



Perron–Frobenius Theory of Nonnegative Matrices

8.1 INTRODUCTION

$\mathbf{A} \in \Re^{m \times n}$ is said to be a **nonnegative matrix** whenever each $a_{ij} \geq 0$, and this is denoted by writing $\mathbf{A} \geq \mathbf{0}$. In general, $\mathbf{A} \geq \mathbf{B}$ means that each $a_{ij} \geq b_{ij}$. Similarly, \mathbf{A} is a **positive matrix** when each $a_{ij} > 0$, and this is denoted by writing $\mathbf{A} > \mathbf{0}$. More generally, $\mathbf{A} > \mathbf{B}$ means that each $a_{ij} > b_{ij}$.

Applications abound with nonnegative and positive matrices. In fact, many of the applications considered in this text involve nonnegative matrices. For example, the connectivity matrix \mathbf{C} in Example 3.5.2 (p. 100) is nonnegative. The discrete Laplacian \mathbf{L} from Example 7.6.2 (p. 563) leads to a nonnegative matrix because $(4\mathbf{I} - \mathbf{L}) \geq \mathbf{0}$. The matrix $e^{\mathbf{A}t}$ that defines the solution of the system of differential equations in the mixing problem of Example 7.9.7 (p. 610) is nonnegative for all $t \geq 0$. And the system of difference equations $\mathbf{p}(k) = \mathbf{A}\mathbf{p}(k-1)$ resulting from the shell game of Example 7.10.8 (p. 635) has a nonnegative coefficient matrix \mathbf{A} .

Since nonnegative matrices are pervasive, it's natural to investigate their properties, and that's the purpose of this chapter. A primary issue concerns the extent to which the properties $\mathbf{A} > \mathbf{0}$ or $\mathbf{A} \geq \mathbf{0}$ translate to spectral properties—e.g., to what extent does \mathbf{A} have positive (or nonnegative) eigenvalues and eigenvectors?

The topic is called the “Perron–Frobenius theory” because it evolved from the contributions of the German mathematicians Oskar (or Oscar) Perron⁸⁹ and

⁸⁹ Oskar Perron (1880–1975) originally set out to fulfill his father's wishes to be in the family busi-

Ferdinand Georg Frobenius.⁹⁰ Perron published his treatment of positive matrices in 1907, and in 1912 Frobenius contributed substantial extensions of Perron's results to cover the case of nonnegative matrices.

In addition to saying something useful, the Perron–Frobenius theory is elegant. It is a testament to the fact that beautiful mathematics eventually tends to be useful, and useful mathematics eventually tends to be beautiful.

ness, so he only studied mathematics in his spare time. But he was eventually captured by the subject, and, after studying at Berlin, Tübingen, and Göttingen, he completed his doctorate, writing on geometry, at the University of Munich under the direction of Carl von Lindemann (1852–1939) (who first proved that π was transcendental). Upon graduation in 1906, Perron held positions at Munich, Tübingen, and Heidelberg. Perron's career was interrupted in 1915 by World War I in which he earned the Iron Cross. After the war he resumed work at Heidelberg, but in 1922 he returned to Munich to accept a chair in mathematics, a position he occupied for the rest of his career. In addition to his contributions to matrix theory, Perron's work covered a wide range of other topics in algebra, analysis, differential equations, continued fractions, geometry, and number theory. He was a man of extraordinary mental and physical energy. In addition to being able to climb mountains until he was in his midseventies, Perron continued to teach at Munich until he was 80 (although he formally retired at age 71), and he maintained a remarkably energetic research program into his nineties. He published 18 of his 218 papers *after* he was 84.

⁹⁰ Ferdinand Georg Frobenius (1849–1917) earned his doctorate under the supervision of Karl Weierstrass (p. 589) at the University of Berlin in 1870. As mentioned earlier, Frobenius was a mentor to and a collaborator with Issai Schur (p. 123), and, in addition to their joint work in group theory, they were among the first to study matrix theory as a discipline unto itself. Frobenius in particular must be considered along with Cayley and Sylvester when thinking of core developers of matrix theory. However, in the beginning, Frobenius's motivation came from Kronecker (p. 597) and Weierstrass, and he seemed oblivious to Cayley's work (p. 80). It was not until 1896 that Frobenius became aware of Cayley's 1857 work, *A Memoir on the Theory of Matrices*, and only then did the terminology "matrix" appear in Frobenius's work. Even though Frobenius was the first to give a rigorous proof of the Cayley–Hamilton theorem (p. 509), he generously attributed it to Cayley in spite of the fact that Cayley had only discussed the result for 2×2 and 3×3 matrices. But credit in this regard is not overly missed because Frobenius's extension of Perron's results are more substantial, and they alone may keep Frobenius's name alive forever.

8.2 POSITIVE MATRICES

The purpose of this section is to focus on matrices $\mathbf{A}_{n \times n} > \mathbf{0}$ with positive entries, and the aim is to investigate the extent to which this positivity is inherited by the eigenvalues and eigenvectors of \mathbf{A} .

There are a few elementary observations that will help along the way, so let's begin with them. First, notice that

$$\mathbf{A} > \mathbf{0} \implies \rho(\mathbf{A}) > 0 \quad (8.2.1)$$

because if $\sigma(\mathbf{A}) = \{0\}$, then the Jordan form for \mathbf{A} , and hence \mathbf{A} itself, is nilpotent, which is impossible when each $a_{ij} > 0$. This means that our discussions can be limited to positive matrices having spectral radius 1 because \mathbf{A} can always be normalized by its spectral radius—i.e., $\mathbf{A} > \mathbf{0} \iff \mathbf{A}/\rho(\mathbf{A}) > \mathbf{0}$, and $\rho(\mathbf{A}) = r \iff \rho(\mathbf{A}/r) = 1$. Other easily verified observations are

$$\mathbf{P} > \mathbf{0}, \mathbf{x} \geq \mathbf{0}, \mathbf{x} \neq \mathbf{0} \implies \mathbf{Px} > \mathbf{0}, \quad (8.2.2)$$

$$\mathbf{N} \geq \mathbf{0}, \mathbf{u} \geq \mathbf{v} \geq \mathbf{0} \implies \mathbf{Nu} \geq \mathbf{Nv}, \quad (8.2.3)$$

$$\mathbf{N} \geq \mathbf{0}, \mathbf{z} > \mathbf{0}, \mathbf{Nz} = \mathbf{0} \implies \mathbf{N} = \mathbf{0}, \quad (8.2.4)$$

$$\mathbf{N} \geq \mathbf{0}, \mathbf{N} \neq \mathbf{0}, \mathbf{u} > \mathbf{v} > \mathbf{0} \implies \mathbf{Nu} > \mathbf{Nv}. \quad (8.2.5)$$

In all that follows, the bar notation $|\star|$ is used to denote a matrix of absolute values—i.e., $|\mathbf{M}|$ is the matrix having entries $|m_{ij}|$. The bar notation will *never* denote a determinant in the sequel. Finally, notice that as a simple consequence of the triangle inequality, it's always true that $|\mathbf{Ax}| \leq |\mathbf{A}||\mathbf{x}|$.

Positive Eigenpair

If $\mathbf{A}_{n \times n} > \mathbf{0}$, then the following statements are true.

$$\bullet \quad \rho(\mathbf{A}) \in \sigma(\mathbf{A}). \quad (8.2.6)$$

$$\bullet \quad \text{If } \mathbf{Ax} = \rho(\mathbf{A})\mathbf{x}, \text{ then } \mathbf{A}|\mathbf{x}| = \rho(\mathbf{A})|\mathbf{x}| \text{ and } |\mathbf{x}| > \mathbf{0}. \quad (8.2.7)$$

In other words, \mathbf{A} has an eigenpair of the form $(\rho(\mathbf{A}), \mathbf{v})$ with $\mathbf{v} > \mathbf{0}$.

Proof. As mentioned earlier, it can be assumed that $\rho(\mathbf{A}) = 1$ without any loss of generality. If (λ, \mathbf{x}) is any eigenpair for \mathbf{A} such that $|\lambda| = 1$, then

$$|\mathbf{x}| = |\lambda||\mathbf{x}| = |\lambda\mathbf{x}| = |\mathbf{Ax}| \leq |\mathbf{A}||\mathbf{x}| = \mathbf{A}|\mathbf{x}| \implies |\mathbf{x}| \leq \mathbf{A}|\mathbf{x}|. \quad (8.2.8)$$

The goal is to show that equality holds. For convenience, let $\mathbf{z} = \mathbf{A}|\mathbf{x}|$ and $\mathbf{y} = \mathbf{z} - |\mathbf{x}|$, and notice that (8.2.8) implies $\mathbf{y} \geq \mathbf{0}$. Suppose that $\mathbf{y} \neq \mathbf{0}$ —i.e.,

suppose that some $y_i > 0$. In this case, it follows from (8.2.2) that $\mathbf{A}\mathbf{y} > \mathbf{0}$ and $\mathbf{z} > \mathbf{0}$, so there must exist a number $\epsilon > 0$ such that $\mathbf{A}\mathbf{y} > \epsilon \mathbf{z}$ or, equivalently,

$$\frac{\mathbf{A}}{1+\epsilon} \mathbf{z} > \mathbf{z}.$$

Writing this inequality as $\mathbf{B}\mathbf{z} > \mathbf{z}$, where $\mathbf{B} = \mathbf{A}/(1+\epsilon)$, and successively multiplying both sides by \mathbf{B} while using (8.2.5) produces

$$\mathbf{B}^2 \mathbf{z} > \mathbf{B}\mathbf{z} > \mathbf{z}, \quad \mathbf{B}^3 \mathbf{z} > \mathbf{B}^2 \mathbf{z} > \mathbf{z}, \quad \dots \implies \mathbf{B}^k \mathbf{z} > \mathbf{z} \quad \text{for all } k = 1, 2, \dots$$

But $\lim_{k \rightarrow \infty} \mathbf{B}^k = \mathbf{0}$ because $\rho(\mathbf{B}) = \sigma(\mathbf{A}/(1+\epsilon)) = 1/(1+\epsilon) < 1$ (recall (7.10.5) on p. 617), so, in the limit, we have $\mathbf{0} > \mathbf{z}$, which contradicts the fact that $\mathbf{z} > \mathbf{0}$. Since the supposition that $\mathbf{y} \neq \mathbf{0}$ led to this contradiction, the supposition must be false and, consequently, $\mathbf{0} = \mathbf{y} = \mathbf{A}|\mathbf{x}| - |\mathbf{x}|$. Thus $|\mathbf{x}|$ is an eigenvector for \mathbf{A} associated with the eigenvalue $1 = \rho(\mathbf{A})$. The proof is completed by observing that $|\mathbf{x}| = \mathbf{A}|\mathbf{x}| = \mathbf{z} > \mathbf{0}$. ■

Now that it's been established that $\rho(\mathbf{A}) > 0$ is in fact an eigenvalue for $\mathbf{A} > \mathbf{0}$, the next step is to investigate the index of this special eigenvalue.

Index of $\rho(\mathbf{A})$

If $\mathbf{A}_{n \times n} > \mathbf{0}$, then the following statements are true.

- $\rho(\mathbf{A})$ is the only eigenvalue of \mathbf{A} on the spectral circle.
- $\text{index}(\rho(\mathbf{A})) = 1$. In other words, $\rho(\mathbf{A})$ is a *semisimple* eigenvalue. Recall Exercise 7.8.4 (p. 596).

Proof. Again, assume without loss of generality that $\rho(\mathbf{A}) = 1$. We know from (8.2.7) on p. 663 that if (λ, \mathbf{x}) is an eigenpair for \mathbf{A} such that $|\lambda| = 1$, then $\mathbf{0} < |\mathbf{x}| = \mathbf{A}|\mathbf{x}|$, so $0 < |x_k| = (\mathbf{A}|\mathbf{x}|)_k = \sum_{j=1}^n a_{kj}|x_j|$. But it's also true that $|x_k| = |\lambda| |x_k| = |(\lambda \mathbf{x})_k| = |(\mathbf{A}\mathbf{x})_k| = \left| \sum_{j=1}^n a_{kj}x_j \right|$, and thus

$$\left| \sum_j a_{kj}x_j \right| = \sum_j a_{kj}|x_j| = \sum_j |a_{kj}x_j|. \quad (8.2.9)$$

For nonzero vectors $\{\mathbf{z}_1, \dots, \mathbf{z}_n\} \subset \mathbb{C}^n$, it's a fact that $\|\sum_j \mathbf{z}_j\|_2 = \sum_j \|\mathbf{z}_j\|_2$ (equality in the triangle inequality) if and only if each $\mathbf{z}_j = \alpha_j \mathbf{z}_1$ for some $\alpha_j > 0$ (Exercise 5.1.10, p. 277). In particular, this holds for scalars, so (8.2.9) insures the existence of numbers $\alpha_j > 0$ such that

$$a_{kj}x_j = \alpha_j(a_{k1}x_1) \quad \text{or, equivalently,} \quad x_j = \pi_j x_1 \quad \text{with } \pi_j = \frac{\alpha_j a_{k1}}{a_{kj}} > 0.$$

In other words, if $|\lambda| = 1$, then $\mathbf{x} = x_1 \mathbf{p}$, where $\mathbf{p} = (1, \pi_2, \dots, \pi_n)^T > \mathbf{0}$, so

$$\lambda \mathbf{x} = \mathbf{A} \mathbf{x} \implies \lambda \mathbf{p} = \mathbf{A} \mathbf{p} = |\mathbf{A} \mathbf{p}| = |\lambda \mathbf{p}| = |\lambda| \mathbf{p} = \mathbf{p} \implies \lambda = 1,$$

and thus 1 is the only eigenvalue of \mathbf{A} on the spectral circle. Now suppose that $\text{index}(1) = m > 1$. It follows that $\|\mathbf{A}^k\|_\infty \rightarrow \infty$ as $k \rightarrow \infty$ because there is an $m \times m$ Jordan block \mathbf{J}_\star in the Jordan form $\mathbf{J} = \mathbf{P}^{-1} \mathbf{A} \mathbf{P}$ that looks like (7.10.30) on p. 629, so $\|\mathbf{J}_\star^k\|_\infty \rightarrow \infty$, which in turn means that $\|\mathbf{J}^k\|_\infty \rightarrow \infty$ and, consequently, $\|\mathbf{J}^k\|_\infty = \|\mathbf{P}^{-1} \mathbf{A}^k \mathbf{P}\|_\infty \leq \|\mathbf{P}^{-1}\|_\infty \|\mathbf{A}^k\|_\infty \|\mathbf{P}\|_\infty$ implies

$$\|\mathbf{A}^k\|_\infty \geq \frac{\|\mathbf{J}^k\|_\infty}{\|\mathbf{P}^{-1}\|_\infty \|\mathbf{P}\|_\infty} \rightarrow \infty.$$

Let $\mathbf{A}^k = [a_{ij}^{(k)}]$, and let i_k denote the row index for which $\|\mathbf{A}^k\|_\infty = \sum_j a_{i_k j}^{(k)}$. We know that there exists a vector $\mathbf{p} > \mathbf{0}$ such that $\mathbf{p} = \mathbf{A} \mathbf{p}$, so for such an eigenvector,

$$\|\mathbf{p}\|_\infty \geq p_{i_k} = \sum_j a_{i_k j}^{(k)} p_j \geq \left(\sum_j a_{i_k j}^{(k)} \right) (\min_i p_i) = \|\mathbf{A}^k\|_\infty (\min_i p_i) \rightarrow \infty.$$

But this is impossible because \mathbf{p} is a constant vector, so the supposition that $\text{index}(1) > 1$ must be false, and thus $\text{index}(1) = 1$. ■

Establishing that $\rho(\mathbf{A})$ is a semisimple eigenvalue of $\mathbf{A} > \mathbf{0}$ was just a steppingstone (but an important one) to get to the following theorem concerning the multiplicities of $\rho(\mathbf{A})$.

Multiplicities of $\rho(\mathbf{A})$

If $\mathbf{A}_{n \times n} > \mathbf{0}$, then $\text{alg mult}_{\mathbf{A}}(\rho(\mathbf{A})) = 1$. In other words, the spectral radius of \mathbf{A} is a *simple* eigenvalue of \mathbf{A} .

So $\dim N(\mathbf{A} - \rho(\mathbf{A})\mathbf{I}) = \text{geo mult}_{\mathbf{A}}(\rho(\mathbf{A})) = \text{alg mult}_{\mathbf{A}}(\rho(\mathbf{A})) = 1$.

Proof. As before, assume without loss of generality that $\rho(\mathbf{A}) = 1$, and suppose that $\text{alg mult}_{\mathbf{A}}(\lambda = 1) = m > 1$. We already know that $\lambda = 1$ is a semisimple eigenvalue, which means that $\text{alg mult}_{\mathbf{A}}(1) = \text{geo mult}_{\mathbf{A}}(1)$ (p. 510), so there are m linearly independent eigenvectors associated with $\lambda = 1$. If \mathbf{x} and \mathbf{y} are a pair of independent eigenvectors associated with $\lambda = 1$, then $\mathbf{x} \neq \alpha \mathbf{y}$ for all $\alpha \in \mathcal{C}$. Select a nonzero component from \mathbf{y} , say $y_i \neq 0$, and set $\mathbf{z} = \mathbf{x} - (x_i/y_i)\mathbf{y}$. Since $\mathbf{A}\mathbf{z} = \mathbf{z}$, we know from (8.2.7) on p. 663 that $\mathbf{A}|\mathbf{z}| = |\mathbf{z}| > \mathbf{0}$. But this contradicts the fact that $z_i = x_i - (x_i/y_i)y_i = 0$. Therefore, the supposition that $m > 1$ must be false, and thus $m = 1$. ■

Since $N(\mathbf{A} - \rho(\mathbf{A})\mathbf{I})$ is a one-dimensional space that can be spanned by some $\mathbf{v} > \mathbf{0}$, there is a *unique* eigenvector $\mathbf{p} \in N(\mathbf{A} - \rho(\mathbf{A})\mathbf{I})$ such that $\mathbf{p} > \mathbf{0}$ and $\sum_j p_j = 1$ (it's obtained by the normalization $\mathbf{p} = \mathbf{v} / \|\mathbf{v}\|_1$ —see Exercise 8.2.3). This special eigenvector \mathbf{p} is called the **Perron vector** for $\mathbf{A} > \mathbf{0}$, and the associated eigenvalue $r = \rho(\mathbf{A})$ is called the **Perron root** of \mathbf{A} .

Since $\mathbf{A} > \mathbf{0} \iff \mathbf{A}^T > \mathbf{0}$, and since $\rho(\mathbf{A}) = \rho(\mathbf{A}^T)$, it's clear that if $\mathbf{A} > \mathbf{0}$, then in addition to the Perron eigenpair (r, \mathbf{p}) for \mathbf{A} there is a corresponding Perron eigenpair (r, \mathbf{q}) for \mathbf{A}^T . Because $\mathbf{q}^T \mathbf{A} = r \mathbf{q}^T$, the vector $\mathbf{q}^T > \mathbf{0}$ is called the **left-hand Perron vector** for \mathbf{A} .

While eigenvalues of $\mathbf{A} > \mathbf{0}$ other than $\rho(\mathbf{A})$ may or may not be positive, it turns out that no eigenvectors other than positive multiples of the Perron vector can be positive—or even nonnegative.

No Other Positive Eigenvectors

There are no nonnegative eigenvectors for $\mathbf{A}_{n \times n} > \mathbf{0}$ other than the Perron vector \mathbf{p} and its positive multiples. (8.2.10)

Proof. If (λ, \mathbf{y}) is an eigenpair for \mathbf{A} such that $\mathbf{y} \geq \mathbf{0}$, and if $\mathbf{x} > \mathbf{0}$ is the Perron vector for \mathbf{A}^T , then $\mathbf{x}^T \mathbf{y} > 0$ by (8.2.2), so

$$\rho(\mathbf{A}) \mathbf{x}^T = \mathbf{x}^T \mathbf{A} \implies \rho(\mathbf{A}) \mathbf{x}^T \mathbf{y} = \mathbf{x}^T \mathbf{A} \mathbf{y} = \lambda \mathbf{x}^T \mathbf{y} \implies \rho(\mathbf{A}) = \lambda. \blacksquare$$

In 1942 the German mathematician Lothar Collatz (1910–1990) discovered the following formula for the Perron root, and in 1950 Helmut Wielandt (p. 534) used it to develop the Perron–Frobenius theory.

Collatz–Wielandt Formula

The Perron root of $\mathbf{A}_{n \times n} > \mathbf{0}$ is given by $r = \max_{\mathbf{x} \in \mathcal{N}} f(\mathbf{x})$, where

$$f(\mathbf{x}) = \min_{\substack{1 \leq i \leq n \\ x_i \neq 0}} \frac{[\mathbf{A}\mathbf{x}]_i}{x_i} \quad \text{and} \quad \mathcal{N} = \{\mathbf{x} \mid \mathbf{x} \geq \mathbf{0} \text{ with } \mathbf{x} \neq \mathbf{0}\}.$$

Proof. If $\xi = f(\mathbf{x})$ for $\mathbf{x} \in \mathcal{N}$, then $\mathbf{0} \leq \xi \mathbf{x} \leq \mathbf{A}\mathbf{x}$. Let \mathbf{p} and \mathbf{q}^T be the respective the right-hand and left-hand Perron vectors for \mathbf{A} associated with the Perron root r , and use (8.2.3) along with $\mathbf{q}^T \mathbf{x} > 0$ (by (8.2.2)) to write

$$\xi \mathbf{x} \leq \mathbf{A}\mathbf{x} \implies \xi \mathbf{q}^T \mathbf{x} \leq \mathbf{q}^T \mathbf{A}\mathbf{x} = r \mathbf{q}^T \mathbf{x} \implies \xi \leq r \implies f(\mathbf{x}) \leq r \quad \forall \mathbf{x} \in \mathcal{N}.$$

Since $f(\mathbf{p}) = r$ and $\mathbf{p} \in \mathcal{N}$, it follows that $r = \max_{\mathbf{x} \in \mathcal{N}} f(\mathbf{x})$. \blacksquare

Below is a summary of the results obtained in this section.

Perron's Theorem

If $\mathbf{A}_{n \times n} > \mathbf{0}$ with $r = \rho(\mathbf{A})$, then the following statements are true.

- $r > 0$. (8.2.11)
- $r \in \sigma(\mathbf{A})$ (r is called the *Perron root*). (8.2.12)
- $\text{alg mult}_{\mathbf{A}}(r) = 1$. (8.2.13)
- There exists an eigenvector $\mathbf{x} > \mathbf{0}$ such that $\mathbf{Ax} = r\mathbf{x}$. (8.2.14)
- The *Perron vector* is the unique vector defined by

$$\mathbf{Ap} = r\mathbf{p}, \quad \mathbf{p} > \mathbf{0}, \quad \text{and} \quad \|\mathbf{p}\|_1 = 1,$$

and, except for positive multiples of \mathbf{p} , there are no other nonnegative eigenvectors for \mathbf{A} , regardless of the eigenvalue.

- r is the only eigenvalue on the spectral circle of \mathbf{A} . (8.2.15)
- $r = \max_{\mathbf{x} \in \mathcal{N}} f(\mathbf{x})$ (*the Collatz–Wielandt formula*),

$$\text{where } f(\mathbf{x}) = \min_{\substack{1 \leq i \leq n \\ x_i \neq 0}} \frac{[\mathbf{Ax}]_i}{x_i} \quad \text{and} \quad \mathcal{N} = \{\mathbf{x} \mid \mathbf{x} \geq \mathbf{0} \text{ with } \mathbf{x} \neq \mathbf{0}\}.$$

Note: Our development is the reverse of that of Wielandt and others in the sense that we first proved the existence of the Perron eigenpair (r, \mathbf{p}) without reference to $f(\mathbf{x})$, and then we used the Perron eigenpair to establish the Collatz–Wielandt formula. Wielandt's approach is to do things the other way around—first prove that $f(\mathbf{x})$ attains a maximum value on \mathcal{N} , and then establish existence of the Perron eigenpair by proving that $\max_{\mathbf{x} \in \mathcal{N}} f(\mathbf{x}) = \rho(\mathbf{A})$ with the maximum value being attained at a positive eigenvector \mathbf{p} .

Exercises for section 8.2

- 8.2.1.** Verify Perron's theorem by computing the eigenvalues and eigenvectors for

$$\mathbf{A} = \begin{pmatrix} 7 & 2 & 3 \\ 1 & 8 & 3 \\ 1 & 2 & 9 \end{pmatrix}.$$

Find the right-hand Perron vector \mathbf{p} as well as the left-hand Perron vector \mathbf{q}^T .

8.2.2. Convince yourself that (8.2.2)–(8.2.5) are indeed true.

8.2.3. Provide the details that explain why the Perron vector is uniquely defined.

8.2.4. Find the Perron root and the Perron vector for

$$\mathbf{A} = \begin{pmatrix} 1 - \alpha & \beta \\ \alpha & 1 - \beta \end{pmatrix},$$

where $\alpha + \beta = 1$ with $\alpha, \beta > 0$.

8.2.5. Suppose that $\mathbf{A}_{n \times n} > \mathbf{0}$ has $\rho(\mathbf{A}) = r$.

- (a) Explain why $\lim_{k \rightarrow \infty} (\mathbf{A}/r)^k$ exists.
- (b) Explain why $\lim_{k \rightarrow \infty} (\mathbf{A}/r)^k = \mathbf{G} > \mathbf{0}$ is the projector onto $N(\mathbf{A} - r\mathbf{I})$ along $R(\mathbf{A} - r\mathbf{I})$.
- (c) Explain why $\text{rank}(\mathbf{G}) = 1$.

8.2.6. Prove that if every row (or column) sum of $\mathbf{A}_{n \times n} > \mathbf{0}$ is equal to ρ , then $\rho(\mathbf{A}) = \rho$.

8.2.7. Prove that if $\mathbf{A}_{n \times n} > \mathbf{0}$, then

$$\min_i \sum_{j=1}^n a_{ij} \leq \rho(\mathbf{A}) \leq \max_i \sum_{j=1}^n a_{ij}.$$

Hint: Recall Example 7.10.2 (p. 619).

8.2.8. To show the extent to which the hypothesis of positivity cannot be relaxed in Perron's theorem, construct examples of square matrices \mathbf{A} such that $\mathbf{A} \geq \mathbf{0}$, but $\mathbf{A} \not> \mathbf{0}$ (i.e., \mathbf{A} has at least one zero entry), with $r = \rho(\mathbf{A}) \in \sigma(\mathbf{A})$ that demonstrate the validity of the following statements. Different examples may be used for the different statements.

- (a) r can be 0.
- (b) $\text{alg mult}_{\mathbf{A}}(r)$ can be greater than 1.
- (c) $\text{index}(r)$ can be greater than 1.
- (d) $N(\mathbf{A} - r\mathbf{I})$ need not contain a positive eigenvector.
- (e) r need not be the only eigenvalue on the spectral circle.

8.2.9. Establish the min-max version of the Collatz–Wielandt formula that says the Perron root for $\mathbf{A} > \mathbf{0}$ is given by $r = \min_{\mathbf{x} \in \mathcal{P}} g(\mathbf{x})$, where

$$g(\mathbf{x}) = \max_{1 \leq i \leq n} \frac{[\mathbf{Ax}]_i}{x_i} \quad \text{and} \quad \mathcal{P} = \{\mathbf{x} \mid \mathbf{x} > \mathbf{0}\}.$$

8.2.10. Notice that $\mathcal{N} = \{\mathbf{x} \mid \mathbf{x} \geq \mathbf{0} \text{ with } \mathbf{x} \neq \mathbf{0}\}$ is used in the max-min version of the Collatz–Wielandt formula on p. 666, but $\mathcal{P} = \{\mathbf{x} \mid \mathbf{x} > \mathbf{0}\}$ is used in the min-max version in Exercise 8.2.9. Give an example of a matrix $\mathbf{A} > \mathbf{0}$ that shows $r \neq \min_{\mathbf{x} \in \mathcal{N}} g(\mathbf{x})$ when $g(\mathbf{x})$ is defined as

$$g(\mathbf{x}) = \max_{\substack{1 \leq i \leq n \\ x_i \neq 0}} \frac{[\mathbf{Ax}]_i}{x_i}.$$

8.3 NONNEGATIVE MATRICES

Now let zeros creep into the picture and investigate the extent to which Perron's results generalize to nonnegative matrices containing at least one zero entry. The first result along these lines shows how to extend the statements on p. 663 to nonnegative matrices by sacrificing the existence of a positive eigenvector for a nonnegative one.

Nonnegative Eigenpair

For $\mathbf{A}_{n \times n} \geq \mathbf{0}$ with $r = \rho(\mathbf{A})$, the following statements are true.

$$\bullet \quad r \in \sigma(\mathbf{A}), \text{ (but } r = 0 \text{ is possible).} \quad (8.3.1)$$

$$\bullet \quad \mathbf{A}\mathbf{z} = r\mathbf{z} \text{ for some } \mathbf{z} \in \mathcal{N} = \{\mathbf{x} \mid \mathbf{x} \geq \mathbf{0} \text{ with } \mathbf{x} \neq \mathbf{0}\}. \quad (8.3.2)$$

$$\bullet \quad r = \max_{\mathbf{x} \in \mathcal{N}} f(\mathbf{x}), \text{ where } f(\mathbf{x}) = \min_{\substack{1 \leq i \leq n \\ x_i \neq 0}} \frac{[\mathbf{A}\mathbf{x}]_i}{x_i} \quad (8.3.3)$$

(i.e., the Collatz–Wielandt formula remains valid).

Proof. Consider the sequence of positive matrices $\mathbf{A}_k = \mathbf{A} + (1/k)\mathbf{E} > \mathbf{0}$, where \mathbf{E} is the matrix of all 1's, and let $r_k > 0$ and $\mathbf{p}_k > \mathbf{0}$ denote the Perron root and Perron vector for \mathbf{A}_k , respectively. Observe that $\{\mathbf{p}_k\}_{k=1}^\infty$ is a bounded set because it's contained in the unit 1-sphere in \mathbb{R}^n . The Bolzano–Weierstrass theorem states that each bounded sequence in \mathbb{R}^n has a convergent subsequence. Therefore, $\{\mathbf{p}_k\}_{k=1}^\infty$ has convergent subsequence

$$\{\mathbf{p}_{k_i}\}_{i=1}^\infty \rightarrow \mathbf{z}, \text{ where } \mathbf{z} \geq \mathbf{0} \text{ with } \mathbf{z} \neq \mathbf{0} \text{ (because } \mathbf{p}_{k_i} > \mathbf{0} \text{ and } \|\mathbf{p}_{k_i}\|_1 = 1).$$

Since $\mathbf{A}_1 > \mathbf{A}_2 > \cdots > \mathbf{A}$, the result in Example 7.10.2 (p. 619) guarantees that $r_1 \geq r_2 \geq \cdots \geq r$, so $\{r_k\}_{k=1}^\infty$ is a monotonic sequence of positive numbers that is bounded below by r . A standard result from analysis guarantees that

$$\lim_{k \rightarrow \infty} r_k = r^* \text{ exists, and } r^* \geq r. \text{ In particular, } \lim_{i \rightarrow \infty} r_{k_i} = r^* \geq r.$$

But $\lim_{k \rightarrow \infty} \mathbf{A}_k = \mathbf{A}$ implies $\lim_{i \rightarrow \infty} \mathbf{A}_{k_i} \rightarrow \mathbf{A}$, so, by using the easily established fact that the limit of a product is the product of the limits (provided that all limits exist), it's also true that

$$\mathbf{A}\mathbf{z} = \lim_{i \rightarrow \infty} \mathbf{A}_{k_i} \mathbf{p}_{k_i} = \lim_{i \rightarrow \infty} r_{k_i} \mathbf{p}_{k_i} = r^* \mathbf{z} \implies r^* \in \sigma(\mathbf{A}) \implies r^* \leq r.$$

Consequently, $r^* = r$, and $\mathbf{A}\mathbf{z} = r\mathbf{z}$ with $\mathbf{z} \geq \mathbf{0}$ and $\mathbf{z} \neq \mathbf{0}$. Thus (8.3.1) and (8.3.2) are proven. To prove (8.3.3), let $\mathbf{q}_k^T > \mathbf{0}$ be the left-hand Perron vector of \mathbf{A}_k . For every $\mathbf{x} \in \mathcal{N}$ and $k > 0$ we have $\mathbf{q}_k^T \mathbf{x} > 0$ (by (8.2.2)), and

$$\begin{aligned} \mathbf{0} \leq f(\mathbf{x})\mathbf{x} \leq \mathbf{A}\mathbf{x} \leq \mathbf{A}_k \mathbf{x} &\implies f(\mathbf{x})\mathbf{q}_k^T \mathbf{x} \leq \mathbf{q}_k^T \mathbf{A}_k \mathbf{x} = r_k \mathbf{q}_k^T \mathbf{x} \implies f(\mathbf{x}) \leq r_k \\ &\implies f(\mathbf{x}) \leq r \quad (\text{because } r_k \rightarrow r^* = r). \end{aligned}$$

Since $f(\mathbf{z}) = r$ and $\mathbf{z} \in \mathcal{N}$, it follows that $\max_{\mathbf{x} \in \mathcal{N}} f(\mathbf{x}) = r$. ■

This is as far as Perron's theorem can be generalized to nonnegative matrices without additional hypothesis. For example, $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ shows that properties (8.2.11), (8.2.13), and (8.2.14) on p. 667 do not hold for general nonnegative matrices, and $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ shows that (8.2.15) is also lost. Rather than accepting that the major issues concerning spectral properties of nonnegative matrices had been settled, Frobenius had the insight to look below the surface and see that the problem doesn't stem just from the existence of zero entries, but rather from the *positions* of the zero entries. For example, (8.2.13) and (8.2.14) are false for

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \text{ but they are true for } \tilde{\mathbf{A}} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}. \quad (8.3.4)$$

Frobenius's genius was to see the difference between \mathbf{A} and $\tilde{\mathbf{A}}$ in terms of reducibility and to relate these ideas to spectral properties of nonnegative matrices. Reducibility and graphs were discussed in Example 4.4.6 (p. 202) and Exercise 4.4.20 (p. 209), but for the sake of continuity they are reviewed below.

Reducibility and Graphs

- $\mathbf{A}_{n \times n}$ is said to be a **reducible matrix** when there exists a permutation matrix \mathbf{P} such that

$$\mathbf{P}^T \mathbf{A} \mathbf{P} = \begin{pmatrix} \mathbf{X} & \mathbf{Y} \\ \mathbf{0} & \mathbf{Z} \end{pmatrix}, \text{ where } \mathbf{X} \text{ and } \mathbf{Z} \text{ are both square.}$$

Otherwise \mathbf{A} is said to be an **irreducible matrix**.

- $\mathbf{P}^T \mathbf{A} \mathbf{P}$ is called a **symmetric permutation** of \mathbf{A} . The effect is to interchange rows in the same way as columns are interchanged.
- The **graph** $\mathcal{G}(\mathbf{A})$ of \mathbf{A} is defined to be the directed graph on n nodes $\{N_1, N_2, \dots, N_n\}$ in which there is a directed edge leading from N_i to N_j if and only if $a_{ij} \neq 0$.
- $\mathcal{G}(\mathbf{P}^T \mathbf{A} \mathbf{P}) = \mathcal{G}(\mathbf{A})$ whenever \mathbf{P} is a permutation matrix—the effect is simply a relabeling of nodes.
- $\mathcal{G}(\mathbf{A})$ is called **strongly connected** if for each pair of nodes (N_i, N_k) there is a sequence of directed edges leading from N_i to N_k .
- \mathbf{A} is an irreducible matrix if and only if $\mathcal{G}(\mathbf{A})$ is strongly connected (see Exercise 4.4.20 on p. 209).

For example, the matrix \mathbf{A} in (8.3.4) is reducible because

$$\mathbf{P}^T \mathbf{A} \mathbf{P} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{for} \quad \mathbf{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and, as can be seen from Figure 8.3.1, $\mathcal{G}(\mathbf{A})$ is not strongly connected because there is no sequence of paths leading from node 1 to node 2. On the other hand, $\tilde{\mathbf{A}}$ is irreducible, and as shown in Figure 8.3.1, $\mathcal{G}(\tilde{\mathbf{A}})$ is strongly connected because each node is accessible from the other.



FIGURE 8.3.1

This discussion suggests that some of Perron's properties given on p. 667 extend to nonnegative matrices when the zeros are in just the right positions to insure irreducibility. To prove that this is in fact the case, the following lemma is needed. It shows how to convert a nonnegative irreducible matrix into a positive matrix in a useful fashion.

Converting Nonnegativity & Irreducibility to Positivity

If $\mathbf{A}_{n \times n} \geq \mathbf{0}$ is irreducible, then $(\mathbf{I} + \mathbf{A})^{n-1} > \mathbf{0}$. (8.3.5)

Proof. Let $a_{ij}^{(k)}$ denote the (i, j) -entry in \mathbf{A}^k , and observe that

$$a_{ij}^{(k)} = \sum_{h_1, \dots, h_{k-1}} a_{ih_1} a_{h_1 h_2} \cdots a_{h_{k-1} j} > 0$$

if and only if there exists a set of indices h_1, h_2, \dots, h_{k-1} such that

$$a_{ih_1} > 0 \quad \text{and} \quad a_{h_1 h_2} > 0 \quad \text{and} \quad \cdots \quad \text{and} \quad a_{h_{k-1} j} > 0.$$

In other words, there is a sequence of k paths $N_i \rightarrow N_{h_1} \rightarrow N_{h_2} \rightarrow \cdots \rightarrow N_j$ in $\mathcal{G}(\mathbf{A})$ that lead from node N_i to node N_j if and only if $a_{ij}^{(k)} > 0$. The irreducibility of \mathbf{A} insures that $\mathcal{G}(\mathbf{A})$ is strongly connected, so for any pair of nodes (N_i, N_j) there is a sequence of k paths (with $k < n$) from N_i to N_j . This means that for each position (i, j) , there is some $0 \leq k \leq n-1$ such that $a_{ij}^{(k)} > 0$, and this guarantees that for each i and j ,

$$[(\mathbf{I} + \mathbf{A})^{n-1}]_{ij} = \left[\sum_{k=0}^{n-1} \binom{n-1}{k} \mathbf{A}^k \right]_{ij} = \sum_{k=0}^{n-1} \binom{n-1}{k} a_{ij}^{(k)} > 0. \quad \blacksquare$$

With the exception of the Collatz–Wielandt formula, we have seen that $\rho(\mathbf{A}) \in \sigma(\mathbf{A})$ is the only property in the list of Perron properties on p. 667 that extends to nonnegative matrices without additional hypothesis. The next theorem shows how adding irreducibility to nonnegativity recovers the Perron properties (8.2.11), (8.2.13), and (8.2.14).

Perron–Frobenius Theorem

If $\mathbf{A}_{n \times n} \geq \mathbf{0}$ is irreducible, then each of the following is true.

- $r = \rho(\mathbf{A}) \in \sigma(\mathbf{A})$ and $r > 0$. (8.3.6)

- $\text{alg mult}_{\mathbf{A}}(r) = 1$. (8.3.7)

- There exists an eigenvector $\mathbf{x} > \mathbf{0}$ such that $\mathbf{Ax} = r\mathbf{x}$. (8.3.8)

- The unique vector defined by

$$\mathbf{Ap} = r\mathbf{p}, \quad \mathbf{p} > \mathbf{0}, \quad \text{and} \quad \|\mathbf{p}\|_1 = 1, \quad (8.3.9)$$

is called the **Perron vector**. There are no nonnegative eigenvectors for \mathbf{A} except for positive multiples of \mathbf{p} , regardless of the eigenvalue.

- The Collatz–Wielandt formula $r = \max_{\mathbf{x} \in \mathcal{N}} f(\mathbf{x})$,

$$\text{where } f(\mathbf{x}) = \min_{\substack{1 \leq i \leq n \\ x_i \neq 0}} \frac{[\mathbf{Ax}]_i}{x_i} \quad \text{and} \quad \mathcal{N} = \{\mathbf{x} \mid \mathbf{x} \geq \mathbf{0} \text{ with } \mathbf{x} \neq \mathbf{0}\}$$

was established in (8.3.3) for all nonnegative matrices, but it is included here for the sake of completeness.

Proof. We already know from (8.3.2) that $r = \rho(\mathbf{A}) \in \sigma(\mathbf{A})$. To prove that $\text{alg mult}_{\mathbf{A}}(r) = 1$, let $\mathbf{B} = (\mathbf{I} + \mathbf{A})^{n-1} > \mathbf{0}$ be the matrix in (8.3.5). It follows from (7.9.3) that $\lambda \in \sigma(\mathbf{A})$ if and only if $(1 + \lambda)^{n-1} \in \sigma(\mathbf{B})$, and $\text{alg mult}_{\mathbf{A}}(\lambda) = \text{alg mult}_{\mathbf{B}}((1 + \lambda)^{n-1})$. Consequently, if $\mu = \rho(\mathbf{B})$, then

$$\mu = \max_{\lambda \in \sigma(\mathbf{A})} |(1 + \lambda)|^{n-1} = \left\{ \max_{\lambda \in \sigma(\mathbf{A})} |(1 + \lambda)| \right\}^{n-1} = (1 + r)^{n-1}$$

because when a circular disk $|z| \leq \rho$ is translated one unit to the right, the point of maximum modulus in the resulting disk $|z + 1| \leq \rho$ is $z = 1 + \rho$ (it's clear if you draw a picture). Therefore, $\text{alg mult}_{\mathbf{A}}(r) = 1$; otherwise $\text{alg mult}_{\mathbf{B}}(\mu) > 1$, which is impossible because $\mathbf{B} > \mathbf{0}$. To see that \mathbf{A} has a positive eigenvector

associated with r , recall from (8.3.2) that there exists a nonnegative eigenvector $\mathbf{x} \geq \mathbf{0}$ associated with r . It's a simple consequence of (7.9.9) that if (λ, \mathbf{x}) is an eigenpair for \mathbf{A} , then $(f(\lambda), \mathbf{x})$ is an eigenpair for $f(\mathbf{A})$ (Exercise 7.9.9, p. 613), so (r, \mathbf{x}) being an eigenpair for \mathbf{A} implies that (μ, \mathbf{x}) is an eigenpair for \mathbf{B} . Hence (8.2.10) insures that \mathbf{x} must be a positive multiple of the Perron vector of \mathbf{B} , and thus \mathbf{x} must in fact be positive. Now, $r > 0$; otherwise $\mathbf{A}\mathbf{x} = \mathbf{0}$, which is impossible because $\mathbf{A} \geq \mathbf{0}$ and $\mathbf{x} > \mathbf{0}$ forces $\mathbf{A}\mathbf{x} > \mathbf{0}$. The argument used to prove (8.2.10) also proves (8.3.9). ■

Example 8.3.1

Problem: Suppose that $\mathbf{A}_{n \times n} \geq \mathbf{0}$ is irreducible with $r = \rho(\mathbf{A})$, and suppose that $r\mathbf{z} \leq \mathbf{A}\mathbf{z}$ for $\mathbf{z} \geq \mathbf{0}$, $\mathbf{z} \neq \mathbf{0}$. Explain why $r\mathbf{z} = \mathbf{A}\mathbf{z}$, and $\mathbf{z} > \mathbf{0}$.

Solution: If $r\mathbf{z} < \mathbf{A}\mathbf{z}$, then by using the Perron vector $\mathbf{q} > \mathbf{0}$ for \mathbf{A}^T we have

$$(\mathbf{A} - r\mathbf{I})\mathbf{z} \geq \mathbf{0} \implies \mathbf{q}^T(\mathbf{A} - r\mathbf{I})\mathbf{z} > \mathbf{0},$$

which is impossible since $\mathbf{q}^T(\mathbf{A} - r\mathbf{I}) = \mathbf{0}$. Thus $r\mathbf{z} = \mathbf{A}\mathbf{z}$, and since \mathbf{z} must be a multiple of the Perron vector for \mathbf{A} by (8.3.9), we also have that $\mathbf{z} > \mathbf{0}$.

The only property in the list on p. 667 that irreducibility is not able to salvage is (8.2.15), which states that there is only one eigenvalue on the spectral circle. Indeed, $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is nonnegative and irreducible, but the eigenvalues ± 1 are both on the unit circle. The property of having (or not having) only one eigenvalue on the spectral circle divides the set of nonnegative irreducible matrices into two important classes.

Primitive Matrices

- A nonnegative irreducible matrix \mathbf{A} having only one eigenvalue, $r = \rho(\mathbf{A})$, on its spectral circle is said to be a **primitive matrix**.
- A nonnegative irreducible matrix having $h > 1$ eigenvalues on its spectral circle is called **imprimitive**, and h is referred to as **index of imprimitivity**.
- A nonnegative irreducible matrix \mathbf{A} with $r = \rho(\mathbf{A})$ is primitive if and only if $\lim_{k \rightarrow \infty} (\mathbf{A}/r)^k$ exists, in which case

$$\lim_{k \rightarrow \infty} \left(\frac{\mathbf{A}}{r} \right)^k = \mathbf{G} = \frac{\mathbf{p}\mathbf{q}^T}{\mathbf{q}^T\mathbf{p}} > \mathbf{0}, \quad (8.3.10)$$

where \mathbf{p} and \mathbf{q} are the respective Perron vectors for \mathbf{A} and \mathbf{A}^T . \mathbf{G} is the (spectral) projector onto $N(\mathbf{A} - r\mathbf{I})$ along $R(\mathbf{A} - r\mathbf{I})$.

Proof of (8.3.10). The Perron–Frobenius theorem insures that $1 = \rho(\mathbf{A}/r)$ is a simple eigenvalue for \mathbf{A}/r , and it's clear that \mathbf{A} is primitive if and only if \mathbf{A}/r is primitive. In other words, \mathbf{A} is primitive if and only if $1 = \rho(\mathbf{A}/r)$ is the only eigenvalue on the unit circle, which is equivalent to saying that $\lim_{k \rightarrow \infty} (\mathbf{A}/r)^k$ exists by the results on p. 630. The structure of the limit as described in (8.3.10) is the result of (7.2.12) on p. 518. ■

The next two results, discovered by Helmut Wielandt (p. 534) in 1950, establish the remarkable fact that the eigenvalues on the spectral circle of an imprimitive matrix are in fact the h^{th} roots of the spectral radius.

Wielandt's Theorem

If $|\mathbf{B}| \leq \mathbf{A}_{n \times n}$, where \mathbf{A} is irreducible, then $\rho(\mathbf{B}) \leq \rho(\mathbf{A})$. If equality holds (i.e., if $\mu = \rho(\mathbf{A}) e^{i\phi} \in \sigma(\mathbf{B})$ for some ϕ), then

$$\mathbf{B} = e^{i\phi} \mathbf{D} \mathbf{A} \mathbf{D}^{-1} \quad \text{for some} \quad \mathbf{D} = \begin{pmatrix} e^{i\theta_1} & & & \\ & e^{i\theta_2} & & \\ & & \ddots & \\ & & & e^{i\theta_n} \end{pmatrix}, \quad (8.3.11)$$

and conversely.

Proof. We already know that $\rho(\mathbf{B}) \leq \rho(\mathbf{A})$ by Example 7.10.2 (p. 619). If $\rho(\mathbf{B}) = r = \rho(\mathbf{A})$, and if (μ, \mathbf{x}) is an eigenpair for \mathbf{B} such that $|\mu| = r$, then

$$r|\mathbf{x}| = |\mu||\mathbf{x}| = |\mu\mathbf{x}| = |\mathbf{B}\mathbf{x}| \leq |\mathbf{B}||\mathbf{x}| \leq \mathbf{A}|\mathbf{x}| \implies |\mathbf{B}||\mathbf{x}| = r|\mathbf{x}|$$

because the result in Example 8.3.1 insures that $\mathbf{A}|\mathbf{x}| = r|\mathbf{x}|$, and $|\mathbf{x}| > \mathbf{0}$. Consequently, $(\mathbf{A} - |\mathbf{B}|)|\mathbf{x}| = \mathbf{0}$. But $\mathbf{A} - |\mathbf{B}| \geq \mathbf{0}$, and $|\mathbf{x}| > \mathbf{0}$, so $\mathbf{A} = |\mathbf{B}|$ by (8.2.4). Since $x_k/|x_k|$ is on the unit circle, $x_k/|x_k| = e^{i\theta_k}$ for some θ_k . Set

$$\mathbf{D} = \begin{pmatrix} e^{i\theta_1} & & & \\ & e^{i\theta_2} & & \\ & & \ddots & \\ & & & e^{i\theta_n} \end{pmatrix}, \quad \text{and notice that} \quad \mathbf{x} = \mathbf{D}|\mathbf{x}|.$$

Since $|\mu| = r$, there is a $\phi \in \Re$ such that $\mu = re^{i\phi}$, and hence

$$\mathbf{B}\mathbf{D}|\mathbf{x}| = \mathbf{B}\mathbf{x} = \mu\mathbf{x} = re^{i\phi}\mathbf{x} = re^{i\phi}\mathbf{D}|\mathbf{x}| \implies e^{-i\phi}\mathbf{D}^{-1}\mathbf{B}\mathbf{D}|\mathbf{x}| = r|\mathbf{x}| = \mathbf{A}|\mathbf{x}|. \quad (8.3.12)$$

For convenience, let $\mathbf{C} = e^{-i\phi}\mathbf{D}^{-1}\mathbf{B}\mathbf{D}$, and note that $|\mathbf{C}| = |\mathbf{B}| = \mathbf{A}$ to write (8.3.12) as $\mathbf{0} = (|\mathbf{C}| - \mathbf{C})|\mathbf{x}|$. Considering only the real part of this equation

yields $\mathbf{0} = (|\mathbf{C}| - \operatorname{Re}(\mathbf{C}))|\mathbf{x}|$. But $|\mathbf{C}| \geq \operatorname{Re}(\mathbf{C})$, and $|\mathbf{x}| > \mathbf{0}$, so it follows from (8.2.4) that $\operatorname{Re}(\mathbf{C}) = |\mathbf{C}|$, and hence

$$\operatorname{Re}(c_{ij}) = |c_{ij}| = \sqrt{\operatorname{Re}(c_{ij})^2 + \operatorname{Im}(c_{ij})^2} \implies \operatorname{Im}(c_{ij}) = 0 \implies \operatorname{Im}(\mathbf{C}) = \mathbf{0}.$$

Therefore, $\mathbf{C} = \operatorname{Re}(\mathbf{C}) = |\mathbf{C}| = \mathbf{A}$, which implies $\mathbf{B} = e^{i\phi} \mathbf{D} \mathbf{A} \mathbf{D}^{-1}$. Conversely, if $\mathbf{B} = e^{i\phi} \mathbf{D} \mathbf{A} \mathbf{D}^{-1}$, then similarity insures that $\rho(\mathbf{B}) = \rho(e^{i\phi} \mathbf{A}) = \rho(\mathbf{A})$. ■

h^{th} Roots of $\rho(\mathbf{A})$ on Spectral Circle

If $\mathbf{A}_{n \times n} \geq \mathbf{0}$ is irreducible and has h eigenvalues $\{\lambda_1, \lambda_2, \dots, \lambda_h\}$ on its spectral circle, then each of the following statements is true.

- $\operatorname{alg mult}_{\mathbf{A}}(\lambda_k) = 1$ for $k = 1, 2, \dots, h$. (8.3.13)

- $\{\lambda_1, \lambda_2, \dots, \lambda_h\}$ are the h^{th} roots of $r = \rho(\mathbf{A})$ given by

$$\{r, r\omega, r\omega^2, \dots, r\omega^{h-1}\}, \quad \text{where } \omega = e^{2\pi i/h}. \quad (8.3.14)$$

Proof. Let $\mathcal{S} = \{r, re^{i\theta_1}, \dots, re^{i\theta_{h-1}}\}$ denote the eigenvalues on the spectral circle of \mathbf{A} . Applying (8.3.11) with $\mathbf{B} = \mathbf{A}$ and $\mu = re^{i\theta_k}$ insures the existence of a diagonal matrix \mathbf{D}_k such that $\mathbf{A} = e^{i\theta_k} \mathbf{D}_k \mathbf{A} \mathbf{D}_k^{-1}$, thus showing that $e^{i\theta_k} \mathbf{A}$ is similar to \mathbf{A} . Since r is a simple eigenvalue of \mathbf{A} (by the Perron–Frobenius theorem), $re^{i\theta_k}$ must be a simple eigenvalue of $e^{i\theta_k} \mathbf{A}$. But similarity transformations preserve eigenvalues and algebraic multiplicities (because the Jordan structure is preserved), so $re^{i\theta_k}$ must be a simple eigenvalue of \mathbf{A} , thus establishing (8.3.13). To prove (8.3.14), consider another eigenvalue $re^{i\theta_s} \in \mathcal{S}$. Again, we can write $\mathbf{A} = e^{i\theta_s} \mathbf{D}_s \mathbf{A} \mathbf{D}_s^{-1}$ for some \mathbf{D}_s , so

$$\mathbf{A} = e^{i\theta_k} \mathbf{D}_k \mathbf{A} \mathbf{D}_k^{-1} = e^{i\theta_k} \mathbf{D}_k (e^{i\theta_s} \mathbf{D}_s \mathbf{A} \mathbf{D}_s^{-1}) \mathbf{D}_k^{-1} = e^{i(\theta_k + \theta_s)} (\mathbf{D}_k \mathbf{D}_s) \mathbf{A} (\mathbf{D}_k \mathbf{D}_s)^{-1}$$

and, consequently, $re^{i(\theta_k + \theta_s)}$ is also an eigenvalue on the spectral circle of \mathbf{A} . In other words, $\mathcal{S} = \{r, re^{i\theta_1}, \dots, re^{i\theta_{h-1}}\}$ is closed under multiplication. This means that $\mathcal{G} = \{1, e^{i\theta_1}, \dots, e^{i\theta_{h-1}}\}$ is closed under multiplication, and it follows that \mathcal{G} is a finite commutative group of order h . A standard result from algebra states that the h^{th} power of every element in a finite group of order h must be the identity element in the group. Therefore, $(e^{i\theta_k})^h = 1$ for each $0 \leq k \leq h-1$, so \mathcal{G} is the set of the h^{th} roots of unity $e^{2\pi ki/h}$ ($0 \leq k \leq h-1$), and thus \mathcal{S} must be the h^{th} roots of r . ■

Combining the preceding results reveals just how special the spectrum of an imprimitive matrix is.

Rotational Invariance

If \mathbf{A} is imprimitive with h eigenvalues on its spectral circle, then $\sigma(\mathbf{A})$ is invariant under rotation about the origin through an angle $2\pi/h$. No rotation less than $2\pi/h$ can preserve $\sigma(\mathbf{A})$. (8.3.15)

Proof. Since $\lambda \in \sigma(\mathbf{A}) \iff \lambda e^{2\pi i/h} \in \sigma(e^{2\pi i/h} \mathbf{A})$, it follows that $\sigma(e^{2\pi i/h} \mathbf{A})$ is $\sigma(\mathbf{A})$ rotated through $2\pi/h$. But (8.3.11) and (8.3.14) insure that \mathbf{A} and $e^{2\pi i/h} \mathbf{A}$ are similar and, consequently, $\sigma(\mathbf{A}) = \sigma(e^{2\pi i/h} \mathbf{A})$. No rotation less than $2\pi/h$ can keep $\sigma(\mathbf{A})$ invariant because (8.3.14) makes it clear that the eigenvalues on the spectral circle won't go back into themselves for rotations less than $2\pi/h$. ■

Example 8.3.2

The Spectral Projector Is Positive. We already know from (8.3.10) that if \mathbf{A} is a primitive matrix, and if \mathbf{G} is the spectral projector associated with $r = \rho(\mathbf{A})$, then $\mathbf{G} > \mathbf{0}$.

Problem: Explain why this is also true for an imprimitive matrix. In other words, establish the fact that *if \mathbf{G} is the spectral projector associated with $r = \rho(\mathbf{A})$ for any nonnegative irreducible matrix \mathbf{A} , then $\mathbf{G} > \mathbf{0}$.*

Solution: Being imprimitive means that \mathbf{A} is nonnegative and irreducible with more than one eigenvalue on the spectral circle. However, (8.3.13) says that each eigenvalue on the spectral circle is simple, so the results concerning Cesàro summability on p. 633 can be applied to \mathbf{A}/r to conclude that

$$\lim_{k \rightarrow \infty} \frac{\mathbf{I} + (\mathbf{A}/r) + \cdots + (\mathbf{A}/r)^{k-1}}{k} = \mathbf{G},$$

where \mathbf{G} is the spectral projector onto $N((\mathbf{A}/r) - \mathbf{I}) = N(\mathbf{A} - r\mathbf{I})$ along $R((\mathbf{A}/r) - \mathbf{I}) = R(\mathbf{A} - r\mathbf{I})$. Since r is a simple eigenvalue the same argument used to establish (8.3.10) (namely, invoking (7.2.12) on p. 518) shows that

$$\mathbf{G} = \frac{\mathbf{p}\mathbf{q}^T}{\mathbf{q}^T \mathbf{p}} > \mathbf{0},$$

where \mathbf{p} and \mathbf{q} are the respective Perron vectors for \mathbf{A} and \mathbf{A}^T .

Trying to determine if an irreducible matrix $\mathbf{A} \geq \mathbf{0}$ is primitive or imprimitive by finding the eigenvalues is generally a difficult task, so it's natural to ask if there's another way. It turns out that there is, and, as the following example shows, determining primitivity can sometimes be trivial.

Example 8.3.3

Sufficient Condition for Primitivity. If a nonnegative irreducible matrix \mathbf{A} has at least one positive diagonal element, then \mathbf{A} is primitive.

Proof. Suppose there are $h > 1$ eigenvalues on the spectral circle. We know from (8.3.15) that if $\lambda_0 \in \sigma(\mathbf{A})$, then $\lambda_k = \lambda_0 e^{2\pi i k/h} \in \sigma(\mathbf{A})$ for $k = 0, 1, \dots, h-1$, so

$$\sum_{k=0}^{h-1} \lambda_k = \lambda_0 \sum_{k=0}^{h-1} e^{2\pi i k/h} = 0 \quad (\text{roots of unity sum to 1—see p. 357}).$$

This implies that the sum of *all* of the eigenvalues is zero. In other words,

- if \mathbf{A} is imprimitive, then $\text{trace}(\mathbf{A}) = 0$. (Recall (7.1.7) on p. 494.)

Therefore, if \mathbf{A} has a positive diagonal entry, then \mathbf{A} must be primitive.

Another of Frobenius's contributions was to show how the powers of a nonnegative matrix determine whether or not the matrix is primitive. The exact statement is as follows.

Frobenius's Test for Primitivity

$$\mathbf{A} \geq \mathbf{0} \text{ is primitive if and only if } \mathbf{A}^m > \mathbf{0} \text{ for some } m > 0. \quad (8.3.16)$$

Proof. First assume that $\mathbf{A}^m > \mathbf{0}$ for some m . This implies that \mathbf{A} is irreducible; otherwise there exists a permutation matrix such that

$$\mathbf{A} = \mathbf{P} \begin{pmatrix} \mathbf{X} & \mathbf{Y} \\ \mathbf{0} & \mathbf{Z} \end{pmatrix} \mathbf{P}^T \implies \mathbf{A}^m = \mathbf{P} \begin{pmatrix} \mathbf{X}^m & \star \\ \mathbf{0} & \mathbf{Z}^m \end{pmatrix} \mathbf{P}^T \text{ has zero entries.}$$

Suppose that \mathbf{A} has h eigenvalues $\{\lambda_1, \lambda_2, \dots, \lambda_h\}$ on its spectral circle so that $r = \rho(\mathbf{A}) = |\lambda_1| = \dots = |\lambda_h| > |\lambda_{h+1}| > \dots > |\lambda_n|$. Since $\lambda \in \sigma(\mathbf{A})$ implies $\lambda^m \in \sigma(\mathbf{A}^m)$ with $\text{alg mult}_{\mathbf{A}}(\lambda) = \text{alg mult}_{\mathbf{A}^m}(\lambda^m)$ (consider the Jordan form—Exercise 7.9.9 on p. 613), it follows that λ_k^m ($1 \leq k \leq h$) is on the spectral circle of \mathbf{A}^m with $\text{alg mult}_{\mathbf{A}}(\lambda_k) = \text{alg mult}_{\mathbf{A}^m}(\lambda_k^m)$. Perron's theorem (p. 667) insures that \mathbf{A}^m has only one eigenvalue (which must be r^m) on its spectral circle, so $r^m = \lambda_1^m = \lambda_2^m = \dots = \lambda_h^m$. But this means that

$$\text{alg mult}_{\mathbf{A}}(r) = \text{alg mult}_{\mathbf{A}^m}(r^m) = h,$$

and therefore $h = 1$ by (8.3.7). Conversely, if \mathbf{A} is primitive with $r = \rho(\mathbf{A})$, then $\lim_{k \rightarrow \infty} (\mathbf{A}/r)^k > \mathbf{0}$ by (8.3.10). Hence there must be some m such that $(\mathbf{A}/r)^m > \mathbf{0}$, and thus $\mathbf{A}^m > \mathbf{0}$. ■

Example 8.3.4

Suppose that we wish to decide whether or not a nonnegative matrix \mathbf{A} is primitive by computing the sequence of powers $\mathbf{A}, \mathbf{A}^2, \mathbf{A}^3, \dots$. Since this can be a laborious task, it would be nice to know when we have computed enough powers of \mathbf{A} to render a judgement. Unfortunately there is nothing in the statement or proof of Frobenius's test to help us with this decision. But Wielandt provided an answer by proving that a nonnegative matrix $\mathbf{A}_{n \times n}$ is primitive if and only if $\mathbf{A}^{n^2-2n+2} > \mathbf{0}$. Furthermore, $n^2 - 2n + 2$ is the smallest such exponent that works for the class of $n \times n$ primitive matrices having all zeros on the diagonal—see Exercise 8.3.9.

Problem: Determine whether or not $\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 3 & 4 & 0 \end{pmatrix}$ is primitive.

Solution: Since \mathbf{A} has zeros on the diagonal, the result in Example 8.3.3 doesn't apply, so we are forced into computing powers of \mathbf{A} . This job is simplified by noticing that if $\mathbf{B} = \beta(\mathbf{A})$ is the Boolean matrix that results from setting

$$b_{ij} = \begin{cases} 1 & \text{if } a_{ij} > 0, \\ 0 & \text{if } a_{ij} = 0, \end{cases}$$

then $[\mathbf{B}^k]_{ij} > 0$ if and only if $[\mathbf{A}^k]_{ij} > 0$ for every $k > 0$. This means that instead of using $\mathbf{A}, \mathbf{A}^2, \mathbf{A}^3, \dots$ to decide on primitivity, we need only compute

$$\mathbf{B}_1 = \beta(\mathbf{A}), \quad \mathbf{B}_2 = \beta(\mathbf{B}_1\mathbf{B}_1), \quad \mathbf{B}_3 = \beta(\mathbf{B}_1\mathbf{B}_2), \quad \mathbf{B}_4 = \beta(\mathbf{B}_1\mathbf{B}_3), \dots,$$

going no further than \mathbf{B}_{n^2-2n+2} , and these computations require only Boolean operations **AND** and **OR**. The matrix \mathbf{A} in this example is primitive because

$$\mathbf{B}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad \mathbf{B}_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad \mathbf{B}_3 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad \mathbf{B}_4 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad \mathbf{B}_5 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

The powers of an irreducible matrix $\mathbf{A} \geq \mathbf{0}$ can tell us if \mathbf{A} has more than one eigenvalue on its spectral circle, but the powers of \mathbf{A} provide no clue to the number of such eigenvalues. The next theorem shows how the index of imprimitivity can be determined without explicitly calculating the eigenvalues.

Index of Imprimitivity

If $c(x) = x^n + c_{k_1}x^{n-k_1} + c_{k_2}x^{n-k_2} + \dots + c_{k_s}x^{n-k_s} = 0$ is the characteristic equation of an imprimitive matrix $\mathbf{A}_{n \times n}$ in which only the terms with nonzero coefficients are listed (i.e., each $c_{k_j} \neq 0$, and $n > (n - k_1) > \dots > (n - k_s)$), then the index of imprimitivity h is the greatest common divisor of $\{k_1, k_2, \dots, k_s\}$.

Proof. We know from (8.3.15) that if $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$ are the eigenvalues of \mathbf{A} (including multiplicities), then $\{\omega\lambda_1, \omega\lambda_2, \dots, \omega\lambda_n\}$ are also the eigenvalues of \mathbf{A} , where $\omega = e^{2\pi i/h}$. It follows from the results on p. 494 that

$$c_{k_j} = (-1)^{k_j} \sum_{1 \leq i_1 < \dots < i_{k_j} \leq n} \lambda_{i_1} \cdots \lambda_{i_{k_j}} = (-1)^{k_j} \sum_{1 \leq i_1 < \dots < i_{k_j} \leq n} \omega \lambda_{i_1} \cdots \omega \lambda_{i_{k_j}} = \omega^{k_j} c_{k_j} \implies \omega^{k_j} = 1.$$

Therefore, h must divide each k_j . If d divides each k_j with $d > h$, then $\gamma^{-k_j} = 1$ for $\gamma = e^{2\pi i/d}$. Hence $\gamma\lambda \in \sigma(\mathbf{A})$ for each $\lambda \in \sigma(\mathbf{A})$ because $c(\gamma\lambda) = 0$. But this means that $\sigma(\mathbf{A})$ is invariant under a rotation through an angle $(2\pi/d) < (2\pi/h)$, which, by (8.3.15), is impossible. ■

Example 8.3.5

Problem: Find the index of imprimitivity of $\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{pmatrix}$.

Solution: Using the principal minors to compute the characteristic equation as illustrated in Example 7.1.2 (p. 496) produces the characteristic equation

$$c(x) = x^4 - 5x^2 + 4 = 0,$$

so that $k_1 = 2$ and $k_2 = 4$. Since $\gcd\{2, 4\} = 2$, it follows that $h = 2$. The characteristic equation is relatively simple in this example, so the eigenvalues can be explicitly determined to be $\{\pm 2, \pm 1\}$. This corroborates the fact that $h = 2$. Notice also that this illustrates the property that $\sigma(\mathbf{A})$ is invariant under rotation through an angle $2\pi/h = \pi$.

More is known about nonnegative matrices than what has been presented here—in fact, there are entire books on the subject. But before moving on to applications, there is a result that Frobenius discovered in 1912 that is worth mentioning because it completely reveals the structure of an imprimitive matrix.

Frobenius Form

For each imprimitive matrix \mathbf{A} with index of imprimitivity $h > 1$, there exists a permutation matrix \mathbf{P} such that

$$\mathbf{P}^T \mathbf{A} \mathbf{P} = \begin{pmatrix} \mathbf{0} & \mathbf{A}_{12} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{A}_{23} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{A}_{h-1,h} \\ \mathbf{A}_{h1} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \end{pmatrix},$$

where the zero blocks on the main diagonal are square.

Example 8.3.6

Leontief's⁹¹ Input–Output Economic Model. Suppose that n major industries in a closed economic system each make one commodity, and let a J -unit be what industry J produces that sells for \$1. For example, the Boeing Company makes airplanes, and the Champion Company makes rivets, so a *BOEING-unit* is only a tiny fraction of an airplane, but a *CHAMPION-unit* might be several rivets. If

$$0 \leq s_j = \# \text{ } J\text{-units produced by industry } J \text{ each year, and if}$$

$$0 \leq a_{ij} = \# \text{ } I\text{-units needed to produce one } J\text{-unit,}$$

then

$$a_{ij}s_j = \# \text{ } I\text{-units consumed by industry } J \text{ each year, and}$$

$$\sum_{j=1}^n a_{ij}s_j = \# \text{ } I\text{-units consumed by all industries each year,}$$

so

$$d_i = s_i - \sum_{j=1}^n a_{ij}s_j = \# \text{ } I\text{-units available to the public (nonindustry) each year.}$$

Consider $\mathbf{d} = (d_1, d_2, \dots, d_n)^T$ to be the public **demand vector**, and think of $\mathbf{s} = (s_1, s_2, \dots, s_n)^T$ as the industrial **supply vector**.

Problem: Determine the supply $\mathbf{s} \geq \mathbf{0}$ that is required to satisfy a given demand $\mathbf{d} \geq \mathbf{0}$.

Solution: At first glance the problem seems to be trivial because the equations $d_i = s_i - \sum_{j=1}^n a_{ij}s_j$ translate to $(\mathbf{I} - \mathbf{A})\mathbf{s} = \mathbf{d}$, so if $\mathbf{I} - \mathbf{A}$ is nonsingular, then $\mathbf{s} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{d}$. The catch is that this solution may have negative components in spite of the fact that $\mathbf{A} \geq \mathbf{0}$. So something must be added. It's not unreasonable to assume that major industries are *strongly connected* in the sense that the commodity of each industry is either directly or indirectly needed to produce all commodities in the system. In other words, it's reasonable to assume that

⁹¹

Wassily Leontief (1906–1999) was the 1973 Nobel Laureate in Economics. He was born in St. Petersburg (now Leningrad), where his father was a professor of economics. After receiving his undergraduate degree in economics at the University of Leningrad in 1925, Leontief went to the University of Berlin to earn a Ph.D. degree. He migrated to New York in 1931 and moved to Harvard University in 1932, where he became Professor of Economics in 1946. Leontief spent a significant portion of his career developing and applying his input–output analysis, which eventually led to the famous “Leontief paradox.” In the U.S. economy of the 1950s, labor was considered to be scarce while capital was presumed to be abundant, so the prevailing thought was that U.S. foreign trade was predicated on trading capital-intensive goods for labor-intensive goods. But Leontief's input–output tables revealed that just the opposite was true, and this contributed to his fame. One of Leontief's secret weapons was the computer. He made use of large-scale computing techniques (relative to the technology of the 1940s and 1950s), and he was among the first to put the Mark I (one of the first electronic computers) to work on nonmilitary projects in 1943.

$\mathcal{G}(\mathbf{A})$ is a strongly connected graph so that in addition to being nonnegative, \mathbf{A} is an *irreducible* matrix. Furthermore, it's not unreasonable to assume that $\rho(\mathbf{A}) < 1$. To understand why, notice that the j^{th} column sum of \mathbf{A} is

$$\begin{aligned} c_j &= \sum_{i=1}^n a_{ij} = \text{total number of all units required to make one } J\text{-unit} \\ &= \text{total number of dollars spent by } J \text{ to create \$1 of revenue.} \end{aligned}$$

In a healthy economy all major industries should have $c_j \leq 1$, and there should be at least one major industry such that $c_j < 1$. This means that there exists a matrix $\mathbf{E} \geq \mathbf{0}$, but $\mathbf{E} \neq \mathbf{0}$, such that each column sum of $\mathbf{A} + \mathbf{E}$ is 1, so

$$\mathbf{e}^T(\mathbf{A} + \mathbf{E}) = \mathbf{e}^T, \quad \text{where } \mathbf{e}^T \text{ is the row of all 1's.}$$

This forces $\rho(\mathbf{A}) < 1$; otherwise the Perron vector $\mathbf{p} > \mathbf{0}$ for \mathbf{A} can be used to write

$$1 = \mathbf{e}^T \mathbf{p} = \mathbf{e}^T(\mathbf{A} + \mathbf{E})\mathbf{p} = 1 + \mathbf{e}^T \mathbf{E} \mathbf{p} > 1$$

because

$$\mathbf{E} \geq \mathbf{0}, \mathbf{E} \neq \mathbf{0}, \mathbf{p} > \mathbf{0} \implies \mathbf{E} \mathbf{p} > \mathbf{0}.$$

(Conditions weaker than the column-sum condition can also force $\rho(\mathbf{A}) < 1$ —see Example 7.10.3 on p. 620.) The assumption that \mathbf{A} is a nonnegative irreducible matrix whose spectral radius is $\rho(\mathbf{A}) < 1$ combined with the Neumann series (p. 618) provides the conclusion that

$$(\mathbf{I} - \mathbf{A})^{-1} = \sum_{k=0}^{\infty} \mathbf{A}^k > \mathbf{0}.$$

Positivity is guaranteed by the irreducibility of \mathbf{A} because the same argument given on p. 672 that is to prove (8.3.5) also applies here. Therefore, for each demand vector $\mathbf{d} \geq \mathbf{0}$, there exists a unique supply vector given by $\mathbf{s} = (\mathbf{I} - \mathbf{A})^{-1} \mathbf{d}$, which is necessarily positive. The fact that $(\mathbf{I} - \mathbf{A})^{-1} > \mathbf{0}$ and $\mathbf{s} > \mathbf{0}$ leads to the interesting conclusion that an increase in public demand by just *one* unit from a *single* industry will force an increase in the output of *all* industries.

Note: The matrix $\mathbf{I} - \mathbf{A}$ is an M-matrix as defined and discussed in Example 7.10.7 (p. 626). The realization that M-matrices are naturally present in economic models provided some of the motivation for studying M-matrices during the first half of the twentieth century. Some of the M-matrix properties listed on p. 626 were independently discovered and formulated in economic terms.

Example 8.3.7

Leslie Population Age Distribution Model. Divide a population of females into age groups G_1, G_2, \dots, G_n , where each group covers the same number of years. For example,

$$\begin{aligned} G_1 &= \text{all females under age 10,} \\ G_2 &= \text{all females from age 10 up to 20,} \\ G_3 &= \text{all females from age 20 up to 30,} \\ &\vdots \end{aligned}$$

Consider discrete points in time, say $t = 0, 1, 2, \dots$ years, and let b_k and s_k denote the birth rate and survival rate for females in G_k . That is, let

b_k = Expected number of daughters produced by a female in G_k ,

s_k = Proportion of females in G_k at time t that are in G_{k+1} at time $t + 1$.

If

$$f_k(t) = \text{Number of females in } G_k \text{ at time } t,$$

then it follows that

$$f_1(t+1) = f_1(t)b_1 + f_2(t)b_2 + \dots + f_n(t)b_n$$

and

$$f_k(t+1) = f_{k-1}(t)s_{k-1} \quad \text{for } k = 2, 3, \dots, n. \quad (8.3.17)$$

Furthermore,

$$F_k(t) = \frac{f_k(t)}{f_1(t) + f_2(t) + \dots + f_n(t)} = \% \text{ of population in } G_k \text{ at time } t.$$

The vector $\mathbf{F}(t) = (F_1(t), F_2(t), \dots, F_n(t))^T$ represents the *population age distribution* at time t , and, provided that it exists, $\mathbf{F}^* = \lim_{t \rightarrow \infty} \mathbf{F}(t)$ is the *long-run (or steady-state) age distribution*.

Problem: Assuming that s_1, \dots, s_n and b_2, \dots, b_n are positive, explain why the population age distribution approaches a steady state, and then describe it. In other words, show that $\mathbf{F}^* = \lim_{t \rightarrow \infty} \mathbf{F}(t)$ exists, and determine its value.

Solution: The equations in (8.3.17) constitute a system of homogeneous difference equations that can be written in matrix form as

$$\mathbf{f}(t+1) = \mathbf{L}\mathbf{f}(t), \quad \text{where } \mathbf{L} = \begin{pmatrix} b_1 & b_2 & \cdots & b_{n-1} & b_n \\ s_1 & 0 & \cdots & \cdots & 0 \\ 0 & s_2 & 0 & & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & s_n & 0 \end{pmatrix}_{n \times n}. \quad (8.3.18)$$

The matrix \mathbf{L} is called the *Leslie matrix* in honor of P. H. Leslie who used this model in 1945. Notice that in addition to being nonnegative, \mathbf{L} is also irreducible when s_1, \dots, s_n and b_2, \dots, b_n are positive because the graph $\mathcal{G}(\mathbf{L})$ is strongly connected. Moreover, \mathbf{L} is primitive. This is obvious if in addition to s_1, \dots, s_n and b_2, \dots, b_n being positive we have $b_1 > 0$ (recall Example 8.3.3 on p. 678). But even if $b_1 = 0$, \mathbf{L} is still primitive because $\mathbf{L}^{n+2} > \mathbf{0}$ (recall (8.3.16) on p. 678). The technique on p. 679 also can be used to show primitivity (Exercise 8.3.11). Consequently, (8.3.10) on p. 674 guarantees that

$$\lim_{t \rightarrow \infty} \left(\frac{\mathbf{L}}{r} \right)^t = \mathbf{G} = \frac{\mathbf{p}\mathbf{q}^T}{\mathbf{q}^T\mathbf{p}} > \mathbf{0},$$

where $\mathbf{p} > \mathbf{0}$ and $\mathbf{q} > \mathbf{0}$ are the respective Perron vectors for \mathbf{L} and \mathbf{L}^T . If we combine this with the fact that the solution to the system of difference equations in (8.3.18) is $\mathbf{f}(t) = \mathbf{L}^t \mathbf{f}(0)$ (p. 617), and if we assume that $\mathbf{f}(0) \neq \mathbf{0}$, then we arrive at the conclusion that

$$\lim_{t \rightarrow \infty} \frac{\mathbf{f}(t)}{r^t} = \mathbf{G}\mathbf{f}(0) = \mathbf{p} \left(\frac{\mathbf{q}^T \mathbf{f}(0)}{\mathbf{q}^T \mathbf{p}} \right) \quad \text{and} \quad \lim_{t \rightarrow \infty} \left\| \frac{\mathbf{f}(t)}{r^t} \right\|_1 = \frac{\mathbf{q}^T \mathbf{f}(0)}{\mathbf{q}^T \mathbf{p}} > 0 \quad (8.3.19)$$

(because $\|\star\|_1$ is a continuous function—Exercise 5.1.7 on p. 277). Now

$$F_k(t) = \frac{f_k(t)}{\|\mathbf{f}(t)\|_1} = \text{\% of population that is in } G_k \text{ at time } t$$

is the quantity of interest, and (8.3.19) allows us to conclude that

$$\begin{aligned} \mathbf{F}^* &= \lim_{t \rightarrow \infty} \mathbf{F}(t) = \lim_{t \rightarrow \infty} \frac{\mathbf{f}(t)}{\|\mathbf{f}(t)\|_1} = \lim_{t \rightarrow \infty} \frac{\mathbf{f}(t)/r^t}{\|\mathbf{f}(t)\|_1/r^t} \\ &= \frac{\lim_{t \rightarrow \infty} \mathbf{f}(t)/r^t}{\lim_{t \rightarrow \infty} \|\mathbf{f}(t)\|_1/r^t} = \mathbf{p} \quad (\text{the Perron vector!}). \end{aligned}$$

In other words, while the numbers in the various age groups may increase or decrease, depending on the value of r (Exercise 8.3.10), the proportion of individuals in each age group becomes stable as time increases. And because the steady-state age distribution is given by the Perron vector of \mathbf{L} , each age group must eventually contain a positive fraction of the population.

Exercises for section 8.3

8.3.1. Let $\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ 3 & 0 & 3 \\ 0 & 2 & 0 \end{pmatrix}$.

- Show that \mathbf{A} is irreducible.
- Find the Perron root and Perron vector for \mathbf{A} .
- Find the number of eigenvalues on the spectral circle of \mathbf{A} .

- 8.3.2. Suppose that the index of imprimitivity of a 5×5 nonnegative irreducible matrix \mathbf{A} is $h = 3$. Explain why \mathbf{A} must be singular with $\text{alg mult}_{\mathbf{A}}(0) = 2$.
- 8.3.3. Suppose that \mathbf{A} is a nonnegative matrix that possesses a positive spectral radius and a corresponding positive eigenvector. Does this force \mathbf{A} to be irreducible?
- 8.3.4. Without computing the eigenvalues or the characteristic polynomial, explain why $\sigma(\mathbf{P}_n) = \{1, \omega, \omega^2, \dots, \omega^{n-1}\}$, where $\omega = e^{2\pi i/n}$ for

$$\mathbf{P}_n = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

- 8.3.5. Determine whether $\mathbf{A} = \begin{pmatrix} 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 7 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 9 & 2 & 0 & 4 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$ is reducible or irreducible.

- 8.3.6. Determine whether the matrix \mathbf{A} in Exercise 8.3.5 is primitive or imprimitive.

- 8.3.7. A matrix $\mathbf{S}_{n \times n} \geq \mathbf{0}$ having row sums less than or equal to 1 with at least one row sum less than 1 is called a **substochastic matrix**.

- (a) Explain why $\rho(\mathbf{S}) \leq 1$ for every substochastic matrix.
 (b) Prove that $\rho(\mathbf{S}) < 1$ for every *irreducible* substochastic matrix.

- 8.3.8. A nonnegative matrix for which each row sum is 1 is called a **stochastic matrix** (some say *row-stochastic*). Prove that if $\mathbf{A}_{n \times n}$ is nonnegative and irreducible with $r = \rho(\mathbf{A})$, then \mathbf{A} is similar to $r\mathbf{P}$ for some ir-

reducible stochastic matrix \mathbf{P} . **Hint:** Consider $\mathbf{D} = \begin{pmatrix} p_1 & 0 & \cdots & 0 \\ 0 & p_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & p_n \end{pmatrix}$,

where the p_k 's are the components of the Perron vector for \mathbf{A} .

- 8.3.9. Wielandt constructed the matrix $\mathbf{W}_n = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 1 & 1 & 0 & \cdots & 0 \end{pmatrix}$ to show

that $\mathbf{W}^{n^2-2n+2} > \mathbf{0}$, but $[\mathbf{W}^{n^2-2n+1}]_{11} = 0$. Verify that this is true for $n = 4$.

- 8.3.10.** In the Leslie population model on p. 683, explain what happens to the vector $\mathbf{f}(t)$ as $t \rightarrow \infty$ depending on whether $r < 1$, $r = 1$, or $r > 1$.
- 8.3.11.** Use the characteristic equation as described on p. 679 to show that the Leslie matrix in (8.3.18) is primitive even if $b_1 = 0$ (assuming all other b_k 's and s_k 's are positive).
- 8.3.12.** A matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ is said to be *essentially positive* if \mathbf{A} is irreducible and $a_{ij} \geq 0$ for every $i \neq j$. Prove that each of the following statements is equivalent to saying that \mathbf{A} is essentially positive.
- There exists some $\alpha \in \mathbb{R}$ such that $\mathbf{A} + \alpha \mathbf{I}$ is primitive.
 - $e^{t\mathbf{A}} > \mathbf{0}$ for all $t > 0$.
- 8.3.13.** Let \mathbf{A} be an essentially positive matrix as defined in Exercise 8.3.12. Prove that each of the following statements is true.
- \mathbf{A} has an eigenpair (ξ, \mathbf{x}) , where ξ is real and $\mathbf{x} > \mathbf{0}$.
 - If λ is any eigenvalue for \mathbf{A} other than ξ , then $\operatorname{Re}(\lambda) < \xi$.
 - ξ increases when any entry in \mathbf{A} is increased.
- 8.3.14.** Let $\mathbf{A} \geq \mathbf{0}$ be an irreducible matrix, and let $a_{ij}^{(k)}$ denote entries in \mathbf{A}^k . Prove that \mathbf{A} is primitive if and only if

$$\rho(\mathbf{A}) = \lim_{k \rightarrow \infty} \left[a_{ij}^{(k)} \right]^{1/k}.$$

8.4 STOCHASTIC MATRICES AND MARKOV CHAINS

One of the most elegant applications of the Perron–Frobenius theory is the algebraic development of the theory of finite Markov chains. The purpose of this section is to present some of the aspects of this development.

A **stochastic matrix** is a nonnegative matrix $\mathbf{P}_{n \times n}$ in which each row sum is equal to 1. Some authors say “row-stochastic” to distinguish this from the case when each column sum is 1.

A **Markov**⁹² **chain** is a stochastic process (a set of random variables $\{X_t\}_{t=0}^{\infty}$ in which X_t has the same range $\{S_1, S_2, \dots, S_n\}$, called the *state space*) that satisfies the *Markov property*

$$P(X_{t+1} = S_j | X_t = S_{i_t}, X_{t-1} = S_{i_{t-1}}, \dots, X_0 = S_{i_0}) = P(X_{t+1} = S_j | X_t = S_{i_t})$$

for each $t = 0, 1, 2, \dots$. Think of a Markov chain as a random chain of events that occur at discrete points $t = 0, 1, 2, \dots$ in time, where X_t represents the state of the event that occurs at time t . For example, if a mouse moves randomly through a maze consisting of chambers S_1, S_2, \dots, S_n , then X_t might represent the chamber occupied by the mouse at time t . The Markov property asserts that the process is *memoryless* in the sense that the state of the chain at the next time period depends only on the current state and not on the past history of the chain. In other words, the mouse moving through the maze obeys the Markov property if its next move doesn’t depend on where in the maze it has been in the past—i.e., the mouse is not using its memory (if it has one).

To emphasize that time is considered discretely rather than continuously the phrase “*discrete-time* Markov chain” is often used, and the phrase “*finite-state* Markov chain” might be used to emphasize that the state space is finite rather than infinite.

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Andrei Andreyevich Markov (1856–1922) was born in Ryazan, Russia, and he graduated from Saint Petersburg University in 1878 where he later became a professor. Markov’s early interest was number theory because this was the area of his famous teacher Pafnuty Lvovich Chebyshev (1821–1894). But when Markov discovered that he could apply his knowledge of continued fractions to probability theory, he embarked on a new course that would make him famous—enough so that there was a lunar crater named in his honor in 1964. In addition to being involved with liberal political movements (he once refused to be decorated by the Russian Czar), Markov enjoyed poetry, and in his spare time he studied poetic style. Therefore, it was no accident that led him to analyze the distribution of vowels and consonants in Pushkin’s work, *Eugene Onegin*, by constructing a simple model based on the assumption that the probability that a consonant occurs at a given position in any word should depend only on whether the preceding letter is a vowel or a consonant and not on any prior history. This was the birth of the “Markov chain.” Markov was wrong in one regard—he apparently believed that the only real examples of his chains were to be found in literary texts. But Markov’s work in 1907 has grown to become an indispensable tool of enormous power. It launched the theory of stochastic processes that is now the foundation for understanding, explaining, and predicting phenomena in diverse areas such as atomic physics, quantum theory, biology, genetics, social behavior, economics, and finance. Markov’s chains serve to underscore the point that the long-term applicability of mathematical research is impossible to predict.

Every Markov chain defines a stochastic matrix, and conversely. Let's see how this happens. The value $p_{ij}(t) = P(X_t = S_j \mid X_{t-1} = S_i)$ is the probability of being in state S_j at time t given that the chain is in state S_i at time $t-1$, so $p_{ij}(t)$ is called the **transition probability** of moving from S_i to S_j at time t . The matrix of transition probabilities $\mathbf{P}_{n \times n}(t) = [p_{ij}(t)]$ is clearly a nonnegative matrix, and a little thought should convince you that each row sum must be 1. Thus $\mathbf{P}(t)$ is a stochastic matrix. When the transition probabilities don't vary with time (say $p_{ij}(t) = p_{ij}$ for all t), the chain is said to be *stationary* (or *homogeneous*), and the **transition matrix** is the constant stochastic matrix $\mathbf{P} = [p_{ij}]$. We will make the assumption of stationarity throughout. Conversely, every stochastic matrix $\mathbf{P}_{n \times n}$ defines an n -state Markov chain because the entries p_{ij} define a set of transition probabilities, which can be interpreted as a stationary Markov chain on n states.

A **probability distribution vector** is defined to be a nonnegative vector $\mathbf{p}^T = (p_1, p_2, \dots, p_n)$ such that $\sum_k p_k = 1$. (Every row in a stochastic matrix is such a vector.) For an n -state Markov chain, the k^{th} **step probability distribution vector** is defined to be

$$\mathbf{p}^T(k) = (p_1(k), p_2(k), \dots, p_n(k)), \quad k = 1, 2, \dots, \quad \text{where } p_j(k) = P(X_k = S_j).$$

In other words, $p_j(k)$ is the probability of being in the j^{th} state after the k^{th} step, but before the $(k+1)^{\text{st}}$ step. The **initial distribution vector** is

$$\mathbf{p}^T(0) = (p_1(0), p_2(0), \dots, p_n(0)), \quad \text{where } p_j(0) = P(X_0 = S_j)$$

is the probability that the chain starts in S_j .

For example, consider the Markov chain defined by placing a mouse in the 3-chamber box with connecting doors as shown in Figure 8.4.1, and suppose that the mouse moves from the chamber it occupies to another chamber by picking a door at random—say that the doors open each minute, and when they do, the mouse is forced to move by electrifying the floor of the occupied chamber.

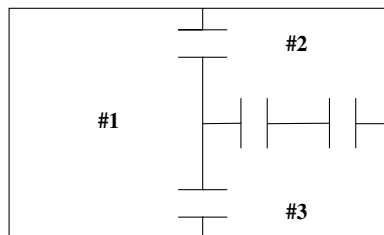


FIGURE 8.4.1

If the mouse is initially placed in chamber #2, then the initial distribution vector is $\mathbf{p}^T(0) = (0, 1, 0) = \mathbf{e}_2^T$. But if the process is started by tossing the mouse into the air so that it randomly lands in one of the chambers, then a reasonable

initial distribution is $\mathbf{p}^T(0) = (.5, .25, .25)$ because the area of chamber #1 is 50% of the box, while chambers #2 and #3 each constitute 25% of the box. The transition matrix for this Markov chain is the stochastic matrix

$$\mathbf{M} = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/3 & 0 & 2/3 \\ 1/3 & 2/3 & 0 \end{pmatrix}. \quad (8.4.1)$$

A standard eigenvalue calculation reveals that $\sigma(\mathbf{M}) = \{1, -1/3, -2/3\}$, so it's apparent that \mathbf{M} is a nonnegative matrix having spectral radius $\rho(\mathbf{M}) = 1$. This is a feature that is shared by all stochastic matrices $\mathbf{P}_{n \times n}$ because having row sums equal to 1 means that $\|\mathbf{P}\|_\infty = 1$ or, equivalently, $\mathbf{P}\mathbf{e} = \mathbf{e}$, where \mathbf{e} is the column of all 1's. Because $(1, \mathbf{e})$ is an eigenpair for every stochastic matrix, and because $\rho(\star) \leq \|\star\|$ for every matrix norm (recall (7.1.12) on p. 497), it follows that

$$1 \leq \rho(\mathbf{P}) \leq \|\mathbf{P}\|_\infty = 1 \implies \rho(\mathbf{P}) = 1.$$

Furthermore, \mathbf{e} is a positive eigenvector associated with $\rho(\mathbf{P}) = 1$. But be careful! This doesn't mean that you necessarily can call \mathbf{e} the Perron vector for \mathbf{P} because \mathbf{P} might not be irreducible—consider $\mathbf{P} = \begin{pmatrix} .5 & .5 \\ 0 & 1 \end{pmatrix}$.

Two important issues that arise in Markovian analysis concern the transient behavior of the chain as well as the limiting behavior. In other words, we want to accomplish the following goals.

- Describe the k^{th} step distribution $\mathbf{p}^T(k)$ for any given initial distribution vector $\mathbf{p}^T(0)$.
- Determine whether or not $\lim_{k \rightarrow \infty} \mathbf{p}^T(k)$ exists, and if it exists, determine the value of $\lim_{k \rightarrow \infty} \mathbf{p}^T(k)$.
- If there is no limiting distribution, then determine the possibility of having a Cesàro limit

$$\lim_{k \rightarrow \infty} \left[\frac{\mathbf{p}^T(0) + \mathbf{p}^T(1) + \cdots + \mathbf{p}^T(k-1)}{k} \right].$$

If such a limit exists, interpret its meaning, and determine its value.

The k^{th} step distribution is easily described by using the laws of elementary probability—in particular, recall that $P(E \vee F) = P(E) + P(F)$ when E and F are mutually exclusive events, and the conditional probability of E occurring given that F occurs is $P(E|F) = P(E \wedge F)/P(F)$ (it's convenient to use \wedge and \vee to denote *AND* and *OR*, respectively). To determine the j^{th} component

$p_j(1)$ in $\mathbf{p}^T(1)$ for a given $\mathbf{p}^T(0)$, write

$$\begin{aligned} p_j(1) &= P(X_1=S_j) = P[X_1=S_j \wedge (X_0=S_1 \vee X_0=S_2 \vee \cdots \vee X_0=S_n)] \\ &= P[(X_1=S_j \wedge X_0=S_1) \vee (X_1=S_j \wedge X_0=S_2) \vee \cdots \vee (X_1=S_j \wedge X_0=S_n)] \\ &= \sum_{i=1}^n P[X_1=S_j \wedge X_0=S_i] = \sum_{i=1}^n P[X_0=S_i] P[X_1=S_j | X_0=S_i] \\ &= \sum_{i=1}^n p_i(0)p_{ij} \quad \text{for } j = 1, 2, \dots, n. \end{aligned}$$

Consequently, $\mathbf{p}^T(1) = \mathbf{p}^T(0)\mathbf{P}$. This tells us what to expect after one step when we start with $\mathbf{p}^T(0)$. But the “no memory” Markov property tells us that the state of affairs at the end of two steps is determined by where we are at the end of the first step—it’s like starting over but with $\mathbf{p}^T(1)$ as the initial distribution. In other words, it follows that $\mathbf{p}^T(2) = \mathbf{p}^T(1)\mathbf{P}$, and $\mathbf{p}^T(3) = \mathbf{p}^T(2)\mathbf{P}$, etc. Therefore, successive substitution yields

$$\mathbf{p}^T(k) = \mathbf{p}^T(k-1)\mathbf{P} = \mathbf{p}^T(k-2)\mathbf{P}^2 = \cdots = \mathbf{p}^T(0)\mathbf{P}^k,$$

and thus the k^{th} step distribution is determined from the initial distribution and the transition matrix by the vector–matrix product

$$\mathbf{p}^T(k) = \mathbf{p}^T(0)\mathbf{P}^k. \quad (8.4.2)$$

Notice that if we adopt the notation $\mathbf{P}^k = [p_{ij}^{(k)}]$, and if we set $\mathbf{p}^T(0) = \mathbf{e}_i^T$ in (8.4.2), then we get $p_j(k) = p_{ij}^{(k)}$ for each $i = 1, 2, \dots, n$, and thus we arrive at the following conclusion.

- The (i, j) -entry in \mathbf{P}^k represents the probability of moving from S_i to S_j in exactly k steps. For this reason, \mathbf{P}^k is often called the ***k*-step transition matrix**.

Example 8.4.1

Let’s go back to the mouse-in-the-box example, and, as suggested earlier, toss the mouse into the air so that it randomly lands somewhere in the box in Figure 8.4.1—i.e., take the initial distribution to be $\mathbf{p}^T(0) = (1/2, 1/4, 1/4)$. The transition matrix is given by (8.4.1), so the probability of finding the mouse in chamber #1 after three moves is

$$[\mathbf{p}^T(3)]_1 = [\mathbf{p}^T(0)\mathbf{M}^3]_1 = 13/54.$$

In fact, the entire third step distribution is $\mathbf{p}^T(3) = (13/54, 41/108, 41/108)$.

To analyze limiting properties of Markov chains, divide the class of stochastic matrices (and hence the class of stationary Markov chains) into four mutually exclusive categories as described below.

- (1) Irreducible with $\lim_{k \rightarrow \infty} \mathbf{P}^k$ existing (i.e., \mathbf{P} is primitive).
- (2) Irreducible with $\lim_{k \rightarrow \infty} \mathbf{P}^k$ not existing (i.e., \mathbf{P} is imprimitive).
- (3) Reducible with $\lim_{k \rightarrow \infty} \mathbf{P}^k$ existing.
- (4) Reducible with $\lim_{k \rightarrow \infty} \mathbf{P}^k$ not existing.

In case (1), where \mathbf{P} is primitive, we know exactly what $\lim_{k \rightarrow \infty} \mathbf{P}^k$ looks like. The Perron vector for \mathbf{P} is \mathbf{e}/n (the uniform distribution vector), so if $\boldsymbol{\pi} = (\pi_1, \pi_2, \dots, \pi_n)^T$ is the Perron vector for \mathbf{P}^T , then

$$\lim_{k \rightarrow \infty} \mathbf{P}^k = \frac{(\mathbf{e}/n)\boldsymbol{\pi}^T}{\boldsymbol{\pi}^T(\mathbf{e}/n)} = \frac{\mathbf{e}\boldsymbol{\pi}^T}{\boldsymbol{\pi}^T\mathbf{e}} = \mathbf{e}\boldsymbol{\pi}^T = \begin{pmatrix} \pi_1 & \pi_2 & \cdots & \pi_n \\ \pi_1 & \pi_2 & \cdots & \pi_n \\ \vdots & \vdots & & \vdots \\ \pi_1 & \pi_2 & \cdots & \pi_n \end{pmatrix} > \mathbf{0} \quad (8.4.3)$$

by (8.3.10) on p. 674. Therefore, if \mathbf{P} is primitive, then a limiting probability distribution exists, and it is given by

$$\lim_{k \rightarrow \infty} \mathbf{p}^T(k) = \lim_{k \rightarrow \infty} \mathbf{p}^T(0)\mathbf{P}^k = \mathbf{p}^T(0)\mathbf{e}\boldsymbol{\pi}^T = \boldsymbol{\pi}^T. \quad (8.4.4)$$

Notice that because $\sum_k p_k(0) = 1$, the term $\mathbf{p}^T(0)\mathbf{e}$ drops away, so we have the conclusion that the value of the limit is *independent* of the value of the initial distribution $\mathbf{p}^T(0)$, which isn't too surprising.

Example 8.4.2

Going back to the mouse-in-the-box example, it's easy to confirm that the transition matrix \mathbf{M} in (8.4.1) is primitive, so $\lim_{k \rightarrow \infty} \mathbf{M}^k$ as well as $\lim_{k \rightarrow \infty} \mathbf{p}^T(0)$ must exist, and their values are determined by the left-hand Perron vector of \mathbf{M} that can be found by calculating any nonzero vector $\mathbf{v} \in N(\mathbf{I} - \mathbf{M}^T)$ and normalizing it to produce $\boldsymbol{\pi}^T = \mathbf{v}^T / \|\mathbf{v}\|_1$. Routine computation reveals that the one solution of the homogeneous equation $(\mathbf{I} - \mathbf{M}^T)\mathbf{v} = \mathbf{0}$ is $\mathbf{v}^T = (2, 3, 3)$, so $\boldsymbol{\pi}^T = (1/8)(2, 3, 3)$, and thus

$$\lim_{k \rightarrow \infty} \mathbf{M}^k = \frac{1}{8} \begin{pmatrix} 2 & 3 & 3 \\ 2 & 3 & 3 \\ 2 & 3 & 3 \end{pmatrix} \quad \text{and} \quad \lim_{k \rightarrow \infty} \mathbf{p}^T(k) = \frac{1}{8}(2, 3, 3).$$

This limiting distribution can be interpreted as meaning that in the long run the mouse will occupy chamber #1 one-fourth of the time, while 37.5% of the time it's in chamber #2, and 37.5% of the time it's in chamber #3, and this is independent of where (or how) the process started. The mathematical justification for this statement is on p. 693.

Now consider the imprimitive case. We know that if \mathbf{P} is irreducible and has $h > 1$ eigenvalues on the unit (spectral) circle, then $\lim_{k \rightarrow \infty} \mathbf{P}^k$ cannot exist (p. 674), and hence $\lim_{k \rightarrow \infty} \mathbf{p}^T(k)$ cannot exist (otherwise taking $\mathbf{p}^T(0) = \mathbf{e}_i^T$ for each i would insure that \mathbf{P}^k has a limit). However, each eigenvalue on the unit circle is simple (p. 676), and this means that \mathbf{P} is Cesàro summable (p. 633). Moreover, \mathbf{e}/n is the Perron vector for \mathbf{P} , and, as pointed out in Example 8.3.2 (p. 677), if $\boldsymbol{\pi}^T = (\pi_1, \pi_2, \dots, \pi_n)$ is the left-hand Perron vector, then

$$\lim_{k \rightarrow \infty} \frac{\mathbf{I} + \mathbf{P} + \cdots + \mathbf{P}^{k-1}}{k} = \frac{(\mathbf{e}/n)\boldsymbol{\pi}^T}{\boldsymbol{\pi}^T(\mathbf{e}/n)} = \frac{\mathbf{e}\boldsymbol{\pi}^T}{\boldsymbol{\pi}^T\mathbf{e}} = \mathbf{e}\boldsymbol{\pi}^T = \begin{pmatrix} \pi_1 & \pi_2 & \cdots & \pi_n \\ \pi_1 & \pi_2 & \cdots & \pi_n \\ \vdots & \vdots & & \vdots \\ \pi_1 & \pi_2 & \cdots & \pi_n \end{pmatrix},$$

which is exactly the same form as the limit (8.4.3) for the primitive case. Consequently, the k^{th} step distributions have a Cesàro limit given by

$$\begin{aligned} \lim_{k \rightarrow \infty} \left[\frac{\mathbf{p}^T(0) + \mathbf{p}^T(1) + \cdots + \mathbf{p}^T(k-1)}{k} \right] &= \lim_{k \rightarrow \infty} \mathbf{p}^T(0) \left[\frac{\mathbf{I} + \mathbf{P} + \cdots + \mathbf{P}^{k-1}}{k} \right] \\ &= \mathbf{p}^T(0)\mathbf{e}\boldsymbol{\pi}^T = \boldsymbol{\pi}^T, \end{aligned}$$

and, just as in the primitive case (8.4.4), this Cesàro limit is independent of the initial distribution.

Let's interpret the meaning of this Cesàro limit. The analysis is essentially the same as the description outlined in the shell game in Example 7.10.8 (p. 635), but for the sake of completeness we will duplicate some of the logic here. The trick is to focus on one state, say S_j , and define a sequence of random variables $\{Z_k\}_{k=0}^\infty$ that count the number of visits to S_j . Let

$$Z_0 = \begin{cases} 1 & \text{if the chain starts in } S_j, \\ 0 & \text{otherwise,} \end{cases}$$

and for $i > 1$,

$$Z_i = \begin{cases} 1 & \text{if the chain is in } S_j \text{ after the } i^{\text{th}} \text{ move,} \\ 0 & \text{otherwise.} \end{cases} \quad (8.4.5)$$

Notice that $Z_0 + Z_1 + \cdots + Z_{k-1}$ counts the number of visits to S_j before the k^{th} move, so $(Z_0 + Z_1 + \cdots + Z_{k-1})/k$ represents the *fraction* of times that S_j is hit before the k^{th} move. The expected (or mean) value of each Z_i is

$$E[Z_i] = 1 \cdot P(Z_i=1) + 0 \cdot P(Z_i=0) = P(Z_i=1) = p_j(i),$$

and, since expectation is linear, the expected fraction of times that S_j is hit before move k is

$$\begin{aligned} E \left[\frac{Z_0 + Z_1 + \cdots + Z_{k-1}}{k} \right] &= \frac{E[Z_0] + E[Z_1] + \cdots + E[Z_{k-1}]}{k} \\ &= \frac{p_j(0) + p_j(1) + \cdots + p_j(k-1)}{k} = \left[\frac{\mathbf{p}^T(0) + \mathbf{p}^T(1) + \cdots + \mathbf{p}^T(k-1)}{k} \right]_j \\ &\rightarrow \pi_j. \end{aligned}$$

In other words, the long-run fraction of time that the chain spends in S_j is π_j , which is the j^{th} component of the Cesàro limit or, equivalently, the j^{th} component of the left-hand Perron vector for \mathbf{P} .

When $\lim_{k \rightarrow \infty} \mathbf{p}^T(k)$ exists, it must be the case that

$$\lim_{k \rightarrow \infty} \mathbf{p}^T(k) = \lim_{k \rightarrow \infty} \left[\frac{\mathbf{p}^T(0) + \mathbf{p}^T(1) + \cdots + \mathbf{p}^T(k-1)}{k} \right] \quad (\text{Exercise 7.10.11, p. 639}),$$

and therefore the interpretation of the limiting distribution $\lim_{k \rightarrow \infty} \mathbf{p}^T(k)$ for the primitive case is exactly the same as the interpretation of the Cesàro limit in the imprimitive case.

Below is a summary of our findings for irreducible chains.

Irreducible Markov Chains

Let \mathbf{P} be the transition probability matrix for an irreducible Markov chain on states $\{S_1, S_2, \dots, S_n\}$ (i.e., \mathbf{P} is an $n \times n$ irreducible stochastic matrix), and let $\boldsymbol{\pi}^T$ denote the left-hand Perron vector for \mathbf{P} . The following statements are true for every initial distribution $\mathbf{p}^T(0)$.

- The k^{th} step transition matrix is \mathbf{P}^k because the (i, j) -entry in \mathbf{P}^k is the probability of moving from S_i to S_j in exactly k steps.
- The k^{th} step distribution vector is given by $\mathbf{p}^T(k) = \mathbf{p}^T(0)\mathbf{P}^k$.
- If \mathbf{P} is primitive, and if \mathbf{e} denotes the column of all 1's, then

$$\lim_{k \rightarrow \infty} \mathbf{P}^k = \mathbf{e}\boldsymbol{\pi}^T \quad \text{and} \quad \lim_{k \rightarrow \infty} \mathbf{p}^T(k) = \boldsymbol{\pi}^T.$$

- If \mathbf{P} is imprimitive, then

$$\begin{aligned} \lim_{k \rightarrow \infty} \frac{\mathbf{I} + \mathbf{P} + \cdots + \mathbf{P}^{k-1}}{k} &= \mathbf{e}\boldsymbol{\pi}^T \\ \text{and} \quad \lim_{k \rightarrow \infty} \left[\frac{\mathbf{p}^T(0) + \mathbf{p}^T(1) + \cdots + \mathbf{p}^T(k-1)}{k} \right] &= \boldsymbol{\pi}^T. \end{aligned}$$

- Regardless of whether \mathbf{P} is primitive or imprimitive, the j^{th} component π_j of $\boldsymbol{\pi}^T$ represents the long-run fraction of time that the chain is in S_j .
- $\boldsymbol{\pi}^T$ is often called the **stationary distribution vector** for the chain because it is the unique distribution vector satisfying $\boldsymbol{\pi}^T \mathbf{P} = \boldsymbol{\pi}^T$.

Example 8.4.3

Periodic Chains. Consider an electronic switch that can be in one of three states $\{S_1, S_2, S_3\}$, and suppose that the switch changes states on regular clock cycles. If the switch is in either S_1 or S_3 , then it must change to S_2 on the next clock cycle, but if the switch is in S_2 , then there is an equal likelihood that it changes to S_1 or S_3 on the next clock cycle. The transition matrix is

$$\mathbf{P} = \begin{pmatrix} 0 & 1 & 0 \\ .5 & 0 & .5 \\ 0 & 1 & 0 \end{pmatrix},$$

and it's not difficult to see that \mathbf{P} is irreducible (because $\mathcal{G}(\mathbf{P})$ is strongly connected) and imprimitive (because $\sigma(\mathbf{P}) = \{\pm 1, 0\}$). Since the left-hand Perron vector is $\boldsymbol{\pi}^T = (.25, .5, .25)$, the long-run expectation is that the switch should be in S_1 25% of the time, in S_2 50% of the time, and in S_3 25% of the time, and this agrees with what common sense tells us. Furthermore, notice that the switch cannot be in just any position at any given clock cycle because if the chain starts in either S_1 or S_3 , then it must be in S_2 on every odd-numbered cycle, and it can occupy S_1 or S_3 only on even-numbered cycles. The situation is similar, but with reversed parity, when the chain starts in S_2 . In other words, the chain is periodic in the sense that the states can be occupied only at periodic points in time. In this example the period of the chain is 2, and this is the same as the index of imprimitivity. This is no accident. The Frobenius form for imprimitive matrices on p. 680 can be used to prove that this is true in general. Consequently, an irreducible Markov chain is said to be a **periodic chain** when its transition matrix \mathbf{P} is imprimitive (with the period of the chain being the index of imprimitivity for \mathbf{P}), and an irreducible Markov chain for which \mathbf{P} is primitive is called an **aperiodic chain**. The shell game in Example 7.10.8 (p. 635) is a periodic Markov chain that is similar to the one in this example.

Because the Perron–Frobenius theorem is not directly applicable to reducible chains (chains for which \mathbf{P} is a reducible matrix), the strategy for analyzing reducible chains is to deflate the situation, as much as possible, back to the irreducible case as described below.

If \mathbf{P} is reducible, then, by definition, there exists a permutation matrix \mathbf{Q} and square matrices \mathbf{X} and \mathbf{Z} such that

$$\mathbf{Q}^T \mathbf{P} \mathbf{Q} = \begin{pmatrix} \mathbf{X} & \mathbf{Y} \\ \mathbf{0} & \mathbf{Z} \end{pmatrix}. \text{ For convenience, denote this by writing } \mathbf{P} \sim \begin{pmatrix} \mathbf{X} & \mathbf{Y} \\ \mathbf{0} & \mathbf{Z} \end{pmatrix}.$$

If \mathbf{X} or \mathbf{Z} is reducible, then another symmetric permutation can be performed to produce

$$\begin{pmatrix} \mathbf{X} & \mathbf{Y} \\ \mathbf{0} & \mathbf{Z} \end{pmatrix} \sim \begin{pmatrix} \mathbf{R} & \mathbf{S} & \mathbf{T} \\ \mathbf{0} & \mathbf{U} & \mathbf{V} \\ \mathbf{0} & \mathbf{0} & \mathbf{W} \end{pmatrix}, \text{ where } \mathbf{R}, \mathbf{U}, \text{ and } \mathbf{W} \text{ are square.}$$

Repeating this process eventually yields

$$\mathbf{P} \sim \begin{pmatrix} \mathbf{X}_{11} & \mathbf{X}_{12} & \cdots & \mathbf{X}_{1k} \\ \mathbf{0} & \mathbf{X}_{22} & \cdots & \mathbf{X}_{2k} \\ \vdots & & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{X}_{kk} \end{pmatrix}, \quad \text{where each } \mathbf{X}_{ii} \text{ is irreducible or } \mathbf{X}_{ii} = [0]_{1 \times 1}.$$

Finally, if there exist rows having nonzero entries *only* in diagonal blocks, then symmetrically permute all such rows to the bottom to produce

$$\mathbf{P} \sim \left(\begin{array}{cccc|cccc} \mathbf{P}_{11} & \mathbf{P}_{12} & \cdots & \mathbf{P}_{1r} & \mathbf{P}_{1,r+1} & \mathbf{P}_{1,r+2} & \cdots & \mathbf{P}_{1m} \\ \mathbf{0} & \mathbf{P}_{22} & \cdots & \mathbf{P}_{2r} & \mathbf{P}_{2,r+1} & \mathbf{P}_{2,r+2} & \cdots & \mathbf{P}_{2m} \\ \vdots & & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{P}_{rr} & \mathbf{P}_{r,r+1} & \mathbf{P}_{r,r+2} & \cdots & \mathbf{P}_{rm} \\ \hline \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{P}_{r+1,r+1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{P}_{r+2,r+2} & \cdots & \mathbf{0} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{P}_{mm} \end{array} \right), \quad (8.4.6)$$

where each $\mathbf{P}_{11}, \dots, \mathbf{P}_{rr}$ is either irreducible or $[0]_{1 \times 1}$, and $\mathbf{P}_{r+1,r+1}, \dots, \mathbf{P}_{mm}$ are irreducible (they can't be zero because each has row sums equal to 1). As mentioned on p. 671, the effect of a symmetric permutation is simply to relabel nodes in $\mathcal{G}(\mathbf{P})$ or, equivalently, to reorder the states in the chain. When the states of a chain have been reordered so that \mathbf{P} assumes the form on the right-hand side of (8.4.6), we say that \mathbf{P} is in the **canonical form for reducible matrices**. When \mathbf{P} is in canonical form, the subset of states corresponding to \mathbf{P}_{kk} for $1 \leq k \leq r$ is called the k^{th} **transient class** (because once left, a transient class can't be reentered), and the subset of states corresponding to $\mathbf{P}_{r+j,r+j}$ for $j \geq 1$ is called the j^{th} **ergodic class**. Each ergodic class is an irreducible Markov chain unto itself that is imbedded in the larger reducible chain. From now on, we will assume that the states in our reducible chains have been ordered so that \mathbf{P} is in canonical form.

The results on p. 676 guarantee that if an irreducible stochastic matrix \mathbf{P} has h eigenvalues on the unit circle, then these h eigenvalues are the h^{th} roots of unity, and each is a simple eigenvalue for \mathbf{P} . The same can't be said for reducible stochastic matrices, but the canonical form (8.4.6) allows us to prove the next best thing as discussed below.

Unit Eigenvalues

The *unit eigenvalues* for a stochastic matrix are defined to be those eigenvalues that are on the unit circle. For every stochastic matrix $\mathbf{P}_{n \times n}$, the following statements are true.

- Every unit eigenvalue of \mathbf{P} is semisimple.
- Every unit eigenvalue has form $\lambda = e^{2k\pi i/h}$ for some $k < h \leq n$.
- In particular, $\rho(\mathbf{P}) = 1$ is always a semisimple eigenvalue of \mathbf{P} .

Proof. If \mathbf{P} is irreducible, then there is nothing to prove because, as proved on p. 676, the unit eigenvalues are roots of unity, and each unit eigenvalue is simple. If \mathbf{P} is reducible, suppose that a symmetric permutation has been performed so that \mathbf{P} is in the canonical form (8.4.6), and observe that

$$\rho(\mathbf{P}_{kk}) < 1 \quad \text{for each } k = 1, 2, \dots, r. \quad (8.4.7)$$

This is certainly true when $\mathbf{P}_{kk} = [0]_{1 \times 1}$, so suppose that \mathbf{P}_{kk} ($1 \leq k \leq r$) is irreducible. Because there must be blocks \mathbf{P}_{kj} , $j \neq k$, that have nonzero entries, it follows that

$$\mathbf{P}_{kk}\mathbf{e} \leq \mathbf{e} \quad \text{and} \quad \mathbf{P}_{kk}\mathbf{e} \neq \mathbf{e}, \quad \text{where } \mathbf{e} \text{ is the column of all } 1\text{'s.}$$

If $\rho(\mathbf{P}_{kk}) = 1$, then the observation in Example 8.3.1 (p. 674) forces $\mathbf{P}_{kk}\mathbf{e} = \mathbf{e}$, which is impossible, and thus $\rho(\mathbf{P}_{kk}) < 1$. Consequently, the unit eigenvalues for \mathbf{P} are the collection of the unit eigenvalues of the irreducible matrices $\mathbf{P}_{r+1, r+1}, \dots, \mathbf{P}_{mm}$. But each unit eigenvalue of $\mathbf{P}_{r+i, r+i}$ is simple and is a root of unity. Consequently, if λ is a unit eigenvalue for \mathbf{P} , then it must be some root of unity, and although it might be repeated because it appears in the spectrum of more than one $\mathbf{P}_{r+i, r+i}$, it must nevertheless be the case that $\text{alg mult}_{\mathbf{P}}(\lambda) = \text{geo mult}_{\mathbf{P}}(\lambda)$, so λ is a semisimple eigenvalue of \mathbf{P} . ■

We know from the discussion on p. 633 that a matrix $\mathbf{A} \in \mathcal{C}^{n \times n}$ is Cesàro summable if and only if $\rho(\mathbf{A}) < 1$ or $\rho(\mathbf{A}) = 1$ with each eigenvalue on the unit circle being semisimple. We just proved that the latter holds for all stochastic matrices \mathbf{P} , so we have in fact established the following powerful statement concerning all stochastic matrices.

All Stochastic Matrices Are Summable

Every stochastic matrix \mathbf{P} is Cesàro summable. That is,

$$\lim_{k \rightarrow \infty} \frac{\mathbf{I} + \mathbf{P} + \cdots + \mathbf{P}^{k-1}}{k} \quad \text{exists for all stochastic matrices } \mathbf{P},$$

and, as discussed on p. 633, the value of the limit is the (spectral) projector \mathbf{G} onto $N(\mathbf{I} - \mathbf{P})$ along $R(\mathbf{I} - \mathbf{P})$.

Since we already know the structure and interpretation of the Cesàro limit when \mathbf{P} is an irreducible stochastic matrix (p. 693), all that remains in order to complete the picture is to analyze the nature of $\lim_{k \rightarrow \infty} (\mathbf{I} + \mathbf{P} + \cdots + \mathbf{P}^{k-1})/k$ for the reducible case.

Suppose that $\mathbf{P} = \begin{pmatrix} \mathbf{T}_{11} & \mathbf{T}_{12} \\ \mathbf{0} & \mathbf{T}_{22} \end{pmatrix}$ is a reducible stochastic matrix that is in the canonical form (8.4.6), where

$$\mathbf{T}_{11} = \begin{pmatrix} \mathbf{P}_{11} & \cdots & \mathbf{P}_{1r} \\ & \ddots & \vdots \\ & & \mathbf{P}_{rr} \end{pmatrix}, \quad \mathbf{T}_{12} = \begin{pmatrix} \mathbf{P}_{1,r+1} & \cdots & \mathbf{P}_{1m} \\ \vdots & & \vdots \\ \mathbf{P}_{r,r+1} & \cdots & \mathbf{P}_{rm} \end{pmatrix},$$

(8.4.8)

and

$$\mathbf{T}_{22} = \begin{pmatrix} \mathbf{P}_{r+1,r+1} & & \\ & \ddots & \\ & & \mathbf{P}_{mm} \end{pmatrix}.$$

We know from (8.4.7) that $\rho(\mathbf{P}_{kk}) < 1$ for each $k = 1, 2, \dots, r$, so it follows that $\rho(\mathbf{T}_{11}) < 1$, and hence

$$\lim_{k \rightarrow \infty} \frac{\mathbf{I} + \mathbf{T}_{11} + \cdots + \mathbf{T}_{11}^{k-1}}{k} = \lim_{k \rightarrow \infty} \mathbf{T}_{11}^k = \mathbf{0} \quad (\text{recall Exercise 7.10.11 on p. 639}).$$

Furthermore, $\mathbf{P}_{r+1,r+1}, \dots, \mathbf{P}_{mm}$ are each irreducible stochastic matrices, so if $\boldsymbol{\pi}_j^T$ is the left-hand Perron vector for \mathbf{P}_{jj} , $r+1 \leq j \leq m$, then our previous results (p. 693) tell us that

$$\lim_{k \rightarrow \infty} \frac{\mathbf{I} + \mathbf{T}_{22} + \cdots + \mathbf{T}_{22}^{k-1}}{k} = \begin{pmatrix} \mathbf{e}\boldsymbol{\pi}_{r+1}^T & & \\ & \ddots & \\ & & \mathbf{e}\boldsymbol{\pi}_m^T \end{pmatrix} = \mathbf{E}. \quad (8.4.9)$$

Furthermore, it's clear from the results on p. 674 that $\lim_{k \rightarrow \infty} \mathbf{T}_{22}^k$ exists if and only if $\mathbf{P}_{r+1,r+1}, \dots, \mathbf{P}_{mm}$ are each primitive, in which case $\lim_{k \rightarrow \infty} \mathbf{T}_{22}^k = \mathbf{E}$.

Therefore, the limits, be they Cesàro or ordinary (if it exists), all have the form

$$\lim_{k \rightarrow \infty} \frac{\mathbf{I} + \mathbf{P} + \cdots + \mathbf{P}^{k-1}}{k} = \begin{pmatrix} \mathbf{0} & \mathbf{Z} \\ \mathbf{0} & \mathbf{E} \end{pmatrix} = \mathbf{G} = \lim_{k \rightarrow \infty} \mathbf{P}^k \text{ (when it exists).}$$

To determine the precise nature of \mathbf{Z} , use the fact that $R(\mathbf{G}) = N(\mathbf{I} - \mathbf{P})$ (because \mathbf{G} is the projector onto $N(\mathbf{I} - \mathbf{P})$ along $R(\mathbf{I} - \mathbf{P})$) to write

$$(\mathbf{I} - \mathbf{P})\mathbf{G} = \mathbf{0} \implies \begin{pmatrix} \mathbf{I} - \mathbf{T}_{11} & -\mathbf{T}_{12} \\ \mathbf{0} & \mathbf{I} - \mathbf{T}_{22} \end{pmatrix} \begin{pmatrix} \mathbf{0} & \mathbf{Z} \\ \mathbf{0} & \mathbf{E} \end{pmatrix} = \mathbf{0} \implies (\mathbf{I} - \mathbf{T}_{11})\mathbf{Z} = \mathbf{T}_{12}\mathbf{E}.$$

Since $\mathbf{I} - \mathbf{T}_{11}$ is nonsingular (because $\rho(\mathbf{T}_{11}) < 1$ by (8.4.7)), it follows that

$$\mathbf{Z} = (\mathbf{I} - \mathbf{T}_{11})^{-1} \mathbf{T}_{12} \mathbf{E},$$

and thus the following results concerning limits of reducible chains are produced.

Reducible Markov Chains

If the states in a reducible Markov chain have been ordered to make the transition matrix assume the canonical form

$$\mathbf{P} = \begin{pmatrix} \mathbf{T}_{11} & \mathbf{T}_{12} \\ \mathbf{0} & \mathbf{T}_{22} \end{pmatrix}$$

that is described in (8.4.6) and (8.4.8), and if π_j^T is the left-hand Perron vector for \mathbf{P}_{jj} ($r+1 \leq j \leq m$), then $\mathbf{I} - \mathbf{T}_{11}$ is nonsingular, and

$$\lim_{k \rightarrow \infty} \frac{\mathbf{I} + \mathbf{P} + \cdots + \mathbf{P}^{k-1}}{k} = \begin{pmatrix} \mathbf{0} & (\mathbf{I} - \mathbf{T}_{11})^{-1} \mathbf{T}_{12} \mathbf{E} \\ \mathbf{0} & \mathbf{E} \end{pmatrix},$$

where

$$\mathbf{E} = \begin{pmatrix} \mathbf{e}\pi_{r+1}^T & & \\ & \ddots & \\ & & \mathbf{e}\pi_m^T \end{pmatrix}.$$

Furthermore, $\lim_{k \rightarrow \infty} \mathbf{P}^k$ exists if and only if the stochastic matrices $\mathbf{P}_{r+1, r+1}, \dots, \mathbf{P}_{mm}$ in (8.4.6) are each primitive, in which case

$$\lim_{k \rightarrow \infty} \mathbf{P}^k = \begin{pmatrix} \mathbf{0} & (\mathbf{I} - \mathbf{T}_{11})^{-1} \mathbf{T}_{12} \mathbf{E} \\ \mathbf{0} & \mathbf{E} \end{pmatrix}. \quad (8.4.10)$$

The preceding analysis shows that every reducible chain eventually gets *absorbed* (trapped) into one of the ergodic classes—i.e., into a subchain defined by $\mathbf{P}_{r+j,r+j}$ for some $j \geq 1$. If $\mathbf{P}_{r+j,r+j}$ is primitive, then the chain settles down to a steady-state defined by the left-hand Perron vector of $\mathbf{P}_{r+j,r+j}$, but if $\mathbf{P}_{r+j,r+j}$ is imprimitive, then the process will oscillate in the j^{th} ergodic class forever. There is not much more that can be said about the limit, but there are still important questions concerning which ergodic class the chain will end up in and how long it takes to get there. This time the answer depends on where the chain starts—i.e., on the initial distribution.

For convenience, let \mathcal{T}_i denote the i^{th} transient class, and let \mathcal{E}_j be the j^{th} ergodic class. Suppose that the chain starts in a particular transient state—say we start in the p^{th} state of \mathcal{T}_i . Since the question at hand concerns only which ergodic class is hit but not what happens after it's entered, we might as well convert every state in each ergodic class into a trap by setting $\mathbf{P}_{r+j,r+j} = \mathbf{I}$ for each $j \geq 1$ in (8.4.6). The transition matrix for this modified chain is $\tilde{\mathbf{P}} = \begin{pmatrix} \mathbf{T}_{11} & \mathbf{T}_{12} \\ \mathbf{0} & \mathbf{I} \end{pmatrix}$, and it follows from (8.4.10) that $\lim_{k \rightarrow \infty} \tilde{\mathbf{P}}^k$ exists and has the form

$$\lim_{k \rightarrow \infty} \tilde{\mathbf{P}}^k = \begin{pmatrix} \mathbf{0} & (\mathbf{I} - \mathbf{T}_{11})^{-1} \mathbf{T}_{12} \\ \mathbf{0} & \mathbf{I} \end{pmatrix} = \left(\begin{array}{cccc|cccc} \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{L}_{1,1} & \mathbf{L}_{1,2} & \cdots & \mathbf{L}_{1s} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{L}_{2,1} & \mathbf{L}_{2,2} & \cdots & \mathbf{L}_{2s} \\ \vdots & & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{L}_{r,1} & \mathbf{L}_{r,2} & \cdots & \mathbf{L}_{rs} \\ \hline \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{I} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{I} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{I} \end{array} \right).$$

Consequently, the (p, q) -entry in block \mathbf{L}_{ij} represents the probability of eventually hitting the q^{th} state in \mathcal{E}_j given that we start from the p^{th} state in \mathcal{T}_i . Therefore, if \mathbf{e} is the vector of all 1's, then the probability of eventually entering somewhere in \mathcal{E}_j is given by

- $P(\text{absorption into } \mathcal{E}_j \mid \text{start in } p^{\text{th}} \text{ state of } \mathcal{T}_i) = \sum_k [\mathbf{L}_{ij}]_{pk} = [\mathbf{L}_{ij} \mathbf{e}]_p.$

If $\mathbf{p}_i^T(0)$ is an initial distribution for starting in the various states of \mathcal{T}_i , then

- $P(\text{absorption into } \mathcal{E}_j \mid \mathbf{p}_i^T(0)) = \mathbf{p}_i^T(0) \mathbf{L}_{ij} \mathbf{e}.$

To determine the expected number of steps required to first hit an ergodic state, proceed as follows. Count the number of times the chain is in transient state S_j given that it starts in transient state S_i by reapplying the argument given in (8.4.5) on p. 692. That is, given that the chain starts in S_i , let

$$Z_0 = \begin{cases} 1 & \text{if } S_i = S_j, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad Z_k = \begin{cases} 1 & \text{if the chain is in } S_j \text{ after step } k, \\ 0 & \text{otherwise.} \end{cases}$$

Since

$$E[Z_k] = 1 \cdot P(Z_k=1) + 0 \cdot P(Z_k=0) = P(Z_k=1) = [\mathbf{T}_{11}^k]_{ij},$$

and since $\sum_{k=0}^{\infty} Z_k$ is the total number of times the chain is in S_j , we have

$$\begin{aligned} E[\# \text{ times in } S_j \mid \text{ start in } S_i] &= E\left[\sum_{k=0}^{\infty} Z_k\right] = \sum_{k=0}^{\infty} E[Z_k] = \sum_{k=0}^{\infty} [\mathbf{T}_{11}^k]_{ij} \\ &= [(\mathbf{I} - \mathbf{T}_{11})^{-1}]_{ij} \quad (\text{because } \rho(\mathbf{T}_{11}) < 1). \end{aligned}$$

Summing this over all transient states produces the expected number of times the chain is in *some* transient state, which is the same as the expected number of times before first hitting an ergodic state. In other words,

$$\bullet \quad E[\# \text{ steps until absorption} \mid \text{ start in } i^{\text{th}} \text{ transient state}] = [(\mathbf{I} - \mathbf{T}_{11})^{-1}\mathbf{e}]_i.$$

Example 8.4.4

Absorbing Markov Chains. It's often the case in practical applications that there is only one transient class, and the ergodic classes are just single *absorbing states* (states such that once they are entered, they are never left). If the single transient class contains r states, and if there are s absorbing states, then the canonical form for the transition matrix is

$$\mathbf{P} = \left(\begin{array}{ccc|ccc} p_{11} & \cdots & p_{1r} & p_{1,r+1} & \cdots & p_{1s} \\ \vdots & & \vdots & \vdots & & \vdots \\ p_{r1} & \cdots & p_{rr} & p_{r,r+1} & \cdots & p_{rs} \\ \hline 0 & \cdots & 0 & 1 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 1 \end{array} \right) \quad \text{and} \quad \mathbf{L}_{ij} = [(\mathbf{I} - \mathbf{T}_{11})^{-1}\mathbf{T}_{12}]_{ij}.$$

The preceding analysis specializes to say that every absorbing chain must eventually reach one of its absorbing states. The probability of being absorbed into the j^{th} absorbing state (which is state S_{r+j}) given that the chain starts in the i^{th} transient state (which is S_i) is

$$P(\text{absorption into } S_{r+j} \mid \text{ start in } S_i \text{ for } 1 \leq i \leq r) = [(\mathbf{I} - \mathbf{T}_{11})^{-1}\mathbf{T}_{12}]_{ij},$$

while the expected time until absorption is

$$E[\# \text{ steps until absorption} \mid \text{ start in } S_i] = [(\mathbf{I} - \mathbf{T}_{11})^{-1}\mathbf{e}]_i,$$

and the amount of time spent in S_j is

$$E[\# \text{ times in } S_j \mid \text{ start in } S_i] = [(\mathbf{I} - \mathbf{T}_{11})^{-1}]_{ij}.$$

Example 8.4.5

Fail-Safe System. Consider a system that has two independent controls, A and B , that can prevent the system from being destroyed. The system is activated at discrete points in time t_1, t_2, t_3, \dots , and the system is considered to be “under control” if either control A or B holds at the time of activation. The system is destroyed if A and B fail simultaneously.

- ▷ For example, an automobile has two independent braking systems—one is operated by a foot pedal, whereas the “emergency brake” is operated by a hand lever. The automobile is “under control” if at least one braking system is operative when you try to stop, but a crash occurs if both braking systems fail simultaneously.

If one of the controls fails at some activation point but the other control holds, then the defective control is repaired before the next activation. If a control holds at time $t = t_k$, then it is considered to be 90% reliable at $t = t_{k+1}$, but if a control fails at time $t = t_k$, then its untested replacement is considered to be only 60% reliable at $t = t_{k+1}$.

Problem: Can the system be expected to run indefinitely without every being destroyed? If not, how long is the system expected to run before destruction occurs?

Solution: This is a four-state Markov chain with the states being the controls that hold at any particular time of activation. In other words the state space is the set of pairs (a, b) in which

$$a = \begin{cases} 1 & \text{if } A \text{ holds,} \\ 0 & \text{if } A \text{ fails,} \end{cases} \quad \text{and} \quad b = \begin{cases} 1 & \text{if } B \text{ holds,} \\ 0 & \text{if } B \text{ fails.} \end{cases}$$

State $(0, 0)$ is absorbing, and the transition matrix (in canonical form) is

$$\mathbf{P} = \begin{matrix} & \begin{matrix} (1, 1) & (1, 0) & (0, 1) & (0, 0) \end{matrix} \\ \begin{matrix} (1, 1) \\ (1, 0) \\ (0, 1) \\ (0, 0) \end{matrix} & \begin{pmatrix} .81 & .09 & .09 & .01 \\ .54 & .36 & .06 & .04 \\ .54 & .06 & .36 & .04 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{matrix}$$

with

$$\mathbf{T}_{11} = \begin{pmatrix} .81 & .09 & .09 \\ .54 & .36 & .06 \\ .54 & .06 & .36 \end{pmatrix} \quad \text{and} \quad \mathbf{T}_{12} = \begin{pmatrix} .01 \\ .04 \\ .04 \end{pmatrix}.$$

The fact that $\lim_{k \rightarrow \infty} \mathbf{P}^k$ exists and is given by

$$\lim_{k \rightarrow \infty} \mathbf{P}^k = \begin{pmatrix} \mathbf{0} & (\mathbf{I} - \mathbf{T}_{11})^{-1} \mathbf{T}_{12} \\ \mathbf{0} & 1 \end{pmatrix}$$

makes it clear that the absorbing state must eventually be reached. In other words, this proves the validity of the popular belief that “if something can go wrong, then it eventually will.” Rounding to three significant figures produces

$$(\mathbf{I} - \mathbf{T}_{11})^{-1} = \begin{pmatrix} 44.6 & 6.92 & 6.92 \\ 41.5 & 8.02 & 6.59 \\ 41.5 & 6.59 & 8.02 \end{pmatrix} \quad \text{and} \quad (\mathbf{I} - \mathbf{T}_{11})^{-1}\mathbf{e} = \begin{pmatrix} 58.4 \\ 56.1 \\ 56.1 \end{pmatrix},$$

so the mean time to failure starting with two proven controls is slightly more than 58 steps, while the mean time to failure starting with one untested control and one proven control is just over 56 steps. The difference here doesn't seem significant, but consider what happens when only one control is used in the system. In this case, there are only two states in the chain, 1 (meaning that the control holds) and 0 (meaning that it doesn't). The transition matrix is

$$\mathbf{P} = \begin{pmatrix} 1 & 0 \\ 0 & .9 \end{pmatrix},$$

so now the mean time to failure is only $(\mathbf{I} - \mathbf{T}_{11})^{-1}\mathbf{e} = 10$ steps. It's interesting to consider what happens when three independent control are used. How much more security does your intuition tell you that you should have? See Exercise 8.4.8.

Exercises for section 8.4

8.4.1. Find the stationary distribution for $\mathbf{P} = \begin{pmatrix} 1/4 & 0 & 0 & 3/4 \\ 3/8 & 1/4 & 3/8 & 0 \\ 1/3 & 1/6 & 1/6 & 1/3 \\ 0 & 0 & 1/2 & 1/2 \end{pmatrix}$. Does

this stationary distribution represent a limiting distribution in the regular sense or only in the Cesàro sense?

8.4.2. A *doubly-stochastic matrix* is a nonnegative matrix $\mathbf{P}_{n \times n}$ having all row sums as well as all column sums equal to 1. For an irreducible n -state Markov chain whose transition matrix is doubly stochastic, what is the long-run proportion of time spent in each state? What form do $\lim_{k \rightarrow \infty} (\mathbf{I} + \mathbf{P} + \cdots + \mathbf{P}^{k-1})/k$ and $\lim_{k \rightarrow \infty} \mathbf{P}^k$ (if it exists) have? **Note:** The purpose of this exercise is to show that doubly-stochastic matrices are not very interesting from a Markov-chain point of view. However, there is an interesting theoretical result (due to G. Birkhoff in 1946) that says the set of $n \times n$ doubly-stochastic matrices forms a convex polyhedron in $\mathbb{R}^{n \times n}$ with the permutation matrices as the vertices.

8.4.3. Explain why $\text{rank}(\mathbf{I} - \mathbf{P}) = n - 1$ for every irreducible stochastic matrix \mathbf{P}_{nn} . Give an example to show that this need not be the case for reducible stochastic matrices.

8.4.4. Prove that the left-hand Perron vector for an irreducible stochastic matrix $\mathbf{P}_{n \times n}$ ($n > 1$) is given by

$$\pi^T = \frac{1}{\sum_{i=1}^n P_i} (P_1, P_2, \dots, P_n),$$

where P_i is the i^{th} principal minor determinant of order $n-1$ in $\mathbf{I} - \mathbf{P}$.
Hint: What is $[\text{adj}(\mathbf{A})]\mathbf{A}$ if \mathbf{A} is singular?

8.4.5. Let $\mathbf{P}_{n \times n}$ be an irreducible stochastic matrix, and let $\mathbf{Q}_{k \times k}$ be a principal submatrix of $\mathbf{I} - \mathbf{P}$, where $1 \leq k < n$. Prove that $\rho(\mathbf{Q}) < 1$.

8.4.6. Let $\mathbf{P}_{n \times n}$ be an irreducible stochastic matrix, and let $\mathbf{Q}_{k \times k}$ be a principal submatrix of $\mathbf{I} - \mathbf{P}$, where $1 \leq k < n$. Explain why \mathbf{Q} is an M-matrix as defined and discussed on p. 626.

8.4.7. Let $\mathbf{P}_{n \times n}$ ($n > 1$) be an irreducible stochastic matrix. Explain why all principal minors of order $1 \leq k < n$ in $\mathbf{I} - \mathbf{P}$ are positive.

8.4.8. Use the same assumptions that are used for the fail-safe system described in Example 8.4.5, but use three controls, A , B , and C , instead of two. Determine the mean time to failure starting with three proven controls, two proven but one untested control, and three untested controls.

8.4.9. A mouse is placed in one chamber of the box shown in Figure 8.4.1 on p. 688, and a cat is placed in another chamber. Each minute the doors to the chambers are opened just long enough to allow movement from one chamber to an adjacent chamber. Half of the time when the doors are opened, the cat doesn't leave the chamber it occupies. The same is true for the mouse. When either the cat or mouse moves, a door is chosen at random to pass through.

- (a) Explain why the cat and mouse must eventually end up in the same chamber, and determine the expected number of steps for this to occur.
- (b) Determine the probability that the cat will catch the mouse in chamber $\#j$ for each $j = 1, 2, 3$.

*Technical skill is mastery of complexity while
creativity is mastery of simplicity.*
— E. Christopher Zeeman (1925–)