AST 1430H
Observational Cosmology

III

Friedmann's Equations

Announcements

1. Website is finally up. https://github.com/jobovy/AST1430-grad-cosmology

2. Assignment #1 out today, due Jan 31st.

3. Presentations:
   - Feb 15th: Early Universe
     - Three, submit topics by next week!
   - Apr 5th: Late Universe

Proposed grading

Grading scheme (TO BE FINALIZED by JAN 20)

- **Assignments:** 40 % over three assignments; ~~see course website for due dates.~~

  Due Jan 31, Feb 28, Mar 28
- **Participation:** 20 %
- **Presentations:** 20 %
- **Take-home final + oral exam:** 20 %

  You are allowed to (and are encouraged to!) work together with classmates on the assignments, but each student must hand in an independent write-up of their solutions. The take-home final should be your own work. Solutions must be written up in a detailed enough manner to demonstrate that you understand each step. The oral exam will consist of a discussion of the take-home final with follow-up questions.

Week Index	Dates	Topics
~~Week 1~~	~~Jan 11, 13~~	~~logistics, introduction, basic observations~~
Week 2	Jan 18, 20	Basic GR, RW metric, Distances, coordinates, Friedmann equations
Week 3	Jan 25, 27	Cosmological models, consistency with observations, Early Hot Universe, BBN
Week 4	Feb 1, 3	Inflation, Perturbations & Structure pre-recombination
Week 5	Feb 8, 10	CMB: basics, polarization, secondaries
Week 6	Feb 15, 17	Early-Universe Presentations + Review
Reading Week – No Class		
Week 7	Mar 1, 3	Post-recombination growth of structure, formation of dark matter halos, halo mass function
Week 8	Mar 8, 10	The relation between dark matter halos and galaxies
Week 9	Mar 15, 17	Probing the cosmic density field / clustering
Week 10	Mar 22, 24	Late-time cosmological observations: BAO, supernovae, weak lensing, etc.
Week 11	Mar 29, 31	H0 controversy: how fast exactly is the Universe expanding today?
Week 12	Apr 5	Late Universe Presentations + Review

Keith

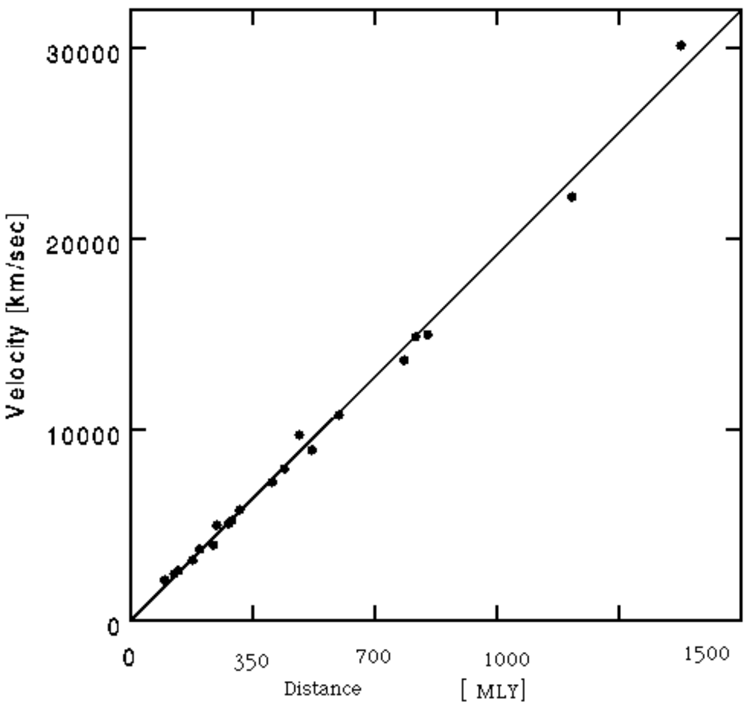
Jo

asst1

asst2

asst3

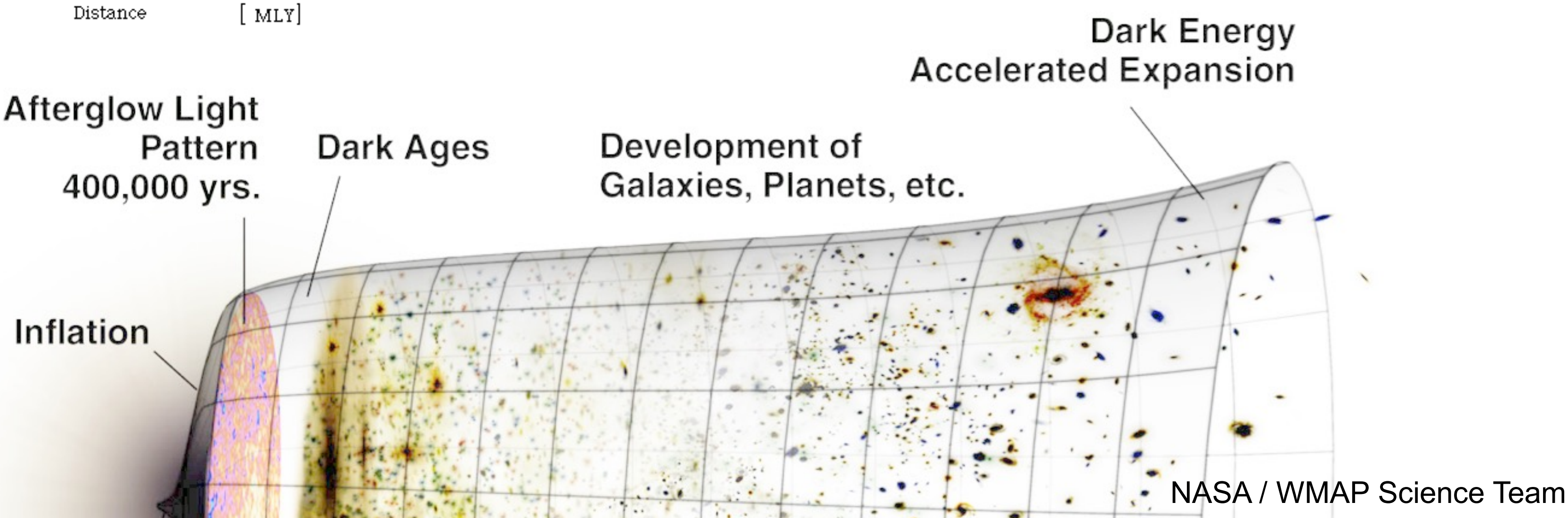
Context



We live in an expanding Universe.

Observations are consistent with a Homogeneous, Isotropic [ie. Curved & Scaled] Spacetime

$$ds^2 = -c^2dt^2 + a(t)^2 \left[dr^2 + S_\kappa(r)^2 d\Omega^2\right]$$



Friedmann Equations

$$ds^2 = -c^2 dt^2 + a(t)^2 \left[dr^2 + S_\kappa(r)^2 d\Omega^2 \right]$$

+

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}$$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3} G\rho - \frac{\kappa c^2}{R_0^2 a^2}$$

From 00 equation

(Dodelson, 2.1.3)

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3} G\left(\rho + \frac{3p}{c^2}\right)$$

From trace (11+22+33)

Density Parameter Ω_0

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}G\rho - \frac{\kappa c^2}{R_0^2 a^2}$$

Flat Universe (κ=0) is produced for a critical density :

$$\rho_c \equiv \frac{3\dot{a}^2}{8a^2\pi G} = \frac{3H_0^2}{8\pi G}$$

We define the Density Parameter :

$$\Omega_0 \equiv \frac{\rho_0}{\rho_c} = \frac{8\pi G\rho_0}{3H_0^2}$$

Solve for R_0 today…

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}G\rho - \frac{\kappa c^2}{R_0^2 a^2} \quad \Rightarrow \quad \left(\frac{\dot{a}_0}{a_0}\right)^2 = \Omega_0 H_0^2 - \frac{\kappa c^2}{R_0^2 a_0^2}$$

Recall

$H_0 = \dot{a}_0/a_0 = \dot{a}_0$ $\Rightarrow$ $H_0^2 = \Omega_0 H_0^2 - \kappa c^2/R_0^2$

$$\Rightarrow R_0^2 = \kappa c^2 H_0^{-2}(\Omega_0 - 1)^{-1}$$

Substituting R_0 back in the Friedmann eqn,

$$\dot{a}^2 = H_0^2(1+\Omega_0(1/a-1))$$

Milne Universe

$$\Omega_0 = 0$$

(Open Universe, κ=-1)

$$\dot{a}^2 = H_0^{\,2}(1+\Omega_0(1/a-1)) = H_0^{\,2}$$

$$R_0 = (H_0/c)^2$$

$$a(t) = H_0 t$$

$$t_0 = 1/H_0$$

H_0 is about 72km/s/Mpc.

If we live in an empty Universe, t_0 = 13.6 Gy

Einstein-de Sitter Universe

$$\Omega_0 = 1$$

(Critical / flat Universe, κ=0)

$$\dot{a}^2 = H_0^2(1+\Omega_0(1/a-1)) = H_0^2/a$$

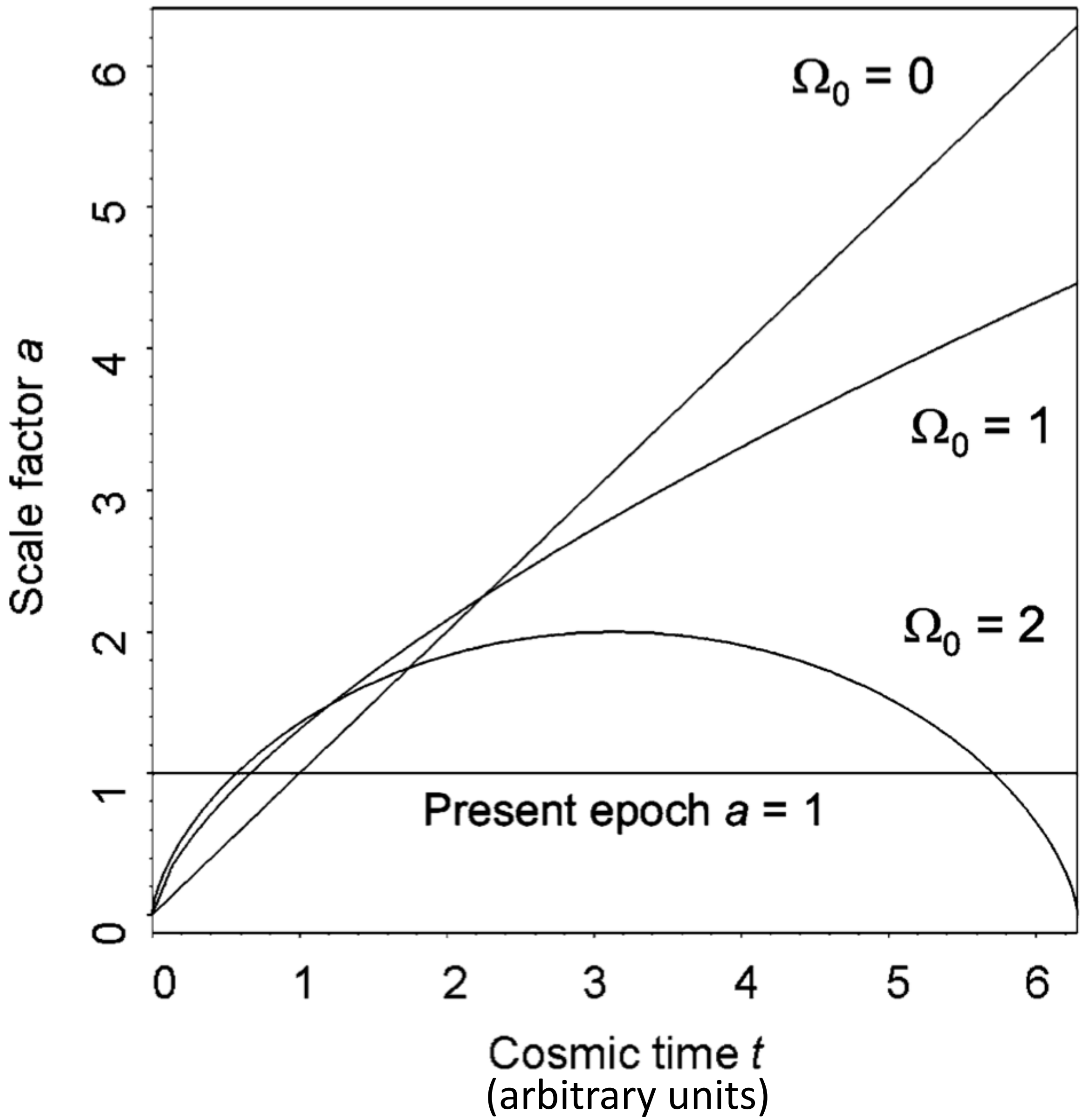
$$a(t) = (t/t_0)^{2/3}$$

where

$$t_0 = 2/3H_0$$

If we live in an EdS Universe, t_0 = 9 Gy

Matter-Only Universe Roundup



Dynamics

First Law of Thermodynamics:

$$\mathrm{d}U = -p\,\mathrm{d}V$$

Internal Energy

$$U = \varepsilon_{\mathrm{tot}} V$$

Differentiate wrt da:

$$\frac{\mathrm{d}}{\mathrm{d}a}(\varepsilon_{\mathrm{tot}} V) = -p\,\frac{\mathrm{d}V}{\mathrm{d}a}$$

$$V \propto a^3 \Rightarrow \frac{dV}{da} = 3V/a$$

$$\boxed{\frac{\mathrm{d}\varepsilon_{\mathrm{tot}}}{\mathrm{d}a} + 3\,\frac{(\varepsilon_{\mathrm{tot}} + p)}{a} = 0}$$

(or, writing $\varepsilon_{tot} = \rho c^2$)

$$\frac{d\rho_i}{da} + 3\frac{\rho_i + p_i/c^2}{a} = 0$$

Non-relativistic Gas (matter)

$$\frac{\mathrm{d}\varepsilon_{\mathrm{tot}}}{\mathrm{d}a} + 3\frac{(\varepsilon_{\mathrm{tot}} + p)}{a} = 0$$

Pressureless: $p = 0$; $\varepsilon = nmc^2$; $n = N/V$

$dn/da = -3n/a$

$\Rightarrow n \propto a^{-3}$

Finite Temp: $p = nkT$; $\varepsilon = nmc^2 + 3nkT/2$

$d(nkT)/da + 5nkT/a = 0$

$\Rightarrow nkT \propto n_0kT_0a^{-5}$

$\Rightarrow T \propto a^{-2}$

Note: Since $T \propto |v|^2$ => $|v| \propto 1/a$.

Expansion causes peculiar velocities to damp out!

Relativistic Gas (γ,ν,…)

$$\frac{\mathrm{d}\varepsilon_{\mathrm{tot}}}{\mathrm{d}a} + 3\frac{(\varepsilon_{\mathrm{tot}} + p)}{a} = 0$$

Relativistic Fluid: $p = \varepsilon / 3$

$$d\varepsilon/da + 4\varepsilon/a = 0$$
$$\Rightarrow \varepsilon \propto a^{-4}$$

As expected! => Number density dilutes by a^{-3},

=> Energy per photon redshifts as $h\nu(t) = h\nu_{em}/a$

Also note: $\varepsilon \propto \sigma T^4$ => $T \propto 1/a$

The Cosmological Constant

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu} - \Lambda g_{\mu\nu}$$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3} G\rho - \frac{\kappa c^2}{R_0^2 a^2} + \frac{\Lambda}{3}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3} G(\rho + \frac{3p}{c^2}) + \frac{\Lambda}{3}$$

Note that Λ is repulsive, accelerating the expansion.

Vacuum Energy

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3}G(\rho + \frac{3p}{c^2}) + \frac{\Lambda}{3}$$

Consider a vacuum energy, $\rho_v = \varepsilon_v/c^2$ = constant

$$\frac{d\varepsilon_{tot}}{da} + 3\frac{(\varepsilon_{tot} + p)}{a} = 0$$

=> $p_v = -\varepsilon_v = -\rho_v c^2$

$$\frac{\ddot{a}}{a} = \underbrace{-\frac{4\pi}{3}G(-2\rho_v)}_{\text{(vacuum energy)}} = \underbrace{\frac{\Lambda}{3}}_{\text{(cosmo const)}}$$

=> $\rho_v = \Lambda/8\pi G$

General Fluids, w_i

More generally, allow other equations of state,

$$p_i = w_i(\mathrm{t})\rho_i c^2$$

$$\frac{\mathrm{d}\varepsilon_{\mathrm{tot}}}{\mathrm{d}a} + 3\frac{(\varepsilon_{\mathrm{tot}} + p)}{a} = 0 \Leftrightarrow \quad \frac{d\rho_i}{da} + 3\frac{\rho_i + p_i/c^2}{a} = 0$$

$$\frac{d\rho_i}{da} = -\frac{3\rho_i(1 + w_i)}{a}$$

$$\rho_i \propto a^{-3(1+w_i)}$$

General Fluids, w_i

$$\rho_i \propto a^{-3(1+w_i)}$$

Cold matter:

$w_i = 0$ => $\rho \propto a^{-3}$

Hot matter / radiation:

$w_i = 1/3$ => $\rho \propto a^{-4}$

Cosmological Constant Λ:

$w_i = -1$ => $\rho = \text{const}$

Single-Component Friedman Eqns

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3}G(\rho + \frac{3p}{c^2}) \qquad p_i = w_i(\mathrm{t})\rho_i c^2$$

For a **single component**: $\ddot{a} = -\frac{4\pi G a \rho_i}{3}(1 + 3w_i(t))$

Recall, $\rho_i \propto a^{-3(1+w_i)}$ $\qquad = -\frac{4\pi G a}{3}\frac{\rho_{i0}}{a^{3(1+w_i)}}(1 + 3w_i)$

We define, $\Omega_{i0} = \frac{8\pi G \rho_{i0}}{3H_0^2}$ $\qquad = -\Omega_{i0} H_0^2 \frac{1 + 3w_i}{2a^{2+3w_i}}$

Going back to the original form, $\dot{a}^2 = \frac{\Omega_{i0} H_0^2}{a^{1+3w_i}} - \frac{\kappa c^2}{R_0^2}$

Multi-Component Universes

$$\dot{a}^2 = \frac{\Omega_{i0} H_0^2}{a^{1+3w_i}} - \frac{\kappa c^2}{R_0^2} \quad \Rightarrow \quad \dot{a}^2 = \sum_i H_0^2 \Omega_{i0} a^{-(1+3w_i)}$$

Where we have defined a “curvature density,”

$$\Omega_{k0} = \frac{\kappa c^2}{R_0^2 H_0^2} = 1 - \sum_{i \neq k} \Omega_{i0}$$

Or, in the more familiar form, assuming the existence of **Matter**, **Radiation**, **Curvature**, and a **Cosmological Constant**:

$$\frac{\dot{a}^2}{a^2} = H_0^2 (\Omega_R a^{-4} + \Omega_M a^{-3} + \Omega_k a^{-2} + \Omega_\Lambda)$$

Summary

$$\dot{a}^2 = \sum_i H_0^2 \Omega_{i0} a^{-(1+3w_i)} \quad \Rightarrow \quad \frac{\dot{a}^2}{a^2} \equiv \boxed{H^2 = \sum_i H_0^2 \Omega_{i0} a^{-3(1+w_i)}}$$

Hubble Expansion goes as the amount of stuff.

$$\frac{H^2}{H_0^2} = (\Omega_R a^{-4} + \Omega_M a^{-3} + \Omega_K a^{-2} + \Omega_\Lambda)$$

Summary, Take 2

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}G\rho - \frac{\kappa c^2}{R_0^2 a^2}$$

H^2

$\rho_R = \rho_{R0} a^{-4}$
$\rho_M = \rho_{M0} a^{-3}$

$$\Omega_\kappa = \frac{-\kappa c^2}{R_0^2 H_0^2}$$

$$\Omega_0 \equiv \frac{\rho_0}{\rho_c} = \frac{8\pi G \rho_0}{3H_0^2}$$

$$\boxed{\frac{H^2}{H_0^2} = \left(\Omega_R a^{-4} + \Omega_M a^{-3} + \Omega_K a^{-2} + \Omega_\Lambda\right)}$$

Back to GR...

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu} - \Lambda g_{\mu\nu}$$

The **left** side describes the **geometry** of spacetime.
The **right** side describes the **contents** of spacetime.

$$\frac{H^2}{H_0^2} = (\Omega_R a^{-4} + \Omega_M a^{-3} + \Omega_K a^{-2} + \Omega_\Lambda)$$

A few words on the expansion

The universe we are considering is one in freefall, each fundamental observer following a geodesic through spacetime.

The expansion observed is not necessarily being “pushed”; more accurately it can be thought of as a kinetic expansion (of space!) – the expansion is an initial condition which then evolves according to GR.

Bound systems (galaxies, people, etc) are not fundamental observers, their internal interactions have “fallen out” of the Hubble flow. Once that initial expanding condition is overcome, there is ***zero*** effect on eg. energy levels in atoms.

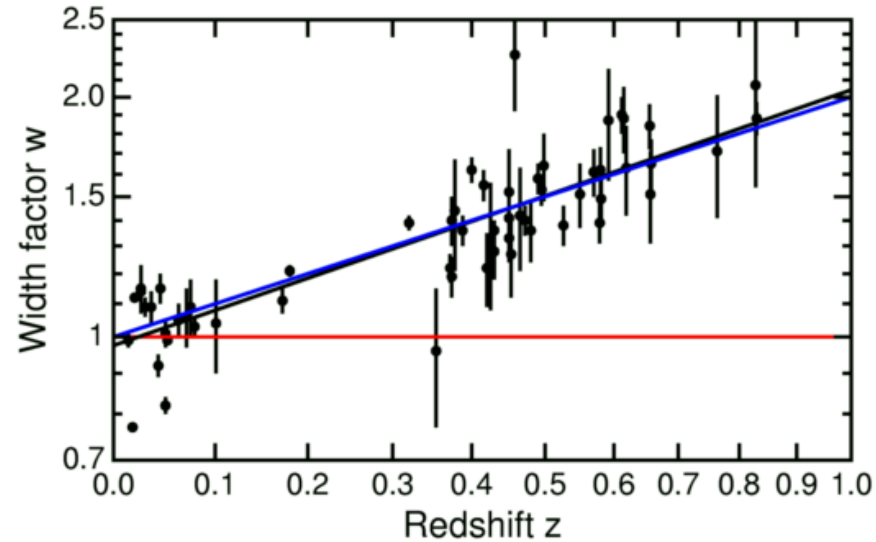
INTERMISSION

Context

$$ds^2 = -c^2dt^2 + a(t)^2\left[dr^2 + S_\kappa(r)^2 d\Omega^2\right]$$

Consistency checks:

- surface brightness vs z
- time dilation vs z

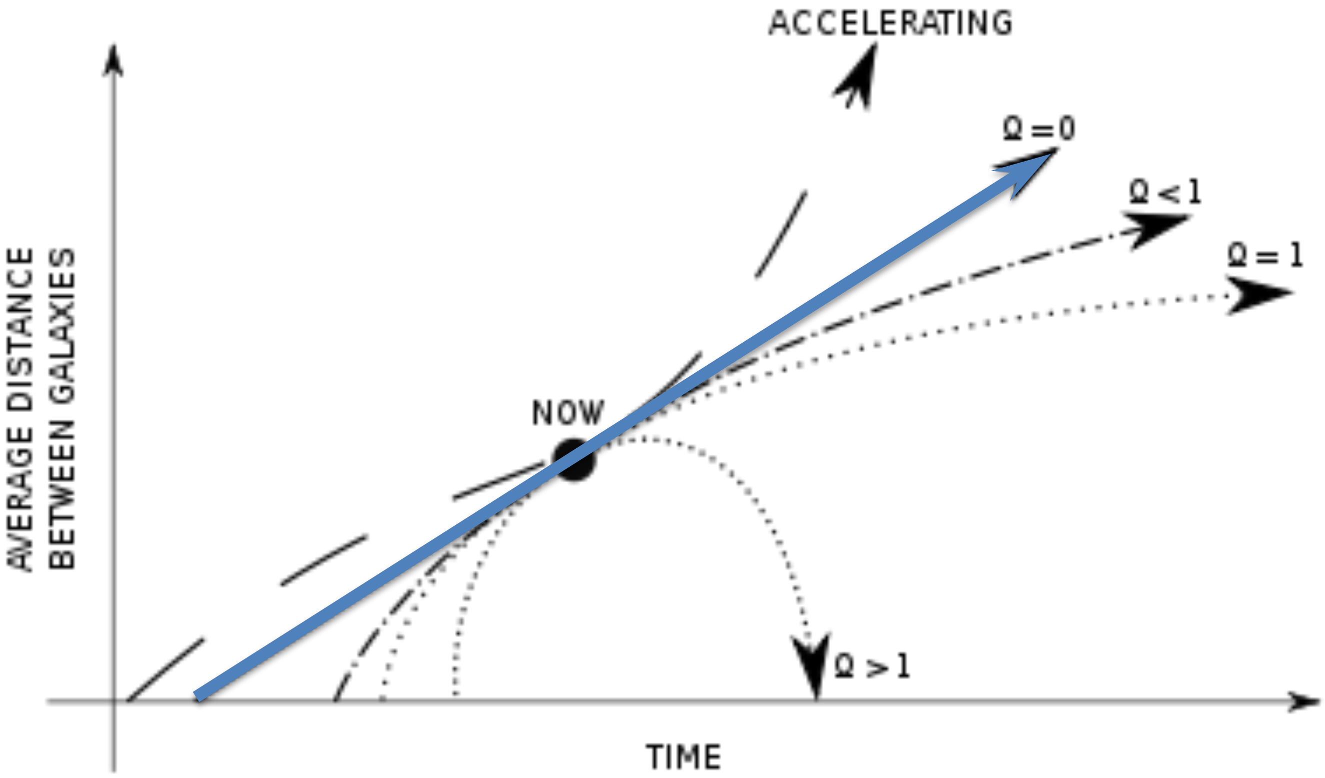


$$\frac{\dot{a}^2}{a^2} = H^2 = H_0^2(\Omega_R a^{-4} + \Omega_M a^{-3} + \Omega_k a^{-2} + \Omega_\Lambda)$$

If we knew a(t), could solve for other parameters (or vice-versa).

Milne Universe

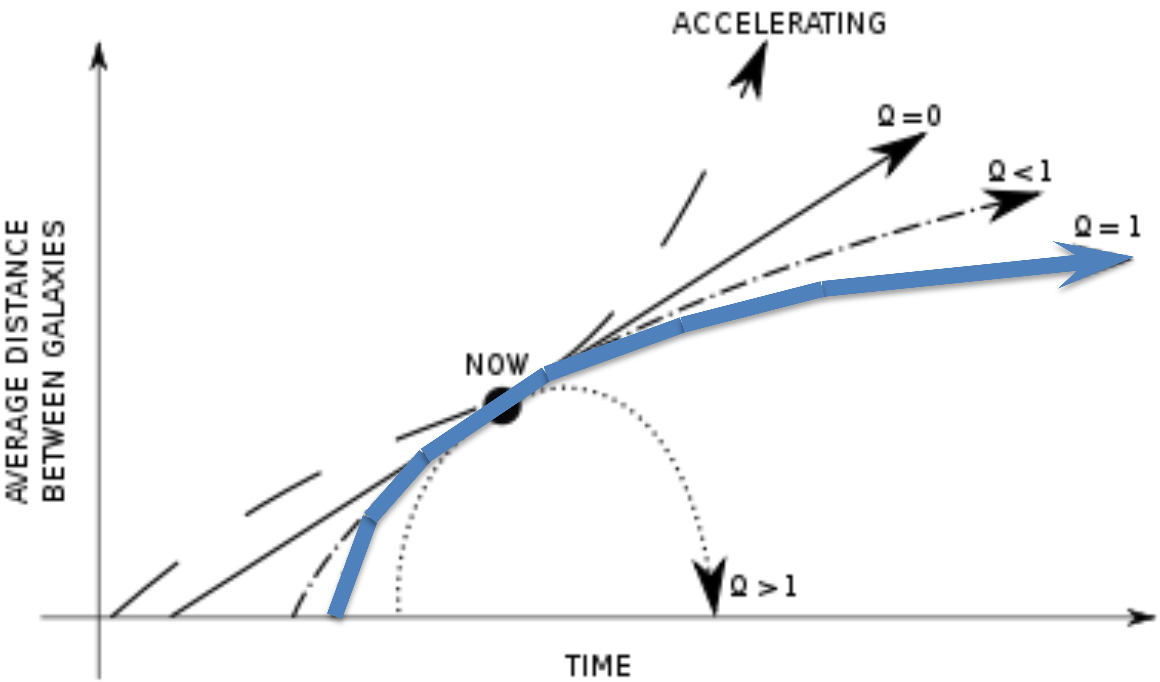
Empty. $\Omega_{tot} = 0$

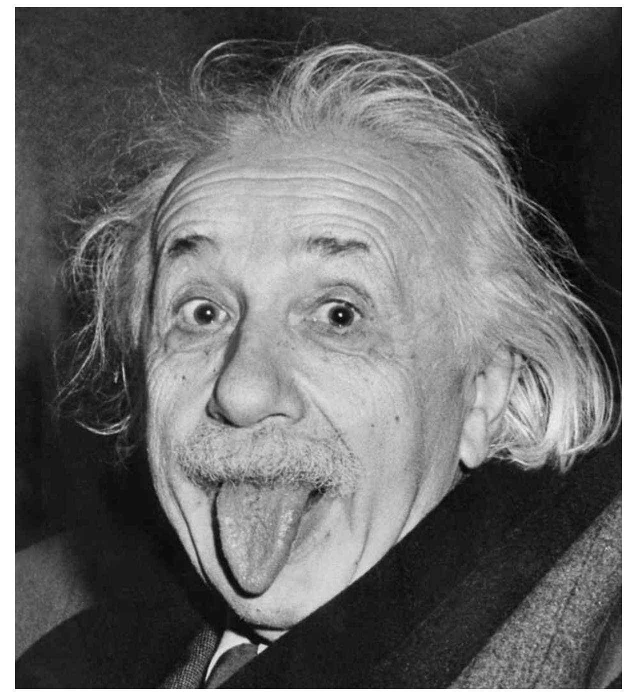


Constant growth rate, a(t) increases linearly with time.

Einstein - de Sitter (EdS) Universe

Matter, Flat. $\Omega_{tot} = \Omega_M = 1$

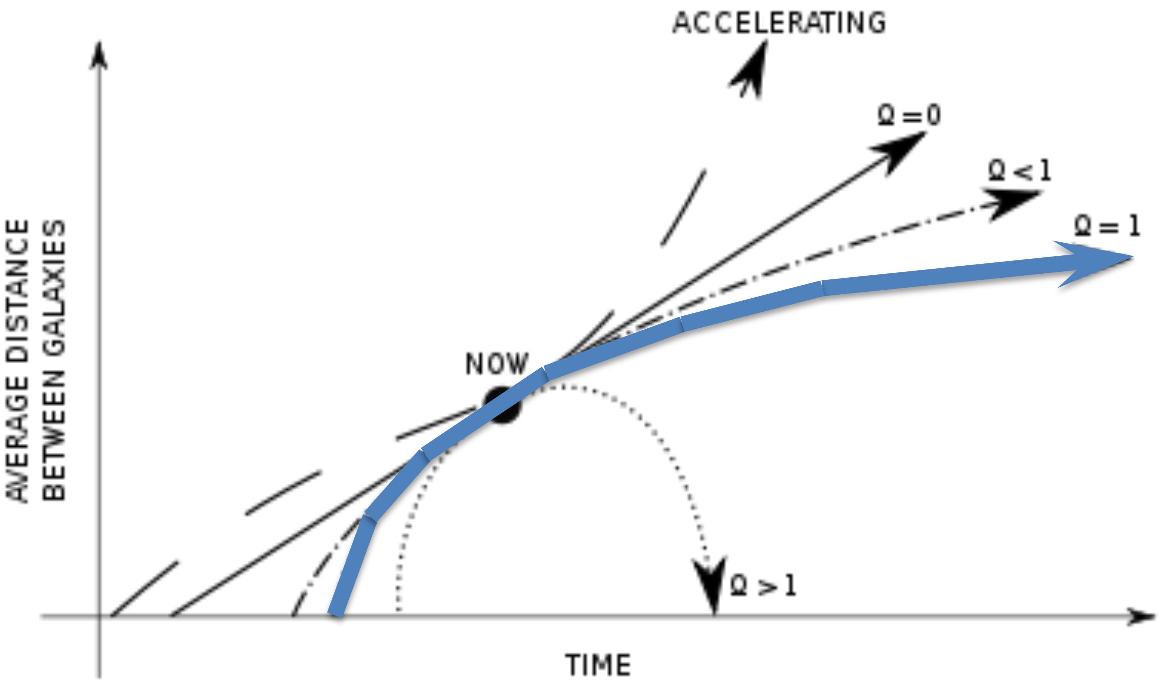


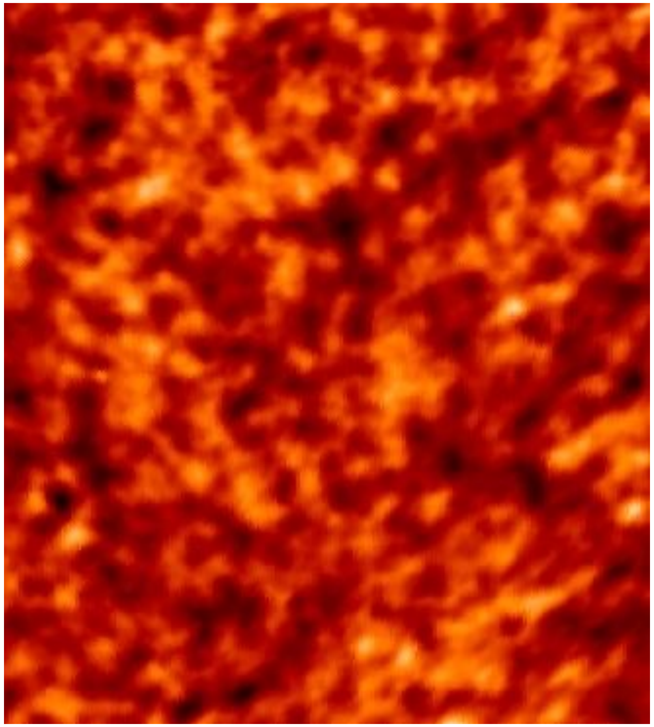


The EdS Universe is a naïve guess at what an evolved Standard Model Universe might look like. It decelerates over time, but grows forever.

Radiation Dominated Universe

Radiation, Flat. $\Omega_{tot} = \Omega_R = 1$

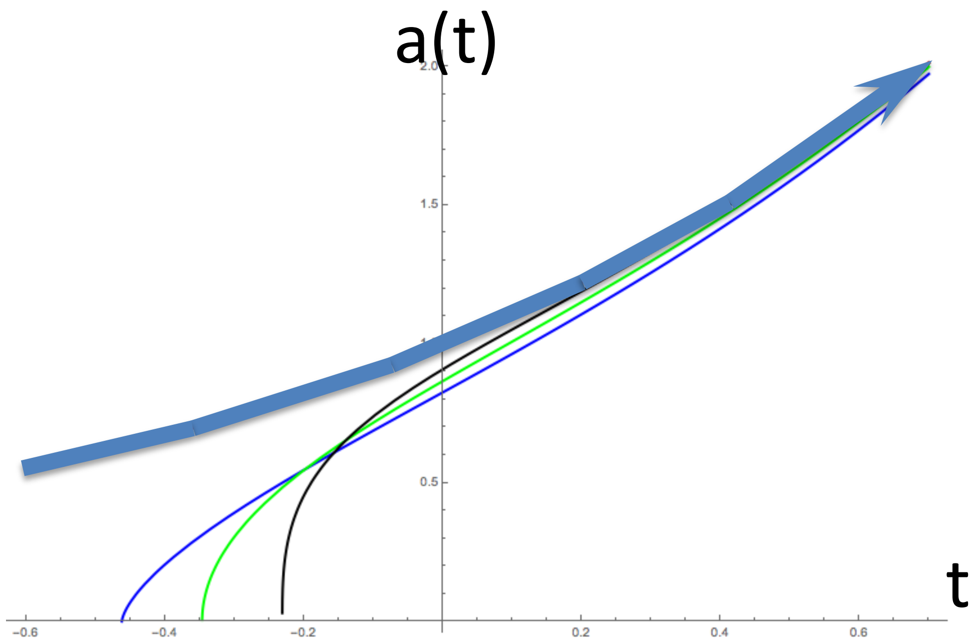


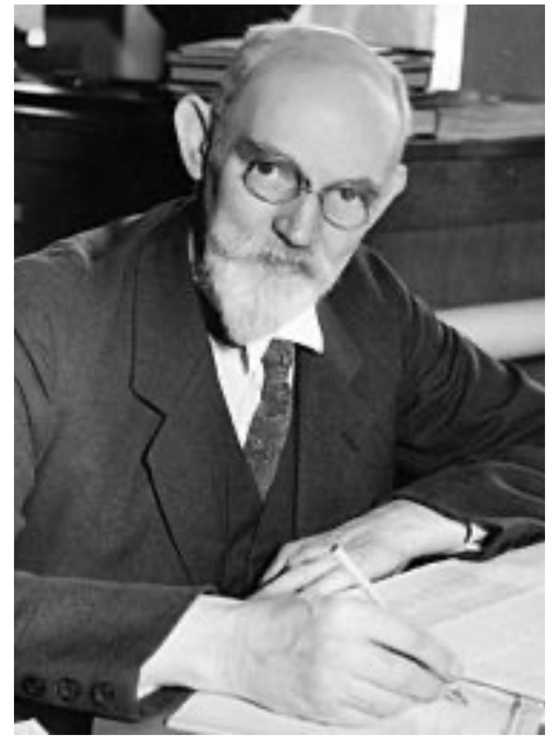


At small scale factors, e.g. shortly after the Big Bang, the energy density in Radiation will dominate other components.

de Sitter Universe

Vacuum, flat. $\Omega_{tot} = \Omega_\Lambda \approx 1$





Including only vacuum energy, the de Sitter Universe grows **exponentially**.

Solving for a(t) in a flat Universe

Radiation Dominated:

$$\frac{\dot{a}^2}{a^2} = H_0^2(\Omega_R a^{-4} + \cancel{\Omega_M a^{-3}} + \cancel{\Omega_k a^{-2}} + \cancel{\Omega_\Lambda})$$

a(t) $\propto$?

Matter Dominated:

$$\frac{\dot{a}^2}{a^2} = H_0^2(\cancel{\Omega_R a^{-4}} + \Omega_M a^{-3} + \cancel{\Omega_k a^{-2}} + \cancel{\Omega_\Lambda})$$

a(t) $\propto$?

Λ Dominated:

$$\frac{\dot{a}^2}{a^2} = H_0^2(\cancel{\Omega_R a^{-4}} + \cancel{\Omega_M a^{-3}} + \cancel{\Omega_k a^{-2}} + \Omega_\Lambda)$$

a(t) $\propto$?

“A Search for Two Numbers”

(Ryden, *Intro to Cosmology*, 2003)

We want to measure a(t). Taylor expanding:

$$a(t) = a(t_0) + \left.\frac{da}{dt}\right|_{t=t_0} (t - t_0) + \frac{1}{2} \left.\frac{d^2a}{dt^2}\right|_{t=t_0} (t - t_0)^2 + \ldots$$

25 years ago, everything was set by two parameters.

$$H_0 = \frac{\dot{a}_0}{a_0} = \dot{a}_0$$

the expansion rate

$$\xcancel{q \equiv -\frac{a\ddot{a}}{\dot{a}} = -\left(1 + \frac{\dot{H}}{H^2}\right)}$$

~~the deceleration parameter~~

Then came Ω_Λ. And $\Omega_M = \Omega_C + \Omega_B$. And structure, EoR, tau, etc

We’re past the days of two parameters.

ΛCDM Parameters

Parameter	Value	Description
t_0	$13.75 \pm 0.11 \times 10^9$ years	Age of the universe
H_0	$70.4^{+1.3}_{-1.4}$ km s^{-1} Mpc^{-1}	Hubble constant
$\Omega_b h^2$	0.02260 ± 0.00053	Physical baryon density
$\Omega_c h^2$	0.1123 ± 0.0035	Physical dark matter density
Ω_b	0.0456 ± 0.0016	Baryon density
Ω_c	0.227 ± 0.014	Dark matter density
Ω_Λ	$0.728^{+0.015}_{-0.016}$	Dark energy density
Δ_R^2	$2.441^{+0.088}_{-0.092} \times 10^{-9}$, $k_0 = 0.002 Mpc^{-1}$	Curvature fluctuation amplitude
σ_8	0.809 ± 0.024	Fluctuation amplitude at $8h^{-1}$Mpc
n_s	0.963 ± 0.012	Scalar spectral index
z_*	$1090.89^{+0.68}_{-0.69}$	Redshift at decoupling
t_*	377730^{+3205}_{-3200} years	Age at decoupling
τ	0.087 ± 0.014	Reionization optical depth
z_{reion}	10.4 ± 1.2	Redshift of reionization

Parameter	Value	Description
Ω_{tot}	$1.0023^{+0.0056}_{-0.0054}$	Total density
w	-0.980 ± 0.053	Equation of state of dark energy
r	< 0.24, $k_0 = 0.002 Mpc^{-1}$ (2σ)	Tensor-to-scalar ratio
$d\, n_s / d \ln k$	-0.022 ± 0.020, $k_0 = 0.002 Mpc^{-1}$	Running of the spectral index
$\Omega_\nu h^2$	< 0.0062	Physical neutrino density
Σm_ν	< 0.58 eV (2σ)	Sum of three neutrino masses

Some Definitions…

Cosmologists tend to use Hubble units:

- Hubble Time = $1/H_0$, the age of a Milne (empty) Universe
- Hubble Length = c/H_0, the distance light travels in a Hubble time
- Hubble Volume, definition varies, but roughly $(c/H_0)^3$
- Observable Universe, the portion of space which is in causal contact with the earth, can plausibly be observed
- $h = H_0$ / 100 km s^{-1} Mpc^{-1}
- $h_{70} = H_0$ / 70 km s^{-1} Mpc^{-1}

Measuring Spacetime, a(t)

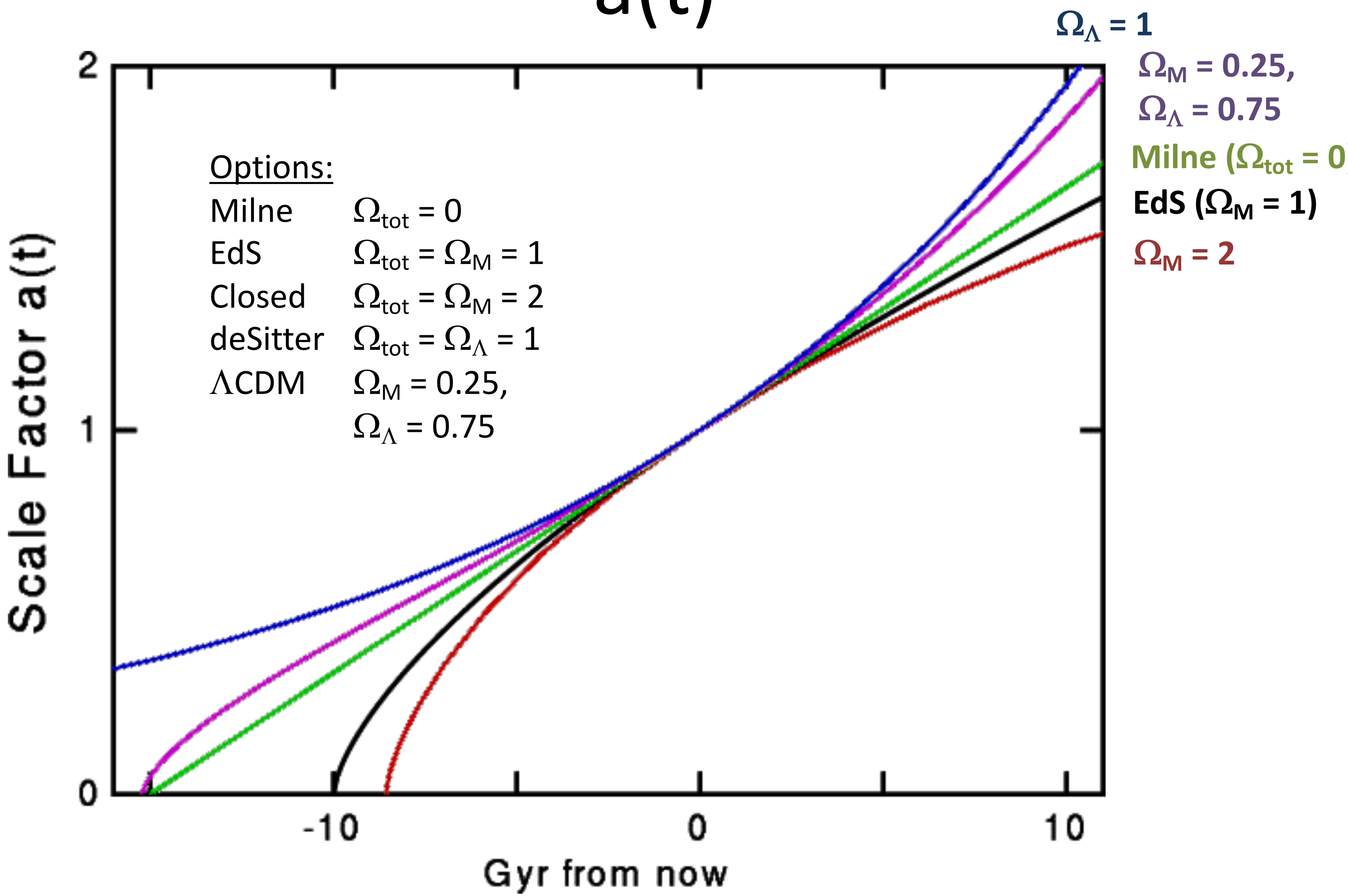
Extend H_0 measurements beyond the local Universe, to measure $H(t) = (da/dt)/a$

Choose objects with some intrinsic property

- luminosity
- number density
- size

Compute a distance measure (D, D_L, D_A), and how it varies with z.

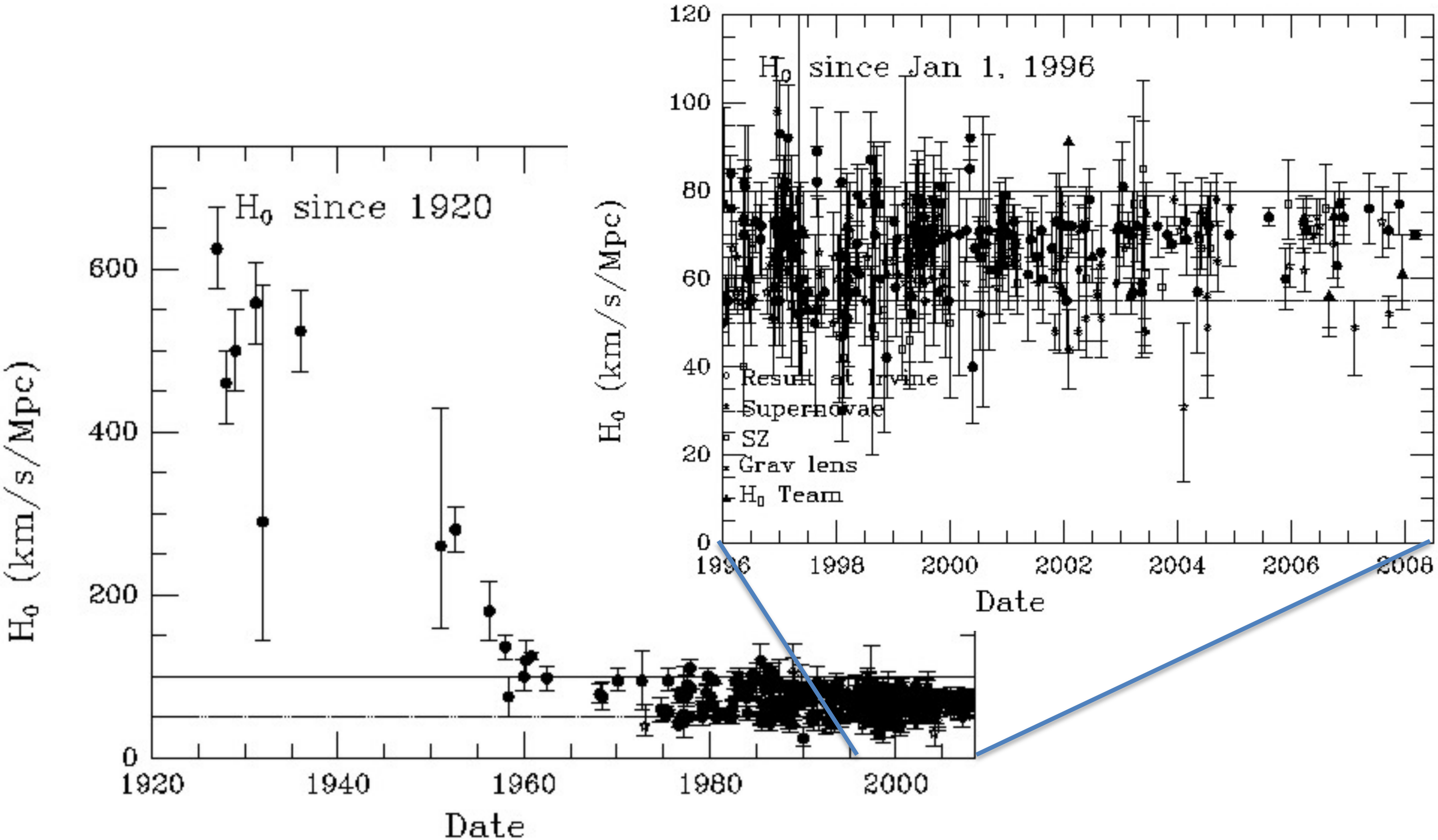
a(t)



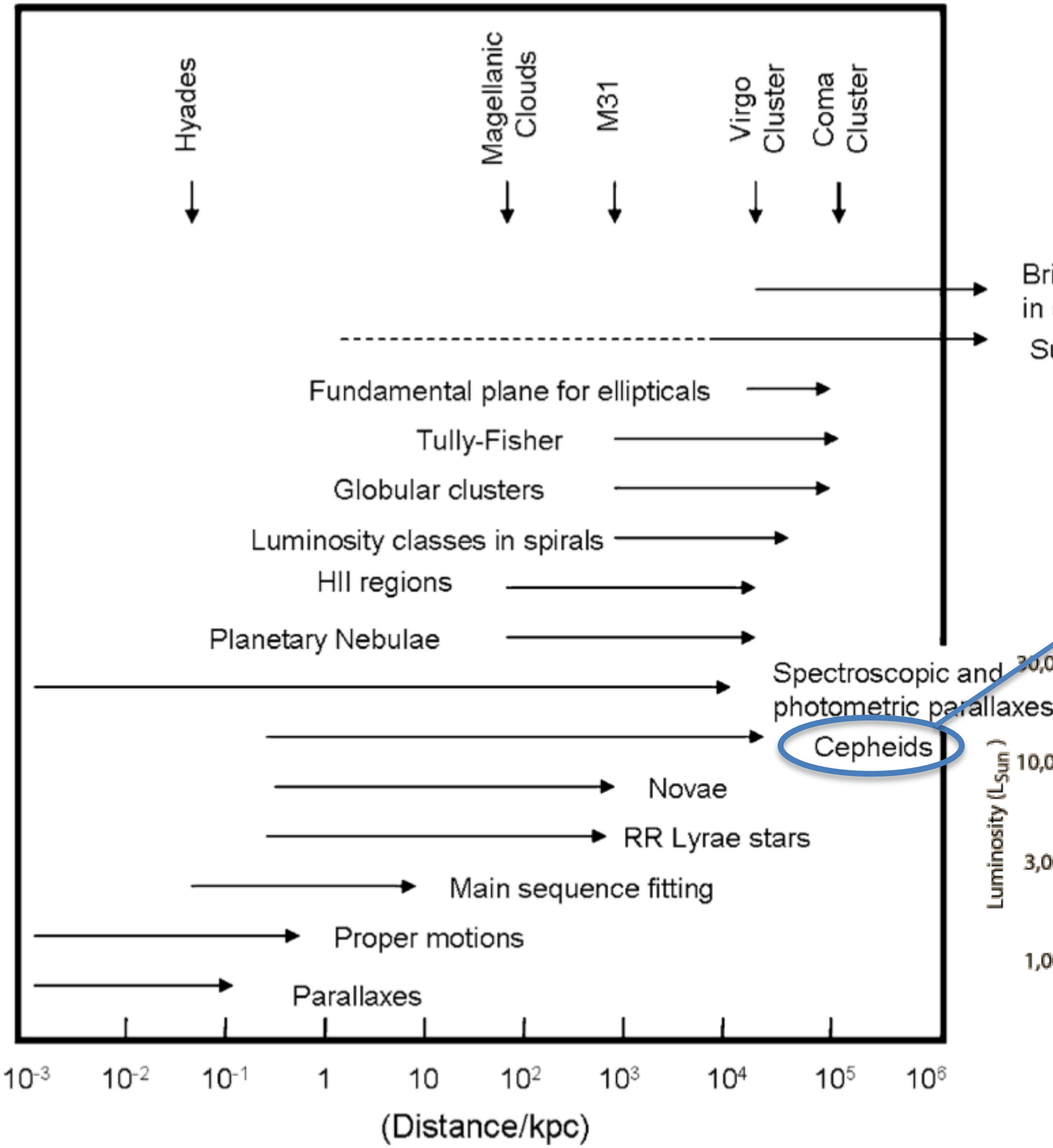
$\Omega_\Lambda = 1$
$\Omega_M = 0.25$,
$\Omega_\Lambda = 0.75$
Milne ($\Omega_{tot} = 0$)
EdS ($\Omega_M = 1$)
$\Omega_M = 2$
Options:
Milne $\Omega_{tot} = 0$
EdS $\Omega_{tot} = \Omega_M = 1$
Closed $\Omega_{tot} = \Omega_M = 2$
deSitter $\Omega_{tot} = \Omega_\Lambda = 1$
ΛCDM $\Omega_M = 0.25$, $\Omega_\Lambda = 0.75$
Scale Factor a(t)
2
1
0
-10
0
10
Gyr from now

The Hubble Constant, H_0

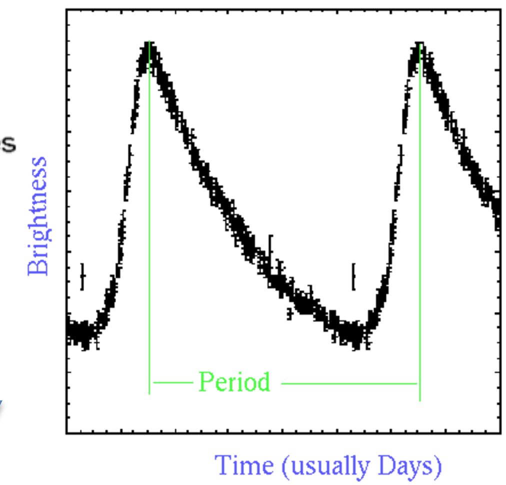
$h = H_0$ / 100 km s^{-1} Mpc^{-1} ; $h_{70} = H_0$ / 70 km s^{-1} Mpc^{-1}

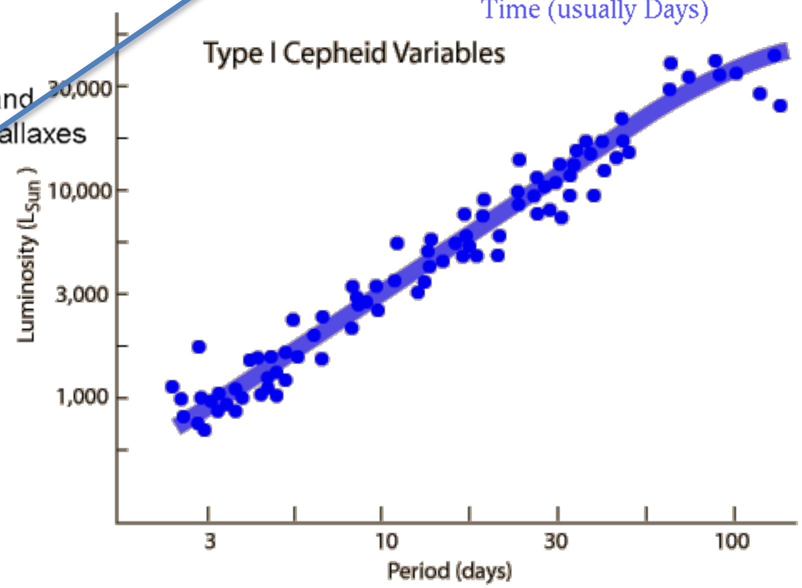


Cosmological Distance Ladder



Data from a Well–Measured Cepheid





Next Time:

The Thermal History:

Radiation & Matter