AST1430 - Assignment 2

Due: Feb 28th, 11:59pm

1 Recombination (30%)

Time to derive the Saha equation! The goal here is to develop an expression for the ionized fraction of Hydrogen gas in thermal equilibrium.

a) To begin, write down expressions for the equilibrium number densities of neutral Hydrogen (n_H), free electrons (n_e), and free protons (n_p). Start from the canonical distribution,

$$n = \frac{4\pi g}{(2\pi\hbar)^3}\int_0^\infty \frac{p^2 dp}{e^{E/kT} \pm 1}$$

Assume a cold Universe ($mc^2 \gg kT$), where you can drop the ± 1 and approximate $\sqrt{1+\epsilon} \approx 1 + \epsilon/2$, with $\epsilon = p^2c^2/m^2c^4$. Note that $g_e = g_p = 2$ and $g_H = 4$.

b) Next, consider the ratio $n_H/(n_p n_e)$. Plug in your answer from (a) and simplify the result, assuming $m_H/m_p \approx 1$, and recognizing that the mass difference is simply the ionization energy of Hydrogen, $(m_H - m_e - m_p)c^2 = -13.6eV$.

c) We ultimately want to solve for the ionization fraction $X \equiv n_p/n_B$. Show that $\frac{1-X}{X^2} = \frac{n_H}{n_e n_p} n_B$, then plug in the previous result, take $n_B = 10^{-9} \times n_\gamma$, and solve for X.

d) Plot $X(T)$. At what temperature, time, and redshift did recombination take place? What was the typical photon energy $h\nu = kT$? Compare this to the ionization energy of Hydrogen, and account for any difference.

2 Massive Neutrinos (10%)

The temperature of the Cosmic Neutrino Background is thought to be $\approx 1.9K$ today. What would the total mass $m_{\nu_e} + m_{\nu_\tau} + m_{\nu_\mu}$ have to be to independently close the Universe (i.e., $\Omega_\nu > 1$)?

3 Nucleosynthesis (30%)

a) Show that the largest possible value of the primordial helium fraction is $Y_{max} = \frac{2f}{1+f}$, regardless of the underlying physics, where the proton to neutron ratio at the time of nucleosythesis $f = n_n/n_p \leqslant 1$.

b) Suppose the neutron decay time was ten times shorter than observed (i.e., $\tau_n = 89$s), with all other physical parameters unchanged. Estimate Y, the helium mass fraction, assuming that all available neutrons are incorporated into helium nuclei.

c) Suppose the difference in rest energy between the neutron and proton was ten times smaller than found (ie. $(m_n - m_p)c^2 = 0.129$ MeV), with all other physical parameters unchanged. Estimate Y, the helium mass fraction, assuming that all available neutrons are incorporated into helium nuclei.

4 CAMB (30%)

Most people studying the CMB use CAMB, the Code for Anisotropies in the Microwave Background (https://camb.readthedocs.io/en/latest/). In this problem, you'll play around with it a bit. (The example ipynb on readthedocs is pretty great reference.)

You're welcome to run your own build / install, but an easy way to get up and running is to use the UofT Jupyter service. You can install CAMB in a notebook by running `%pip install camb` in an otherwise empty cell.

a) Compute a basic vanilla-ΛCDM set of power spectra using *Planck* best-fit parameters. Plot D_ℓ^{TT}, D_ℓ^{EE}, D_ℓ^{BB} on the same set of axes. (You'll need to include lensing to get anything in the BB!)

b) Let's try messing around with parameters a bit. Add some curvature, $\Omega_k = 0.1$, and compare the new spectra. Try again, moving all the CDM mass into HDM (i.e. $\Omega_C \rightarrow \Omega_\nu$).

c) Let's test the neutrino mass hierarchy! As background, we know all three neutrinos have mass, but not how much. From solar neutrino oscillation measurements, we have measured the mass *splitting*, which tells us only a lower limit of $\sum m_\nu > 58$meV, not the absolute value of masses. Figure 1 which displays the two possible scenarios, "normal" and "inverted" mass hierarchies. The CMB can distinguish between these by placing stringent constraints on $\sum m_\nu$.

Generate CMB spectra with normal and inverted hierarchies, taking $\sum m_\nu = 105$meV. Plot the difference in temperature and polarization spectra.

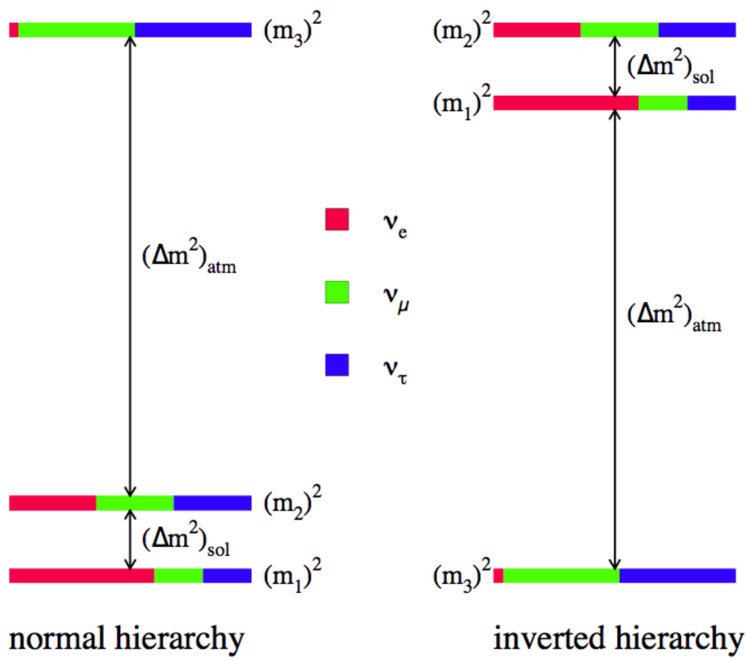

Figure 1: Solar neutrino experiments tell us that neutrinos have masses, by telling us the mass gaps between eigenstates as shown here. We can distinguish between the two possible scenarios by measuring the total neutrino mass, which allows for either one or two high-mass states.