CMP

COMMERCE MENTORSHIP PROGRAM

MIDTERM REVIEW SESSION

MATH 100

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1 Asymptotics

Have you ever wondered what happens to a function at ∞ or $-\infty$? Does it blow up to infinity? Does it approach 0? What happens at a certain point like $x = 0$? Asymptotics will help us answer these questions, as well as provide an intuitive framework when evaluating limits.

1.1 Properties of Functions

Exponentials ($c > 1$):

- $c^x \to \infty$ (as $x \to \infty$) : $7^x \to \infty$ as $x \to \infty$
- $c^x \to 0$ (as $x \to -\infty$) : $2^x \to 0$ as $x \to -\infty$
- $c^x \to 1$ (as $x \to 0$) : $4^x \to 1$ as $x \to 0$

Exponentials ($0 < c < 1$):

- $c^x \to 0$ (as $x \to \infty$) : $(0.999)^x \to 0$ as $x \to \infty$
- $c^x \to \infty$ (as $x \to -\infty$) : $(0.5)^x \to \infty$ as $x \to -\infty$
- $c^x \to 1$ (as $x \to 0$) : $(0.2)^x \to 1$ as $x \to 0$

Trigonometric:

- $\tan(x) \to \infty \left(\text{as } x \to \frac{\pi}{2}\right)$
- $\tan(x) \to -\infty \left(\text{as } x \to \frac{-\pi}{2}\right)$

1.2 Asymptotics of Functions

Polynomials ($b > a \geq 0$):

- $x^b + x^a + c \sim x^b$ (as $x \to \infty$) : $x^{17} + x^{14} + 3 \sim x^{17}$ as $x \to \infty$
- $x^b + x^a + c \sim x^b$ (as $x \to -\infty$) : $x^5 + x^2 + 9 \sim x^5$ as $x \to -\infty$
- $x^b + x^a + c \sim c$ (as $x \to 0$) : $x^{33} + x^{22} + 12 \sim 12$ as $x \to 0$
- $x^b + x^a \sim x^a$ (as $x \to 0$) : $x^{22} + x^9 \sim x^9$ as $x \to 0$

Domination at infinity (as $x \to \infty, n > 0, c > 1$):

- $\log(x) \ll x^n \ll c^x \ll x^x$: $x^{20} + 7^x + 52 \sim 7^x$ as $x \to \infty$

Note that $\ll$ means "much less than"

1.3 Bounds of Functions

Trigonometric:

- $-1 \leq \sin(x) \leq 1$ for all x
- $-1 \leq \cos(x) \leq 1$ for all x

1.4 Practice Problems

1.4.1 Using Asymptotics

Discuss the behaviour of the following functions when x gets large, close to 0, and large but negative.

a) $e^x - \dfrac{1}{x^2} - x^4$

***Solution*:**

- As $x \to \infty$, $\dfrac{1}{x^2} \to 0$. Moreover, $x^4 \ll e^x$, so e^x will be the dominant term.

$$e^x - \frac{1}{x^2} - x^4 \sim e^x \text{ (as } x \to \infty).$$

- As $x \to -\infty$, $\dfrac{1}{x^2} \to 0$. Moreover, $e^x \to 0$, so x^4 will be the dominant term.

$$e^x - \frac{1}{x^2} - x^4 \sim -x^4 \text{ (as } x \to -\infty).$$

- As $x \to 0$, $e^x \to 1$ and $x^4 \to 0$. We also observe that $\dfrac{1}{x^2}$ blows up to infinity near 0 so it is the only dominant term.

$$e^x - \frac{1}{x^2} - x^4 \sim -\frac{1}{x^2} \text{ (as } x \to 0).$$

b) $\pi^x + 100000 \sin\left(x^{2022}\right)$

***Solution*:**

- As $x \to \infty$, $100000 \sin\left(x^{2022}\right)$ is bounded between -100000 and 100000. Simultaneously, π^x blows up to infinity, and will be dominant over any constant.

$$\pi^x + 100000 \sin\left(x^{2022}\right) \sim \pi^x \text{ (as } x \to \infty).$$

- As $x \to -\infty$, $\pi^x \to 0$. The dominant term will be $100000 \sin\left(x^{2022}\right)$.

$$\pi^x + 100000 \sin\left(x^{2022}\right) \sim 100000 \sin(x^{2022}) \text{ (as } x \to -\infty).$$

- As $x \to 0$, $\pi^x \to 1$ and $100000 \sin\left(x^{2022}\right) \to 0$.

$$\pi^x + 100000 \sin\left(x^{2022}\right) \sim 1 \text{ (as } x \to 0).$$

2 Evaluating Limits

A *limit* of a function $f(x)$ is the value it **approaches** as x approaches a. This value is often called L.

We can write it as $\lim_{x \to a} f(x) = L$.

To evaluate a limit, the simplest way is to plug in a. If the output is **finite**, ∞, or $-\infty$, you are done!

We also encounter limits that result in **indeterminate forms** ($\frac{0}{0}, \frac{\infty}{\infty}, 1^\infty, \infty^0, 0^0, \infty - \infty, 0 \cdot \infty$). Some may even result in the form $\frac{A}{0}$, where $A \neq 0$. All these forms are inconclusive and need to be manipulated, but don't worry, below are some methods to evaluate these types of limits!

2.1 Methods to Evaluate Limits

- Factor an expression to reveal a cancellation
- Use the conjugate of an expression
- Look at one-sided limits
- Find a common denominator and combine fractions
- Use trigonometric identities
- Replace the absolute value ($|f(x)| = -f(x)$ when $f(x) < 0$, and $|f(x)| = f(x)$ when $f(x) > 0$)
- Factor the highest denominator power out of the denominator and numerator
- Interpret the limit as a definition of a derivative (**see section 3**)
- Use asymptotics
- Be prepared to use any combination of the above!

2.2 One-sided Limits

When we evaluate one-sided limits, we need to look at $\lim_{x \to a^+} f(x)$ and $\lim_{x \to a^-} f(x)$.

$\lim_{x \to a^+} f(x)$ is the limit as x approaches a from the **right-hand side**.

$\lim_{x \to a^-} f(x)$ is the limit as x approaches a from the **left-hand side**.

If $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = L$ and both exist, then $\lim_{x \to a} f(x) = L$. Otherwise, $\lim_{x \to a} f(x)$ DNE.

2.3 Practice Problems

2.3.1 Limits at Finite Values

a) $\displaystyle\lim_{x\to 17} \frac{2x+\sqrt{|x-17|}}{\log(x)+\sqrt{17}}$

Solution:

- Although this limit contains an absolute value sign, one may just plug in 17.

$$\lim_{x\to 17} \frac{2x+\sqrt{|x-17|}}{\log(x)+\sqrt{17}} = \frac{2(17)+\sqrt{|17-17|}}{\log(17)+\sqrt{17}} = \frac{34}{\log(17)+\sqrt{17}}.$$

b) $\displaystyle\lim_{x\to -1} \frac{\sqrt{8-x}-3}{4-\sqrt{15-x}}$

Solution:

- We first take the conjugate of the top expression and bottom expression.

$$\lim_{x\to -1} \frac{\sqrt{8-x}-3}{4-\sqrt{15-x}} = \lim_{x\to -1} \frac{\sqrt{8-x}-3}{4-\sqrt{15-x}} \cdot \frac{(\sqrt{8-x}+3)(4+\sqrt{15-x})}{(4+\sqrt{15-x})(\sqrt{8-x}+3)} = \lim_{x\to -1} \frac{(8-x-9)(4+\sqrt{15-x})}{(16-(15-x))(\sqrt{8-x}+3)}.$$

- Then, we can simplify the expression.

$$\lim_{x\to -1} \frac{(8-x-9)(4+\sqrt{15-x})}{(16-(15-x))(\sqrt{8-x}+3)} = \lim_{x\to -1} \cancelto{-1}{\frac{(-1-x)}{(1+x)}} \cdot \frac{(4+\sqrt{15-x})}{(\sqrt{8-x}+3)} = \lim_{x\to -1} \frac{-(4+\sqrt{15-x})}{(\sqrt{8-x}+3)}.$$

- Finally, we plug in -1 to find the limit.

$$\frac{-\left(4+\sqrt{15-(-1)}\right)}{\left(\sqrt{8-(-1)}+3\right)} = \frac{-(4+4)}{(3+3)} = \frac{-4}{3}.$$

c) If $a \neq 0$, find $\displaystyle\lim_{x\to a} \frac{2x^2-2a^2}{x^4-a^4}$.

Solution:

- We will factor using a common factor and a difference of squares.

$$\lim_{x\to a} \frac{2x^2-2a^2}{x^4-a^4} = \lim_{x\to a} \frac{2\cancel{(x+a)(x-a)}}{\cancel{(x+a)(x-a)}(x^2+a^2)} = \lim_{x\to a} \frac{2}{x^2+a^2} = \frac{2}{a^2+a^2} = \frac{2}{2a^2} = \frac{1}{a^2}.$$

d) $\displaystyle\lim_{x\to -3} \frac{(\sqrt{-x+1})(x-9)}{x+3}$

***Solution*:**

- We notice that this limit is in the form $\frac{-24}{0}$ (constant divided by zero) when -3 is plugged in, so we must use one-sided limits.

Right-hand Limit: $x \to -3^+$

$$\lim_{x\to -3^+} \frac{\overbrace{(\sqrt{-x+1})(x-9)}^{\approx -24}}{\underbrace{(x+3)}_{>0}} = \frac{-24}{\text{very small positive number}} = -\infty.$$

Left-hand Limit: $x \to -3^-$

$$\lim_{x\to -3^-} \frac{\overbrace{(\sqrt{-x+1})(x-9)}^{\approx -24}}{\underbrace{(x+3)}_{<0}} = \frac{-24}{\text{very small negative number}} = \infty.$$

Conclusion:

$$\lim_{x\to -3^+} \frac{(\sqrt{-x+1})(x-9)}{x+3} \neq \lim_{x\to -3^-} \frac{(\sqrt{-x+1})(x-9)}{x+3}, \text{ so } \lim_{x\to -3} \frac{(\sqrt{-x+1})(x-9)}{x+3} \text{ DNE.}$$

2.3.2 Limits at Infinity

a) $\displaystyle\lim_{x\to\infty} \frac{|-x+7000|}{9x+\pi}$

***Solution*:**

- Since $|-x+7000| = -(-x+7000)$ for $-x+7000<0$, we can write the limit as follows.

$$\lim_{x\to\infty} \frac{|-x+7000|}{9x+\pi} = \lim_{x\to\infty} \frac{-(-x+7000)}{9x+\pi}.$$

- **Method 1** (Highest Power): We then factor out the highest denominator power (x) out of the numerator and denominator.

$$\lim_{x\to\infty} \frac{-(-x+7000)}{9x+\pi} = \lim_{x\to\infty} \frac{-\cancel{x}(-1+7000/x)}{\cancel{x}(9+x/\pi)} = \frac{-(-1+\cancel{7000/x}^{\,0})}{(9+\cancel{\pi/x}^{\,0})} = \frac{1+0}{9+0} = \frac{1}{9}.$$

- **Method 2** (Asymptotics): Observe that $-(-x+7000) \sim x$ as $x\to\infty$ and $9x+\pi \sim 9x$ as $x\to\infty$. We get the following conclusion:

$$\frac{-(-x+7000)}{9x+\pi} \sim \frac{x}{9x} = \frac{1}{9}.$$

b) $\lim_{x\to-\infty} \sqrt{3x^2+5x} + x\sqrt{3}$

***Solution*:**

- We will first take the conjugate of the expression.

$$\lim_{x\to-\infty} \sqrt{3x^2+5x}+x\sqrt{3} = \lim_{x\to-\infty} \frac{\sqrt{3x^2+5x}+x\sqrt{3}}{1}\cdot\frac{\sqrt{3x^2+5x}-x\sqrt{3}}{\sqrt{3x^2+5x}-x\sqrt{3}} = \lim_{x\to-\infty}\frac{\cancel{3x^2}+5x-\cancel{3x^2}}{\sqrt{3x^2+5x}-x\sqrt{3}}.$$

- **Method 1** (Highest Power): Then we factor x^2 out of $\sqrt{3x^2+5x}$, and let $\sqrt{x^2}=|x|$.

$$\lim_{x\to-\infty}\frac{5x}{\sqrt{3x^2+5x}-x\sqrt{3}} = \lim_{x\to-\infty}\frac{5x}{(\sqrt{x^2})\left(\sqrt{3+5/x}\right)-x\sqrt{3}} = \lim_{x\to-\infty}\frac{5x}{\left(|x|\sqrt{3+5/x}\right)-x\sqrt{3}}.$$

- Since $|x|=-x$ when $x<0$, we can the rewrite the limit, then factor $-x$ out of the denominator, and simplify as follows.

$$\lim_{x\to-\infty}\frac{5x}{\left(|x|\sqrt{3+5/x}\right)-x\sqrt{3}} = \lim_{x\to-\infty}\frac{5x}{\left(-x\sqrt{3+5/x}\right)-x\sqrt{3}} = \lim_{x\to-\infty}\frac{5\cancel{x}}{-\cancel{x}\left(\sqrt{3+5/x}+\sqrt{3}\right)}$$

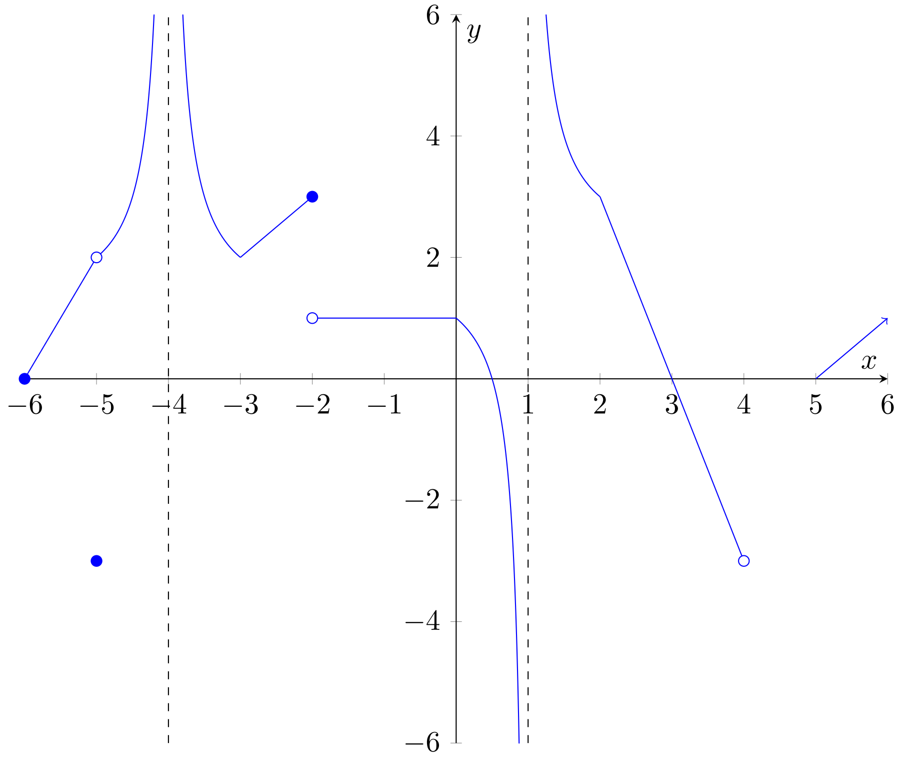
$$= \lim_{x\to-\infty}\frac{-5}{\sqrt{3+5/x}+\sqrt{3}} = \frac{-5}{\sqrt{3+\cancelto{0}{5/x}}+\sqrt{3}} = \frac{-5}{\sqrt{3+0}+\sqrt{3}} = \frac{-5}{2\sqrt{3}}.$$

- **Method 2** (Asymptotics): Observe that $3x^2+5x\sim 3x^2$ as $x\to\infty$. It follows that $\sqrt{3x^2}=|x|\sqrt{3}$, and using the fact that $|x|=-x$ when $x<0$, we get the following conclusion:

$$\frac{5x}{\sqrt{3x^2+5x}-x\sqrt{3}} \sim \frac{5x}{\sqrt{3x^2}-x\sqrt{3}} = \frac{5x}{|x|\sqrt{3}-x\sqrt{3}} = \frac{5x}{-x\sqrt{3}-x\sqrt{3}} = \frac{5x}{-2x\sqrt{3}} = \frac{-5}{2\sqrt{3}}.$$

2.3.3 Graphical Limits

Answer the following questions based on the graph of $f(x)$ below. The dotted lines are asymptotes.



a) $\lim_{x\to-6^+} f(x) = 0$	i) $\lim_{x\to-4^+} f(x) = \infty$	q) $\lim_{x\to1^+} f(x) = \infty$	y) $\lim_{x\to4^+} f(x) = \text{DNE}$
b) $\lim_{x\to-6^-} f(x) = \text{DNE}$	j) $\lim_{x\to-4^-} f(x) = \infty$	r) $\lim_{x\to1^-} f(x) = -\infty$	z) $\lim_{x\to4^-} f(x) = -3$
c) $\lim_{x\to-6} f(x) = \text{DNE}$	k) $\lim_{x\to-4} f(x) = \infty$	s) $\lim_{x\to1} f(x) = \text{DNE}$	aa) $\lim_{x\to4} f(x) = \text{DNE}$
d) $f(-6) = 0$	l) $f(-4) = \text{DNE}$	t) $f(1) = \text{DNE}$	bb) $f(4) = \text{DNE}$
e) $\lim_{x\to-5^+} f(x) = 2$	m) $\lim_{x\to-2^+} f(x) = 1$	u) $\lim_{x\to3^+} f(x) = 0$	cc) $\lim_{x\to6^+} f(x) = 1$
f) $\lim_{x\to-5^-} f(x) = 2$	n) $\lim_{x\to-2^-} f(x) = 3$	v) $\lim_{x\to3^-} f(x) = 0$	dd) $\lim_{x\to6^-} f(x) = 1$
g) $\lim_{x\to-5} f(x) = 2$	o) $\lim_{x\to-2} f(x) = \text{DNE}$	w) $\lim_{x\to3} f(x) = 0$	ee) $\lim_{x\to6} f(x) = 1$
h) $f(-5) = -3$	p) $f(-2) = 3$	x) $f(3) = 0$	ff) $f(6) = 1$

3 Continuity

In this section, we will review the definition of continuity, building on our use of limits past evaluation.

3.1 The Definition of Continuity

To informally define continuity for a function $f(x)$ at $x = a$, we have 3 conditions.

1. $f(a)$ **exists**.
2. $\lim_{x\to a} f(x)$ **exists** and is **finite** (This encodes the fact that $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x)$).
3. $f(a) = \lim_{x\to a} f(x)$.

Graphically speaking, we can visualize continuity with respect to a function being **unbroken** at a point. Remember the last time you drew $y = x^2$? How about $y = x$? You didn't need to lift off your pencil since they are continuous functions across their entire domain!

3.2 Practice Problems

3.2.1 Continuity

a) For what value(s) of b and c make $f(x)$ continuous at $x = c$?

$$f(x) = \begin{cases} \sqrt{-c+2}\,, & \text{if } x > c \\ c+4\,, & \text{if } x < c \\ b\,, & \text{if } x = c\,. \end{cases}$$

***Solution*:**

- We will first find c by setting $\lim_{x\to c^+} f(x) = \lim_{x\to c^-} f(x)$.

$$\begin{aligned} \lim_{x\to c^+} \sqrt{-c+2} &= \lim_{x\to c^-} c+4 \\ \sqrt{-c+2} &= c+4 \\ -c+2 &= c^2+8c+16 \\ 0 &= c^2+9c+14 \\ 0 &= (c+7)(c+2) \end{aligned}$$

From the equation, $c = -7$ or -2. However, $\sqrt{-(-7)+2} \neq -7+4$, so $c = -2$.

- To find b, we will set $f(c) = \lim_{x\to c} f(x)$.

$$b = f(-2) = \lim_{x\to -2} f(x) = -2+4 = 2.$$

b) For what value of p makes $f(x)$ continuous at $x = \frac{\pi}{4}$?

$$f(x) = \begin{cases} [\cos(x) - \sin(x)][\sec(2x)]\,, & \text{if } x \neq \frac{\pi}{4} \\ p\,, & \text{if } x = \frac{\pi}{4}. \end{cases}$$

***Solution*:**

- Since $\lim\limits_{x \to \frac{\pi}{4}^+} f(x) = \lim\limits_{x \to \frac{\pi}{4}^-} f(x)$, we can first use trig identities to evaluate $\lim\limits_{x \to \frac{\pi}{4}} f(x)$.

$$\lim_{x \to \frac{\pi}{4}} [\cos(x) - \sin(x)][\sec(2x)] = \lim_{x \to \frac{\pi}{4}} \frac{\cos(x) - \sin(x)}{\cos(2x)} = \lim_{x \to \frac{\pi}{4}} \frac{\cos(x) - \sin(x)}{\cos^2(x) - \sin^2(x)}$$

$$= \lim_{x \to \frac{\pi}{4}} \frac{\cancel{\cos(x) - \sin(x)}}{[\cos(x) + \sin(x)]\cancel{[\cos(x) - \sin(x)]}} = \lim_{x \to \frac{\pi}{4}} \frac{1}{\cos(x) + \sin(x)} = \frac{1}{\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}} = \frac{1}{\sqrt{2}}.$$

- To find p, we will set $f(\frac{\pi}{4}) = \lim\limits_{x \to \frac{\pi}{4}} f(x)$.

$$f(\tfrac{\pi}{4}) = p = \lim_{x \to \frac{\pi}{4}} f(x) = \frac{1}{\sqrt{2}}. \text{ Thus, } p = \frac{1}{\sqrt{2}}.$$

4 The Definition of a Derivative

4.1 Average and Instantaneous Rates of Change

The definition of a derivative isn't as scary as it seems. If we start working from the slope formula we learned in high school, things will seem more clear.

Recall that: Slope $= \dfrac{\text{rise}}{\text{run}} = \dfrac{y_2 - y_1}{x_2 - x_1} =$ **Average rate of change** between (x_1, y_1) and (x_2, y_2).

I'm sure we are more comfortable with this formula, but let's be more formal in notation! Remember that in function notation, $f(a)$ can mean the y-value at the x-coordinate, a. We can represent this as the point $(a, f(a))$, analogous to (x_1, y_1).

If we venture further - precisely h units right from a, we will be at the x-coordinate of $(a + h)$. At this x-coordinate, the y-value will be $f(a+h)$. We can represent this as the point $((a+h), f(a+h))$, analogous to (x_2, y_2).

Let's now revisit our slope formula with new notation. Since (x_1, y_1) is analogous to $(a, f(a))$, and (x_2, y_2) is analogous to $((a + h), f(a + h))$, we can modify the slope formula like this:

$$\text{Slope} = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(a+h) - f(a)}{(a+h) - (a)} = \frac{f(a+h) - f(a)}{h}.$$

Seem familiar? This is still a formula for the **average rate of change**, but remember we are in a Calculus course. Let's apply a *limit* and "close" the horizontal distance (h) between the two points to get a tangent line.

`Click here for a live visual of this "limiting" process!`

Definition of a Derivative: $\displaystyle\lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = f'(a)$, provided the limit exists.

This is the formula for the **instantaneous rate of change** and encodes the slope of the tangent line at $x = a$. Moreover, recall that there is an alternate definition as well!

Alternate Definition 1: $\displaystyle\lim_{x\to a} \frac{f(x) - f(a)}{x - a} = f'(a)$, provided the limit exists.

How are they related? Replace h with $x - a$ in the first formula to get the result! Notice that if you remove the limit sign, you get back another formula for the average rate of change.

Regardless of which instantaneous formula you use, notice that the aim is to "close" the distance between two points through a "limiting" process to get from a secant line to a tangent line.

Alternate Definition 2: $f(a + h) \approx f(a) + f'(a)h$ (as h gets very small)

If we algebraically manipulate the first definition by multiplying by h and adding $f(a)$ on both sides, we get the result above.

4.2 Practice Problems

4.2.1 Computing Derivatives With the Definition

a) Find $f'(a)$ if $f(x) = x^2$ using **all** definitions of a derivative.

Solution:

- We will first use $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ to find $f'(a)$.

$$\lim_{h\to 0} \frac{(a+h)^2 - a^2}{h} = \lim_{h\to 0} \frac{(\cancel{a^2} + 2ah + h^2) - \cancel{a^2}}{h} = \lim_{h\to 0} \frac{2ah + h^2}{h} = \lim_{h\to 0} 2a + h = 2a.$$

- Now we will use $\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$.

$$\lim_{x\to a} \frac{x^2 - a^2}{x-a} = \lim_{x\to a} \frac{\cancel{(x-a)}(x+a)}{\cancel{x-a}} = \lim_{x\to a} x + a = a + a = 2a.$$

- Finally we will use $f(a+h) \approx f(a) + f'(a)h$.

$$f(a+h) = a^2 + 2ah + h^2, \text{ so } f'(a) = 2a.$$

b) Find $f'(a)$ if $f(x) = \frac{2x}{3-x}$ using any definition of the derivative.

Solution:

- We will use $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ to find $f'(a)$, use a common denominator to simplify the numerator.

$$\lim_{h\to 0} \frac{\frac{2(a+h)}{3-(a+h)} - \frac{2a}{3-a}}{h} = \lim_{h\to 0} \frac{\frac{2(a+h)}{3-a-h} - \frac{2a}{3-a}}{h} \cdot \frac{(3-a-h)(3-a)}{(3-a-h)(3-a)}.$$

- Now we will use algebra to simplify the complex fraction and plug in $h = 0$.

$$\lim_{h\to 0} \frac{\frac{2(a+h)\cancel{(3-a-h)}(3-a)}{\cancel{3-a-h}} - \frac{(2a)(3-a-h)\cancel{(3-a)}}{\cancel{3-a}}}{h(3-a-h)(3-a)} = \lim_{h\to 0} \frac{2(a+h)(3-a) - [(2a)(3-a-h)]}{h(3-a-h)(3-a)}$$

$$= \lim_{h\to 0} \frac{(2a+2h)(3-a) - (6a - 2a^2 - 2ah)}{h(a^2 - 6a - 3h + ah + 9)} = \lim_{h\to 0} \frac{\cancel{6a} + 6h - \cancel{2a^2} - \cancel{2ah} - \cancel{6a} + \cancel{2a^2} + \cancel{2ah}}{h(a^2 - 6a - 3h + ah + 9)}$$

$$= \lim_{h\to 0} \frac{6\cancel{h}}{\cancel{h}(a^2 - 6a - 3h + ah + 9)} = \frac{6}{a^2 - 6a - 3(0) + a(0) + 9}$$

$$= \frac{6}{a^2 - 6a + 9} = \frac{6}{(a-3)^2}.$$

4.2.2 Finding the Tangent Slope at $x = a$ With the Definition

a) Find $f'(0)$ if $f(x) = \dfrac{x}{\sqrt{x^2+4}}$ using any definition. What does $f'(0)$ mean?

***Solution*:**

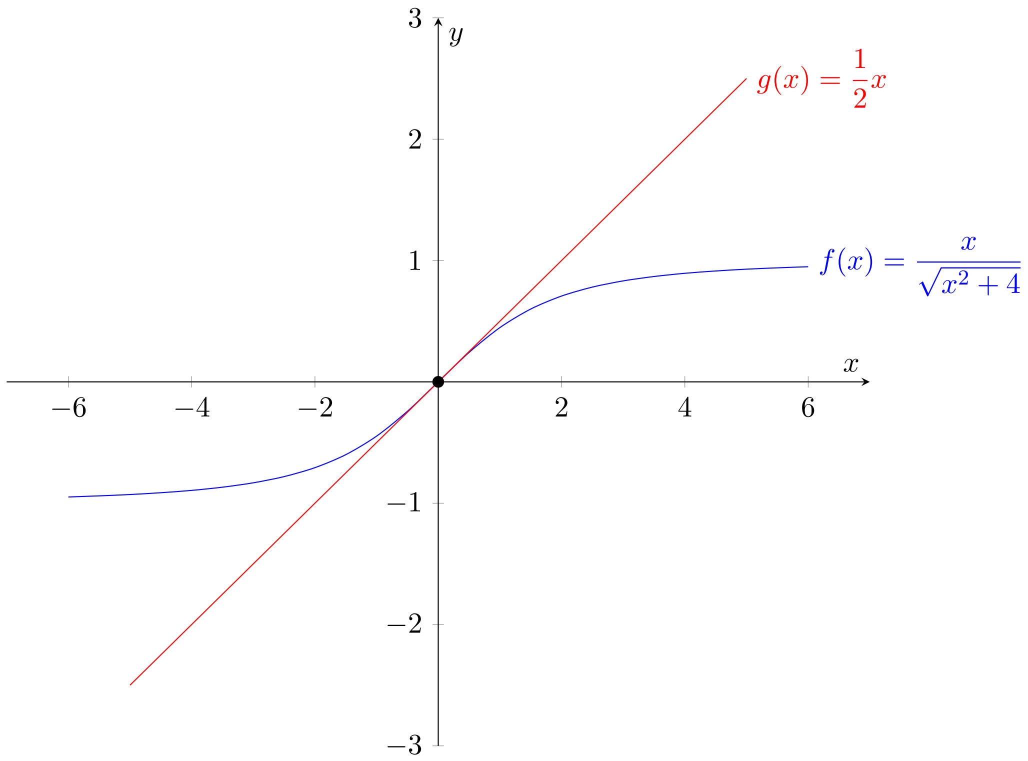
- We will use $\lim\limits_{h\to 0} \dfrac{f(a+h)-f(a)}{h}$ to find $f'(a)$. To save some hassle, first plug in $a = 0$!

$$\lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{\frac{0+h}{\sqrt{(0+h)^2+4}} - \frac{0}{\sqrt{(0)^2+4}}}{h} = \lim_{h\to 0} \frac{\frac{h}{\sqrt{h^2+4}} - 0}{h} = \lim_{h\to 0} \frac{\cancel{h}}{\cancel{h}\sqrt{h^2+4}}.$$

- Then we take the limit and plug in $h = 0$ to find $f'(0)$.

$$\lim_{h\to 0} \frac{1}{\sqrt{h^2+4}} = \frac{1}{\sqrt{0+4}} = \frac{1}{2}.$$

$f'(0) = \dfrac{1}{2}$ means that the tangent slope to the graph of $f(x) = \dfrac{x}{\sqrt{x^2+4}}$ at $x = 0$ is $\dfrac{1}{2}$.



b) Find $f'(1)$ if $f(x) = \sqrt{5+x^2}$ using any definition. What does $f'(1)$ mean?

Solution:

- Let's use $\lim\limits_{x\to a} \frac{f(x)-f(a)}{x-a}$ this time to find $f'(a)$. Remember to first plug in $a = 1$!

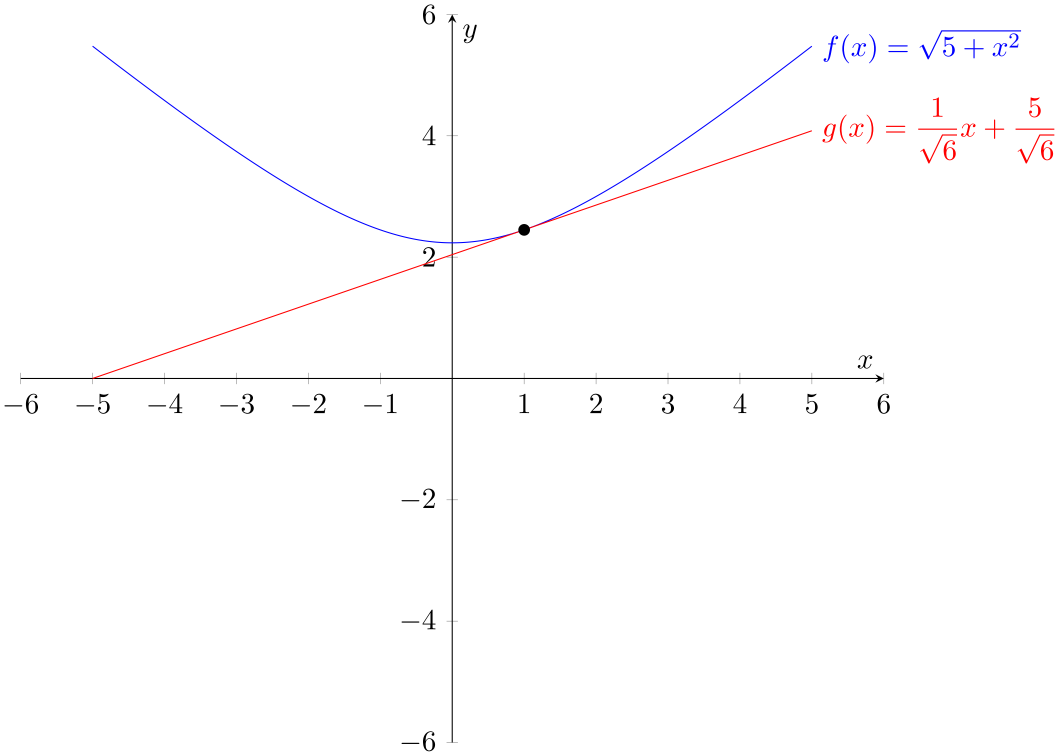
$$\lim_{x\to 1} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1} \frac{\sqrt{5+x^2}-\sqrt{5+1^2}}{x-1} = \lim_{x\to 1} \frac{\sqrt{5+x^2}-\sqrt{6}}{x-1}.$$

- To solve the limit, we then take the conjugate of the numerator, simplify, then factor.

$$\lim_{x\to 1} \frac{\sqrt{5+x^2}-\sqrt{6}}{x-1} \cdot \frac{\sqrt{5+x^2}+\sqrt{6}}{\sqrt{5+x^2}+\sqrt{6}} = \lim_{x\to 1} \frac{5+x^2-6}{(x-1)(\sqrt{5+x^2}+\sqrt{6})} = \lim_{x\to 1} \frac{x^2-1}{(x-1)(\sqrt{5+x^2}+\sqrt{6})}$$

$$= \lim_{x\to 1} \frac{\cancel{(x-1)}(x+1)}{\cancel{(x-1)}(\sqrt{5+x^2}+\sqrt{6})} = \lim_{x\to 1} \frac{x+1}{\sqrt{5+x^2}+\sqrt{6}} = \frac{1+1}{\sqrt{5+(1)^2}+\sqrt{6}} = \frac{2}{2\sqrt{6}} = \frac{1}{\sqrt{6}}.$$

$f'(1) = \dfrac{1}{\sqrt{6}}$ means that the tangent slope to the graph of $f(x) = \sqrt{5+x^2}$ at $x = 1$ is $\dfrac{1}{\sqrt{6}}$.



5 Derivative Rules and Tangent Lines

The definition of a derivative is inefficient at calculating derivatives. Though it should not be neglected, there are many rules that help speed up derivative calculations. We shall review these rules, and continue our discussion of $f'(a)$ with tangent lines and their use in linear approximations.

5.1 Derivative Rules

$\frac{d}{dx}\ c = 0$	$\frac{d}{dx}\ \cos(x) = -\sin(x)$
$\frac{d}{dx}\ x^n = nx^{n-1}$	$\frac{d}{dx}\ \tan(x) = \sec^2(x)$
$\frac{d}{dx}\ cx^n = cnx^{n-1}$	$\frac{d}{dx}\ \sec(x) = \sec(x)\tan(x)$
$* \ \frac{d}{dx}\ \log(x) = \frac{1}{x},\ x > 0$	$\frac{d}{dx}\ \csc(x) = -\csc(x)\cot(x)$
$\frac{d}{dx}\ \log_c(x) = \frac{1}{x\log(c)},\ x > 0,\ c > 0,\ c \neq 1$	$\frac{d}{dx}\ \cot(x) = -\csc^2(x)$
$\frac{d}{dx}\ e^x = e^x$	$\frac{d}{dx}\ \sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$
$\frac{d}{dx}\ c^x = c^x\log(c),\ c > 0$	$\frac{d}{dx}\ \cos^{-1}(x) = \frac{-1}{\sqrt{1-x^2}}$
$\frac{d}{dx}\ \sin(x) = \cos(x)$	$\frac{d}{dx}\ \tan^{-1}(x) = \frac{1}{1+x^2}$

Sum/Difference Rule: $\frac{d}{dx}\ [f(x) \pm g(x)] = f'(x) \pm g'(x)$

Product Rule: $\frac{d}{dx}\ f(x)g(x) = f'(x)g(x) + f(x)g'(x)$

Quotient Rule: $\frac{d}{dx}\ \frac{f(x)}{g(x)} = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$

Chain Rule: $\frac{d}{dx}\ f(g(x)) = f'(g(x))g'(x)$

* $\log(x)$ refers to the natural logarithm $\ln(x)$ while $\log_c(x)$ refers to a logarithm with base c, where c is a constant.

5.2 Tangent Lines

In section 3, we mentioned that $f'(a)$ encoded the slope of the tangent line at $x = a$ for a function, $f(x)$. What is the equation of that tangent line?

Let's start with a formula from high school - the equation of a line in point-slope form.

$$y - y_1 = m(x - x_1)$$

To make this more formal, let's replace x_1 with a and y_1 with $f(a)$ as done in section 3. Furthermore, let's replace m with $f'(a)$, another measure for slope. We now get the following equation:

$$y - f(a) = f'(a)(x - a)$$

Moving $f(a)$ onto the right-hand side, we get the following conclusion at $x = a$.

Equation of a Tangent Line: $y = f'(a)(x - a) + f(a)$.

When solving tangent line questions, I like splitting the equation into 3 separate components. That way, I can see what I know, what I don't know, and what I want to find.

1. $f'(a)$: the slope of the tangent line at $x = a$.
2. $(a, f(a))$: the coordinates of the point that exists on the tangent line AND $f(x)$.
3. (x, y): any other point (if needed) on the tangent line that may or may not be on $f(x)$.

5.3 Linear Approximation

Suppose we wanted to find $\sqrt{7}$ or even $(26)^{\frac{1}{3}}$ without a calculator. Though it is hard to get the true value, we could make a rough approximation.

Linearization of $f(x)$ **at** $x = a$: $L(x) = f'(a)(x - a) + f(a)$

Think about x as the number you want to estimate, and a as a value that results in a "nice" number when plugged into $f(x)$.

Example: Approximate $\sqrt{7}$

Choose $f(x) = \sqrt{x}$, $a = 9$ and $x = 7$. Note that $f'(a) = \dfrac{1}{2\sqrt{a}}$.

$L(7) = \left(\dfrac{1}{2\sqrt{9}}\right)(7 - 9) + \sqrt{9} = \dfrac{8}{3} \approx 2.667$

The real value of $\sqrt{7}$ is approximately 2.646, so we have a pretty good approximation above.

5.4 Practice Problems

5.4.1 Take the Derivative

a) Find $\frac{d}{dx}\left(e^e + \pi^\pi + e^\pi + \pi^e + e^2 + \pi^2\right)$.

***Solution*:**

- Note that every term is a constant, so the derivative of each term is zero.

$$\frac{d}{dx}\left(e^e + \pi^\pi + e^\pi + \pi^e + e^2 + \pi^2\right) = 0+0+0+0+0+0 = 0.$$

b) Find $\frac{d}{dx}\frac{x}{(x+1)^2}$.

***Solution*:**

- Since this is in the form $\frac{f(x)}{g(x)}$, we will use the quotient rule.

$$\frac{d}{dx}\frac{x}{(x+1)^2} = \frac{(x+1)^2\cdot\left[\frac{d}{dx}(x)\right] - x\cdot\left[\frac{d}{dx}(x+1)^2\right]}{[(x+1)^2]^2} = \frac{[(x+1)^2(1)] - [(x)(2(x+1))]}{(x+1)^4}$$

$$= \frac{(x^2+2x+1)-(2x^2+2x)}{(x+1)^4} = \frac{x^2+\cancel{2x}+1-2x^2-\cancel{2x}}{(x+1)^4} = \frac{-x^2+1}{(x+1)^4} = \frac{-(x+1)(x-1)}{(x+1)^4} = -\frac{(x-1)}{(x+1)^3}.$$

c) Find $\frac{d}{dx}\log\left(\sec\left(e^{x^2} + x^{2^e} + 2^{e^x}\right)\right)$.

***Solution*:**

- We need to use the chain rule three times.

$$\frac{d}{dx}\log\left(\sec\left(e^{x^2} + x^{2^e} + 2^{e^x}\right)\right) = \frac{1}{\sec\left(e^{x^2} + x^{2^e} + 2^{e^x}\right)}\cdot\left[\frac{d}{dx}\sec\left(e^{x^2} + x^{2^e} + 2^{e^x}\right)\right]$$

$$= \frac{1}{\cancel{\sec\left(e^{x^2} + x^{2^e} + 2^{e^x}\right)}}\cdot\left[\cancel{\sec\left(e^{x^2} + x^{2^e} + 2^{e^x}\right)}\tan\left(e^{x^2} + x^{2^e} + 2^{e^x}\right)\right]\cdot\left[\frac{d}{dx}\left(e^{x^2} + x^{2^e} + 2^{e^x}\right)\right]$$

$$= \tan\left(e^{x^2} + x^{2^e} + 2^{e^x}\right)\left(e^{x^2}\cdot\left[\frac{d}{dx}x^2\right] + (2^e)x^{(2^e-1)} + 2^{e^x}\log(2)\cdot\left[\frac{d}{dx}e^x\right]\right)$$

$$= \tan\left(e^{x^2} + x^{2^e} + 2^{e^x}\right)\left((2x)e^{x^2} + (2^e)x^{(2^e-1)} + (2^{e^x})(e^x)\log(2)\right).$$

d) Find $\frac{d}{dx}\,(x^2+3)^3(2x+5)^4$.

***Solution*:**

- We need to use the product rule with $f(x) = (x^2+3)^3$ and $g(x) = (2x+5)^4$, along with the chain rule.

$$\frac{d}{dx}\,(x^2+3)^3(2x+5)^4 = \left[\frac{d}{dx}\,(x^2+3)^3\right]\cdot(2x+5)^4 + (x^2+3)^3\cdot\left[\frac{d}{dx}\,(2x+5)^4\right]$$

$$= 3(x^2+3)^2\cdot\left[\frac{d}{dx}\,(x^2+3)\right](2x+5)^4 + (x^2+3)^3\cdot 4(2x+5)^3\cdot\left[\frac{d}{dx}\,(2x+5)\right]$$

$$= 3(x^2+3)^2(2x)(2x+5)^4 + 4(x^2+3)^3(2x+5)^3(2)$$

$$= (6x)(x^2+3)^2(2x+5)^4 + 8(x^2+3)^3(2x+5)^3.$$

- To clean this up a bit, we will factor out $2(x^2+3)^2(2x+5)^3$.

$$(6x)(x^2+3)^2(2x+5)^4 + 8(x^2+3)^3(2x+5)^3 = [2(x^2+3)^2(2x+5)^3][(3x)(2x+5)+4(x^2+3)]$$

$$= [2(x^2+3)^2(2x+5)^3][6x^2+15x+4x^2+12] = 2(x^2+3)^2(2x+5)^3(10x^2+15x+12).$$

5.4.2 More Derivatives

a) Find $\lim_{x\to 1}\frac{\log(x)}{x-1}$.

***Solution*:**

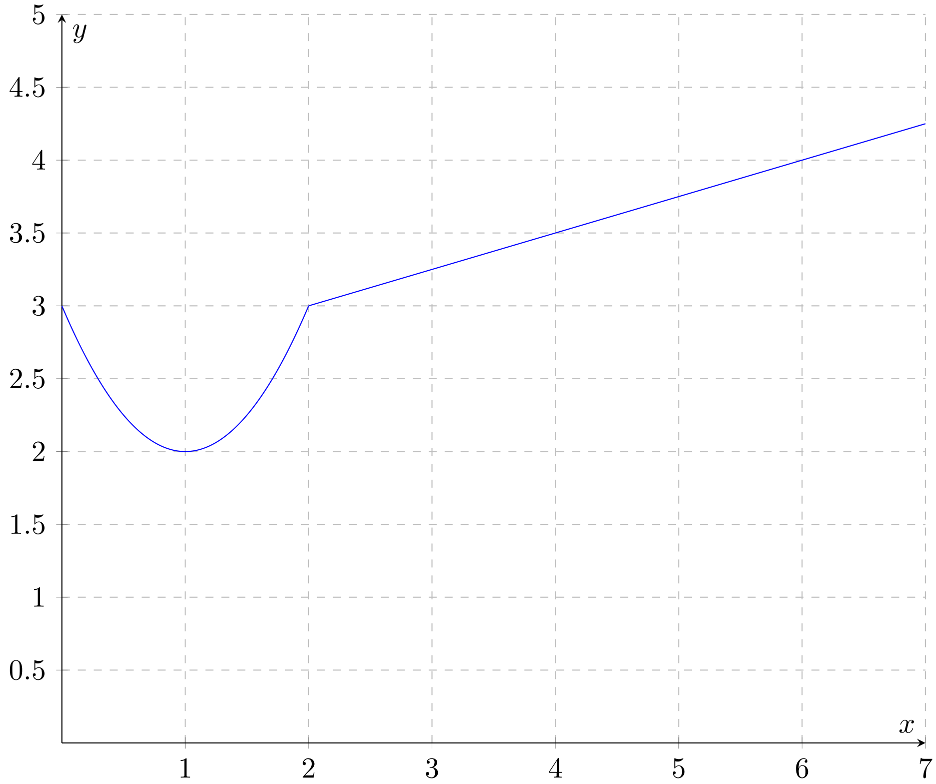
- We first interpret the limit as the definition of the derivative using $\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$ with $f(x) = \log(x)$ and $a = 1$.

$$\lim_{x\to a}\frac{f(x)-f(a)}{x-a} = \lim_{x\to 1}\frac{\log(x)-\log(1)}{x-1} = \lim_{x\to 1}\frac{\log(x)-0}{x-1} = \lim_{x\to 1}\frac{\log(x)}{x-1}.$$

- We cannot solve this limit using algebra, but note that $\lim_{x\to 1}\frac{\log(x)-\log(1)}{x-1}$ is the same as $f'(1)$ with $f(x) = \log(x)$.

$$\frac{d}{dx}\log(x) = \frac{1}{x},\text{ so } f'(1) = \frac{1}{1} = 1.$$

b) Below is the graph of $f(x)$. If $g(x) = \dfrac{f(x)}{x^2} + f(2x-11)$, find $g'(6)$.



Solution:

- First take the derivative of $g(x)$ using the quotient and chain rule.

$$\frac{d}{dx}\left(\frac{f(x)}{x^2} + f(2x-11)\right) = \frac{x^2 \cdot \left[\frac{d}{dx} f(x)\right] - f(x) \cdot \left[\frac{d}{dx}(x^2)\right]}{(x^2)^2} + f'(2x-11) \cdot \left[\frac{d}{dx}(2x-11)\right]$$

$$= \frac{[x^2 f'(x)] - [2x f(x)]}{x^4} + 2f'(2x-11).$$

- Then we plug in $x = 6$ to find $g'(6)$.

$$g'(6) = \frac{[(6)^2 f'(6)] - [(2)(6) f(6)]}{(6)^4} + 2f'(2(6)-11) = \frac{[(36)(\frac{1}{4})] - [(12)(4)]}{1296} + 2f'(1)$$

$$= \frac{9-48}{1296} + (2)(0) = -\frac{39}{1296} + 0 = -\frac{13}{432}.$$

When evaluating $g'(6)$ above, note that $f(6)$ can be directly read off the graph as 4. To find $f'(1)$, imagine a tangent line to $f(x)$ at $x = 1$. It's a horizontal line, so $f'(1) = 0$. For $f'(6)$, do something similar, and you will realize that the tangent slope at $x = 6$ is just the slope of the line in the graph. Thus, $f'(6) = \frac{1}{4}$.

5.4.3 Finding Tangent Lines

a) Find all values of b such that the line $y = bx + b$ is tangent to the graph of $x^2 + 2x + 5$.

Solution:

- Since the equation of the tangent line is $y = bx + b$ we can first find $f'(a)$, $f(a)$, and a in terms of b.

Finding $f'(a)$:

The equation of the tangent line is $bx + b$, which denotes a slope of b, so $f'(a)$ must be b.

Finding a:

From the last part, $f'(a) = b$. Since $f(x) = x^2 + 2x + 5$ and $f'(x) = 2x + 2$, $f'(a)$ is $2a + 2$. We set this expression for $f'(a)$ equal to b and solve for a.

$$\begin{aligned} 2a + 2 &= b \\ 2a &= b - 2 \\ a &= \frac{b}{2} - 1 \end{aligned}$$

Finding $f(a)$:

Since $a = \frac{b}{2} - 1$, $f(a) = f\left(\frac{b}{2} - 1\right)$.

If $f(x) = x^2 + 2x + 5$, $f\left(\frac{b}{2} - 1\right) = \left(\frac{b}{2} - 1\right)^2 + 2\left(\frac{b}{2} - 1\right) + 5 = \frac{1}{4}b^2 + 4$.

- Now that we have found $f'(a)$, a, and $f(a)$ in terms of b, use the equation of a tangent line to solve for b. Note that $y = bx + b$.

$$\begin{aligned} y &= f'(a)(x - a) + f(a) \\ bx + b &= b\left[x - \left(\frac{b}{2} - 1\right)\right] + \left(\frac{1}{4}b^2 + 4\right) \\ bx + b &= bx - b\left(\frac{b}{2} - 1\right) + \frac{1}{4}b^2 + 4 \\ \cancel{bx} + b &= \cancel{bx} - \frac{1}{2}b^2 + b + \frac{1}{4}b^2 + 4 \\ \cancel{b} &= -\frac{1}{4}b^2 + \cancel{b} + 4 \\ 0 &= -\frac{1}{4}b^2 + 4 \\ 16 &= b^2, \text{ so } b \text{ is } 4 \text{ or } -4. \end{aligned}$$

5.4.4 Approximating Values

Approximate the following three values using linear approximation.

a) $\sqrt[4]{85}$

***Solution*:**

- Let $f(x) = x^{\frac{1}{4}}$, $x = 85$, $a = 81$. $f'(a)$ will be $\frac{1}{4\sqrt[4]{a^3}}$.

$$L(85) = \frac{1}{4\sqrt[4]{81^3}}(85 - 81) + \sqrt[4]{81} = \frac{1}{3^3}(4) + 3 = \frac{85}{27}.$$

b) $\log(1.2)$

***Solution*:**

- Let $f(x) = \log(x)$, $x = 1.2$, $a = 1$. $f'(a)$ will be $\frac{1}{a}$.

$$L(1.2) = \frac{1}{1}(1.2 - 1) + \log(1) = 0.2 + 0 = 0.2.$$

c) $\cos(0.3)$

***Solution*:**

- Let $f(x) = \cos(x)$, $x = 0.3$, $a = 0$. $f'(a)$ will be $-\sin(a)$.

$$L(0.3) = -\sin(0)(0.3 - 0) + \cos(0) = 1.$$

d) Let $f(x)$ be a differentiable function such that the linearization of $f(x)$ at $a = 3$ is $5x + 7$. What is $f'(3)$?

***Solution*:**

- If $L(x) = f'(a)(x - a) + f(a)$, it follows that $L'(x) = f'(a)$.

$$L'(x) = \frac{d}{dx}(5x + 7) = 5, \text{ so } f'(3) = 5.$$

6 Differentiability

6.1 Continuity's Older Cousin

Differentiability implies continuity. For $f(x)$ to be **differentiable** at $x = a$, one major condition needs to be satisfied.

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \quad \left(\text{or } \lim_{x\to a} \frac{f(x) - f(a)}{x-a}\right) \textbf{ exists}.$$

When does the limit exist? Look at left and right-sided limits!

$\displaystyle\lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$ exists when $\displaystyle\lim_{h\to 0^+} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0^-} \frac{f(a+h) - f(a)}{h}$, and both exist.

$\displaystyle\lim_{x\to a} \frac{f(x) - f(a)}{x-a}$ exists when $\displaystyle\lim_{x\to a^+} \frac{f(x) - f(a)}{x-a} = \lim_{x\to a^-} \frac{f(x) - f(a)}{x-a}$, and both exist.

One important comment: If $f'(a)$ exists, $f(a)$ must exist since differentiatiability implies continuity!

However, if we only know that $f(a)$ exists, $f'(a)$ may or may not exist. Continuity does not imply differentiability - think $f(x) = |x|$ at $x = 0$.

6.2 Practice Problems

6.2.1 Is it Differentiable?

a) Determine if $f(x)$ is differentiable at $x = 0$ using the definition of a derivative.

$$f(x) = \begin{cases} 5x^5 + 4x^4 + 3x^3 + 2x^2 + x\,, & \text{if } x \le 0 \\ x^{2022}\cos\left(\dfrac{1}{x^{2022}}\right)\,, & \text{if } x > 0. \end{cases}$$

***Solution*:**

- We need to check the one-sided limits of $\displaystyle\lim_{x\to a} \frac{f(x) - f(a)}{x-a}$ at $a = 0$.

Right-hand Limit: $x \to 0^+$

$$\lim_{x\to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^+} \frac{5x^5 + 4x^4 + 3x^3 + 2x^2 + x - 0}{x}$$

$$= \lim_{x\to 0^+} 5x^4 + 4x^3 + 3x^2 + 2x + 1 = 0 + 0 + 0 + 0 + 1 = 1.$$

(Continued on Next Page)

Left-hand Limit: $x \to 0^-$

$$\lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-} \frac{x^{2022}\cos\left(\dfrac{1}{x^{2022}}\right) - 0}{x} = \lim_{x\to 0^-} x^{2021}\cos\left(\frac{1}{x^{2022}}\right).$$

- **Method 1** (Asymptotics): Observe that x^{2021} approaches 0 when $x \to 0$. Moreover, $\cos\left(\dfrac{1}{x^{2022}}\right)$ is bounded between -1 and 1. So,

$$x^{2021}\cos\left(\frac{1}{x^{2022}}\right) \sim 0 \text{ (as } x \to 0).$$

- **Method 2** (Squeeze Theorem): We can use the Squeeze Theorem to evaluate the limit.

We know $-1 \le \cos\left(\dfrac{1}{x^{2022}}\right) \le 1$.

Multiplying by x^{2021}, we get $-x^{2021} \le x^{2021}\cos\left(\dfrac{1}{x^{2022}}\right) \le x^{2021}$.

Furthermore, $\lim_{x\to 0^-} -x^{2021} = \lim_{x\to 0^-} x^{2021} = 0$.

Thus, $\lim_{x\to 0^-} x^{2021}\cos\left(\dfrac{1}{x^{2022}}\right) = 0$ by the Squeeze Theorem.

Conclusion:

$$\lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} \ne \lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0}, \text{ so } f(x) \text{ is not differentiable at } x = 0.$$