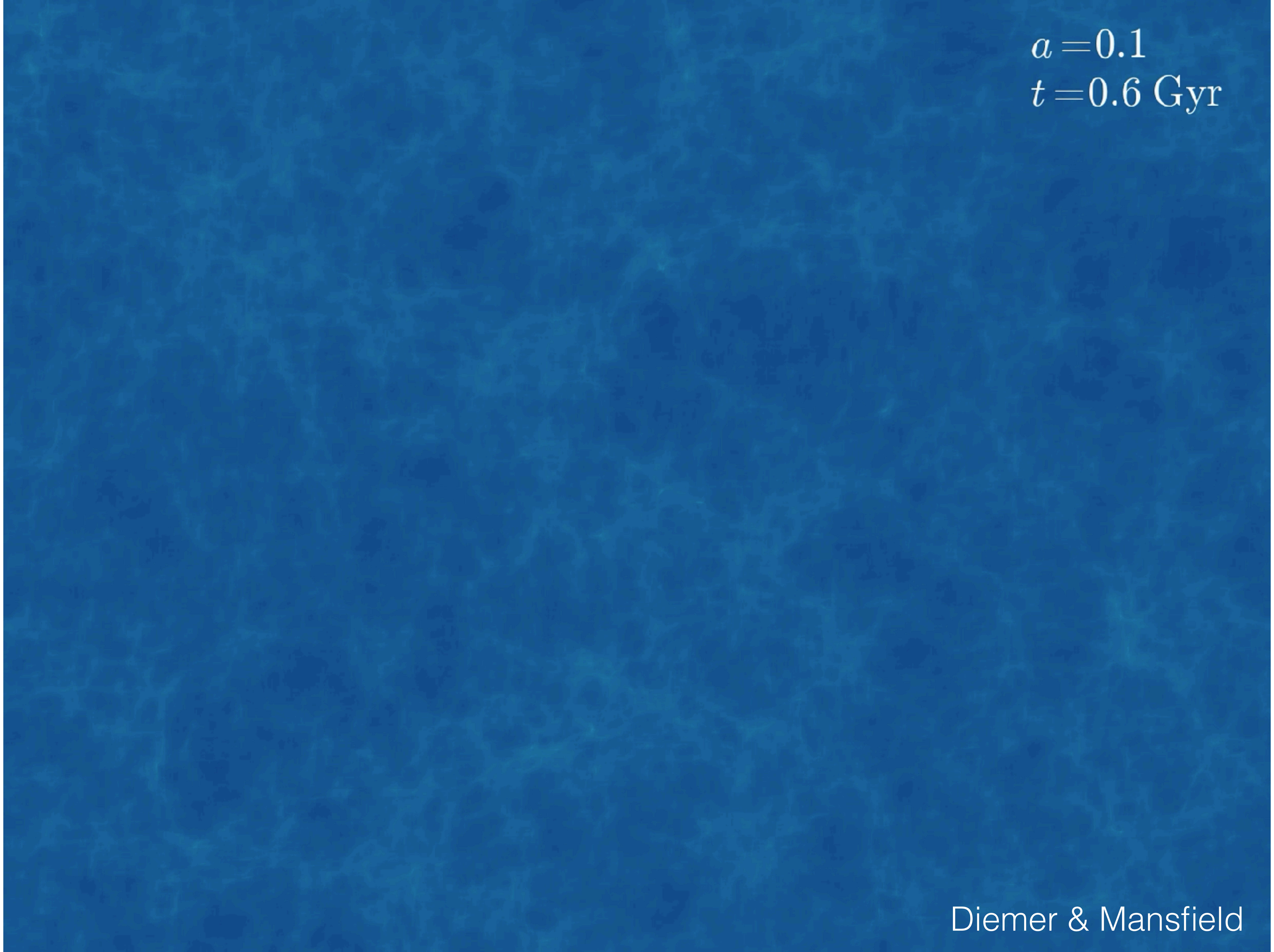
- Scale-free growth is usually encapsulated in something known as the (linear) **growth factor $D_+(a)$**
- Defined such that for EdS: $$D_+(a) = a$$
- General expression

$$D_+(a) = \frac{5\Omega_{0,m}}{2} \frac{H(a)}{H_0} \int_0^a \frac{\mathrm{d}a'}{(a' \, H(a')/H_0)^3} \, .$$

- $D_+(a)/a$ turns out to only depend on *a* through the density parameters Ω_i

The growth factor for our Universe




$a = 0.1$
$t = 0.6$ Gyr
Diemer & Mansfield

The growth of structure in the early Universe

See Dodelson & Schmidt Chapter 8

The growth of structure in the early Universe

- So far we have described the growth of structure in the matter-dominated and dark-energy dominated eras and saw it was scale-independent (for DM and large-scale baryons)

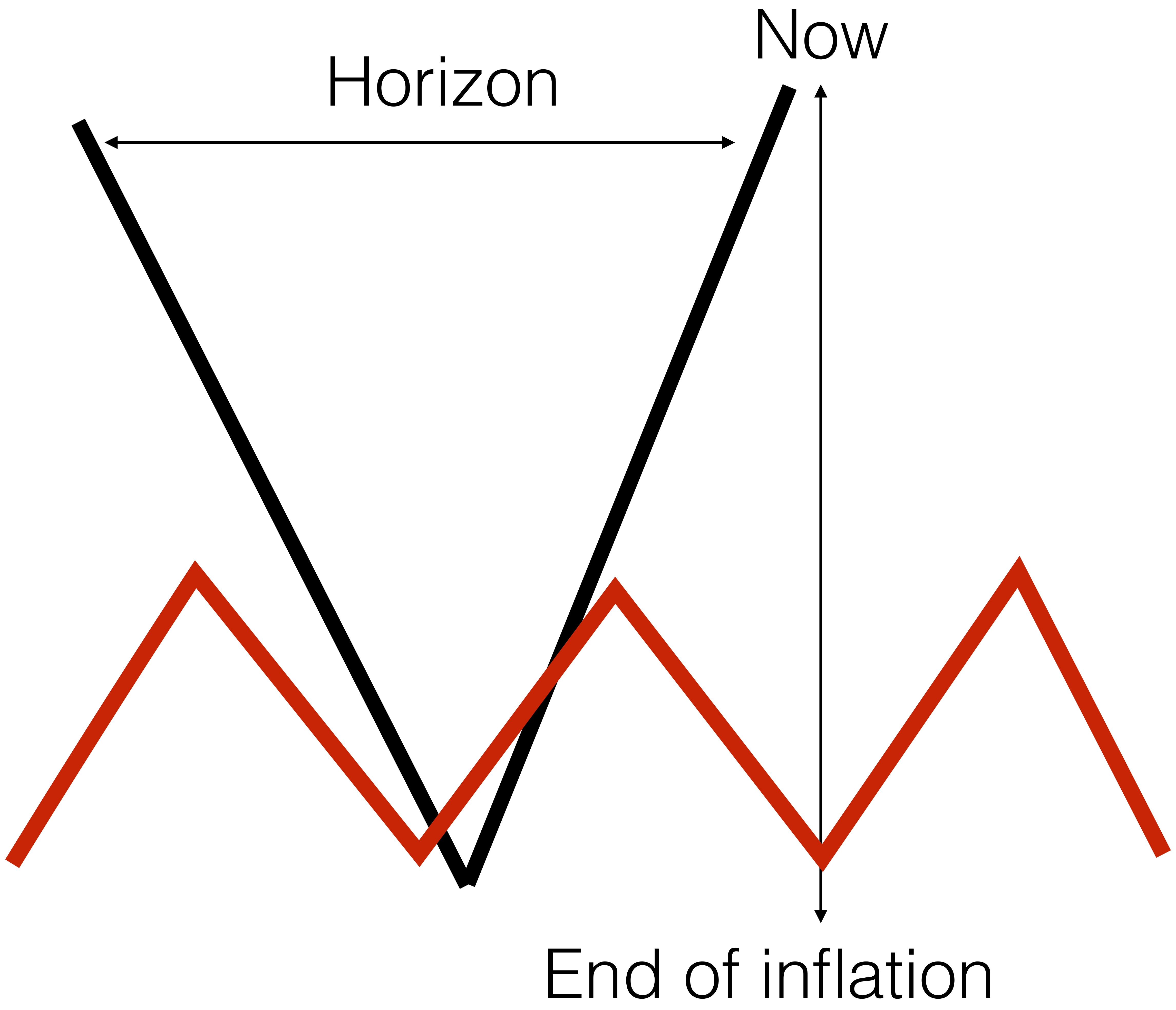
- To fully understand the growth of structure, we also need to discuss the growth of structure before matter domination:

    - Inflation lays down initial pattern of fluctuations

    - These evolve in a *scale-dependent* manner until matter domination starts

Evolution in Fourier space

- All *k* modes start out >> the horizon
- Horizon grows, so they enter the horizon at some point
- No physics beyond the expanding Universe on super-horizon scales, so little evolution possible



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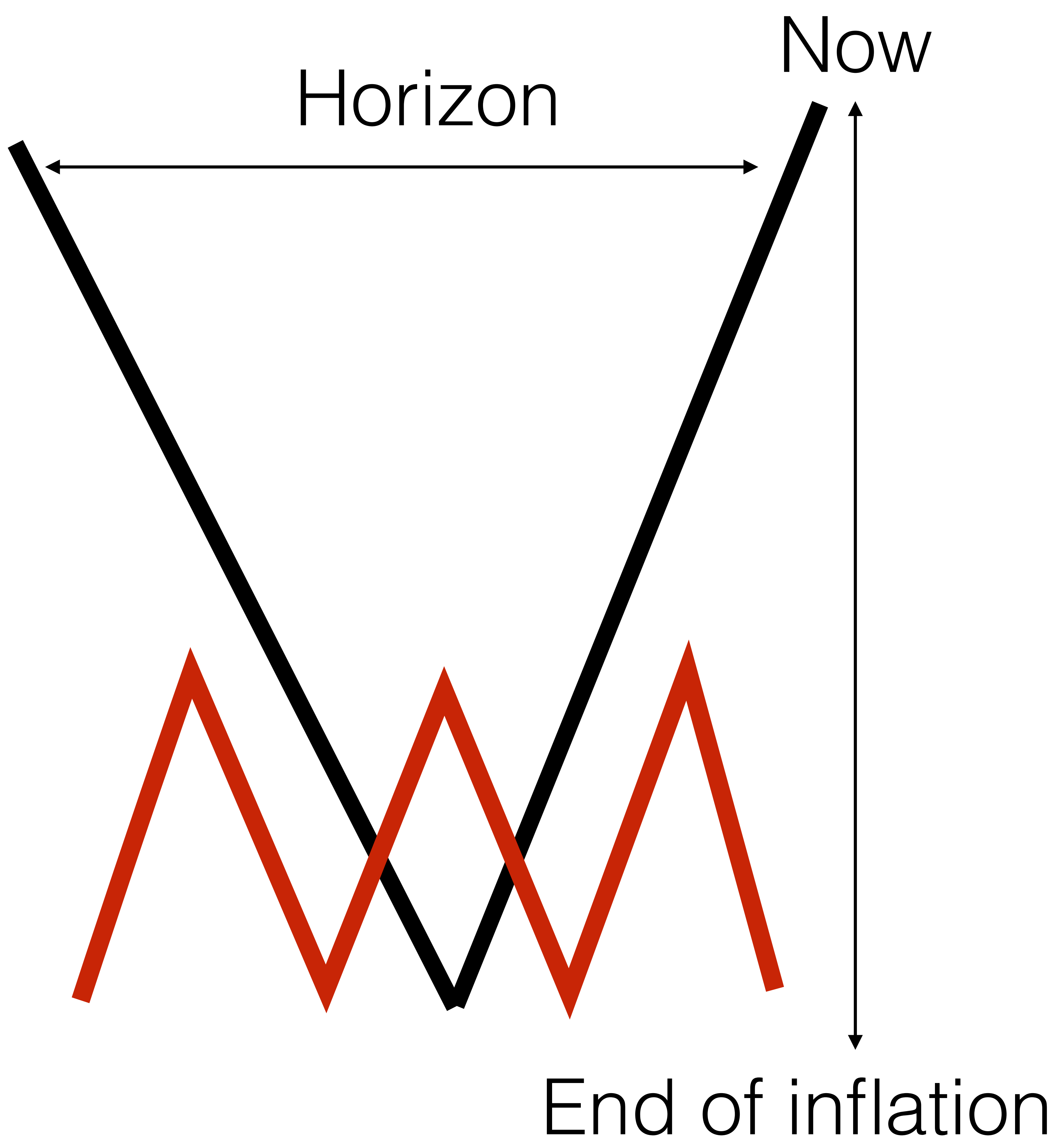

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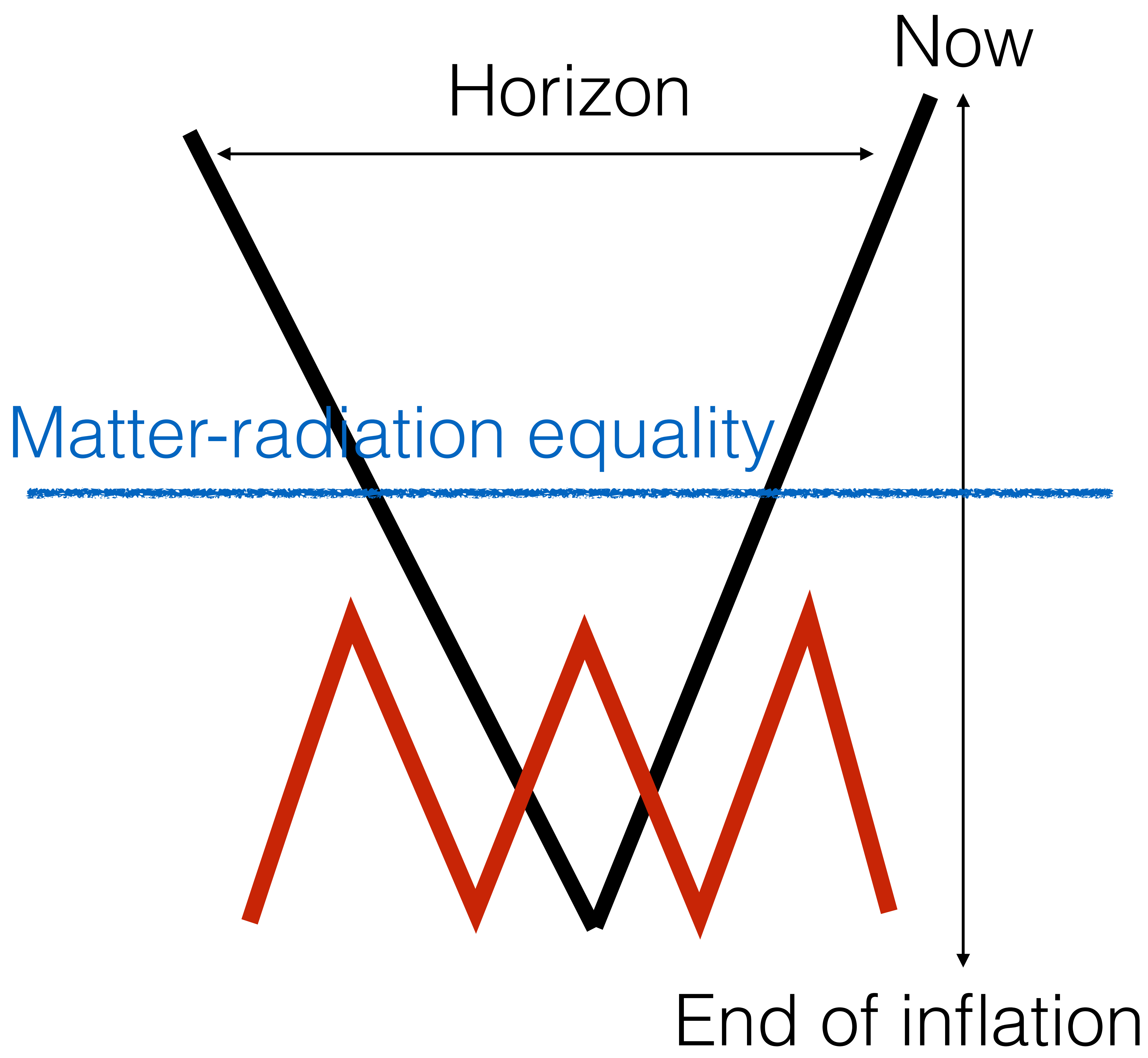


The evolution of the gravitational potential

- Because the gravitational potential at late times is entirely sourced by matter, we can understand the evolution of matter overdensities at late times through the evolution of the gravitational potential

$$\delta(\mathbf{k}, a) = -\frac{2}{3}\frac{k^2 a}{\Omega_{0,m} H_0^2}\,\Phi(\mathbf{k}, a)\,,$$

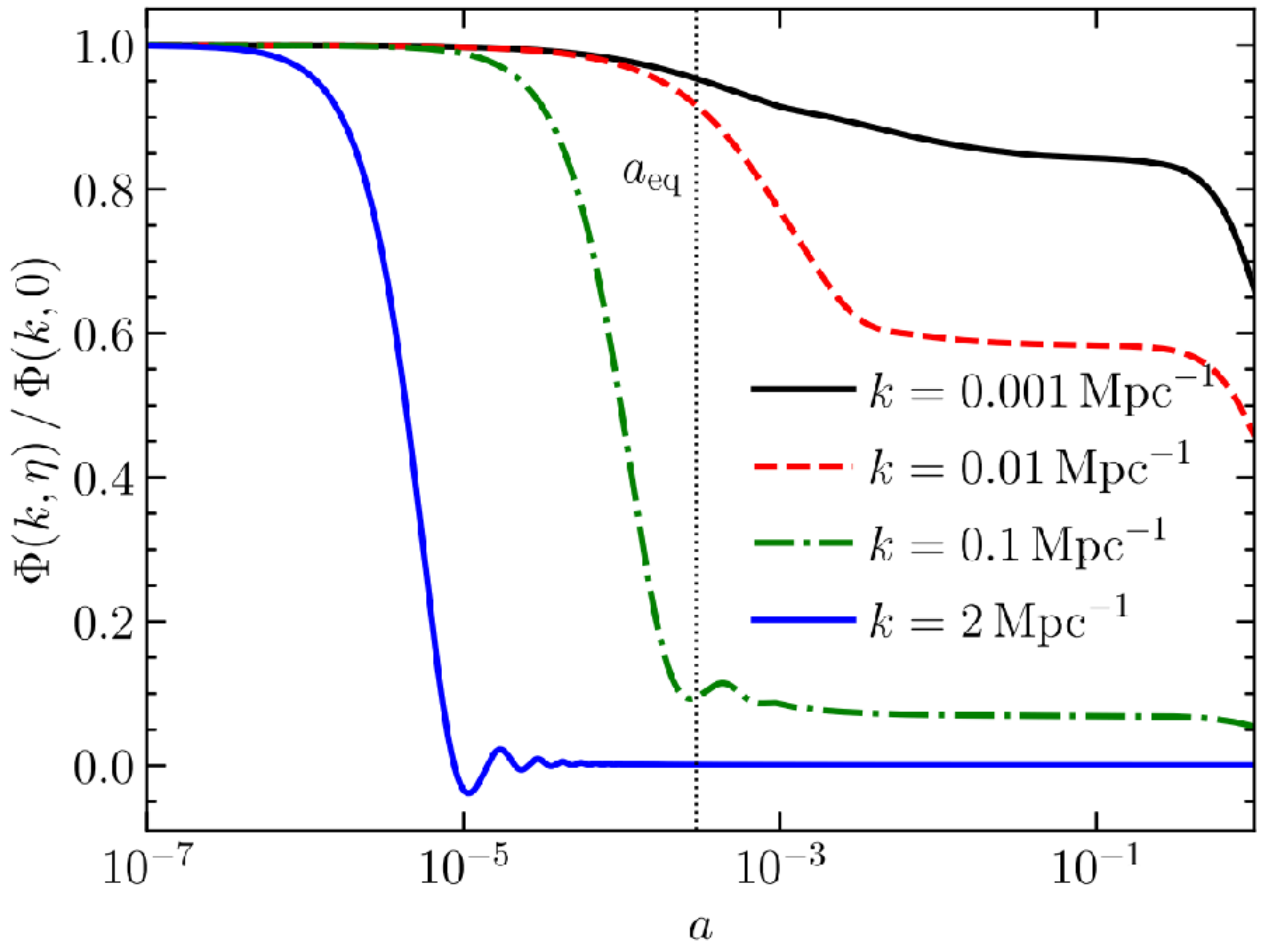
The evolution of the gravitational potential

- Very large scales (super-horizon until well into matter-domination):
  - potential is constant on super-horizon scales during radiation domination
  - Potential drops by 10% during the radiation —> matter transition
  - Potential constant during matter domination
  - Potential decays during dark-energy domination (scale-independent)

The evolution of the gravitational potential

- Small scales (enter the horizon during radiation domination):
  - Potential is constant on super-horizon scales
  - Potential decays after the mode enters the horizon
    - Potential decays because pressure stabilizes density fluctuations and expansion dilutes the overdensities
  - Potential remains constant during matter domination
  - Potential decays during dark-energy domination (scale-independent)

The evolution of the gravitational potential



Dodelson & Schmidt 2020; Fig. 8.1

The transfer function

- Scale-dependent, early Universe growth of the potential usually encapsulated in the *transfer function* T(k)

$$\Phi(\mathbf{k}, a) = \frac{3}{5}\, c^2\, \mathcal{R}(\mathbf{k})\, T(k)\, \frac{D_+(a)}{a}\,.$$

- Potential given in terms of
  - Initial condition 2/3 R(k) of the curvature perturbation R(k)
    [D&S Section 7.5]
  - Early-Universe evolution T(k)
  - Late-Universe evolution $D_+(a)/a$
- 3/5 = 2/3 x 9/10 such that T(large scale) = 1

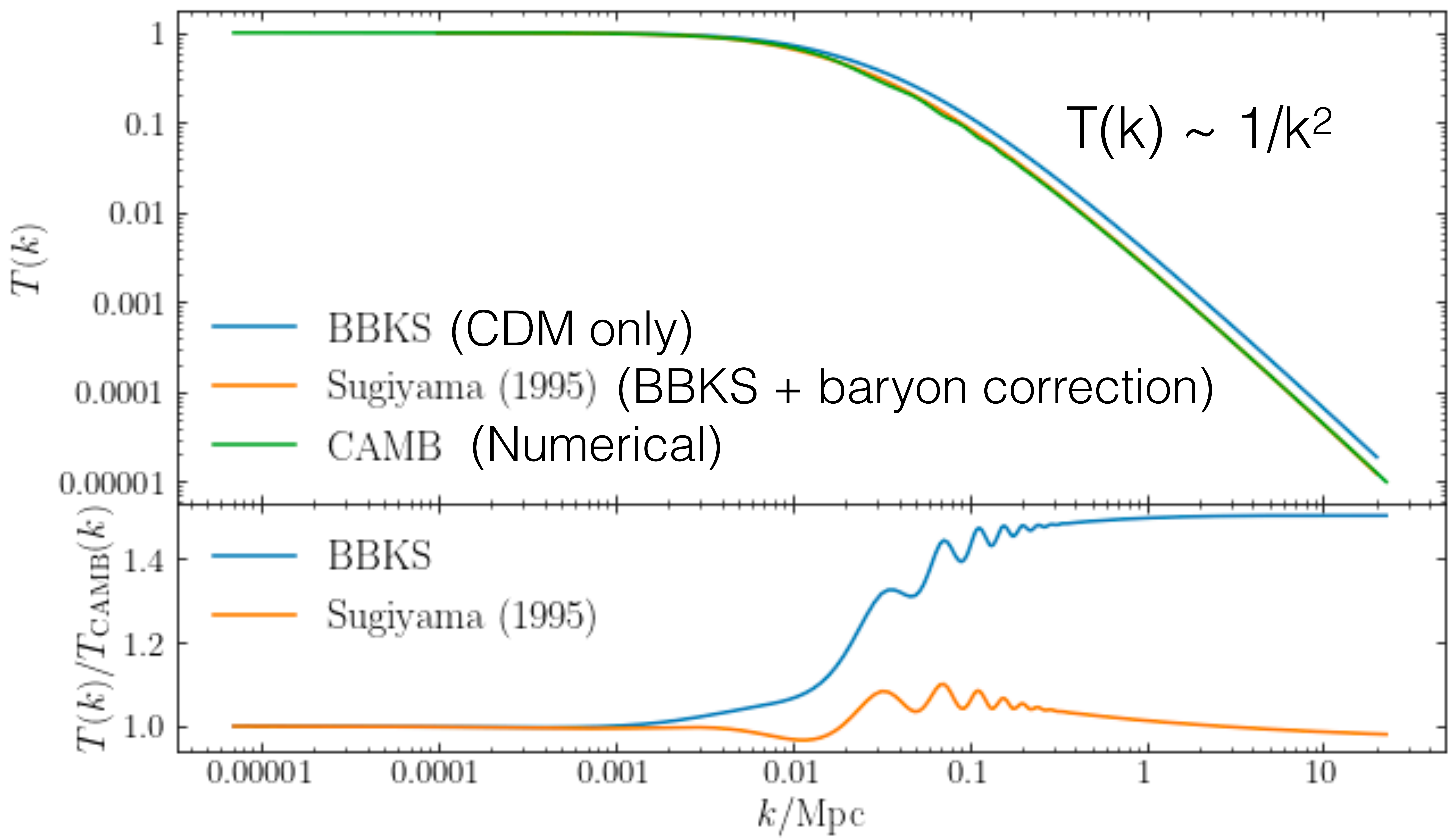
The transfer function

- From previous discussion:
  - Large scales (small *k*): T(k) = 1
  - Small scales (large *k*): T(k) drops with k (more potential decay the earlier a mode enters the horizon)
  - Transition at k_{eq} that enters the horizon at matter-radiation equality

$$k_{\rm eq} = 7.32 \times 10^{-2}\,\Omega_{0,m}\,h^2\,{\rm Mpc}^{-1}\,.$$

- Ignoring baryons (which are sub-dominant), T only depends on k/k_{eq}

The transfer function



The matter power spectrum

The matter power spectrum

- Initial fluctuations are a realization of a Gaussian random field, characterized by its power spectrum (of curvature perturbations R)

$$\langle \mathcal{R}(\mathbf{k})\mathcal{R}^*(\mathbf{k}')\rangle = (2\pi)^3\, P_{\mathcal{R}}(k)\, \delta^{(3)}(\mathbf{k}-\mathbf{k}')\,,$$

- During linear evolution, potential and density fluctuations are a similar Gaussian field, characterized by a power spectrum as well

$$\langle \delta(\mathbf{k})\delta^*(\mathbf{k}')\rangle = (2\pi)^3\, P_m(k)\, \delta^{(3)}(\mathbf{k}-\mathbf{k}')\,.$$

- $P_m(k)$ is the *linear matter power spectrum*

The initial curvature power spectrum

- Produced at the end of inflation

$$P_{\mathcal{R}}(k) = 2\pi^2 A_s \, k^{-3} \left(\frac{k}{k_p}\right)^{n_s - 1},$$

- Planck 2018:

$$n_s = 0.9665 \pm 0.0038\,,$$
$$\ln(10^{10} A_s) = 3.047 \pm 0.014\,.$$

The matter power spectrum

$$P_{\mathcal{R}}(k) = 2\pi^2 A_s k^{-3} \left(\frac{k}{k_p}\right)^{n_s - 1}$$ + $$\delta(\mathbf{k}, a) = -\frac{2}{3} \frac{k^2 a}{\Omega_{0,m} H_0^2} \Phi(\mathbf{k}, a)\,,$$

+

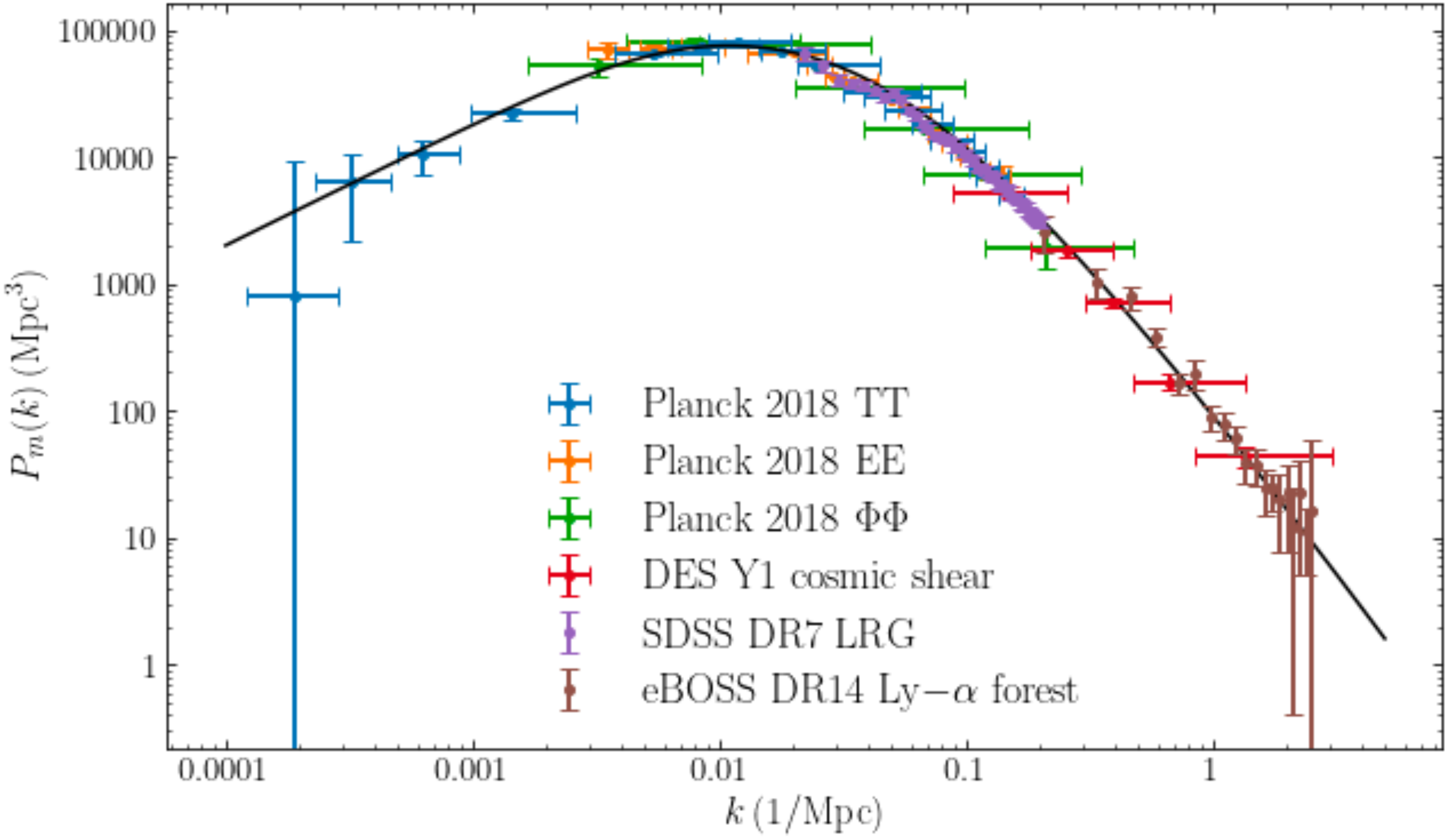
$$\Phi(\mathbf{k}, a) = \frac{3}{5} c^2 \mathcal{R}(\mathbf{k}) T(k) \frac{D_+(a)}{a}\,.$$

$$\downarrow$$

$$P_m(k, a) = \frac{8\pi^2}{25} \frac{A_s k_p c^4}{\Omega_{0,m}^2 H_0^4} T^2(k) D_+^2(a) \left(\frac{k}{k_p}\right)^{n_s}\,.$$

The matter power spectrum

Peak at k~k_{eq}



Halo formation

Halo formation

- Halos form when $\delta > 1$: over densities become non-linear, gravitationally collapse and form a bound, virialized object
- Equations derived before therefore no longer apply —> complicated process that needs to be studied with numerical simulations in detail
- Can make some headway with simple models of collapse

Spherical top-hat collapse

- Simplest model for halo formation: tophat overdensity ___|‾|___ that collapses radially (zero angular momentum)

- Consider this system as a series of spherical shells that expand and then collapse, equation of motion:

$$\ddot{r} = -\frac{GM}{r^2}$$

- Assuming that shells do not cross —> M(<r) is constant —> simple Keplerian orbits!

- This is in fact the Newtonian derivation of the GR Friedmann equation, which describes the evolution of a spherical patch of the Universe

Spherical top-hat collapse

___|‾|___

- Solution for bound shells (E < 0):

$$r = \frac{r_{\max}}{2}\left(1 - \cos\eta\right) \ ,$$
$$t = \frac{t_c}{2\pi}\left(\eta - \sin\eta\right) \ ,$$

- Collapse happens at t_c

- Density within the shell is $\rho(t_i) = \bar{\rho}_m(t_i)\,(1+\delta_i),$

- Starting from some initial time t_i at which $\delta_i << 1$, can then write r_{max} and t_c in terms of initial quantities

$$\frac{r_i}{r_{\max}} = 1 - \left(1 - \frac{5}{3}\,\delta_i\right)\,\Omega_{m,i}^{-1} \ . \qquad t_c^2 = \left(\frac{3\pi}{2}\right)^2 \frac{t_i^2}{\left(1 - \left[1 - \frac{5}{3}\,\delta_i\right]\,\Omega_{m,i}^{-1}\right)^3}\,\frac{1}{\Omega_{m,i}\,(1+\delta_i)}$$

Spherical top-hat collapse

___|‾‾|___

- Can compute the over density assuming linear evolution at the time of collapse, e.g., in Einstein—de Sitter

$$
\begin{aligned}
\delta_c &= \delta(t_c) \\
&= \delta_i \left(\frac{t_c}{t_i}\right)^{2/3} \\
&= \delta_i \left(\frac{3\pi}{2}\right)^{2/3} \frac{1}{\left(1 - \left[1 - \frac{5}{3}\,\delta_i\right]\right)} \frac{1}{([1 + \delta_i])^{1/3}} \\
&= \frac{3}{5}\left(\frac{3\pi}{2}\right)^{2/3} \approx 1.686\,,
\end{aligned}
$$

- For general Universes, always get ~1.686

- Incredibly powerful result! Can use linear-perturbation-theory to compute δ and know that if δ > 1.686 —> collapsed halo has formed!

Virialization

- In reality, due to collisionless relaxation, a virialized object forms at the time of collapse
- Can estimate the actual, physical overdensity using the virial theorem
- In virial equilibrium: E = W/2 (W potential energy)
- At maximum expansion of shells: E = W_{max}
- Approximate W as homogeneous density sphere:

$$W_{\max} = -\frac{3}{5}\frac{GM^2}{r_{\max}} \quad \text{—> virial radius} \quad r_\nu = \frac{r_{\max}}{2}$$

Virialization

- Assume virialization happens at t_c
- Physical overdensity then is $$\Delta_v = \frac{\rho(t_v)}{\bar{\rho}_m(t_v)} = \frac{M/[4\pi r_v^3/3]}{\bar{\rho}_m(t_v)} ,$$
- or

$$\Delta_v = 8\left(1 - \left[1 - \frac{5}{3}\,\delta_i\right]\,\Omega_{m,i}^{-1}\right)^3 (1+\delta_i)\,\frac{\bar{\rho}_m(t_i)}{\bar{\rho}_m(t_v)} .$$

- For ~Einstein—de Sitter

$$\Delta_v = 8\left(1 - \left[1 - \frac{5}{3}\,\delta_i\right]\,\Omega_{m,i}^{-1}\right)^3 (1+\delta_i)\left(\frac{t_c}{t_i}\right)^2 . \qquad \Delta_v = 18\,\pi^2\,\Omega_{m,i}^{-1} .$$

  - Collapsed overdensity is therefore ~ 178 >> 1.686

Virial radius

- Because of this calculation, the virial radius of a DM halo is typically defined as that radius where the mean density is Δ_v times the background density with ~ 180 or ~200

- In detail: overdensity depends on content of Universe and time of collapse, for flat Universe better calculation gives (Bryan & Normal 1998)

$$\Delta_v = \left(18\,\pi^2 + 82 - 39x^2\right)\ \Omega_m^{-1}(t_c)\,, \qquad x = \Omega_m(t_c) - 1$$

The halo mass function

Press-Schechter formalism

- Now understand collapse of a single object, but like to understand the mass function of objects formed
- Can make use of result that collapsed objects have formed when $\delta_{lin} \sim 1.686$
- Press-Schechter formalism: smooth density field on different scales using a kernel W

$$\delta_S(\mathbf{x}, t; M) = \int \mathrm{d}\mathbf{x}' \, \delta(\mathbf{x}', t) \, W(\mathbf{x} - \mathbf{x}'; R) \, .$$

- Cosmology gives statistics of different over densities —> compute probability distribution of smoothed overdensities
- Cumulative mass function N(>M) is then given by the number of regions that have δ_s >~ 1.686

Kernels / window functions

- Press-Schechter can be done for a variety of window functions W
- Most physically-plausible one is a spherical top-hat _|-|_
- But often use others (e.g., Gaussian)
- To construct the halo *mass* function, we need to relate the size R of the window to the mass M of the halo it corresponds to, depends on specific window
- E.g., spherical top hat, comoving size R vs. Mass M:

$$R = \left(\frac{2\,GM}{\Omega_{0,m}\,H_0^2}\right)^{1/3},$$
$$= 1.82\,\mathrm{Mpc}\left(\frac{M}{10^{12}\,M_\odot}\right)^{1/3}.$$

Statistics of smoothed overdensities

- Remember: statistics of overdensities given by the matter power spectrum $P_m(k)$

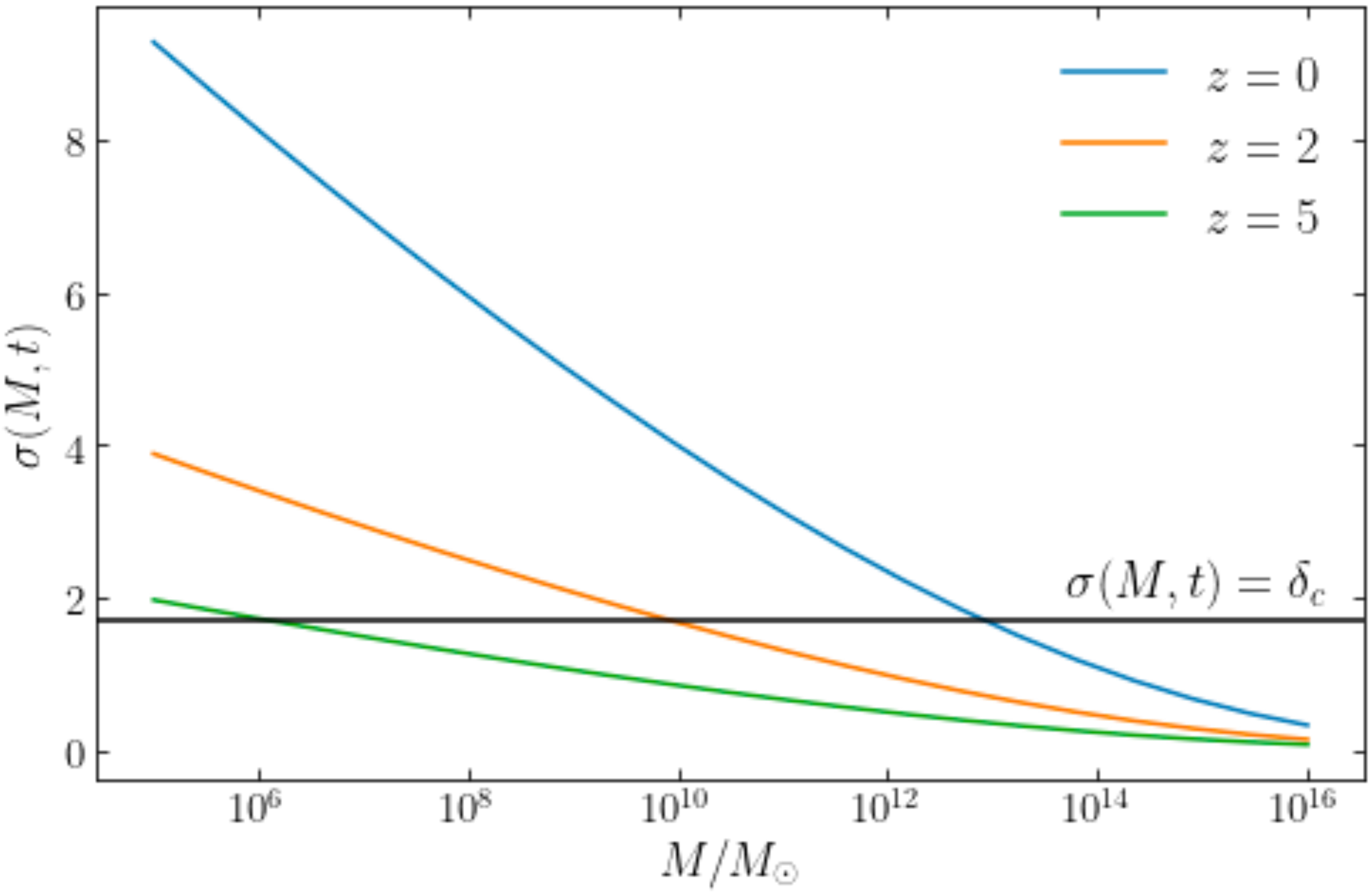
- Smoothed overdensities are just a linear combination of the raw overdensities, so also Gaussian field

- Variance $$\sigma^2(M,t) = \frac{1}{2\pi^2}\int_0^\infty \mathrm{d}k\, k^2\, P_m(k,a)\, W_k^2(R)\,,$$

- Where $W_k(R)$ is the Fourier transform of the Window, e.g., for top-hat

$$W_k(R) = 3\,\frac{j_1(kR)}{kR} = 3\,\frac{\sin kR - kR\,\cos kR}{(kR)^3}\,,$$

Statistics of smoothed overdensities



Press-Schechter halo mass function

- Ansatz: mass fraction of halos with mass > M is *twice* the probability that $\delta_S(M) > \delta_c \sim 1.686$

$$P[\delta_S(\mathbf{x},t) > \delta_c] = \frac{1}{2}\,\mathrm{erfc}\left[\frac{\delta_c}{\sqrt{2}\,\sigma(M,t)}\right] .$$

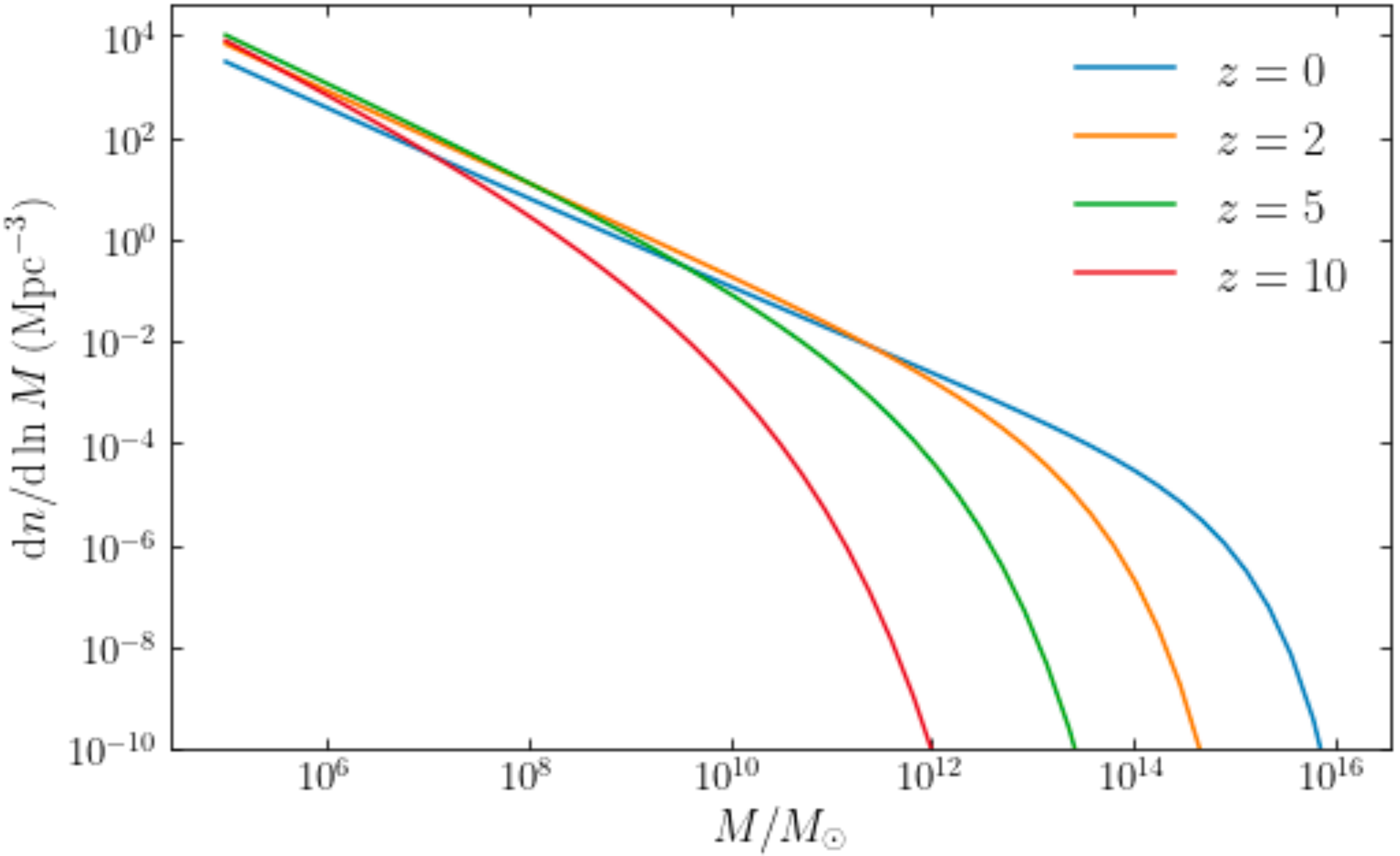
- Converting to a number density of halos

$$\frac{\mathrm{d}n}{\mathrm{d}\ln M} = \sqrt{\frac{2}{\pi}}\,\frac{\bar{\rho}_{0,m}}{M}\,\frac{\delta_c}{\sigma}\,\exp\left(-\frac{\delta_c^2}{2\sigma^2}\right)\frac{\mathrm{d}\ln\sigma^{-1}}{\mathrm{d}\ln M}$$

- 'Twice' necessary, because otherwise only half the mass in the Universe is in halos

- We will soon discuss a more rigorous derivation of this result that naturally includes the factor of two

Press-Schechter halo mass function



Sheth-Tormen halo mass function

- The Press-Schechter halo mass function is in reasonable agreement with halo mass functions obtained using N-body simulations, but one can do better

- First, write the PS function as

$$\frac{\mathrm{d}n}{\mathrm{d}\ln M} = \frac{\bar{\rho}_{0,m}}{M} f_{\mathrm{PS}}(\nu) \frac{\mathrm{d}\ln\sigma^{-1}}{\mathrm{d}\ln M}\,, \quad \text{where } \nu = \delta_c/\sigma(M,t) \text{ and } \quad f_{\mathrm{PS}}(\nu) = \sqrt{\frac{2}{\pi}}\,\nu\, e^{-\nu^2/2}\,,$$

- Sheth & Tormen showed that a better fit to N-body simulations in a variety of Universes is given by

$$f_{\mathrm{ST}}(\nu) = 0.3222\sqrt{\frac{2}{\pi}}\,\nu'\, e^{-\nu'^2/2}\,\left(1+\nu'^{-0.6}\right)\,, \qquad \text{where } \; \nu' = 0.707\nu.$$

Sheth-Tormen halo mass function