MATH 104/184

2018W1 Midterm 1

Review Package

Solutions

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[15] **1**. **Short Problems**. Each question is worth 3 points. Put your answer in the box provided and show your work. No credit will be given for the answer without the correct accompanying work.

(a) Evaluate $\lim_{x \to -1} \frac{\sqrt{4x+20}-4}{x+1}$.

Solution: If you plug in $x = -1$, you get an expression $\frac{0}{0}$ which is undefined, so you need to manipulate the expression to find the limit.

Try getting rid of the radical:

$$\begin{aligned}
\lim_{x \to -1} \frac{\sqrt{4x+20}-4}{x+1} &= \lim_{x \to -1} \frac{\sqrt{4x+20}-4}{x+1} \cdot \frac{\sqrt{4x+20}+4}{\sqrt{4x+20}+4} \\
&= \lim_{x \to -1} \frac{4x+20-16}{(\sqrt{4x+20}+4)(x+1)} \\
&= \lim_{x \to -1} \frac{4}{(\sqrt{4x+20}+4)} \\
&= \frac{4}{8} \\
&= \frac{1}{2}
\end{aligned}$$

Answer:
$\frac{1}{2}$

(b) Evaluate $\lim_{x \to 2} \frac{2x-4}{x^2+x-6}$.

Solution: If you plug in $x = 2$, you get an expression $\frac{0}{0}$ which is undefined, so you need to manipulate the expression to find the limit.

Try finding a common factor. We know that $(x-2)$ is a factor of both top and bottom polynomials since 2 is a root of both.

$$\begin{aligned}
\lim_{x \to 2} \frac{2x-4}{x^2+x-6} &= \lim_{x \to 2} \frac{2(x-2)}{(x-2)(x+3)} \\
&= \lim_{x \to 2} \frac{2}{x+3} \\
&= \frac{2}{5}
\end{aligned}$$

Answer:
$\frac{2}{5}$

(c) Suppose $f(x)$ and $g(x)$ are continuous functions for all real numbers and $\lim_{x\to 2} f(x) = -2$, $\lim_{x\to -2} g(x) = 4$, and $\lim_{x\to 2} g(x) = 3$.

Evaluate $\lim_{x\to 2} \frac{f(x)}{2(g(x))^2 + 4}$.

Solution: We know that when the limits exist, then the limits of the sums are sums of the limits, and limits of products are products of the limits, and we can manipulate the expressions as if they were numbers. So, we can plug in the given limits of $\lim_{x\to 2} f(x)$ and $\lim_{x\to 2} g(x)$ into the unknown expression:

$$\begin{aligned}
\lim_{x\to 2} \frac{f(x)}{2(g(x))^2 + 4} &= \frac{\lim_{x\to 2} f(x)}{\lim_{x\to 2}(2(g(x))^2) + \lim_{x\to 2} 4} \\
&= \frac{\lim_{x\to 2} f(x)}{2\left(\lim_{x\to 2}(g(x))\right)^2 + \lim_{x\to 2} 4} \\
&= -\frac{2}{2(3^2) + 4} \\
&= -\frac{1}{11}
\end{aligned}$$

Answer:
$-\frac{1}{11}$

(d) Solve for x if $\ln 2^{2x+4} = \ln 5$.

Solution: We just manipulate the expression using the rules about ln. Remember that $\ln 2$ and $\ln 5$ are just numbers.

$$\begin{aligned}
\ln 2^{2x+4} = \ln 5 &\Rightarrow (2x+4)\ln 2 = \ln 5 \\
&\Rightarrow (2x+4) = \frac{\ln 5}{\ln 2} \\
&\Rightarrow x = \frac{1}{2}\Big(\frac{\ln 5}{\ln 2} - 4\Big)
\end{aligned}$$

Answer:
$\frac{1}{2}\Big(\frac{\ln 5}{\ln 2} - 4\Big)$

(e) Find the inverse function for $f(x) = \dfrac{1}{4x+3}$ and state where it has an inverse. Explain your answer.

Solution: The original function is one-one everywhere it is defined.

In order to find an inverse function, we can exchange x and y, and then solve for y.

$$\begin{aligned} x = \frac{1}{4y+3} &\Rightarrow 4y + 3 = \frac{1}{x} \\ &\Rightarrow 4y = \frac{1}{x} - 3 = \frac{1-3x}{x} \\ &\Rightarrow y = \frac{1-3x}{4x} \end{aligned}$$

This inverse function is defined everywhere except at $x \neq 0$.

Answer:
$f^{-1}(x) = \dfrac{1-3x}{4x}, x \neq 0$

Long Problems. In questions 2 - 6, show your work. No credit will be given for the answer without the correct accompanying work.

[10] **2.** Prove that the equation

$$x^3 - x^2 + 2x = 1 - 2\cos x$$

has a solution.

Solution: We should use IVT here. Recall that one application of IVT says that a continuous function has a root (i.e. some point a where $f(a) = 0$) in any interval whose endpoints has function values of opposite signs.
Let $f(x) = x^3 - x^2 + 2x - 1 + 2\cos x$. We want to show that $f(x) = 0$ has a solution. We will try to find a couple of points in the domain where the function takes opposite signs.
First, let's try 0 and see what we get.

$$f(0) = 0 - 0 + 0 - 1 + 2 \cdot 1 = 1 > 0.$$

Now we need a point where the function will be negative. We know that for any negative x, all the terms $x^3 - x^2 + 2x - 1$ will be negative, so let's try $x = -\pi/2$:

$$f\left(\frac{-\pi}{2}\right) = \left(\frac{-\pi}{2}\right)^3 - \left(\frac{-\pi}{2}\right)^2 + \left(\frac{-\pi}{2}\right) - 1 + 2 \cdot 0 < 0.$$

Since f is a continuous function on $[-\frac{\pi}{2}, 0]$ and $f\left(\frac{-\pi}{2}\right) < 0$ and $f(0) > 0$, there is a number c in $(-\frac{\pi}{2}, 0)$ such that $f(c) = 0$ by the Intermediate Value Theorem. This proves that the equation $x^3 - x^2 + 2x = 1 - 2\cos x$ has the solution $x = c$.

[10] **3.** Use the definition of the derivative as a limit to find $f'(4)$ for the following function. No marks will be given for the use of differentiation rules.

$$f(x) = \frac{x}{2x+5}.$$

Solution:

$$\begin{aligned}
f'(4) &= \lim_{x\to 4} \frac{f(x) - f(4)}{x-4} \\
&= \lim_{x\to 4} \left(\frac{x}{2x+5} - \frac{4}{13}\right) \cdot \frac{1}{x-4} \\
&= \lim_{x\to 4} \left(\frac{13x - 4\cdot(2x+5)}{(2x+5)\cdot 13}\right) \cdot \frac{1}{x-4} \\
&= \lim_{x\to 4} \frac{5x-20}{(2x+5)\cdot 13\cdot(x-4)} \\
&= \lim_{x\to 4} \frac{5}{(2x+5)\cdot 13} \\
&= \frac{5}{169}
\end{aligned}$$

Hence: $f'(4) = \dfrac{5}{169}$.

[15] **4.** You manufacture chocolate teapots. The demand for your product as a function of price is given by the equation $q(p) = 200 - \sqrt{p}$.

(a) What is your revenue as a function of p?

Solution: Revenue R is $R = p \cdot q$, so

$R = p \cdot q = 200p - p\sqrt{p} = 200p - p^{\frac{3}{2}}$.

(b) What is your revenue as a function of q?

Solution: We first have to express $p(q)$ in terms of q.

Since, $q = 200 - \sqrt{p}$,

$$\Rightarrow \sqrt{p} = 200 - q$$
$$\Rightarrow p = (200 - q)^2$$
$$\Rightarrow p(q) = q^2 - 400q + 40000$$

Again using that $R = p \cdot q$, we get:

$R(q) = p(q) \cdot q = (q^2 - 400q + 40000)q = q^3 - 400q^2 + 40000q$.

(c) You have have fixed costs of 360000 KPW (North Korean won) and the variable cost of producing teapots is q^3. What is your profit as a function of quantity? What are your break-even points?

Solution: We have that the Cost function is:

$C(q) = 360000 + q^3$.

Since the profit is $P(q) = R(q) - C(q)$,

$$\begin{aligned} P(q) &= q^3 - 400q^2 + 40000q - 360000 - q^3 \\ &= -400q^2 + 40000q - 360000 \\ &= -400(q^2 - 100q + 900) \\ &= -400((q-50)^2 - 1600) \end{aligned}$$

So the break-even points where the profit $P = 0$, are the roots of this quadratic:

$P = 0 \Rightarrow -400((q-50)^2 - 1600) = 0 \Rightarrow q - 50 = \pm\sqrt{1600} \Rightarrow q = 50 \pm 40$.

The break-even points are $q = 10$ and $q = 90$.

(d) At what price should you sell your chocolate teapots to make the maximum profit?

Solution: The maximum profit is at the peak of this parabolic profit graph, i.e. when $q = 50$. If you didn't complete the square in the previous step, you can also solve $\frac{dP}{dq} = 0 \Rightarrow 2q - 100 = 0 \Rightarrow q = 50$ to get the same answer.

Now you need to convert this q and express it in terms of the price p. So to get the maximum profit, you should set $p = p(50) = 50^2 - 400 \cdot 50 + 40000 = 22500$ KPW.

[10] **5.** Find the equation of the tangent line to the graph $f(x) = \dfrac{x}{\tan x + 1}$ at $x = 0$.

Solution: To find the slope of the tangent line, we need to find $f'(x)$ at $x = 0$.
Using the quotient rule, and derivatives for trig functions,

$$f'(x) = \frac{(\tan x + 1) \cdot 1 - x \cdot \sec^2 x}{(\tan x + 1)^2}.$$

Evaluating this at $x = 0$, and remembering the trig values and relations,

$$\sin 0 = 0, \quad \cos 0 = 1, \quad \tan x = \frac{\sin x}{\cos x}, \quad \sec x = \frac{1}{\cos x},$$

we have that the slope of the tangent line at $x = 0$ is:

$$f'(0) = \frac{(0+1) - 0}{(0+1)^2} = 1.$$

Next, since at $x = 0$, we have $f(0) = 0$, so using the point slope form of the line, we see that the equation of the tangent line at $x = 0$ is given by

$$y = x.$$

[10] **6.** Find numbers a and b that makes

$$f(x) = \begin{cases} \ln x + a & \text{if } x > 1 \\ x^2 + x - 2 & \text{if } 0 \leq x \leq 1 \\ 3x^3 - 4b\cos x & \text{if } x < 0 \end{cases}$$

$f(x)$ continuous for all real numbers. With these values, is $f(x)$ differentiable at $x = 0$? Is $f(x)$ differentiable at $x = 0$?

Solution: The piecewise function $f(x)$ is continuous on the intervals $(-\infty, 0) \cup (0, 1) \cup (1, \infty)$, since there it is given by standard functions that we know are continuous. In order for $f(x)$ to be continuous everywhere on the real line, we must check the function at $x = 0$ and at $x = 1$.

At $x = 0$, in order for $f(x)$ to be continuous, we must have

$$\lim_{x \to 0^-} 3x^3 - 4b\cos x = \lim_{x \to 0^+} x^2 + x - 2$$
$$\Rightarrow 0 - 4b = -2$$
$$\Rightarrow b = \frac{1}{2}.$$

At $x = 1$, in order for $f(x)$ to be continuous, we must have

$$\lim_{x \to 1^-} x^2 + x - 2 = \lim_{x \to 1^+} \ln x + a$$
$$\Rightarrow 1 + 1 - 2 = 0 + a$$
$$\Rightarrow a = 0.$$

So if $a = 0$ and $b = \frac{1}{2}$, then $f(x)$ is continuous for all real numbers.

At $x = 0$, in order for $f'(0)$ to exist, we must have

$$\lim_{x \to 0^-} (3x^3 - 2\cos x)' = \lim_{x \to 0^+} (x^2 + x - 2)'$$
$$\Rightarrow \lim_{x \to 0^-} (9x^2 + 2\sin x) = \lim_{x \to 0^+} (2x + 1)$$
$$\text{But } 0 \neq 1.$$

So $f(x)$ is not differentiable at $x = 0$.
At $x = 1$, in order for $f'(1)$ to exist, we must have

$$\lim_{x \to 1^-} (x^2 + x - 2)' = \lim_{x \to 1^+} (\ln x)'$$
$$\Rightarrow \lim_{x \to 1^-} (2x + 1) = \lim_{x \to 1^+} (\frac{1}{x})$$
$$\text{But } 3 \neq 1.$$

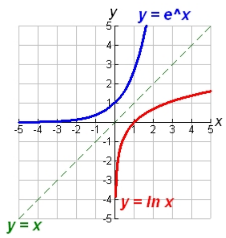
So $f(x)$ is not differentiable at $x = 1$ either.
So if $a = 0$ and $b = \frac{1}{2}$, then $f(x)$ is continuous for all real numbers. However, $f(x)$ is not differentiable at $x = 0$ or $x = 1$.

1. Cheat Sheet

1.1. **Preliminaries.**

- exp and log

  negative exponents $\rightarrow$ reciprocal
  fractional exponent $\rightarrow$ roots
  $b^x b^y = b^{x+y}$
  $\log(xy) = \log x + \log y$
  $(b^x)^y = b^{xy}$
  $\log(e^x) = x = e^{\log x}$ (where defined)
  $e^0 = 1, \log 1 = 0$
  graph:

- inverse functions, horizontal line test

1.2. **Business problem:**

- **quantity** q (number of widgets)
- **price** p (price per widget)
- **Revenue** R (money you receive when you sell q widgets for price p)
- **Cost** $C(q)$ (what it costs to make widgets)
  - **Fixed cost** F (costs that don't depend on the number of widgets)
  - **Variable cost** $V(q)$ (costs that depend on the number of widgets)
- **Break-Even Points** q such that $C(q) = R(q)$
- **Profit** P (the financial gain; the difference between amount earned and spent)
- **Demand** (equation that expresses the relation between quantity and price) - assumed linear for the MT1

In math:

$$R = pq$$

$$C(q) = F + V(q)$$

$$P = R - C$$

$$m = \frac{\Delta p}{\Delta q}$$

$$p - P_0 = m(q - Q_0)$$

given the data point (P_0, Q_0)

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1.3. **Limits.**

Limits may not exist for many reasons. If they do, define:

- **Left-hand limit:** $\lim_{x \to a^-} f(x) = K$
  $f(x)$ gets closer to K as x approaches a from the left, i.e. $x < a$
- **Right-hand limit:** $\lim_{x \to a^+} f(x) = L$
  $f(x)$ gets closer to L as x approaches a from the right, i.e. $x > a$
- **Limit:** $\lim_{x \to a} f(x) = L$
  $f(x)$ gets closer to L as x approaches a both from the right or the left

Some strategies

- If left limit and right limit exist but are not equal, the limit DNE.
- The limit does not have to be the value of the function.
- Rational function (polynomial divided by polynomial) - if you get 0/0, then factor
- Rationalize denominator
- Limit laws, e.g. the sum of limits is the limit of sums, etc
- If you know the function is continuous at the point, then we can 'plug in'. Continuous functions include polynomials, exp, sin, cos, rational functions where the denominator is not 0. Also sums, products, compositions of such functions.
- Some of the ways functions can go bad: dividing by zero, roots of negatives, log of $x \leq 0$, going outside the defined domain.
- If a rational function $\to \pm\infty$, decide sign by checking signs of numerator and denominator

1.4. **Continuity.**

- A function $f(x)$ is **continuous** at a if $\lim_{x \to a} f(x) = f(a)$

1.5. **Intermediate Value Theorem.**

Memorize this theorem!

- Let f be continuous on $[a, b]$. Then for any Y between $f(a)$ and $f(b)$, we can find a $c \in [a, b]$ with $f(c) = Y$.
  This just means f takes every value (y-coord) in between its endpoint values

1.6. **Derivatives.**

- Rate of change: think slope. Secants give average rate of change; tangents are instantaneous rates of change.

Memorize this definition!

- Let $a \in \mathbf{R}$ and $f(x)$ be defined on an open interval containing a. The **derivative** of $f(x)$ at x=a is defined by

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

  if the limit exists.

1.7. **Differentiation.**

$$\frac{d}{dx}x^n = nx^{n-1}$$

$$\frac{d}{dx}\Big(f(x) + g(x)\Big) = f'(x) + g'(x)$$

$$\frac{d}{dx}\Big(cf(x)\Big) = cf'(x)$$

$$\frac{d}{dx}\Big(f(x)g(x)\Big) = f'(x)g(x) + f(x)g'(x)$$

$$\frac{d}{dx}\Big(\frac{f(x)}{g(x)}\Big) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$$

$$\frac{d}{dx}e^x = e^x$$

- Finding equations of tangent lines