# Stellar kinematics

Jo Bovy


## ABSTRACT

This document is the appendix of my PhD thesis, available **here**. We present the transformations from observed kinematical quantities ($\alpha$,$\delta$,$\varpi$,$\mu_\alpha$,$\mu_\delta$) into Galactic quantities ($U$,$V$,$W$) and ($\mu_l$,$\mu_b$).


## 1. Introduction

## 2. From ($\alpha$,$\delta$) to ($l$,$b$)

The equatorial coordinates right ascension ($\alpha$) and declination ($\delta$) are defined as follows: Consider a very large sphere centered at the center of the Earth (see Figure 1). The intersection of the Earth's axis of rotation with this sphere is labeled as NCP (which we sometimes abbreviate even further as P), the North Celestial Pole. On this sphere there are parallels and meridians completely analogous to those defined on the surface of the Earth. Consider a star at a position $X$ on this sphere. The declination of that star is then defined as the complement of the arc between NCP and $X$ on the meridian connecting these two points. The right ascension of the star is defined as the angle between the meridian through NCP and $X$ and a reference meridian. This reference meridian is defined as the meridian which intersects the celestial equator (the intersection of the plane in which the Earth's equator lies and the celestial sphere) at the point in which the intersection of the ecliptic, i.e., the plane in which the Earth's orbit around the Sun lies, with the celestial sphere, the "celestial ecliptic'" if you will, intersects with the celestial equator; this defines two points, the point chosen is the point at which the Sun crosses the equatorial plane from South to North. This event is called the (northern) vernal equinox and it happens around March 21. The location of the vernal equinox on the celestial sphere is denoted by $\Upsilon$ (Green 1985, pp.14-6). Therefore, $\alpha$ and $\delta$ are formally defined as

$$\delta = 90^\circ - \mathrm{P}X\,, \tag{1}$$

$$\alpha = \Upsilon\mathrm{P}X\,. \tag{2}$$

Declinations run from $-90^\circ$ to $90^\circ$ whereas right ascensions can take on any value between $0^\circ$ and $360^\circ$.

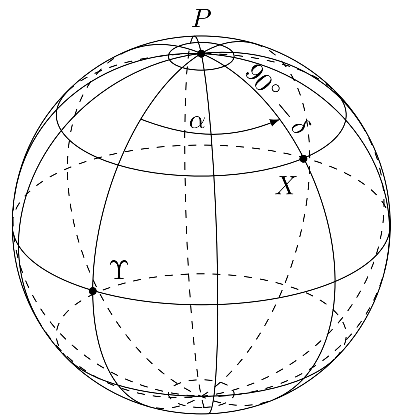


Fig. 1.— Definition of the equatorial coordinate system.

Galactic coordinates, i.e., a system of coordinates that has the Galactic plane as its equatorial plane, were introduced as early as the late eighteenth century by William Herschel and a variety of definitions existed based mainly on ambiguities in the definition of the Galactic poles. A standard system was introduced early in the twentieth century (Ohlsson 1932) and the zero of the longitude was set, rather arbitrarily, at the intersection of the Galactic plane and the equatorial plane for the equinox 1900.0. After the second World War, new radio observations revealed that the position of the Galactic poles needed revision and a new system of Galactic coordinates was introduced (Blaauw et al. 1960). New in this system was that radio observations had also shown that the compact radio source Sagittarius A* is at the center of the Galaxy and its position was used to define the zero of the Galactic longitude.

In detail the Galactic coordinate system is defined as follows (see Figure 2): Let NGP be the point on the celestial sphere corresponding to the North Galactic Pole (we will also sometimes just call this G, when brevity is required). The Galactic latitude $b$ is the defined as the complement of the arc along the great circle connecting the star's position $X$ to NGP. Let GC be the point on the Galactic equator (that is, the intersection of the Galactic plane with the celestial sphere) corresponding to the Galactic center (sometimes simply C). The Galactic longitude $l$ of the star $X$ is then the angle on the celestial sphere between the meridian through the NGP and $X$ and the meridian through the NGP and the GC. That is

$$b = 90^\circ - \mathrm{G}X\,, \tag{3}$$

$$l = \mathrm{CG}X\,. \tag{4}$$

Galactic latitudes run from $-90^\circ$ to $90^\circ$ whereas Galactic longitudes can take on any value between $0^\circ$ and $360^\circ$.

To derive the transformation between equatorial coordinates of a star and the Galactic coordinates we need the coordinates of the NGP and the GC in the equatorial frame, i.e., we need $\alpha_{\rm NGP}$, $\delta_{\rm NGP}$, and the position angle of the Galactic center $\theta$, or equivalently, the Galactic longitude of the north celestial pole. These quantities were defined for the epoch 1950.0 as follows: (Blaauw et al. 1960)

$$\alpha_{\rm NGP} = 12^{\rm h}49^{\rm m} = 192^\circ.25\,, \tag{5}$$

$$\delta_{\rm NGP} = 27^\circ.4\,, \tag{6}$$

$$\theta = 123^\circ\,. \tag{7}$$

We also need some formulas from spherical trigonometry, which are given in the following subsection.

### 2.1. Some formulas from spherical trigonometry

To derive the transformation between equatorial coordinates and Galactic coordinates we need three formulas from spherical trigonometry: the cosine formula, the sine formula, and the analogue formula. Let $ABC$ be a spherical triangle and let $a$ be the arc between $B$ and $C$ (that is, the arc on the great circle connecting $B$ and $C$) and analogous for $b$ and $c$. Also let, through a slight abuse of notation, $A$ be the angle between the meridians on which $B$ respectively $C$ lie, where the meridians are defined with respect to the pole $A$, and similarly for $B$ and $C$. Then the following holds:

$$\sin a \cos B = \cos b \sin c - \sin b \cos c \cos A \qquad \text{(analogue formula)} \tag{8}$$

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c} \qquad \text{(sine formula)} \tag{9}$$

$$\cos a = \cos b \cos c + \sin b \sin c \cos A \qquad \text{(cosine formula)}\,. \tag{10}$$

### 2.2. The transformation from $(\alpha,\delta)$ to $(l,b)$

We can now derive the transformation from equatorial coordinates to Galactic coordinates. We will consider the spherical triangle $\mathrm{G}PX$ on the celestial sphere (see Figure 2). If the point $X$ has Galactic coordinates $(l,b)$ and equatorial coordinates $(\alpha,\delta)$ then we have

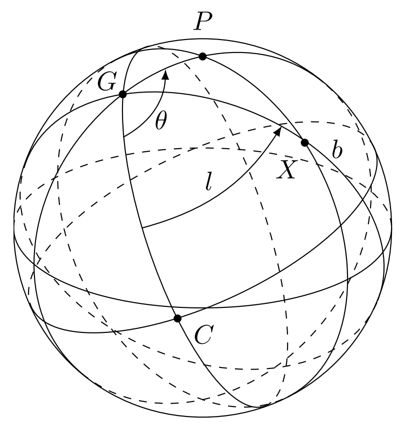


Fig. 2.— Transformation between the equatorial and Galactic coordinate systems.

the following (Green 1985):

$$\mathrm{P}X = 90^\circ - \delta\,, \tag{11}$$
$$\mathrm{G}X = 90^\circ - b\,, \tag{12}$$
$$\mathrm{GP} = 90^\circ - \delta_{\mathrm{NGP}}\,, \tag{13}$$
$$\mathrm{GP}X = \alpha - \alpha_{\mathrm{NGP}}\,, \tag{14}$$
$$\mathrm{PG}X = \theta - l\,. \tag{15}$$

A simple application of the analogue formula (8) gives

$$\sin \mathrm{G}X \cos \mathrm{PG}X = \cos \mathrm{P}X \sin \mathrm{GP} - \sin \mathrm{P}X \cos \mathrm{GP} \cos \mathrm{GP}X\,. \tag{16}$$

Substituting in (11) this becomes

$$\sin(90^\circ - b)\cos(\theta - l) = \cos(90^\circ - \delta)\sin(90^\circ - \delta_{\mathrm{NGP}}) - \sin(90^\circ - \delta)\cos(90^\circ - \delta_{\mathrm{NGP}})\cos(\alpha - \alpha_{\mathrm{NGP}})\,, \tag{17}$$

which becomes

$$\cos b \cos(\theta - l) = \sin\delta\cos\delta_{\mathrm{NGP}} - \cos\delta\sin\delta_{\mathrm{NGP}}\cos(\alpha - \alpha_{\mathrm{NGP}})\,. \tag{18}$$

The sine formula (9) gives in this case:

$$\frac{\sin \mathrm{GP}X}{\sin \mathrm{G}X} = \frac{\sin \mathrm{PG}X}{\sin \mathrm{P}X}\,, \tag{19}$$

and this becomes using (11)

$$\frac{\sin(\alpha - \alpha_{\mathrm{NGP}})}{\sin(90^\circ - b)} = \frac{\sin(\theta - l)}{\sin(90^\circ - \delta)}\,, \tag{20}$$

which is

$$\cos b \sin(\theta - l) = \cos\delta \sin(\alpha - \alpha_{\rm NGP}) \,. \tag{21}$$

And lastly we can apply the cosine formula (10) as follows

$$\cos {\rm G}X = \cos {\rm GP} \cos {\rm P}X + \sin {\rm GP} \sin {\rm P}X \cos {\rm GP}X \,, \tag{22}$$

which becomes using (11)

$$\cos(90^\circ - b) = \cos(90^\circ - \delta_{\rm NGP}) \cos(90^\circ - \delta) + \sin(90^\circ - \delta_{\rm NGP}) \sin(90^\circ - \delta) \cos(\alpha - \alpha_{\rm NGP}) \,, \tag{23}$$

which simplifies to

$$\sin b = \sin\delta_{\rm NGP} \sin\delta + \cos\delta_{\rm NGP} \cos\delta \cos(\alpha - \alpha_{\rm NGP}) \,. \tag{24}$$

The way to use these transformation equations (18),(21), and (24) is to first use (24) to solve for $b$. Then use (18) and (21) to unambiguously solve for $l$.

The transformation given here, using the definitions (5), is only valid for the epoch 1950.0. We will describe why this is and how to deal with measurements of $(\alpha,\delta)$ at different epochs below.

#### 2.2.1. *The transformation from $(\alpha,\delta)$ to $(l,b)$ in matrix form*

The transformation equations (18),(21), and (24) can be written in matrix form which shows that the full transformation is the combination of three simple transformations. This will also allow us to easily extract the inverse transformation, i.e from $(l,b)$ to $(\alpha,\delta)$.

For convenience we will list the transformation equations (18),(21), and (24) here together:

$$\begin{aligned} \cos b \cos(\theta - l) &= \sin\delta \cos\delta_{\rm NGP} - \cos\delta \sin\delta_{\rm NGP} \cos(\alpha - \alpha_{\rm NGP}) \,. \\ \cos b \sin(\theta - l) &= \cos\delta \sin(\alpha - \alpha_{\rm NGP}) \,. \\ \sin b &= \sin\delta_{\rm NGP} \sin\delta + \cos\delta_{\rm NGP} \cos\delta \cos(\alpha - \alpha_{\rm NGP}) \,. \end{aligned} \tag{25}$$

We will show now that these transformation equations can be written as

$$\begin{bmatrix} \cos b \cos l \\ \cos b \sin l \\ \sin b \end{bmatrix} = \mathbf{T} \begin{bmatrix} \cos\delta \cos\alpha \\ \cos\delta \sin\alpha \\ \sin\delta \end{bmatrix} \,, \tag{26}$$

where $\mathbf{T} \equiv \mathbf{T}(\alpha_{\rm NGP}, \delta_{\rm NGP}, \theta)$, and thus can be evaluated for the epoch at which the ($\alpha$,$\delta$) are measured.

The right hand sides of (25) can be written in matrix form as follows:

$$\begin{bmatrix} \cos b \cos(\theta - l) \\ \cos b \sin(\theta - l) \\ \sin b \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ \sin\theta & -\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos b \cos l \\ \cos b \sin l \\ \sin b \end{bmatrix} . \tag{27}$$

The left hand sides of (25) can be similarly be written first as:

$$\begin{bmatrix} \sin\delta \cos\delta_{\rm NGP} - \cos\delta \sin\delta_{\rm NGP} \cos(\alpha - \alpha_{\rm NGP}) \\ \cos\delta \sin(\alpha - \alpha_{\rm NGP}) \\ \sin\delta_{\rm NGP} \sin\delta + \cos\delta_{\rm NGP} \cos\delta \cos(\alpha - \alpha_{\rm NGP}) \end{bmatrix} = \begin{bmatrix} -\sin\delta_{\rm NGP} & 0 & \cos\delta_{\rm NGP} \\ 0 & 1 & 0 \\ \cos\delta_{\rm NGP} & 0 & \sin\delta_{\rm NGP} \end{bmatrix} \begin{bmatrix} \cos\delta \cos(\alpha - \alpha_{\rm NGP}) \\ \cos\delta \sin(\alpha - \alpha_{\rm NGP}) \\ \sin\delta \end{bmatrix} . \tag{28}$$

The final vector in the previous equation can itself be written as

$$\begin{bmatrix} \cos\delta \cos(\alpha - \alpha_{\rm NGP}) \\ \cos\delta \sin(\alpha - \alpha_{\rm NGP}) \\ \sin\delta \end{bmatrix} = \begin{bmatrix} \cos\alpha_{\rm NGP} & \sin\alpha_{\rm NGP} & 0 \\ -\sin\alpha_{\rm NGP} & \cos\alpha_{\rm NGP} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\delta \cos\alpha \\ \cos\delta \sin\alpha \\ \sin\delta \end{bmatrix} . \tag{29}$$

Combining (27), (28), and (29) we can write

$$\begin{aligned} \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ \sin\theta & -\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos b \cos l \\ \cos b \sin l \\ \sin b \end{bmatrix} &= \begin{bmatrix} -\sin\delta_{\rm NGP} & 0 & \cos\delta_{\rm NGP} \\ 0 & 1 & 0 \\ \cos\delta_{\rm NGP} & 0 & \sin\delta_{\rm NGP} \end{bmatrix} \\ &\times \begin{bmatrix} \cos\alpha_{\rm NGP} & \sin\alpha_{\rm NGP} & 0 \\ -\sin\alpha_{\rm NGP} & \cos\alpha_{\rm NGP} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\delta \cos\alpha \\ \cos\delta \sin\alpha \\ \sin\delta \end{bmatrix} , \end{aligned} \tag{30}$$

which means that we can write the matrix $\mathbf{T}$ in (26) as

$$\mathbf{T} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ \sin\theta & -\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\sin\delta_{\rm NGP} & 0 & \cos\delta_{\rm NGP} \\ 0 & 1 & 0 \\ \cos\delta_{\rm NGP} & 0 & \sin\delta_{\rm NGP} \end{bmatrix} \begin{bmatrix} \cos\alpha_{\rm NGP} & \sin\alpha_{\rm NGP} & 0 \\ -\sin\alpha_{\rm NGP} & \cos\alpha_{\rm NGP} & 0 \\ 0 & 0 & 1 \end{bmatrix} , \tag{31}$$

which is slightly different from but agrees with the representation given by Johnson & Soderblom (1987).

### 2.3. The transformation from ($\alpha$,$\delta$) to ($l$,$b$) for general epochs

We can evaluate the transformation matrix $\mathbf{T}$ given in (31) for the epoch 1950.0 by plugging in the values for $\alpha_{\rm NGP}$, $\delta_{\rm NGP}$, and $\theta$ from the definition of the Galactic coordinate system (3). In order to evaluate the matrix $\mathbf{T}$ for epochs other than 1950.0 one needs to use the values for $\alpha_{\rm NGP}$, $\delta_{\rm NGP}$, and $\theta$ for that epoch. These values are different from the 1950.0 values in general because of the effects of Luni-solar precession, planetary precession and nutation. The definition of the equatorial coordinate system is fully specified by the Earth's rotation axis, i.e., the North celestial pole, and the position of the ecliptic, which together with the equatorial plane defines the vernal equinox. Therefore, any changes to the orientation of the Earth's rotation axis or the position of the ecliptic will change the equatorial coordinates of a star.

Luni-solar precession comes about due to the torque on the Earth's rotation axis due to the gravitational interaction with the Moon and the Sun. If the Earth were a perfect sphere, the gravitational attraction between the Earth, the Moon and the Sun would produce no such torque, but, mostly due to its axial rotation, the Earth has developed an equatorial oblateness which violates the perfect spherical symmetry. The torque resulting from this depends on the configuration of the three bodies and the changing distances between them and produces therefore a rather complicated movement. Broadly this movement can be characterized by a long-term precessional movement, which is what is meant by Luni-solar precession, of the Earth's rotation axis, modulated by short-term variations, which is called the nutation.

The other planets in the Solar system have a negligible influence on the orientation of the Earth's rotational axis, however, planetary perturbations do influence the Earth's orbit around the Sun, i.e., they induce small changes in the orbital elements describing the orbit of the Earth around the Sun. In particular, the inclination of the ecliptic is not fixed. This gives rise to the so-called planetary precession, which corresponds to the change in the equatorial coordinate system due to the changing inclination of the ecliptic.

In order to derive high precision Galactic coordinates for objects on the sky, the complicated phenomena of Luni-Solar precession, nutation, and planetary precession need to be known at a high accuracy and the corresponding changes to an objects equatorial coordinates need to be modeled at high accuracy as well. We will not derive these formulas here, but simply state the result to second order (as given in Green 1985) relative to the epoch J2000.0 (where the "J" indicates that this is a Julian date; the epoch 1950.0 is actually B1950.0, where the "B" indicates that this is a Besselian date). To second order the equatorial coordinates at a time $t$ from J2000.0, expressed in Julian centuries, are given in terms of those at J2000.0 as follows:

$$\begin{aligned}
\alpha &= \alpha_{\rm J2000.0} + M + N \sin\alpha_m \tan\delta_m \\
\delta &= \delta_{\rm J2000.0} + N \cos\alpha_m \\
\alpha_m &= \alpha_{\rm J2000.0} + \frac{1}{2}(M + N \sin\alpha_{\rm J2000.0} \tan\delta_{\rm J2000.0}) \\
\delta_m &= \delta_{\rm J2000.0} + \frac{1}{2} N \cos\alpha_m \\
M &= 1^\circ.281232\ t + 0^\circ.000388\ t^2 \\
N &= 0^\circ.556753\ t - 0^\circ.000119\ t^2 \,.
\end{aligned} \tag{32}$$

Given the transformation equations (32) we can evaluate the transformation matrix $\mathbf{T}$ at any epoch $t$ by calculating the values of the defining parameters (3) at that epoch. The value of $\theta$ at a given epoch can be calculated from the position of the NGP ($\alpha_{\rm NGP}$,$\delta_{\rm NGP}$) and the GC ($\alpha_{\rm GC}$,$\delta_{\rm GC}$) at that epoch. To see how this is done we need to apply some basic spherical trigonometry again. We consider the spherical triangle spanned by the NGP, the GC, and the NCP: An application of the sine rule (9) gives

$$\frac{\sin {\rm CGP}}{\sin {\rm CP}} = \frac{\sin {\rm GPC}}{\sin {\rm GC}}\,, \tag{33}$$

or

$$\frac{\sin\theta}{\cos\delta_{\rm GC}} = \frac{\sin(\alpha_{\rm GC} - \alpha_{\rm NGP})}{\cos b_{\rm GC}}\,, \tag{34}$$

in which $b_{\rm GC} = 0$. Similarly the cosine rule (10) gives

$$\cos {\rm CP} = \cos {\rm CG} \cos {\rm GP} + \sin {\rm CG} \sin {\rm GP} \cos {\rm CGP}\,, \tag{35}$$

which becomes

$$\sin\delta_{\rm GC} = \sin b_{\rm GC} \sin\delta_{\rm NGP} + \cos b_{\rm GC} \cos\delta_{\rm NGP} \cos\theta\,. \tag{36}$$

Therefore, we can find $\theta$ by solving the following set of equations

$$\sin\theta = \sin(\alpha_{\rm GC} - \alpha_{\rm NGP}) \cos\delta_{\rm GC} \tag{37}$$

$$\cos\theta = \frac{\sin\delta_{\rm GC}}{\cos\delta_{\rm NGP}}\,. \tag{38}$$

The positions of the NGP and the GC for the epoch J2000.0 are given by (Binney & Merrifield 1998)

$$\begin{aligned}
(\alpha_{\rm NGP}, \delta_{\rm NGP}) &= (192^\circ.85948, 27^\circ.12825) \\
(\alpha_{\rm GC}, \delta_{\rm GC}) &= (266^\circ.405, -28^\circ.936)\,.
\end{aligned} \tag{39}$$

From these we can calculate $\theta$ from (37) at epoch J2000.0

$$\theta = 122^\circ.932\,. \tag{40}$$

The procedure to transform the position of a star given in ($\alpha$,$\delta$) at a given epoch to its Galactic coordinates ($l$,$b$) is now as follows: First calculate the position of the NGP and the GC in equatorial coordinates using (32) with the values given in (39); then use (37) to calculate the position angle of the GC for the given epoch; finally evaluate the transformation matrix $\mathbf{T}$ (31) for the calculated values ($\alpha_{\rm NGP}$,$\delta_{\rm NGP}$,$\theta$) and apply this transformation as in (26).

## 3. Parallax

The annual motion of the Earth around the Sun gives rise to an apparent displacement of a star relative to background objects that is inversely proportional to the distance to the star. Measurements of this apparent shift, or parallax, can thus be used to determine the distance to stars. Parallaxes are traditionally reported in units of arcseconds; a star with a parallax of 1 arcsecond is defined to be at a distance of 1 parsec (pc), which is approximately $3 \times 10^{16}$ m. The intrinsic motion of a star also gives rise to a systematic shift in its position relative to background sources, such that its angular motion—known as its proper motion—can be measured. Combining the distance and angular velocity gives the components of the space velocity of a star that are perpendicular to the line of sight.

When all three position coordinates of an object are known, i.e., its position on the celestial sphere and the radial distance $1/\varpi$ to it, its position can be specified in a rectangular coordinate system. We consider the following two rectangular coordinate systems: (1) the rectangular equatorial coordinate system and (2) the rectangular Galactic coordinate system. The rectangular equatorial coordinate system is defined as follows: the $z_{\rm eq}$-axis is the axis that goes from the center of the celestial sphere to the NCP; the $x_{\rm eq}$-axis goes from the center to the vernal equinox $\Upsilon$; and the $y_{\rm eq}$-axis completes the set such that the coordinate system is an orthogonal, right-handed system. The rectangular equatorial coordinates of a star are then given in terms of ($1/\varpi$,$\alpha$,$\delta$) as:

$$x_{\rm eq} = \frac{1}{\varpi}\cos\delta\cos\alpha\,, \tag{41}$$

$$y_{\rm eq} = \frac{1}{\varpi}\cos\delta\sin\alpha\,, \tag{42}$$

$$z_{\rm eq} = \frac{1}{\varpi}\sin\delta\,. \tag{43}$$

The rectangular Galactic coordinate system, which is more useful for studies of the dynamics in the Solar neighborhood, is similarly defined as follows: the $z$-axis is defined as the axis that goes from the center of the celestial sphere to the NGP; the $x$-axis is defined as the axis from the center to the GC; the $y$-axis which completes this set to form a right-handed set is in the direction of the Galactic rotation. The rectangular Galactic coordinates of a star are then given by

$$x = \frac{1}{\varpi} \cos b \cos l \,, \tag{44}$$

$$y = \frac{1}{\varpi} \cos b \sin l \,, \tag{45}$$

$$z = \frac{1}{\varpi} \sin b \,. \tag{46}$$

## 4. From ($v_r$,$\mu_\alpha$,$\mu_\delta$) to (U,V,W) and to ($v_r$,$\mu_l$,$\mu_b$)

We will now describe the transformation of the observed motion of a star to the motion of the star in the rectangular Galactic coordinate system. The velocity of a star in the rectangular Galactic coordinate system is conventially denoted as

$$(U, V, W) \equiv (\dot{x}, \dot{y}, \dot{z}) \,. \tag{47}$$

Since observations of the movement of a star are generally done by comparing the ($\alpha$,$\delta$) of the star at different epochs, while measurements of its motion in the radial direction are done using the Doppler shift displayed in the star's spectrum, which measures the change in the distance $d = 1/\varpi$ directly, we will relate the velocity components $U$, $V$, and $W$ to the changes $\dot{d} \equiv v_r$, $\dot{\alpha}$, and $\dot{\delta}$. In the following we will be dealing with the distance $d$, with the tacit assumption that actual distances are measured as inverse parallaxes, while changes in distances are directly measured.

The transformation between the rectangular equatorial coordinate system and the rectangular Galactic coordinate system is given similar to (26) as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{T} \begin{bmatrix} x_{\mathrm{eq}} \\ y_{\mathrm{eq}} \\ z_{\mathrm{eq}} \end{bmatrix} , \tag{48}$$

in which $\mathbf{T}$ is the same transformation matrix as given in (31). Since the matrix $\mathbf{T}$ is constant, we can write

$$\begin{bmatrix} U \\ V \\ W \end{bmatrix} = \mathbf{T} \frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} x_{\mathrm{eq}} \\ y_{\mathrm{eq}} \\ z_{\mathrm{eq}} \end{bmatrix} , \tag{49}$$

or

$$\begin{bmatrix} U \\ V \\ W \end{bmatrix} = \mathbf{T} \frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} d\cos\delta\cos\alpha \\ d\cos\delta\sin\alpha \\ d\sin\delta \end{bmatrix} , \tag{50}$$

which becomes

$$\begin{bmatrix} U \\ V \\ W \end{bmatrix} = \mathbf{T} \begin{bmatrix} \cos\delta\cos\alpha\,\dot{d} - d\sin\delta\cos\alpha\,\dot{\delta} - d\cos\delta\sin\alpha\,\dot{\alpha} \\ \cos\delta\sin\alpha\,\dot{d} - d\sin\delta\sin\alpha\,\dot{\delta} + d\cos\delta\cos\alpha\,\dot{\alpha} \\ \sin\delta\,\dot{d} + d\cos\delta\,\dot{\delta} \end{bmatrix} . \tag{51}$$

This can be simplified by writing

$$\begin{bmatrix} \cos\delta\cos\alpha\,\dot{d} - d\sin\delta\cos\alpha\,\dot{\delta} - d\cos\delta\sin\alpha\,\dot{\alpha} \\ \cos\delta\sin\alpha\,\dot{d} - d\sin\delta\sin\alpha\,\dot{\delta} + d\cos\delta\cos\alpha\,\dot{\alpha} \\ \sin\delta\,\dot{d} + d\cos\delta\,\dot{\delta} \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\delta\,\dot{d} - d\sin\delta\,\dot{\delta} \\ d\cos\delta\,\dot{\alpha} \\ \sin\delta\,\dot{d} + d\cos\delta\,\dot{\delta} \end{bmatrix} , \tag{52}$$

which can be further simplified as

$$\begin{bmatrix} \cos\delta\cos\alpha\,\dot{d} - d\sin\delta\cos\alpha\,\dot{\delta} - d\cos\delta\sin\alpha\,\dot{\alpha} \\ \cos\delta\sin\alpha\,\dot{d} - d\sin\delta\sin\alpha\,\dot{\delta} + d\cos\delta\cos\alpha\,\dot{\alpha} \\ \sin\delta\,\dot{d} + d\cos\delta\,\dot{\delta} \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\delta & 0 & -\sin\delta \\ 0 & 1 & 0 \\ \sin\delta & 0 & \cos\delta \end{bmatrix} \begin{bmatrix} \dot{d} \\ d\,\dot{\alpha}\,\cos\delta \\ d\dot{\delta} \end{bmatrix} . \tag{53}$$

We define

$$\mathbf{A} \equiv \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\delta & 0 & -\sin\delta \\ 0 & 1 & 0 \\ \sin\delta & 0 & \cos\delta \end{bmatrix} , \tag{54}$$

such that we can write

$$\begin{bmatrix} U \\ V \\ W \end{bmatrix} = \mathbf{T}\,\mathbf{A} \begin{bmatrix} v_r \\ \frac{1}{\varpi}\,\dot{\alpha}\,\cos\delta \\ \frac{1}{\varpi}\dot{\delta} \end{bmatrix} . \tag{55}$$

Here we used the fact that $\dot{d} \equiv v_r$. Note that the matrix $\mathbf{A}$ depends on the position of the star, that is $\mathbf{A} \equiv \mathbf{A}(\alpha, \delta)$. Let us define $\mu_\alpha \equiv \dot{\alpha}$ and $\mu_\delta \equiv \dot{\delta}$. Generally, the proper motions reported in astrometrical catalogues are such that $\mu_\alpha$ has already been multiplied with the $\cos\delta$ factor which features in (55).

Assuming that parallaxes are given in units of arcseconds (as), such that the distance is given in parsec, and that proper motions are given in units of as yr$^{-1}$, we need a constant of proportionality in the transformation (55) if we want $(U,V,W)$ in units of km s$^{-1}$. We have that

$$1\,\mathrm{as} = \frac{\pi}{180 \times 3600}\ \mathrm{rad} \tag{56}$$

$$1\ \mathrm{pc} = \frac{180 \times 3600}{\pi}\ \mathrm{AU}\,, \tag{57}$$

where AU is one astronomical unit. Therefore the constant of proportionality is given by $k = 1\mathrm{AU}\ \mathrm{yr}^{-1}$ expressed in units of km s$^{-1}$. The year here is the tropical year, and we have

$$\text{Tropical year } = 365.242198\ \text{days}\,, \tag{58}$$

such that

$$k = 4.74047\,. \tag{59}$$

Therefore, we summarize

$$\begin{bmatrix} U \\ V \\ W \end{bmatrix} = \mathbf{T}\,\mathbf{A} \begin{bmatrix} v_r \\ \frac{k}{\varpi}\,\mu_\alpha\,\cos\delta \\ \frac{k}{\varpi}\mu_\delta \end{bmatrix}, \tag{60}$$

where $[v_r]$ = km s$^{-1}$, $[\varpi]$ = as, and $[\mu_\alpha]$=$[\mu_\delta]$= as yr$^{-1}$.

To go to a description of the motion in terms of proper motions in the Galactic coordinates we can go through a similar argument as the one used in (51)-(53) and write

$$\begin{bmatrix} U \\ V \\ W \end{bmatrix} = \begin{bmatrix} \cos l & -\sin l & 0 \\ \sin l & \cos l & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos b & 0 & -\sin b \\ 0 & 1 & 0 \\ \sin b & 0 & \cos b \end{bmatrix} \begin{bmatrix} \dot{d} \\ d\,\dot{l}\,\cos b \\ d\dot{b} \end{bmatrix}. \tag{61}$$

We define

$$\mathbf{R}^\top \equiv \begin{bmatrix} \cos l & -\sin l & 0 \\ \sin l & \cos l & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos b & 0 & -\sin b \\ 0 & 1 & 0 \\ \sin b & 0 & \cos b \end{bmatrix}, \tag{62}$$

$\dot{l} \equiv \mu_l$, and $\dot{b} \equiv \mu_b$, such that

$$\begin{bmatrix} v_r \\ \frac{k}{\varpi}\,\mu_l\,\cos b \\ \frac{k}{\varpi}\mu_b \end{bmatrix} = \mathbf{R} \begin{bmatrix} U \\ V \\ W \end{bmatrix}, \tag{63}$$

where we have included the $k$ factor defined above to allow parallaxes and proper motions to be specified in as and as yr$^{-1}$, repectively. Defining $v_l \equiv \frac{k}{\varpi}\,\mu_l\,\cos b$ and $v_b \equiv \frac{k}{\varpi}\mu_b$ this becomes

$$\begin{bmatrix} v_r \\ v_l \\ v_b \end{bmatrix} = \mathbf{R} \begin{bmatrix} U \\ V \\ W \end{bmatrix}. \tag{64}$$

The full transformation from $(v_r,\mu_\alpha,\mu_\delta)$ to $(v_r,v_l,v_b)$ can then be written by combining (60) and (64), such that

$$\begin{bmatrix} v_r \\ v_l \\ v_b \end{bmatrix} = \mathbf{R}\,\mathbf{T}\,\mathbf{A} \begin{bmatrix} v_r \\ \frac{1}{\varpi}\,\mu_\alpha\,\cos\delta \\ \frac{1}{\varpi}\mu_\delta \end{bmatrix} . \tag{65}$$

### 4.1. Straight from $(\mu_\alpha,\mu_\delta)$ to $(\mu_l,\mu_b)$

Equation (65) can be simplified to a single rotation such that

$$\begin{bmatrix} v_r \\ \frac{1}{\varpi}\,\mu_l\,\cos b \\ \frac{1}{\varpi}\mu_b \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} v_r \\ \frac{1}{\varpi}\,\mu_\alpha\,\cos\delta \\ \frac{1}{\varpi}\mu_\delta \end{bmatrix} . \tag{66}$$

One can show by working out the matrix product $\mathbf{R}\,\mathbf{T}\,\mathbf{A}$ and using the identities from Equation (25) that the *Galactic parallactic angle* $\phi$ is the solution of

$$\cos\phi = \frac{\sin\delta_{\mathrm{NGP}} - \sin\delta\sin b}{\cos\delta\cos b}\,, \tag{67}$$

$$\sin\phi = \frac{\sin(\alpha-\alpha_{\mathrm{NGP}})\cos\delta_{\mathrm{NGP}}}{\cos b}\,. \tag{68}$$

Therefore, one can directly transform measured proper motions in celestial coordinates into proper motions in Galactic coordinates as

$$\begin{bmatrix} \mu_l\,\cos b \\ \mu_b \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \mu_\alpha\,\cos\delta \\ \mu_\delta \end{bmatrix} . \tag{69}$$

We will denote the transformation matrix in this Equation as $\mathbf{P}$.

## 5. Uncertainty propagation

Propagation of observational uncertainties is simple using the matrix formalism employed above. We assume that uncertainties in the direction $(\alpha,\delta)$ of an object are negligible, but we make no other assumptions about the independence of the uncertainties of the other quantities since for example the scanning strategy of astrometric satellites can give rise to correlated proper motion measurements.

In what follows we give formulas for first-order uncertainty propagation. We assume that we know the distance uncertainty $\sigma_d$. If the distance is measured as a parallax, this distance uncertainty is given by $\sigma_d = \sigma_\varpi/\varpi^2$. Proper accounting for the non-linear nature of the transformations above should go beyond this first-order calculation or employ a Monte Carlo uncertainty propagation.

### 5.1. Cov($d$,$v_r$,$\mu_\alpha$,$\mu_\delta$) to Cov($U$,$V$,$W$)

The propagation of uncertainty Cov($d$,$v_r$,$\mu_\alpha$,$\mu_\delta$) to Cov($U$,$V$,$W$) can be broken up into two parts: (i) Cov($d$,$\mu_\alpha$,$\mu_\delta$) to Cov($v_\alpha$,$v_\delta$) and (ii) Cov($v_r$,$v_\alpha$,$v_\delta$) to Cov($U$,$V$,$W$), where $v_\alpha \equiv \frac{k}{\varpi}\,\mu_\alpha\,\cos\delta$ and $v_\delta \equiv \frac{k}{\varpi}\mu_\delta$.

The transformation (i) is non-linear, so we compute the Jacobian as

$$\frac{\partial\,(v_\alpha, v_\delta)}{\partial\,(d, \mu_\alpha, \mu_\delta)} = k \left[ \begin{array}{ccc} \mu_\alpha \cos\delta & d & 0 \\ \mu_\delta & 0 & d \end{array} \right]. \tag{70}$$

Then we have that

$$\mathrm{Cov}(v_\alpha, v_\delta) = \frac{\partial\,(v_\alpha, v_\delta)}{\partial\,(d, \mu_\alpha, \mu_\delta)}\mathrm{Cov}(d, \mu_\alpha, \mu_\delta)\,\frac{\partial\,(v_\alpha, v_\delta)}{\partial\,(d, \mu_\alpha, \mu_\delta)}^{\top}. \tag{71}$$

The transformation (ii) is given by Equation (55) such that we find

$$\mathrm{Cov}(U, V, W) = \mathbf{T}\,\mathbf{A}\,\mathrm{Cov}(v_r, v_\alpha, v_\delta)\,\mathbf{A}^\top\,\mathbf{T}^\top\,. \tag{72}$$

### 5.2. Cov($d$,$v_r$,$\mu_l$,$\mu_b$) to Cov($U$,$V$,$W$)

This uncertainty propagation is similar to that in the § above. We again split the transformation into (i) Cov($d$,$\mu_l$,$\mu_b$) to Cov($v_l$,$v_b$) and (ii) Cov($v_r$,$v_l$,$v_b$) to Cov($U$,$V$,$W$), where $v_l \equiv \frac{k}{\varpi}\,\mu_l\,\cos b$ and $v_b \equiv \frac{k}{\varpi}\mu_b$.

The transformation (i) is non-linear with Jacobian

$$\frac{\partial\,(v_l, v_b)}{\partial\,(d, \mu_l, \mu_b)} = k \left[ \begin{array}{ccc} \mu_l \cos b & d & 0 \\ \mu_b & 0 & d \end{array} \right]. \tag{73}$$

Then we have that

$$\mathrm{Cov}(v_l, v_b) = \frac{\partial\,(v_l, v_b)}{\partial\,(d, \mu_l, \mu_b)}\mathrm{Cov}(d, \mu_l, \mu_b)\,\frac{\partial\,(v_l, v_b)}{\partial\,(d, \mu_l, \mu_b)}^{\top}. \tag{74}$$

The transformation (ii) is given by Equation (64) such that we find

$$\mathrm{Cov}(U, V, W) = \mathbf{R}^{-1}\,\mathrm{Cov}(v_r, v_l, v_b)\,\mathbf{R}^{-\top}\,. \tag{75}$$

### 5.3. Cov($\mu_\alpha$,$\mu_\delta$) to Cov($\mu_l$,$\mu_b$)

To propagate the proper motion uncertainty in $\alpha$ and $\delta$ to that in Galactic coordinates (see § 4.1) we do

$$\mathrm{Cov}(\mu_l \cos b, \mu_b) = \mathbf{P}\,\mathrm{Cov}(\mu_\alpha \cos\delta, \mu_\delta)\,\mathbf{P}^\top\,, \tag{76}$$

where $\mathbf{P}$ is defined below Equation 69.

## REFERENCES

Binney, J. & Merrifield, M. 1998, Galactic Astronomy (Princeton University Press)

Blaauw, A., Gum, C. S., Pawsey, J. L., & Westerhout, G. 1960, MNRAS, 121, 123

Green, R. M. 1985, Spherical astronomy (Cambridge University Press)

Johnson, D. R. H. & Soderblom, D. R. 1987, AJ, 93, 864

Ohlsson, J. 1932, Ann. Lund Obs., 6