CMP

COMMERCE MENTORSHIP PROGRAM

FINAL REVIEW SESSION

# MATH 100

Prepared by: Samuel Cheng

# Final Review Package

By: Samuel Cheng

## Contents

# 1 More on Differentiation

## 1.1 Implicit Differentiation

Most of the time, you have been given $f(x)$ as an explicit function. For example, $f(x) = x^2 + 3x + 4$.

What if $f(x)$ was an implicit function? For example, $xf(x) + f(x)^2 + x = 2$.

Note that $f(x) = y$, so let's rewrite this in a more friendlier form.

$$xf(x) + f(x)^2 + x = 2 \Rightarrow xy + y^2 + x = 2$$

To differentiate, we do the following:

1. Treat $y$ as a variable and apply differentiation rules on both sides
2. Whenever you touch $y$ during differentiation, put a $\frac{dy}{dx}$ (or $y'$ or $f'(x)$)
3. Isolate for $\frac{dy}{dx}$ (or $y'$ or $f'(x)$)

Note that these 3 steps are more specific to when $y$ is a function of $x$. We could have $x$ as a function of $t$, and would need $\frac{dx}{dt}$ instead of $\frac{dy}{dx}$ when differentiating. More generally, the bottom letter you see is the independent variable and the top variable you see is a dependent variable.

**Example**: Find $\frac{dy}{dx}$ if $xy + y^2 + x = 2$.

First differentiate both sides to obtain

$$\begin{aligned}\frac{d}{dx}\left(xy + y^2 + x\right) &= \frac{d}{dx}\,2 \\ y + x \cdot \frac{dy}{dx} + 2y \cdot \frac{dy}{dx} + 1 &= 0.\end{aligned}$$

Group the terms with $\frac{dy}{dx}$ onto one side to get

$$(x + 2y) \cdot \frac{dy}{dx} = -1 - y.$$

Solve for $\frac{dy}{dx}$ by dividing $x + 2y$ on both sides

$$\frac{dy}{dx} = \frac{-1 - y}{x + 2y} = -\frac{y + 1}{x + 2y}.$$

## 1.2 Logarithmic Differentiation

Suppose we wanted to differentiation $y = \dfrac{2x^2(x-2)(x-3)}{\sin(x)(x-4)}$. How would we do it?

If we used quotient rule, it will certainly "blow up", so what if we could separate each expression using a "log" operation?

$$\textbf{Logarithmic Differentiation}: \frac{d}{dx}f(x) = f(x) \cdot \frac{d}{dx}\log(f(x))$$

Simply put, you do the following:

1. Take the natural logarithm of both sides
2. Differentiate both sides
3. Multiply both sides by $y$ (or $f(x)$ if you like to call $y$ that way)

**Example**: Differentiate $y = \dfrac{2x^2(x-2)(x-3)}{\sin(x)(x-4)}$.

First take the natural logarithm of both sides

$$\log(y) = \log\left(\frac{2x^2(x-2)(x-3)}{\sin(x)(x-4)}\right) = \log(2x^2) + \log(x-2) + \log(x-3) - \log(\sin(x)) - \log(x-4)$$

Differentiate both sides (Note how we used implicit differentiation on the left-hand side)

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{1}{2x^2} \cdot (4x) + \frac{1}{x-2} + \frac{1}{x-3} - \frac{1}{\sin(x)} \cdot \cos(x) - \frac{1}{x-4}$$

Multiply both sides by $y$ to get

$$\frac{dy}{dx} = y \cdot \left(\frac{2}{x} + \frac{1}{x-2} + \frac{1}{x-3} - \cot(x) - \frac{1}{x-4}\right)$$

$$\frac{dy}{dx} = \left(\frac{2x^2(x-2)(x-3)}{\sin(x)(x-4)}\right) \cdot \left(\frac{2}{x} + \frac{1}{x-2} + \frac{1}{x-3} - \cot(x) - \frac{1}{x-4}\right).$$

Once again, remember that $\dfrac{dy}{dx}$ is just another way of saying $f'(x)$ in this scenario.

## 1.3 Practice Problems

### 1.3.1 Using Implicit Differentiation

**a)** Find the tangent line to the graph of $x^3 + y^3 = 8xy$ at $(4, 4)$.

***Solution*:**

- First differentiate to find an expression in terms of $\frac{dy}{dx}$.

$$\frac{d}{dx}x^3 + y^3 = \frac{d}{dx}8xy \Rightarrow 3x^2 + 3y^2 \cdot \frac{dy}{dx} = 8y + 8x \cdot \frac{dy}{dx}$$

- Plug in $x = 4$ and $y = 4$ to find $\frac{dy}{dx}$.

$$3(4)^2 + 3(4)^2 \cdot \frac{dy}{dx} = 8(4) + 8(4) \cdot \frac{dy}{dx}, \text{ so } \frac{dy}{dx} = -1$$

- Therefore, our tangent line is the following.

$$y = -(x - 4) + 4 = -x + 8$$

**b)** Two cars start from the same point at the same time. One car travels north in the positive $x$ direction while the other travels east in the positive $y$ direction. If both $x$ and $y$ depend on the variable $t$, time, and their distance apart is modeled by the equation $z^2 = x^2 + y^2$, what is the rate of change between the distance of the two cars? Your answer will be in terms of $x$, $y$, and their rates of change.

***Solution*:**

- First differentiate $z^2 = x^2 + y^2$ with respect to $t$.

$$\frac{d}{dt}z^2 = \frac{d}{dt}(x^2 + y^2) \Rightarrow 2z \cdot \frac{dz}{dt} = 2x \cdot \frac{dx}{dt} + 2y \cdot \frac{dy}{dt}$$

- Now isolate for $\frac{dz}{dt}$.

$$\frac{dz}{dt} = \frac{x \cdot \frac{dx}{dt} + y \cdot \frac{dy}{dt}}{z}$$

### 1.3.2 Using Logarithmic Differentiation

**a)** Find $f'(x)$ if $f(x) = \dfrac{\sin^2(x)\cos(x)\log(x)}{x^2 - 2x}$.

***Solution*:**

- First take the natural logarithm to both sides.

$$\log(y) = \log(\sin^2(x)) + \log(\cos(x)) + \log(\log(x)) - \log(x^2 - 2x)$$

- Now differentiate and isolate for $\dfrac{dy}{dx}$.

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{\sin^2(x)} \cdot 2\sin(x)\cos(x) + \frac{1}{\cos(x)} \cdot (-\sin(x)) + \frac{1}{\log(x)} \cdot \frac{1}{x} - \frac{1}{x^2 - 2x} \cdot (2x - 2)$$

$$\frac{dy}{dx} = \left(\frac{\sin^2(x)\cos(x)\log(x)}{x^2 - 2x}\right)\left(2\cot(x) - \tan(x) + \frac{1}{x\log(x)} - \frac{2x - 2}{x^2 - 2x}\right)$$

**b)** Find $f'(x)$ if $f(x) = x^{\tan^{-1}(x)}$, where $\tan^{-1}(x) = \arctan(x)$.

***Solution*:**

- First take the natural logarithm to both sides.

$$\log(f(x)) = \tan^{-1}(x)\log(x)$$

- Now differentiate on both sides and isolate for $f'(x)$.

$$\frac{1}{f(x)}f'(x) = \frac{1}{1 + x^2}\log(x) + \frac{\tan^{-1}(x)}{x}$$

$$f'(x) = \left(x^{\tan^{-1}(x)}\right) \cdot \left(\frac{\log(x)}{1 + x^2} + \frac{\tan^{-1}(x)}{x}\right)$$

# 2 Higher Degree Approximations

## 2.1 Taylor Polynomials

Linear approximations are a nice way to approximate functions, but we can increase the accuracy of our approximation if we take it to a higher degree.

**Taylor Polynomial**: $f(x) \approx T_n(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + ... + c_n(x-a)^n$ for $x$ near $a$

**Coefficients of Taylor Polynomial**: $c_n = \frac{f^n(a)}{n!}$, where $n! = n(n-1)(n-2)...(3)(2)(1)$

We call a Taylor polynomial a Maclaurin polynomial if it is centered at 0. That is, $a = 0$.

## 2.2 Common Maclaurin Polynomials

$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + ...$ for all $-1 < x < 1$

$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - ...$ for all $-1 < x \leq 1$

$e^x = x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + ...$ for all $-\infty < x < \infty$

$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - ...$ for all $-\infty < x < \infty$

$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - ...$ for all $-\infty < x < \infty$

$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - ...$ for all $-1 \leq x \leq 1$

Given these, we can make all sorts of manipulations, and approximate sums, differences, and compositions of functions.

**Example**: Find the Taylor polynomial of $f(x) = \sin(x^2) - \cos(x)$ to 6th order at $a = 0$.

$$\sin(x^2) \approx x^2 - \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!} - ... = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - ...$$

$$\cos(x) \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ...$$

$$\sin(x^2) - \cos(x) \approx \left(x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!}\right) - \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}\right) = -1 + \frac{3x^2}{2!} - \frac{x^4}{4!} - \frac{119x^6}{6!} + \frac{x^{10}}{5!}$$

To 6th order, $\sin(x^2) - \cos(x) \approx -1 + \frac{3x^2}{2!} - \frac{x^4}{4!} - \frac{119x^6}{6!}$.

When approximating a function to $n$th order, make sure to include all terms that are degree $n$ or less, and discard terms that are higher than degree $n$.

## 2.3 Practice Problems

### 2.3.1 Manipulating Taylor Polynomials

**a)** Find the Taylor polynomial of $\dfrac{1}{5-2x}$ to third order centered at $a = 3$. Use it to approximate $f(3.1)$.

> ***Solution*:**
>
> - First manipulate the equation so that $(x-3)$ shows up in some form.
>
> $$\frac{1}{5-2x} = \frac{1}{5-2(x-3)-6} = \frac{1}{-1-2(x-3)} = \frac{1}{-(1+2(x-3))} = \frac{-1}{1+2(x-3)}$$
>
> - Now replace $x$ with $-2(x-3)$ for every $x$ in the Maclaurin series for $\dfrac{1}{1-x}$ and multiply by $-1$.
>
> $$\frac{1}{5-2x} \approx -(1+(-2(x-3))+(-2(x-3))^2+(-2(x-3))^3) = -1+2(x-3)-4(x-3)^2+8(x-3)^3$$
>
> - Plug in $x$ as 3.1 to approximate $f(3.1)$.
>
> $$f(3.1) \approx -1+2(3.1-3)-4(3.1-3)^2+8(3.1-3)^3 = \frac{-104}{125}$$

**b)** If $f(x) = x^8 e^x$, find $f^{12}(0)$.

> ***Solution*:**
>
> - First multiply $x^8$ with the Maclaurin series of $e^x$.
>
> $$x^8 \cdot \left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+...\right) = x^8+x^9+\frac{x^{10}}{2!}+\frac{x^{11}}{3!}+\frac{x^{12}}{4!}+...$$
>
> - Note that $c_{12} = \dfrac{f^{12}(0)}{12!}$, and $c_{12} = \dfrac{1}{4!}$ from the expansion we created.
>
> $$\frac{1}{4!} = \frac{f^{12}(0)}{12!} \Rightarrow f^{12}(0) = \frac{12!}{4!} = 19958400$$

**c)** Find $f^3(3)$ if its 4th order Taylor polynomial at $a = 3$ is $2(x-3)-8(x-3)^3+10(x-3)^4$.

> ***Solution*:**
>
> - For a $n$ degree polynomial $T_n(x)$, we have $T_n^k(a) = f^k(a)$ for all integers from 1 to $k$.
>
> $$T_4^3(x) = -48+240(x-3)$$
>
> - At $x = a = 3$, we have the following.
>
> $$T_4^3(3) = -48+240(3-3) = -48 \Rightarrow f^3(3) = -48$$

# 3 Curve Sketching

## 3.1 Terminology and First Derivative Test

We introduce the following: local minimum, local maximum, absolute minimum, absolute maximum, critical point, singular point.

(1) Let $a \leq b$ and $f(x)$ be defined over $[a, b]$. Also let $a < c < b$.

- **Local Minimum** at $x = c$ : There exists $a'$ and $b'$ where $a \leq a' < c < b' \leq b$ such that $f(x) \geq f(c)$ for all $x$ that satisfy $a' < x < b'$

This states that at $x = c$, there exists an interval smaller or equal to $(a, b)$ containing $c$ such that all $y$ values in that interval are **greater than or equal** to $f(c)$.

- **Local Maximum** at $x = c$ : There exists $a'$ and $b'$ where $a \leq a' < c < b' \leq b$ such that $f(x) \leq f(c)$ for all $x$ that satisfy $a' < x < b'$

This states that at $x = c$, there exists an interval smaller or equal to $(a, b)$ containing $c$ such that all $y$ values in that interval are **less than or equal** to $f(c)$.

⚠ Note the inequality signs on "$a < c < b$". Local minima/maxima do not include the endpoints of a function!

(2) Let $a \leq b$ and $f(x)$ be defined over $[a, b]$. Also let $a \leq c \leq b$.

- **Global Minimum** at $x = c$ : $f(x) \geq f(c)$ for all $a \leq x \leq b$

This states that at $x = c$, $f(c)$ is **less than or equal** to any other $y$ value obtained in $[a, b]$.

- **Global Maximum** at $x = c$ : $f(x) \leq f(c)$ for all $a \leq x \leq b$

This states that at $x = c$, $f(c)$ is **greater than or equal** to any other $y$ value obtained in $[a, b]$.

(3) Let $f(x)$ be any function and $c$ be a point *in its domain*.

- **Critical Point** at $x = c$ : $f'(c) = 0$
- **Singular Point** at $x = c$ : $f'(c)$ does not exist

(4) Let $f(x)$ be differentiable over $(a, b)$ and continuous over $[a, b]$.

- If $f'(x) > 0$ for all $a < x < b$, $f(x)$ is **increasing** on $(a, b)$
- If $f'(c) < 0$ for all $a < x < b$, $f(x)$ is **decreasing** on $(a, b)$

Using the above, we can find whether a critical point or singular point is a local extrema

(5) Let $f(x)$ be differentiable around $x = c$ and let $x = c$ be a critical point or singular point.

- If $f'(x) < 0$ to the left of $c$ and $f'(x) > 0$ to the right of $c$, a *local minimum* is at $x = c$
- If $f'(x) > 0$ to the left of $c$ and $f'(x) < 0$ to the right of $c$, a *local maximum* is at $x = c$

⚠ If neither of the above is satisfied, the critical point or singular point is not a local maximum or minimum. Think of the critical point at $x = 0$ of $f(x) = x^3$. It is *neither* a local maximum or local minimum.

### 3.2 Second Derivative Test and Inflection Points

We introduce concavity and inflection points, as well as another way to check for the existence of local maxima and minima at critical points.

(1) Let $f(x)$ be a twice differentiable function on $[a, b]$.

- If $f''(x) > 0$ for all $a < x < b$ then the graph is **concave up**
- If $f''(x) < 0$ for all $a < x < b$ then the graph is **concave down**
- If $f''(c) = 0$ for some $a < c < b$, and the concavity of $f(c)$ changes across $x = c$, the point at $x = c$ is an **inflection point**. Note that this guarantees the existence of $f(c)$.

⚠ When checking for concavity during curve sketching, make sure to check for the sign of $f''(x)$ to the left and right of points where $f''(x) = 0$ **AND** also where $f''(x)$ does not exist.

(2) Let $f(x)$ be a twice differentiable function on $[a, b]$

- If $f''(c) > 0$ and $f'(c) = 0$, the point at $x = c$ is a *local minimum*
- If $f''(c) < 0$ and $f'(c) = 0$, the point at $x = c$ is a *local maximum*

The above conditions are another way to check whether a critical point is a local maximum or minimum. Note that we cannot apply these rules to singular points as $f'(c)$ does not exist.

### 3.3 Curve Sketching Checklist

(1) Using $f(x)$

- Check for the $x$ and $y$ intercepts if they are easy to find
- Find all vertical asymptotes where $f(x)$ blows up to $\pm\infty$
- Find all horizontal asymptotes by finding $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$

(2) Using $f'(x)$

- Find all critical points where $f'(x) = 0$ and all singular points where $f'(x)$ does not exist
- Find all intervals where $f(x)$ is increasing or decreasing by checking the sign of $f'(x)$ to the left and right of all critical points **AND** singular points

(3) Using $f''(x)$

- Find all points where $f''(x) = 0$ or where $f''(x)$ does not exist
- Find all intervals where $f(x)$ is concave up or concave down by checking the sign of $f''(x)$ to the left and right of all points where $f''(x) = 0$ **AND** also where $f''(x)$ does not exist
- Find all points where $f''(x) = 0$ and distinguish whether they are inflection points or not. Remember that concavity has to change, so $f''(x) = 0$ does not always guarantee a inflection point. For example, at $x = 0$ for $f(x) = x^4$, $f''(x) = 0$, but concavity doesn't change across $x = 0$, so it isn't an inflection point.

## 3.4 Practice Problems

### 3.4.1 Sketch the Graph

**a)** Sketch the graph of $f(x) = e^{-x}(x+2)(x+1)$.

***Solution*:**

Using $f(x)$:

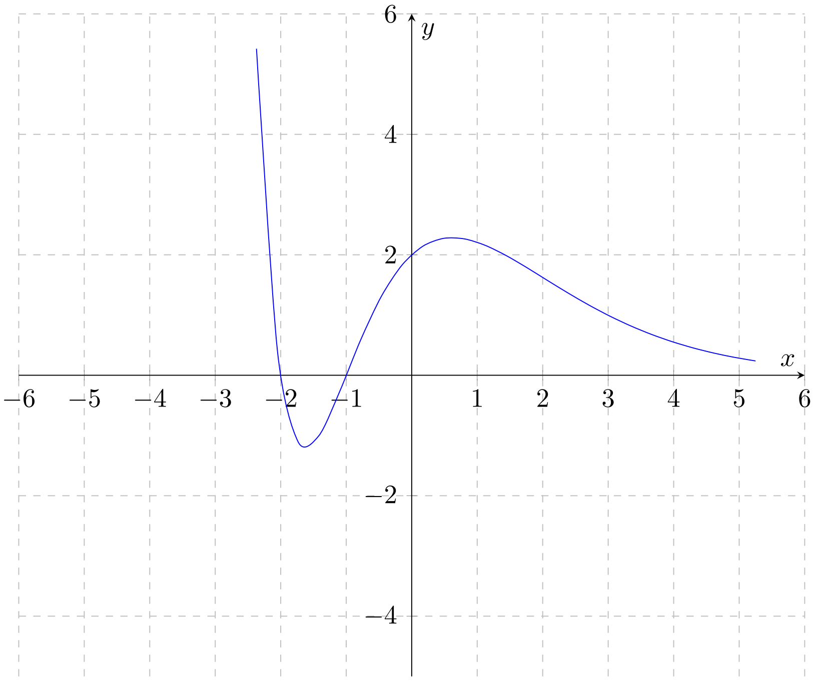
- The $x$ intercepts are $(-2,0)$ and $(-1,0)$ and the y intercept is $(0,2)$.
- $\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to-\infty} f(x) = \infty$ so $f(x)$ has a horizontal asymptote of $y = 0$ as it approaches positive infinity. There are no vertical asymptotes.

Using $f'(x)$:

- $f'(x) = -e^{-x}(x^2 + x - 1)$.
- There are no singular points, but the critical points occur at $x = \dfrac{-1 \pm \sqrt{5}}{2}$.
- $f(x)$ increases on $\left(\dfrac{-1-\sqrt{5}}{2}, \dfrac{-1+\sqrt{5}}{2}\right)$ and decreases on $\left(-\infty, \dfrac{-1-\sqrt{5}}{2}\right) \cup \left(\dfrac{-1+\sqrt{5}}{2}, \infty\right)$.

Using $f''(x)$:

- $f''(x) = e^{-x}(x-2)(x+1)$.
- There are no points where $f''(x)$ does not exist, but $f''(x) = 0$ at $x = 2$ and $x = -1$.
- $f(x)$ is concave up on $(-\infty, -1) \cup (2, \infty)$ and concave down on $(-1, 2)$.
- By analyzing the concavity of $f(x)$, the points at $x = 2$ and $x = -1$ are inflection points.



**b)** Suppose you only have the following information.

$$f'(x) = \frac{e^{-x}(1-x)}{(x+2)^2}\ ,\ \lim_{x\to\infty} f(x) = 0\ ,\ f(0) = 0$$

**i)** Find all possible critical and singular point(s) of $f(x)$ and the interval(s) where $f(x)$ is increasing or decreasing. Classify each critical and singular point as a local minimum, maximum, or neither.

***Solution*:**

- By inspection, $x = 1$ is a critical point and $x = -2$ is a singular point.
- $f(x)$ increases on $(-\infty, -2) \cup (-2, 1)$ and decreases on $(1, \infty)$.
- Since $f(x)$ only increases around $x = -2$, the point at $x = -2$ is neither a local maximum or minimum. At $x = 1$, $f(x)$ increases to the left of 1 and decreases to the right of 1, so there is a local minimum at $x = 1$.

**ii)** Suppose you now know that $f(x)$ has a vertical asymptote at $x = -2$. Find $\lim_{x\to -2^+} f(x)$.

***Solution*:**

- From part i), $f(x)$ is increasing to the right of $-2$ as the $x$ value grows bigger than $-2$, so the function must come up from $-\infty$. Therefore, $\lim_{x\to -2^+} f(x) = -\infty$.

Note that with the new information given in this question, $x = -2$ is ruled out as a singular point as $f(x)$ is not defined at $x = -2$.

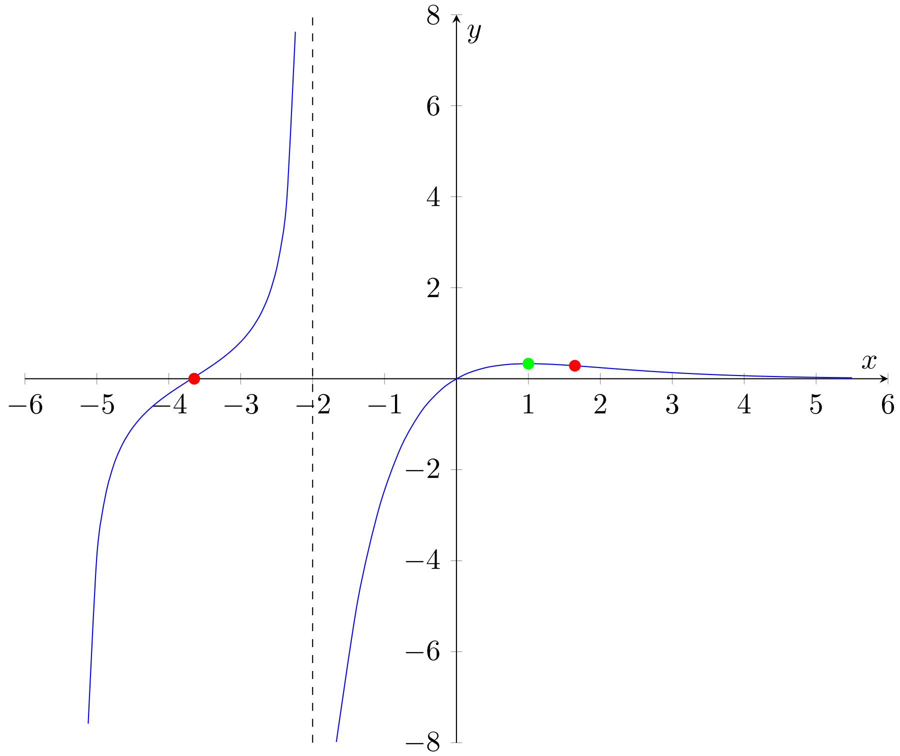
**iii)** Find all inflection points of $f(x)$ and the interval(s) where $f(x)$ is concave up or down.

***Solution*:**

- $f''(x) = \dfrac{e^{-x}(x^2+2x-6)}{(x+2)^3}$ [Bring $(x+2)^2$ up in $f'(x)$, use product/chain rule, then factor].
- If we apply the quadratic formula to $x^2 + 2x - 6$, $x = -1 \pm \sqrt{7}$ are **possible** inflection points (we check if they are with concavity below). Also note that $x = -2$ makes $f''(x)$ not exist.
- $f(x)$ is concave up on $(-1-\sqrt{7}, -2) \cup (-1+\sqrt{7}, \infty)$ and concave down on $(-\infty, -1-\sqrt{7}) \cup (-2, -1+\sqrt{7})$. Since concavity changes across $x = -1 \pm \sqrt{7}$, both of these are inflection points.

**iv)** Sketch a *rough* graph of $f(x)$ based on all the information given so far. Label the critical and inflection points, but do not worry about their $y$ values.

***Solution*:**




This is one example of what $f(x)$ may look like. The inflection points are labeled in red, while the critical point is labeled in green.

# 4 Related Rates and Single-variable Optimization

## 4.1 How to Tackle Related Rates Problems

Related rates extend the concept of implicit differentiation. Instead of $x$ being the independent variable, it is $t$, time. All of the dependent variables we encounter here will depend on time. Below are some general tips to follow when encountering these types of problems.

1. Sketch a picture if possible
2. Find a formula that allows you get the "rates" you are interesting in obtaining given the information in the question
3. Manipulate the formula as necessary e.g., use similar triangles in "cone" type problems and then differentiate it respect to time
4. Use the information in the question to find your answer, and interpret it in the context of your question (don't forget units)

## 4.2 How to Tackle Single-variable Optimization Problems

Optimization problems can be broad in scope, but here are some general tips you want to follow when you encounter these types of problems.

1. Sketch a picture if possible
2. Find the objection function you want to maximize/minimize and the appropriate constraints
3. If the objection function is in more than 1 variable, make the appropriate substitution to knock it down to 1 variable
4. Use $f'(x)$ to find all critical and singular points, and then check if they are local maxima or minima
5. Check the value of $f(x)$ at the local maxima/minima, **AND** check the value of $f(x)$ at the endpoints
6. Determine the maximum/minimum value of the objective function and interpret this in the context of the question. Make sure your answer makes sense e.g., you can't have negative area!

## 4.3 Practice Problems

### 4.3.1 Related Rates

**a)** A conical tank with a diameter of 8m and a height of 10m is being filled with water at a rate of $60\pi$ m$^3$/min. Find the rate of change in the surface area at the top of the water, when the water's surface has a radius of 2m.

***Solution*:**

- The surface area at the top of the water is the area of a circle, so we differentiate $A = \pi r^2$ with respect to $t$, time, on both sides.

$$\frac{d}{dt}A = \frac{d}{dt}\pi r^2 \Rightarrow \frac{dA}{dt} = 2\pi r \cdot \frac{dr}{dt}$$

- To get $\frac{dA}{dt}$, we need $\frac{dr}{dt}$. Though $\frac{dr}{dt}$ is not given, we can find it by differentiating the formula for the volume of a cone.

By similar triangles: $\frac{4}{10} = \frac{r}{h} \Rightarrow h = \frac{5r}{2}$, so $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^2 \left(\frac{5r}{2}\right) = \frac{5}{6}\pi r^3$.

$$\frac{dV}{dt} = \frac{5}{2}\pi r^2 \cdot \frac{dr}{dt}$$

- Using $\frac{dV}{dt} = 60\pi$ and $r = 2$, we solve for $\frac{dr}{dt}$.

$$60\pi = \frac{5}{2}\pi(2)^2 \cdot \frac{dr}{dt} \Rightarrow \frac{dr}{dt} = 6\text{m/min}$$

- Now use the differentiated area formula and substitute in $\frac{dr}{dt}$ and $r = 2$ to find $\frac{dA}{dt}$.

If $\frac{dr}{dt} = 6$m/min and $\frac{dA}{dt} = 2\pi r \cdot \frac{dr}{dt}$, then $\frac{dA}{dt} = 2\pi(2) \cdot (6) = 24\pi$ m$^2$/min.

### 4.3.2 Single-variable Optimization

**a)** Suppose you have a passion for baking and wanted to make chocolate ($c$) and vanilla ($v$) cookies. The cost to make one chocolate cookie is $p_c$ and the cost to make one vanilla cookie is $p_v$. You want to achieve a certain level of utility ($U$) defined by $U = 2\sqrt{cv}$ while minimizing costs. How many chocolate and vanilla cookies do you bake in order to achieve this? Assume $c > 0$ and $v > 0$.

***Solution*:**

- Let $E(c, v) = p_c c + p_v v$ be the cost function we want to minimize subject to the constraint $U = 2\sqrt{cb}$. We use the constraint so our cost function is in terms of 1 variable.

$$U = 2\sqrt{cv} \Rightarrow c = \frac{U^2}{4v}, \text{ so } E(c, v) = p_c c + p_v v \text{ becomes } E(v) = \frac{U^2 p_c}{4v} + p_v v.$$

- Now differentiate to get $E'(v)$ and find all critical points.

$$E'(v) = \frac{-U^2 p_c}{4v^2} + p_v, \text{ so setting } E'(v) = 0 \text{ and solving for } v, \text{ we get } v = \frac{U}{2}\sqrt{\frac{p_c}{p_v}}.$$

- As $0 < v < \infty$, we check the "endpoints" of $E(v)$ by finding $\lim_{v \to 0^+} E(v)$ and $\lim_{v \to \infty} E(v)$.

$$\lim_{v \to 0^+} E(v) = \infty + 0 = \infty \text{ and } \lim_{v \to \infty} E(v) = 0 + \infty = \infty$$

- Since both tail-ends of $E(v)$ blow up to infinity and we only have 1 critical point, that critical point must be the global minimum. We then find $c$ using our constraint and expression for $v$ at the critical point.

$$\text{If } c = \frac{U^2}{4v} \text{ and } v = \frac{U}{2}\sqrt{\frac{p_c}{p_v}}, \text{ then } c = \frac{U}{2}\sqrt{\frac{p_v}{p_c}}.$$

We therefore produce $\frac{U}{2}\sqrt{\frac{p_v}{p_c}}$ chocolate cookies and $\frac{U}{2}\sqrt{\frac{p_c}{p_v}}$ vanilla cookies to achieve the smallest cost while satisfying the amount of utility needed.

# 5 Differential Equations

## 5.1 Verifying Solutions

A *differential equation* is an equation that relates one or more derivatives of an unknown function to the function itself.

When verifying a solution, it is as simple as finding its derivatives and plugging them in into the differential equation.

**Example**: Is $f(x) = 3x + x^2$ a solution to the differential equation $xy' - y = x^2$?

We first note that $y = 3x + x^2$, and by taking a derivative, we find that $y' = 3 + 2x$.

Substituting them into the differential equation yields $x(3 + 2x) - (3x + x^2) = x^2$.

Simplifying the left-hand side, we obtain $3x+2x^2-3x-x^2$, which is equal to $x^2$. Thus, $y = 3x+x^2$ is a solution to the differential equation.

## 5.2 Using Ansatzes to Solve Differential Equations

Sometimes we can make an educated guess on the general form of the solution to a differential equation. This guess is called an **ansatz**. They are parameterized by constants, and a goal is to find the value(s) of these constants that satisfy the differential equation.

**Example**: Using the ansatz $y = \cos(at)$, find all values of $a$ such that $y'' + 16y = 0$.

Observe that $y'' = -a^2 \cos(at)$, so we have $-a^2 \cos(at) + 16 \cos(at) = \cos(at)(16 - a^2) = 0$.

Thus, $a = 4$ or $-4$.

## 5.3 General Solutions to First-order Differential Equations

When we solve differential equations, we often get a *family* of solutions, parameterized by a constant e.g., $C$. This is a general solution of a differential equation. Below are some general solutions of **first-order** differential equations. Note that I'm using $t$ as my independent variable, not $x$.

Let $a$, $b$, $k$, and $C$ be constants, then we have the following:

1. $\dfrac{dy}{dt} = ky$ has the general solution $y = Ce^{kt}$
2. $\dfrac{dy}{dt} = -ky$ has the general solution $y = Ce^{-kt}$
3. $\dfrac{dy}{dt} = a(b - y)$ has the general solution $y = Ce^{at} + b$

How we derive these is either by inspection, or integral calculus (outside the scope of this course). I normally would not encourage memorization, but memorize these!

## 5.4 Initial Value Problems and the Steady State

What if we wanted a specific solution? Then we need an **initial value** $y(t) = y_0$, where $t$ and $y_0$ are constants. Most of what you will encounter will have $t = 0$, so the initial condition is $y(0) = y_0$.

Let $a$, $b$, and $k$ be constants, and $y(0) = y_0$ be the initial condition. Then we have the following:

1. $\dfrac{dy}{dt} = ky$ has solution $y = y_0 e^{kt} \quad (C = y_0)$
2. $\dfrac{dy}{dt} = -ky$ has the solution $y = y_0 e^{-kt} \quad (C = y_0)$
3. $\dfrac{dy}{dt} = a(b - y)$ has the solution $y = (y_0 - b)e^{at} + b \quad (C = y_0 - b)$

These are the results you get if you set $t = 0$, $y = y_0$, and then solve for $C$.

There is one type of special solution where there is no change over time. If we have initial condition $y(t) = y_0$ satisfying $\dfrac{dy}{dt} = 0$, then a solution to the differential equation is $y = y_0$. The solution $y = y_0$ is called the **steady state** solution of the differential equation. It is a constant function.

**Example**: If we start with initial condition $y(0) = b$, the steady state solution to $\dfrac{dy}{dt} = a(b - y)$ is $y = b$ since $a(b - b) = 0$.

## 5.5 Slope Fields of First-order Differential Equations

Whether we can solve a first-order differential equation or not, we can draw a **slope field** to understand its family of solutions.

The idea:

The first derivative represents *slope*, so $\dfrac{dy}{dx} = f(x, y)$ means we have a formula to find *slope*, $\dfrac{dy}{dx} = m$ at any point $(x, y)$ by plugging it into $f(x, y)$.

How do we do this?

1. Make a graph with many *little* lines that represent slopes at points
2. Given a point (initial value) we can find an approximate curve whose slopes fit

**Example**: Draw the slope field corresponding to the differential equation $\dfrac{dy}{dx} = x + y$. Based on the slope field, draw a few solution curves.

First notice that $m = x + y$, so we can construct a table of values for slope given $(x, y)$.

| Slope $= x + y$ | $x = -4$ | $x = -3$ | $x = -2$ | $x = -1$ | $x = -0$ | $x = 1$ | $x = 2$ | $x = 3$ | $x = 4$ |
|---|---|---|---|---|---|---|---|---|---|
| $y = -4$ | $-8$ | $-7$ | $-6$ | $-5$ | $-4$ | $-3$ | $-2$ | $-1$ | $0$ |
| $y = -3$ | $-7$ | $-6$ | $-5$ | $-4$ | $-3$ | $-2$ | $-1$ | $0$ | $1$ |
| $y = -2$ | $-6$ | $-5$ | $-4$ | $-3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ |
| $y = -1$ | $-5$ | $-4$ | $-3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ |
| $y = 0$ | $-4$ | $-3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ | $4$ |
| $y = 1$ | $-3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ | $4$ | $5$ |
| $y = 2$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ | $4$ | $5$ | $6$ |
| $y = 3$ | $-1$ | $0$ | $1$ | $2$ | $3$ | $4$ | $5$ | $6$ | $7$ |
| $y = 4$ | $0$ | $1$ | $2$ | $3$ | $4$ | $5$ | $6$ | $7$ | $8$ |

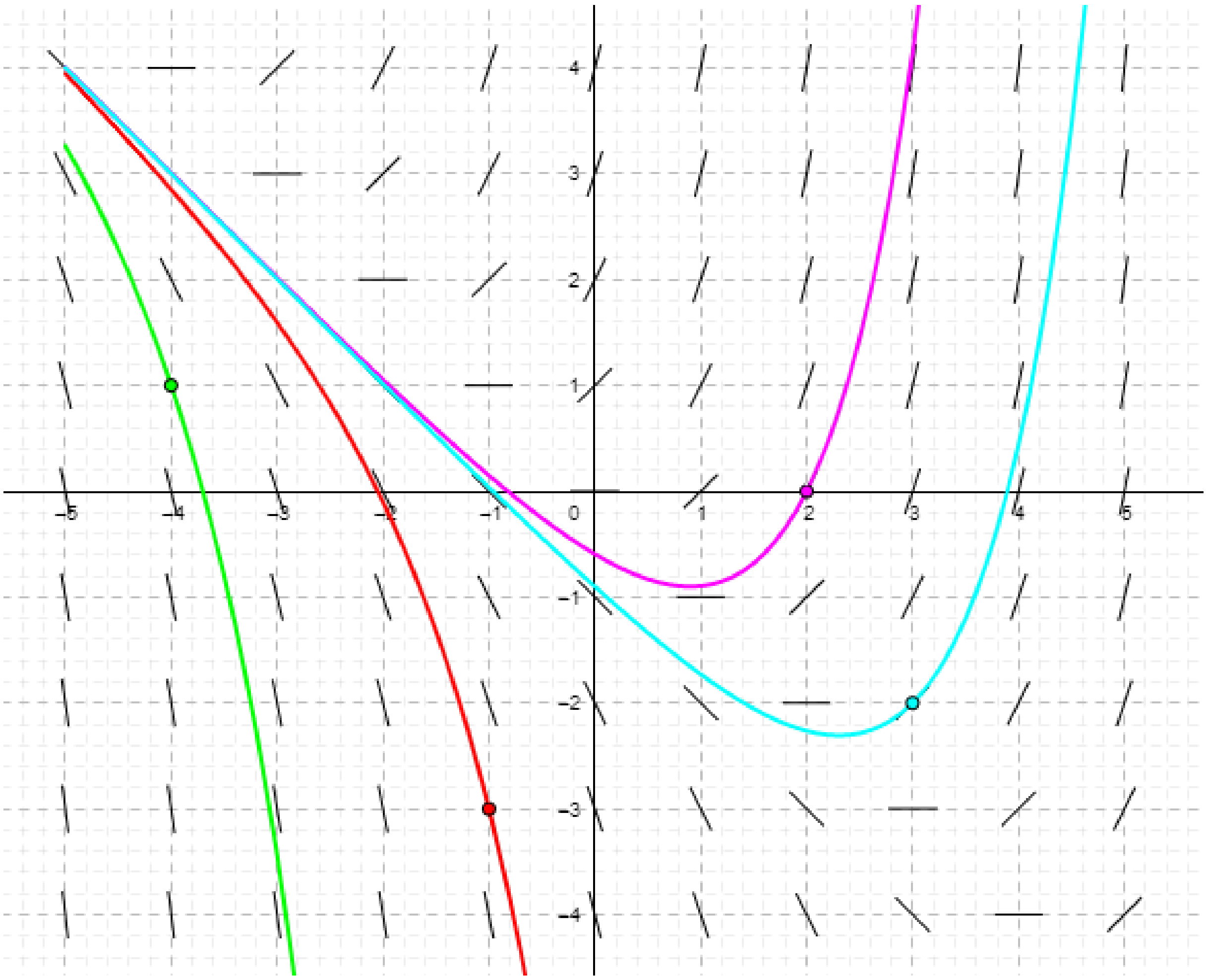



Figure 1: Slope field of $\dfrac{dy}{dx} = x + y$

## 5.6 Practice Problems

### 5.6.1 Solutions and Applications of Differential Equations

**a)** Is $y = \sqrt{\dfrac{2x^3}{3} + 16}$ is a solution to the differential equation $y \cdot \dfrac{dy}{dx} = -x^2$?

> ***Solution*:**
>
> - Note that $y = \left(\dfrac{2x^3}{3} + 16\right)^{\frac{1}{2}}$ and $\dfrac{dy}{dx} = x^2 \left(\dfrac{2x^3}{3} + 16\right)^{\frac{-1}{2}}$
>
>   Multiplying the 2 terms together, we get $x^2$. This is not equal to $-x^2$, so $y = \sqrt{\dfrac{2x^3}{3} + 16}$ is **not** a solution to the differential equation.

**b)** A colony of bacteria doubles every 1600 seconds. If we start with 100 bacteria, how many more bacteria will there be in 100 seconds?

> ***Solution*:**
>
> - Bacteria growth has the differential equation $\dfrac{dy}{dt} = ky$. If we start with 100 bacteria, the initial condition is $y(0) = 100$. We use these facts to solve an expression for $y$.
>
> $$\frac{dy}{dt} = ky, \text{ given } y(0) = y_0 \text{ is } y = y_0 e^{kt}.\ y_0 = 100, \text{ so we have } y = 100e^{kt}.$$
>
> - Now we solve for $k$, knowing that at $t = 1600$, $y = 200$ since the initial amount (100) doubles every 1600 seconds.
>
> $$200 = 100e^{1600k} \Rightarrow 2 = e^{1600k} \Rightarrow \log(2) = 1600k \Rightarrow k = \frac{\log(2)}{1600}$$
>
> - Now we find how many bacteria there are if 100 seconds pass. That is, $t = 100$.
>
> $$y = 100e^{100 \cdot \frac{\log(2)}{1600}} = y = 100e^{\frac{\log(2)}{16}} \approx 104.43, \text{ so about } 104.43 - 100 = 4.33 \text{ bacteria grow.}$$

**c)** Consider the differentiation equation $2y'' - 3y' = 5y$. Using the ansatz $y = e^{rt}$, find the values of $r$ that satisfy this differentiation equation. In general, what are the values of $r$ such that $ay'' - by' = cy$? Assume $a \neq 0$.

> ***Solution*:**
>
> - We have $y = e^{rt}$, $y' = re^{rt}$, and $y'' = r^2 e^{rt}$. Plug these quantities into the differential equation and solve for $r$.
>
> $$2r^2e^{rt} - 3re^{rt} = 5e^{rt} \Rightarrow e^{rt}(2r^2 - 3r - 5) = 0, \text{ so } r = \frac{3 \pm \sqrt{(-3)^2 - (4)(2)(-5)}}{4} = \frac{5}{2} \text{ and } -1.$$
>
> - For any $a$, $b$, $c$ the solution to $ay'' - by' = cy$ is $r = \dfrac{b \pm \sqrt{b^2 - (4)(a)(-c)}}{2a}$.

# 6 Numerical Computation

## 6.1 Euler's Method

Suppose we wanted to estimate the value of a function, $f(b)$, at $x = b$ without the function. If we were just given a differential equation, we can do so starting at initial value $f(x_0) = y_0$.

**Euler's Method**: $y_n = y_{n-1} + \Delta h \cdot f'(x_{n-1}, y_{n-1})$, where $\Delta h = \dfrac{b - x_0}{n}$, the step size with $n$ steps.

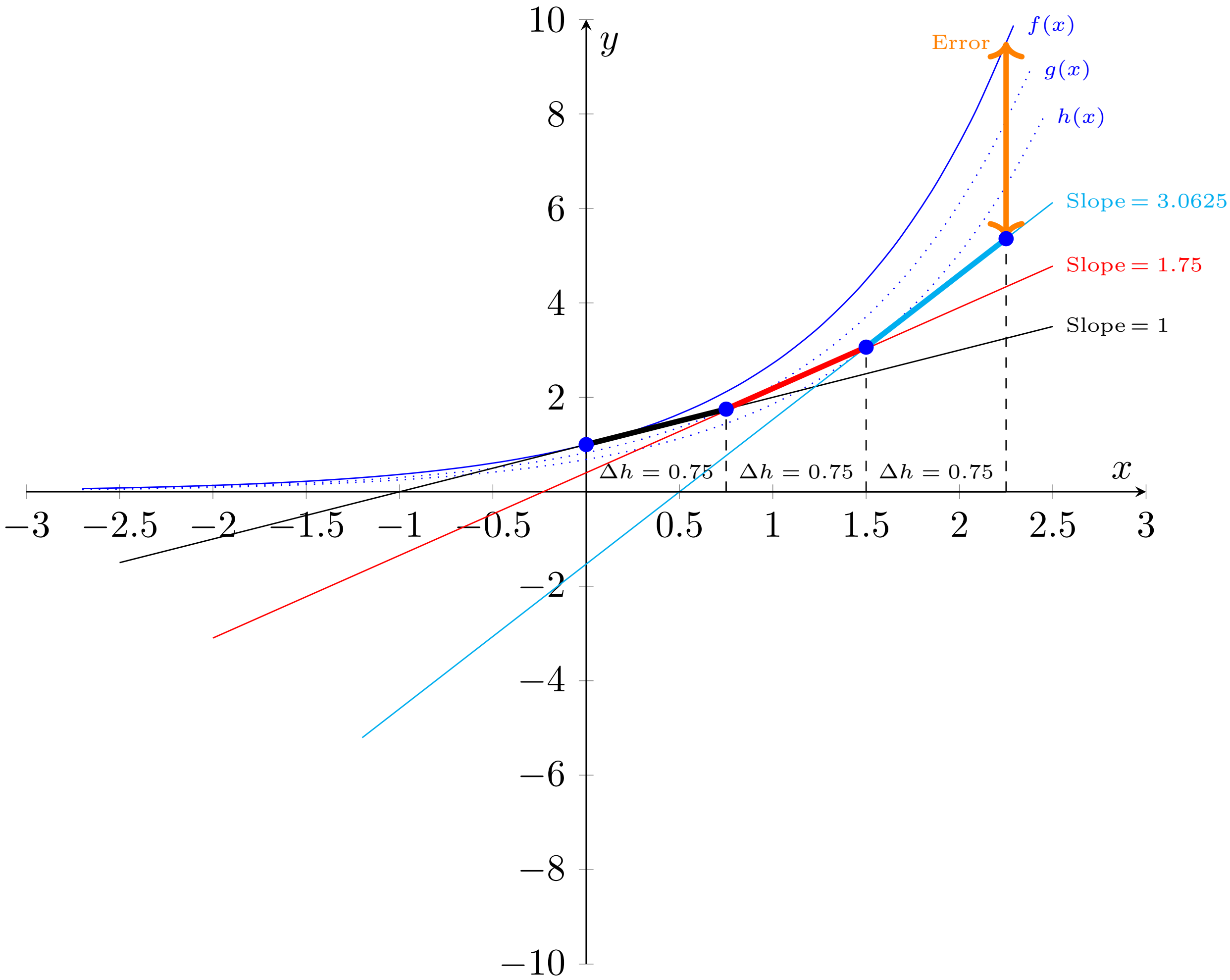
Geometric Interpretation:

When applying Euler's method, you are moving along tangent lines of curves in the family of solutions to the differential equation. In the diagram, the curves are $f(x)$, $g(x)$, and $h(x)$.

Every time you finish one iteration (go from one blue dot to the next), you increase (+) the current $y$ value by $\Delta h \cdot f'(x_{n-1}, y_{n-1})$. Why? Think back to what slope means.

If slope = 2, this means that for every 1 unit I travel horizontally, I increase the function value $y$ by 2. If I only move halfway by 0.5 units, I increase my function value by $1 = 0.5(2)$.

$f'(x_{n-1}, y_{n-1})$ is the **slope** of the tangent line at your current point, and $\Delta h$ is how much I move horizontally within each iteration. The function value thus increases (+) by $\Delta h \cdot f'(x_{n-1}, y_{n-1})$.



**Example**: If $\dfrac{dy}{dx} = y$, estimate $f(2.25)$ given initial value $y(0) = 1$ with a step size $n$ of 3.

Given $x_0 = 0$, $b = 2.25$, and $n = 3$, the step size, $\Delta h$ is $\dfrac{2.25 - 0}{3} = 0.75$.

Slope $= m = \dfrac{dy}{dx} = f'(x_{n-1}, y_{n-1}) = y_{n-1}$

**Iteration 1** ($x_0 = 0$, $y_0 = 1$, $\Delta h = 0.75$): Moving along **black** tangent line to $f(x)$.

$y_1 = y_0 + \Delta h \cdot f'(x_0, y_0) \Rightarrow y_1 = 1 + 0.75 \cdot f'(0, 1) = 1 + 0.75(1) = 1.75.$

Since we moved 0.75 horizontally $x_1 = 0 + 0.75 = 0.75$.

**Iteration 2** ($x_1 = 0.75$, $y_1 = 1.75$, $\Delta h = 0.75$): Moving along **red** tangent line to $g(x)$.

$y_2 = y_1 + \Delta h \cdot f'(x_1, y_1) \Rightarrow y_2 = 1.75 + 0.75 \cdot f'(0.75, 1.75) = 1.75 + 0.75(1.75) = 3.0625.$

Since we moved 0.75 horizontally $x_2 = 0.75 + 0.75 = 1.5$.

**Iteration 3** ($x_2 = 1.5$, $y_2 = 3.0625$, $\Delta h = 0.75$): Moving along **blue** tangent line to $h(x)$.

$y_3 = y_2 + \Delta h \cdot f'(x_2, y_2) \Rightarrow y_3 = 3.0625 + 0.75 \cdot f'(1.5, 3.0625) = 3.0625 + 0.75(3.0625) \approx 5.36.$

Since we moved 0.75 horizontally $x_3 = 1.5 + 0.75 = 2.25$.

From our approximation, we conclude that $f(2.25) \approx 5.36$. Is this a good approximation? Not really. The step size is quite big, and $f(2.25) = 9.477$. However, the large step size was for illustration purposes. The smaller you make $\Delta h$ by having a larger $n$, the better the approximation will be.

When performing Euler's method, you won't have access to any visual, but understanding the inner workings of what actually happens is beneficial.

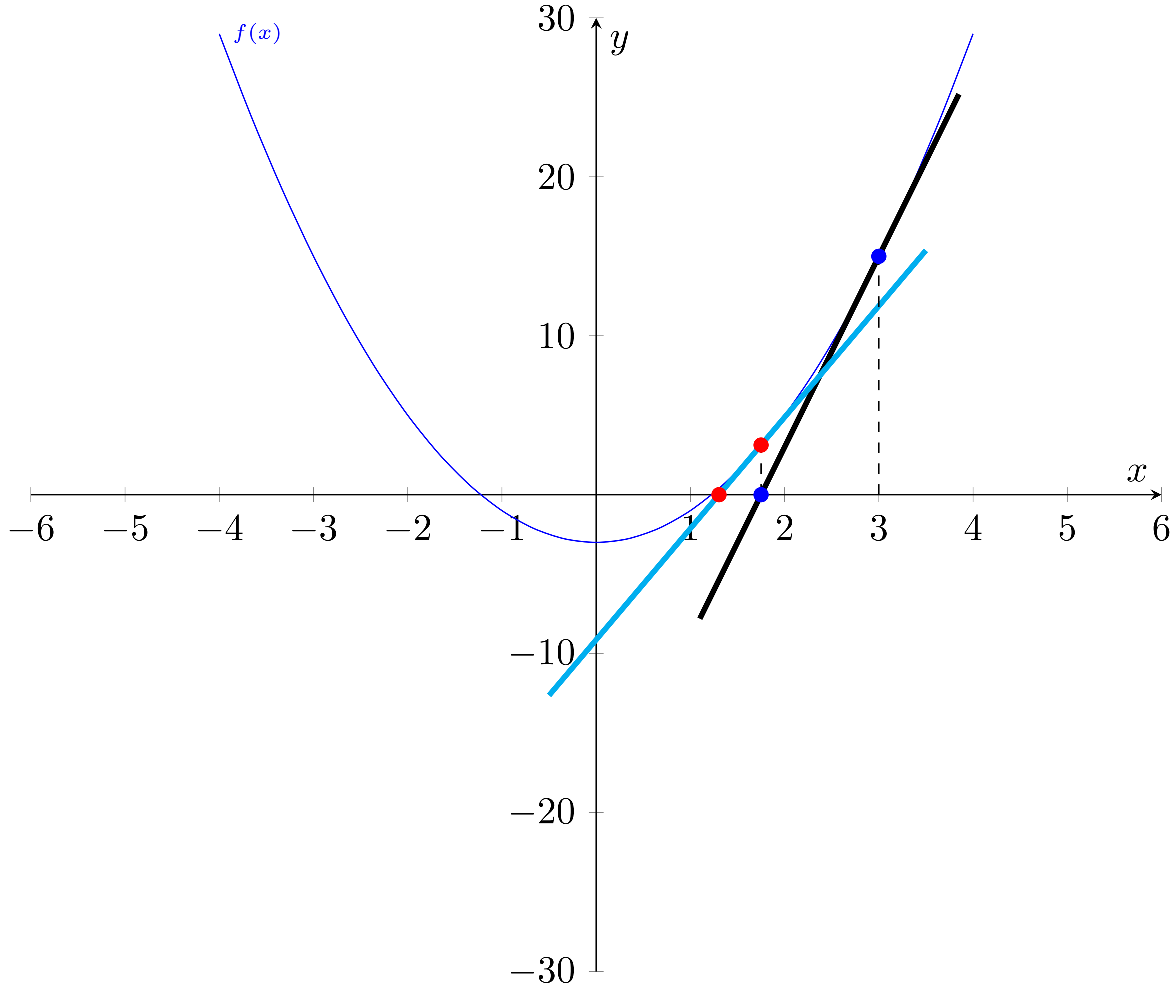
### 6.2 Newton's Method

Instead of approximating any function value, what if we wanted to approximate a root of a function?

**Newton's Method**: $x_n = x_{n-1} - \dfrac{f(x_{n-1})}{f'(x_{n-1})}$.

Geometric Interpretation:

When applying Newton's method, you find a tangent line at an intial point $(x_0, y_0)$ and find where it intersects the $x$-axis. You then find the tangent line at the new $x$, and repeat the process again and again.




If we wanted to approximate one root of the function $f(x) = 2x^2 - 3$, starting at $x_0 = 3$ provides a decent estimate even after 2 iterations! The first iteration refers to the black tangent line, and the second iteration refers to the blue tangent line.

## 6.3 Practice Problems

### 6.3.1 Euler's Method

**a)** Approximate $f(0.3)$ if $\dfrac{dy}{dx} = y + xy$ with step size 0.1 and initial condition $y(0) = 1$.

***Solution:***

- We note that $\Delta h = 0.1$ and $f'(x_{n-1}, y_{n-1}) = y_{n-1} + x_{n-1}y_{n-1}$.

**Iteration 1** $(x_0 = 0,\ y_0 = 1,\ \Delta h = 0.1)$ :

$y_1 = y_0 + \Delta h \cdot f'(x_0, y_0) \Rightarrow y_1 = 1 + 0.1 \cdot f'(0, 1) = 1 + 0.1(1) = 1.1.$

Since we moved 0.1 horizontally $x_1 = 0 + 0.1 = 0.1$.

**Iteration 2** $(x_1 = 0.1,\ y_1 = 1.1,\ \Delta h = 0.1)$ :

$y_2 = y_1 + \Delta h \cdot f'(x_1, y_1) \Rightarrow y_2 = 1.1 + 0.1 \cdot f'(0.1, 1.1) = 1.1 + 0.1(1.21) = 1.221.$

Since we moved 0.1 horizontally $x_2 = 0.1 + 0.1 = 0.2$.

**Iteration 3** $(x_2 = 0.2,\ y_2 = 1.221,\ \Delta h = 0.1)$ :

$y_3 = y_2 + \Delta h \cdot f'(x_2, y_2) \Rightarrow y_3 = 1.221 + 0.1 \cdot f'(0.2, 1.221) = 1.221 + 0.1(1.4652) \approx 1.368.$

Since we moved 0.1 horizontally $x_3 = 0.2 + 0.1 = 0.3$.

Therefore $f(0.3) \approx 1.368$.

### 6.3.2 Newton's Method

**a)** Find one root of the function $3x^4 - 8x^3 + 1 = 0$ starting at $x_0 = 3$. Use 3 iterations.

***Solution*:**

- We note that $f(x_{n-1}) = 3(x_{n-1})^4 - 8(x_{n-1})^3 + 1$ and $f'(x_{n-1}) = 12(x_{n-1})^3 - 24(x_{n-1})^2$.

**Iteration 1** $(x_0 = 3)$ :

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 3 - \frac{3(3)^4 - 8(3)^3 + 1}{12(3)^3 - 24(3)^2} \approx 2.741.$$

**Iteration 2** $(x_1 = 2.741)$ :

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2.741 - \frac{3(2.741)^4 - 8(2.741)^3 + 1}{12(2.741)^3 - 24(2.741)^2} \approx 2.657.$$

**Iteration 3** $(x_2 = 2.657)$ :

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 2.657 - \frac{3(2.657)^4 - 8(2.657)^3 + 1}{12(2.657)^3 - 24(2.657)^2} \approx 2.649.$$

Therefore one root of the function is $\approx 2.649$.

# 7 Sketching in 3 Dimensions

## 7.1 Traces

Upon entering the 3rd dimension, we have functions $z = f(x, y)$ corresponding to 3D objects. Though we cannot readily draw them like 2D graphs, we can gain a better sense of what they look like by observing its cross-sections. This brings our graphical intuition back into the 2D world.

A **trace** is a 2D drawing of the intersection between the 3D object and one of the coordinate planes. That is, if we set $x$, $y$, or $z$ to be a constant, we get a 2D cross-section of the 3D object.

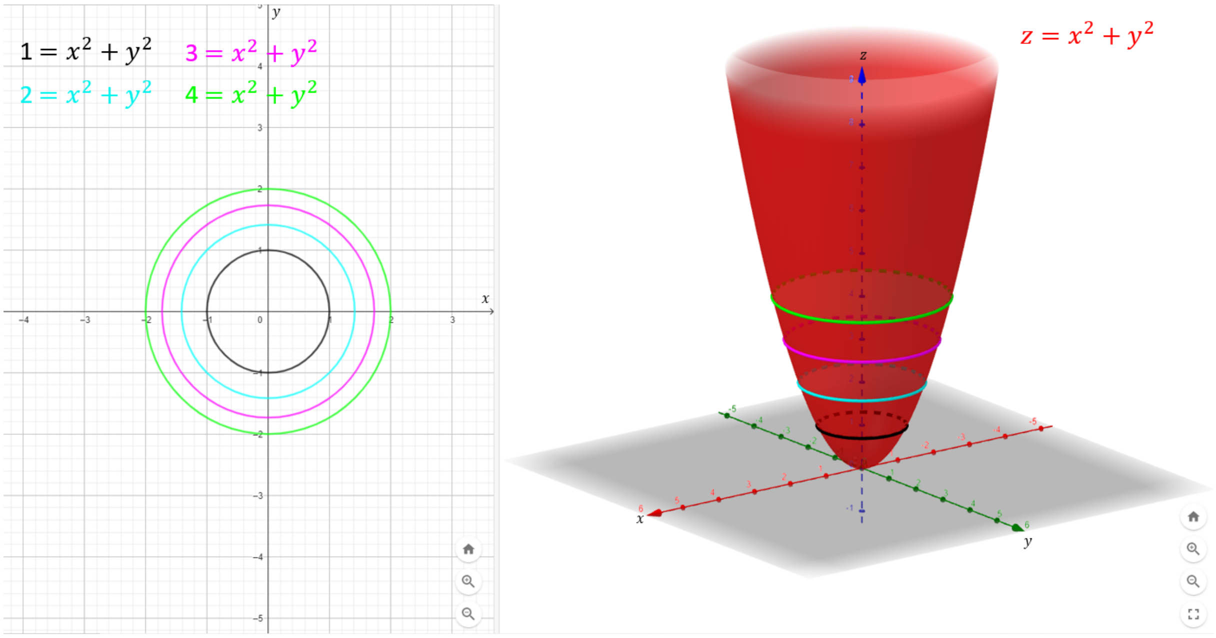



Figure 2: Traces of $z = x^2 + y^2$ with $z = 1, 2, 3, 4$ ($z$ is a constant)

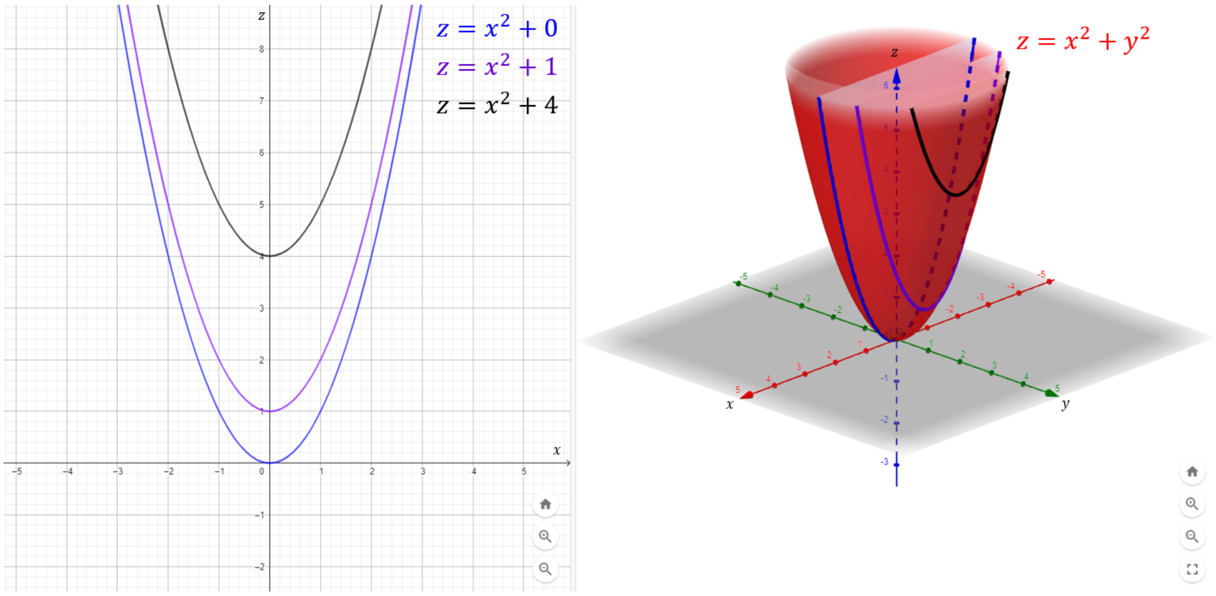



Figure 3: Traces of $z = x^2 + y^2$ with $y = 0, 1, 2$ ($y$ is a constant)

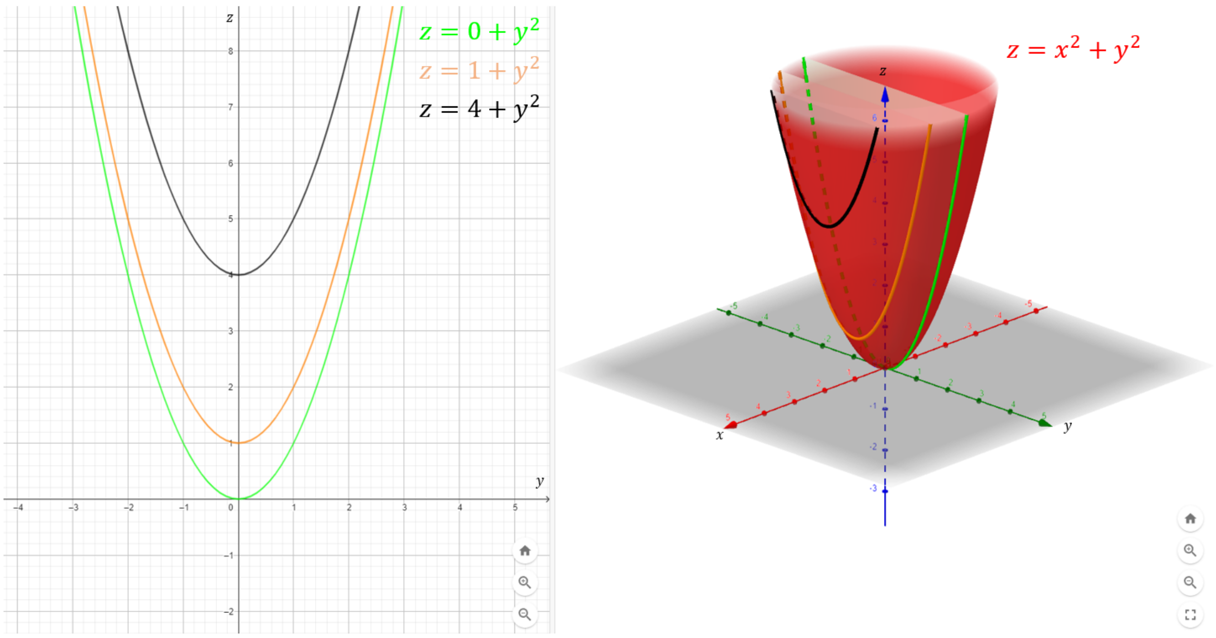



Figure 4: Traces of $z = x^2 + y^2$ with $x = 0,\ 1,\ 2$ ($x$ is a constant)

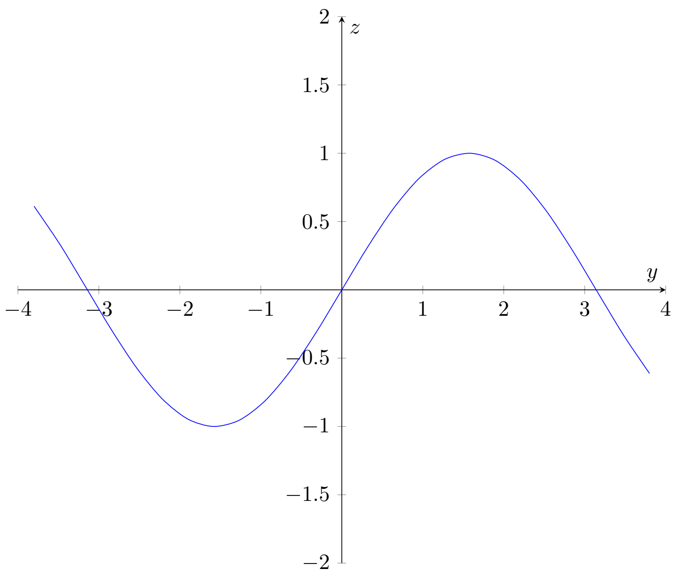
## 7.2 Practice Problems

### 7.2.1 Drawing Traces

**a)** Let $f(x, y) = \cos(x)\sin(y)$. Sketch the trace when i) $x = 0$, ii) $y = \dfrac{\pi}{2}$, and iii) $z = 0$. For iii), let $-\pi < x < \pi$ and $-\pi < y < \pi$.

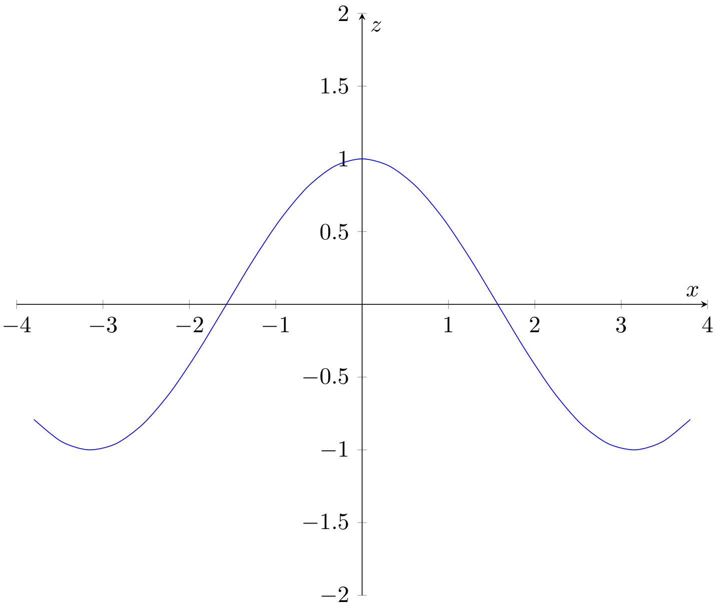
***Solution*:**

- i) When $x = 0$, we obtain the graph of $z = \cos(0)\sin(y) = \sin(y)$.

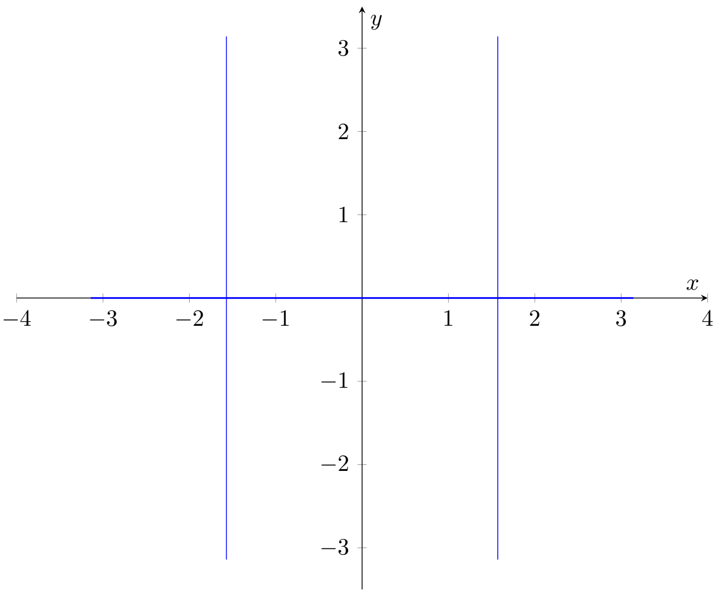


(Continued on Next Page)

- ii) When $y = \frac{\pi}{2}$, we obtain the graph of $z = \cos(x)\sin\left(\frac{\pi}{2}\right) = \cos(x)$.




- iii) When $z = 0$, we obtain the graph of $0 = \cos(x)\sin(y)$. With the restrictions $-\pi < x < \pi$ and $-\pi < y < \pi$, the equation is satisfied only if $x = \frac{\pi}{2}, -\frac{\pi}{2}$, or $y = 0$.



# 8 Partial Derivatives

## 8.1 Treating Variables as Constants

Partial derivatives are similar to regular derivatives. If $z = f(x, y)$, we have $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$. The partial derivatives can also be denoted as $f_x$ and $f_y$.

The second-order partials are the following: $\frac{\partial^2 z}{\partial x^2}$, $\frac{\partial^2 z}{\partial y^2}$, $\frac{\partial^2 z}{\partial y \partial x}$, and $\frac{\partial^2 z}{\partial x \partial y}$. They can also be respectively denoted as $f_{xx}$, $f_{yy}$, $f_{xy}$, and $f_{yx}$.

⚠ Note that $f_{xy}$ means $\frac{\partial^2 z}{\partial y \partial x}$ and $f_{yx}$ means $\frac{\partial^2 z}{\partial x \partial y}$. The order matters!

Intuition:

When taking a partial derivative with respect to a variable, you are holding all other variables constant. As a *starting point*, think of those other variables as your favourite number.

**Example**: If $z = x^2y - xy + \sin^2(x)$, find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

Let's pretend that $y = 7$, a constant. Then we have $z = 7x^2 - 7x + \sin^2(x)$.

Taking a derivative with respect to $x$, we get $2(7)(x) - 7 + 2\sin(x)\cos(x)$

Therefore, $\frac{\partial z}{\partial x} = 2xy - y + 2\sin(x)\cos(x)$

Let's pretend that $x = 7$, a constant. Then we have $z = 7^2y - 7y + \sin^2(7)$.

Taking a derivative with respect to $y$, we get $7^2 - 7 + 0$

Therefore, $\frac{\partial z}{\partial y} = x^2 - x$

By no means do you have to substitute a constant to take partial derivatives. The way I have demonstrated above is just a way of seeing what is going on. I encourage you to not make any substitutions as you get better at taking partial derivatives.

## 8.2 Practice Problems

### 8.2.1 Finding Partial Derivatives

**a)** Find both first-order partial derivatives if $z = ye^{xy}$.

***Solution*:**

- For the partial with respect to $x$, we focus on the $e^{xy}$ part.

$$\frac{\partial z}{\partial x} = ye^{xy} \cdot \frac{\partial}{\partial x}(xy) = y^2 e^{xy}.$$

- For the partial with respect to $y$, we use product rule.

$$\frac{\partial z}{\partial y} = e^{xy} + y \cdot \frac{\partial}{\partial y}(e^{xy}) = e^{xy} + y(xe^{xy}) = e^{xy}(1 + xy).$$

**b)** Find both first-order partial derivatives if $z = \arctan\left(\frac{y}{x}\right)$.

***Solution*:**

- For both partials, we use chain rule.

$$\frac{\partial z}{\partial x} = \frac{1}{1 + \left(\frac{y}{x}\right)^2} \cdot \frac{\partial}{\partial x}\left(\frac{y}{x}\right) = \frac{1}{1 + \left(\frac{y^2}{x^2}\right)} \cdot \frac{-y}{x^2} = \frac{-y}{x^2 + y^2}.$$

$$\frac{\partial z}{\partial y} = \frac{1}{1 + \left(\frac{y}{x}\right)^2} \cdot \frac{\partial}{\partial y}\left(\frac{y}{x}\right) = \frac{1}{1 + \left(\frac{y^2}{x^2}\right)} \cdot \frac{1}{x} = \frac{1}{x + \left(\frac{y^2}{x}\right)} = \frac{x}{x^2 + y^2}.$$

# 9 Multivariable Optimization

## 9.1 Critical Points and Extrema

Let $z = f(x, y)$ be a function of 2 variables. If $(a, b)$ is a critical point of $f(x, y)$, then one of the following conditions is satisfied.

1. $f_x(a, b) = f_y(a, b) = 0$
2. $f_x(a, b)$ DNE
3. $f_y(a, b)$ DNE

If $(a, b)$ is a critical point of $f$, it can either be a local minimum, local maximum, or a saddle point.

Second Derivative Test:

- If $f_{xx}f_{yy} - (f_{xy})^2 > 0$ and $f_{xx} > 0$, then $(a, b)$ is a local minimum
- If $f_{xx}f_{yy} - (f_{xy})^2 > 0$ and $f_{xx} < 0$, then $(a, b)$ is a local maximum
- If $f_{xx}f_{yy} - (f_{xy})^2 < 0$, then $(a, b)$ is a saddle point
- If $f_{xx}f_{yy} - (f_{xy})^2 = 0$, the test is inconclusive

## 9.2 Constrained Optimization and Boundaries

Let $f(x, y)$ be a continuous function defined on a closed and bounded set $D$. Then a global maximum and minimum must occur at one of the following.

- At a critical point of $f(x, y)$
- At an endpoint of $f(x, y)$
- On the boundary of $f(x, y)$

We've learned how to find critical points, but how do we check the endpoints and boundary?

The first way is to plug the boundary into the function and check the function's endpoints. This is what you have done in single-variable optimization, where you reduce your 2 variable problem down into a 1 variable problem.

For example, suppose we wanted to find the maximum and minimum value of $f(x, y) = x^2y - y + xy$ subjected to the triangle formed by $y = 8 - x$, $y = 1$, and $x = 1$. We would need to plug in each boundary as a separate case and use single-variable optimization techniques.

## 9.3 Lagrange Multipliers

What if we our boundary was not "nice" and we couldn't explicitly plug it into the function? Fortunately we have a technique called Lagrange Multipliers that helps us get around this problem.

Let $f(x, y)$ be the function we want to optimized, subject to the constraint $g(x, y) = 0$. Then $(a, b)$ is a candidate for a global maximum or global minimum if it satisfies all the following properties for some constant $\lambda$.

1. $f_x(a, b) = \lambda g_x(a, b)$
2. $f_y(a, b) = \lambda g_y(a, b)$
3. $g(a, b) = 0$

⚠ Note that this only deals with the boundary! If we had constraints with $\leq$ or $\geq$ signs, we also have to find the critical points located in the interior of the constraint (**see section 9.1**).

## 9.4 Practice Problems

### 9.4.1 Unconstrained Optimization

**a)** Find the minimum value of $f(x, y) = x^2 + y^2 + \dfrac{1}{x^2 + 9y^2 + 1}$.

***Solution*:**

- We first take partial derivatives and find the critical points of $f$.

Equation 1: $\dfrac{\partial z}{\partial x} = 2x - \dfrac{2x}{(x^2 + 9y^2 + 1)^2} = 0$, Equation 2: $\dfrac{\partial z}{\partial y} = 2y - \dfrac{18y}{(x^2 + 9y^2 + 1)^2} = 0$.

$$\text{Equation 1 implies } 2x(x^2 + 9y^2 + 1)^2 - 2x = 0 \Rightarrow 2x[(x^2 + 9y^2 + 1)^2 - 1] = 0$$

$$\text{This implies } x = 0 \textbf{ or } (x^2 + 9y^2 + 1)^2 - 1 = 0 \Rightarrow x^2 + 9y^2 = 0 \Rightarrow x = 0 \text{ and } y = 0.$$

$$\text{Equation 2 implies } 2y(x^2 + 9y^2 + 1)^2 - 18y = 0 \Rightarrow 2y[(x^2 + 9y^2 + 1)^2 - 9] = 0$$

$$\text{This implies } y = 0 \textbf{ or } (x^2 + 9y^2 + 1)^2 - 9 = 0 \Rightarrow x^2 + 9y^2 = 2.$$

From equation 1, $x$ must be 0, so plugging $x = 0$ into $x^2 + 9y^2 = 2$ yields $9y^2 = 2$.

$$\text{Therefore, } 9y^2 = 2 \Rightarrow y^2 = \frac{2}{9} \Rightarrow y = \pm\frac{\sqrt{2}}{3}.$$

- We have 3 critical points: $(0, 0)$, $\left(0, \dfrac{\sqrt{2}}{3}\right)$, and $\left(0, -\dfrac{\sqrt{2}}{3}\right)$.

$$f(0,0) = 0 + 0 + \frac{1}{0 + 9(0) + 1} = 1, \text{ and } f\left(0, \pm\frac{\sqrt{2}}{3}\right) = 0 + \frac{2}{9} + \frac{1}{0 + 9 \cdot \frac{2}{9} + 1} = \frac{5}{9}$$

$$1 > \frac{5}{9}, \text{ so the minimum value of } f(x, y) \text{ is } \frac{5}{9}.$$

### 9.4.2 Constrained Optimization

**a)** Find the maximum and minimum values of $f(x, y) = x^2 + y^2$ subject to the curve $17x^2 + 12xy + 8y^2 = 100$.

***Solution*:**

- Let $g(x, y) = 17x^2 + 12xy + 8y^2 - 100 = 0$. We will use Lagrange Multipliers.

$$\begin{aligned} &2x = \lambda(34x + 12y) && \text{[Equation 1]} \\ &2y = \lambda(12x + 16y) && \text{[Equation 2]} \\ &17x^2 + 12xy + 8y^2 - 100 = 0 && \text{[Equation 3]} \end{aligned}$$

- Isolate for $\lambda$ in the first 2 equations, and equate them.

$$\frac{x}{17x + 6y} = \frac{y}{6x + 8y} \Rightarrow 6x^2 + 8xy = 17xy + 6y^2 \Rightarrow 6x^2 - 6y^2 = 9xy \Rightarrow 8x^2 - 8y^2 = 12xy$$

- Substitute $8x^2 - 8y^2$ into equation 3 to solve for $x$.

$$17x^2 + 12xy + 8y^2 - 100 = 17x^2 + (8x^2 - 8y^2) + 8y^2 - 100 = 25x^2 - 100 = 0 \Rightarrow x = \pm 2$$

- Use the relationship $8x^2 - 8y^2 = 12xy$ to solve for $y$, given $x = \pm 2$.

$$x = 2\text{: } 32 - 8y^2 = 24y \Rightarrow 8y^2 + 24y - 32 = 8(y + 4)(y - 1) = 0 \Rightarrow y = -4 \text{ or } 1.$$
$$x = -2\text{: } 32 - 8y^2 = -24y \Rightarrow 8y^2 - 24y - 32 = 8(y - 4)(y + 1) = 0 \Rightarrow y = 4 \text{ or } -1.$$

- Now find the value of $f(x, y)$ given the candidate points $(2, -4)$, $(2, 1)$, $(-2, 4)$, $(-2, -1)$.

$$f(2, -4) = 20,\ f(2, 1) = 5,\ f(-2, 4) = 20,\ f(-2, -1) = 5$$

The maximum value of $f(x, y)$ is 20, and the minimum value of $f(x, y)$ is 5.