MATH 104/184

2017W1 Midterm 1

Review Package

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[36] **1. Short Problems**. Each question is worth 3 points. Put your answer in the box provided and show your work. No credit will be given for the answer without the correct accompanying work.

(a) Find the equation to the tangent line to $f(x) = x\cos x$ at $x = \pi/2$.

Answer:

(b) Evaluate $\lim_{x\to 1}\left(\frac{1}{x-1} - \frac{1}{x^2-x}\right)$.

Answer:

(c) Evaluate $\lim_{x\to -2}\frac{2x^2+x-6}{2x+4}$.

Answer:

(d) Evaluate $\lim_{h\to 0^-}\frac{1}{e^{2h}-1}$.

Answer:

(e) Let $f(x) = \dfrac{2x+6}{x+1}$.

Find the equations of all tangent lines to f which are parallel to the line $x + y = 2$.

Answer:

(f) Let $f(x)$ be the same function as above, $f(x) = \dfrac{2x+6}{x+1}$.

Find the equations of all tangent lines to f which are perpendicular to the line $x+y = 2$.

Answer:

(g) If $f(x)$ is a function satisfying $f(0) = 1$ and $f'(0) = 4$, find the equation of the tangent line to the graph of $g(x) = f(x)e^x$ at $x = 0$.

Answer:

(h) Let $f(x)$ be defined by

$$f(x) = \begin{cases} \dfrac{|x|^2}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Is $f(x)$ continuous at $x = 0$? Is it differentiable at $x = 0$?

Answer:

(i) Find numbers a and b that makes

$$f(x) = \begin{cases} e^x + a & \text{if } x > 2 \\ bx^2 + 1 & \text{if } 1 \leq x \leq 2 \\ 3x^3 - b & \text{if } x < 1 \end{cases}$$

$f(x)$ continuous for all real numbers. With these values, is $f(x)$ differentiable at $x = 1$?

Answer:

(j) Evaluate $\lim_{x \to 3} \dfrac{x - 3}{\sqrt{5x + 1} - 4}$.

Answer:

(k) Evaluate $\lim_{x \to 0} \dfrac{\sin 2x}{\sqrt{x^2 + 2}}$.

Answer:

(l) Evaluate $\lim_{x \to 2} \dfrac{x^4 - 16}{x^2 - 4}$.

Answer:

Long Problems. In questions 2 - 6, show your work. No credit will be given for the answer without the correct accompanying work.

[10] **2.**

(a) Carefully state what it means for a function $f(x)$ to be continuous at $x = a$.[3pts]

(b) Carefully state the definition of the derivative $f'(a)$ of a function $f(x)$ at $x = a$.[3pts]

(c) Let $f(x)$ be defined as

$$f(x) = \begin{cases} -3x^2 + |x| & \text{if } x < 0 \\ x^3 - x & \text{if } x \geq 0 \end{cases}$$

Is $f(x)$ continuous at $x = 0$? If so, what is $f(0)$? Is it differentiable at $x = 0$? If so, what is $f'(0)$?

Justify your answers. [4pts]

[10] **3**. Use the definition of the derivative as a limit to find $f'(x)$ for the following functions. No marks will be given for the use of differentiation rules.

(a) $f(x) = \dfrac{1}{\sqrt{x+1}}$. [5pts]

(b) $f(x) = \dfrac{x}{3x+2}$. [5pts]

[8] **4**. True or False. Justify your answer or give a counterexample.

(a) If $f(x)$ and $g(x)$ are differentiable at $x = a$, then $\dfrac{f(x)}{g(x)}$ is also differentiable at $x = a$. [2pts]

(b) If $\lim_{x\to a^+} f(x)$ and $\lim_{x\to a^-} f(x)$ exist, then $\lim_{x\to a}(f(x) + x)$ exists. [2pts]

(c) Let $f(x)$ be continuous on $[0, 1]$ and $f(0) = 0$, $f(1) = -1$. Then there is a $c \in [0, 1]$ such that $f(x) = -1/2$. [2pts]

(d) Let $f(x)$ be continuous on $[0, 1]$ and $f(0) = 0$, $f(1) = -1$. Then for all $c \in [0, 1]$, $\lim_{x\to c} \dfrac{f(x) - f(c)}{x - c} > 1$. [2pts]

[8] **5**.

(a) Let $f(x)$ be defined by

$$f(x) = \begin{cases} \cos x & \text{if } x < c \\ 2x + 7 & \text{if } x \geq c \end{cases}$$

for some number c. Show that there is a number c that makes $f(x)$ continuous on all real numbers. [4pts]

(b) Show that $f(x) = 2^x - 7x$ has a root in between 5 and 6. [4pts]

[8] **6**.

(a) Show that there are at least two solutions to $x^4 - 2x^3 + 3x^2 - 1 = 0$ in $[-1, 1]$. [4pts]

(b) Give an example of a function defined on all real numbers such that $f(0) = 0$, $f(1) = -1$ and $f(x) \neq 1/2$ for any $x \in [0, 1]$. [4pts]

[12] **7**. The CMP sells 200 review packs a term at \$60 each. They pay \$200 per term to rent their office and each pack costs \$10 to print. They discover that rival Perp001 are charging \$59.99 for review packs. So, they estimate that for every \$20 cheaper they can sell their packs, their sales will increase by 100 per term.

(a) Find the linear demand equation using p for price and q for for quantity

(b) What should they charge to maximize profit?

[12] **8**. The following year, the CMP notice their sales are still 200 review packs per term, even though they are only charging \$45 per pack. They find that sneaky rival Perp001 have slashed their pack price to \$40. In order to stay competitive, they cut a deal with a local printer to reduce printing costs to \$5 per pack and move into a broom closet office that costs \$100 per term.

Now they estimate that for every \$10 more they charge for their packs, their sales will decrease by 20 every term.

(a) Find the linear demand equation using p for price and q for for quantity

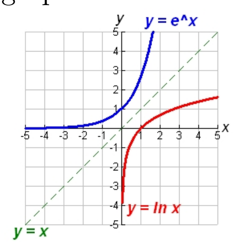
(b) What should they charge to maximize profit?

1. Cheat Sheet

1.1. **Preliminaries.**

- exp and log

  negative exponents $\rightarrow$ reciprocal
  fractional exponent $\rightarrow$ roots
  $b^x b^y = b^{x+y}$
  $\log(xy) = \log x + \log y$
  $(b^x)^y = b^{xy}$
  $\log(e^x) = x = e^{\log x}$ (where defined)
  $e^0 = 1, \log 1 = 0$
  graph:



- inverse functions, horizontal line test

1.2. **Business problem:**

- **quantity** q (number of widgets)
- **price** p (price per widget)
- **Revenue** R (money you receive when you sell q widgets for price p)
- **Cost** $C(q)$ (what it costs to make widgets)
  - **Fixed cost** F (costs that don't depend on the number of widgets)
  - **Variable cost** $V(q)$ (costs that depend on the number of widgets)
- **Break-Even Points** q such that $C(q) = R(q)$
- **Profit** P (the financial gain; the difference between amount earned and spent)
- **Demand** (equation that expresses the relation between quantity and price) - assumed linear for the MT1

In math:

$$R = pq$$
$$C(q) = F + V(q)$$
$$P = R - C$$
$$m = \frac{\Delta p}{\Delta q}$$
$$p - P_0 = m(q - Q_0)$$

given the data point (P_0, Q_0)

1.3. **Limits.**

Limits may not exist for many reasons. If they do, define:

- **Left-hand limit:** $\lim_{x \to a^-} f(x) = K$
  $f(x)$ gets closer to K as x approaches a from the left, i.e. $x < a$
- **Right-hand limit:** $\lim_{x \to a^+} f(x) = L$
  $f(x)$ gets closer to L as x approaches a from the right, i.e. $x > a$
- **Limit:** $\lim_{x \to a} f(x) = L$
  $f(x)$ gets closer to L as x approaches a both from the right or the left

Some strategies

- If left limit and right limit exist but are not equal, the limit DNE.
- The limit does not have to be the value of the function.
- Rational function (polynomial divided by polynomial) - if you get 0/0, then factor
- Rationalize denominator
- Limit laws, e.g. the sum of limits is the limit of sums, etc
- If you know the function is continuous at the point, then we can 'plug in'. Continuous functions include polynomials, exp, sin, cos, rational functions where the denominator is not 0. Also sums, products, compositions of such functions.
- Some of the ways functions can go bad: dividing by zero, roots of negatives, log of $x \leq 0$, going outside the defined domain.
- If a rational function $\to \pm\infty$, decide sign by checking signs of numerator and denominator

1.4. **Continuity.**

- A function $f(x)$ is **continuous** at a if $\lim_{x \to a} f(x) = f(a)$

1.5. **Intermediate Value Theorem.**

Memorize this theorem!

- Let f be continuous on $[a, b]$. Then for any Y between $f(a)$ and $f(b)$, we can find a $c \in [a, b]$ with $f(c) = Y$.
  This just means f takes every value (y-coord) in between its endpoint values

1.6. **Derivatives.**

- Rate of change: think slope. Secants give average rate of change; tangents are instantaneous rates of change.

Memorize this definition!

- Let $a \in \mathbf{R}$ and $f(x)$ be defined on an open interval containing a. The **derivative** of $f(x)$ at x=a is defined by

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

  if the limit exists.

1.7. **Differentiation.**

$$\frac{d}{dx}x^n = nx^{n-1}$$

$$\frac{d}{dx}\Big(f(x)+g(x)\Big) = f'(x)+g'(x)$$

$$\frac{d}{dx}\Big(cf(x)\Big) = cf'(x)$$

$$\frac{d}{dx}\Big(f(x)g(x)\Big) = f'(x)g(x)+f(x)g'(x)$$

$$\frac{d}{dx}\Big(\frac{f(x)}{g(x)}\Big) = \frac{f'(x)g(x)-f(x)g'(x)}{g(x)^2}$$

$$\frac{d}{dx}e^x = e^x$$

- Finding equations of tangent lines