commerce undergraduate society

COMMERCE MENTORSHIP PROGRAM

MIDTERM REVIEW SESSION

MATH 104

PREPARED BY

ETHAN BITUN

TABLE OF CONTENTS

Limits:

The limit of a function is a value that the function approaches as the input approaches a value.

$\lim\limits_{x \to a} f(x) = c$ is an example of what a limit equation looks like.

- To interpret this equation, as ***x*** approaches ***a***, the **limit** of ***f(x)*** approaches ***c.***

For this limit to exist at ***a***, f(x) must approach the same value from both sides of ***a.***

$$\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x)$$

- a^+ means we are approaching a from the right (positive) side
- a^- means we are approaching a from the left (negative) side

To evaluate a limit, you can try evaluating ***f(x)*** when ***x = a***. If ***f(a)*** gives you a valid numerical answer, then no further work is needed. Sometimes you may end up with an invalid answer such as a number divided by 0. This is an **indeterminate form**, that can be solved with algebraic manipulation. We will go through them in the examples.

One more thing, when we say

- ***"***As ***x*** approaches ***a***, the **limit** of ***f(x)*** approaches ***c."***

This is **not** the same thing as proving

- *"When **x = a, f(x) = c.**"*

f(x) does **not have to** equal ***c*** to approach it, hence the existence of indeterminate forms.

Problems with Limits:

1. Evaluate:

a) $\lim\limits_{x \to 11} \frac{2x^2 - 21x - 11}{x^2 - 6x - 55}$

Answer:

b) $\lim\limits_{x \to 0} \frac{x}{5 - \sqrt{x + 25}}$

Answer:

c) $\lim\limits_{x \to 2} \frac{2x^2 - 7x - 4}{x + 4}$

Answer:

d) $\lim\limits_{x \to \sqrt{2}} \frac{x-2}{x(x-\sqrt{2})}$

Answer:

2. Evaluate the following limit **<u>if</u>** it exists. If it does not exist, state so.

$$f(x) = \begin{cases} -2(x+1)+2 & x < 0 \\ e^x + 1 & x \geq 0 \end{cases}$$

<u>Continuity:</u>

A function ***f(x)*** is continuous at ***x = a*** when

1) ***f(a)*** is defined — ***f(x)*** is a function, and ***a*** exists in the domain of ***f(x)***
2) $\lim_{x \to a} f(x)$ exists — As ***x*** approaches ***a***, the limit of ***f(x)*** **exists**
3) $\lim_{x \to a} f(x) = f(a)$ — The limit of ***f(x)*** as ***x*** approaches ***a*** is the same as ***f(a)***

When we say continuous, we mean that the graph of the function is **unbroken**. A common explanation is that you can draw any two points on the graph without lifting your pen from your paper.

Some types of polynomial functions such as ***sin(x)*** or ***cos(x)*** are continuous throughout their whole domains. Proving this is outside the scope of MATH 104, so you can just say they are continuous without providing proofs.

Continuity Problem:

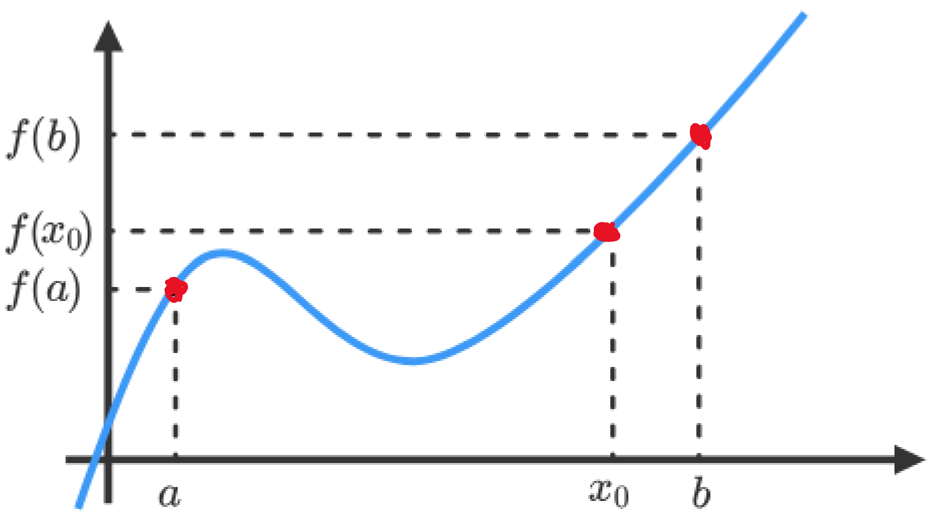
3. Given the restriction of ***a* > 1,** find the values for ***a*** and ***b*** such that ***f(x)*** is continuous for all real numbers

$$\boldsymbol{f(x)\begin{cases}(x-2)^2+b & x>2\\ \dfrac{1}{2}x+a & -2\le x\le 2\\ -(x+a)^2+3 & x<-2\end{cases}}$$

Intermediate Value Theorem (IVT):

Formal Definition: If ***f*** is a continuous function whose domain is in the interval [***a***,***b***], then there exists at least one ***c*** within [***a***,***b***] such that ***f(a)*** < ***f(c)*** < ***f(b)***.

This theorem explains a simple concept. If we have two points, (***a, f(a)***) and (***b, f(b)***) on a continuous function, then the function must go through every ***y-value*** between ***f(a)*** and ***f(b).*** This means you can find any ***y-value*** between ***f(a)*** and ***f(b)***, with at least one corresponding ***x-value*** between ***a*** and ***b***.



In this graph, we can see that ***f(x0)*** is a value between ***f(a)*** and ***f(b)***, and it exists at ***x0***, which is between ***a*** and ***b***.

IVT tells us, without using the graph, that there is at least one ***x*** value between ***a*** and ***b*** where ***f(x) = f(x0).***

Using IVT:

4. Prove that this equation has a root using IVT

$$f(x) = -2x^3 - 3x^2 + 4$$

Derivatives:

The derivative is the **instantaneous rate of change** of a function at a specific point.

The limit definition of a derivative of ***f(x)*** is

$$\boldsymbol{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

f(x) is differentiable, where this limit **exists**.

f'(a), aka the derivative of ***f(x)*** at ***x = a*** can be described by

$$\boldsymbol{f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}}$$

Also note: The derivative at a point (in this case, ***a***) is equivalent to the slope of the tangent line of ***f(x)*** at ***x = a***.

You can also use differentiation rules to find the derivatives of functions depending on what the question allows you to do.

One notable rule is the power rule because of its widespread use. Other rules can be found in the appendix of this package.

$\boldsymbol{\frac{d}{dx}(x^n) = nx^{n-1}}$, Power rule.

Problems involving Differentiation:

5.

a) Carefully state the limit definition of the derivative of a function $\boldsymbol{y = f(x)}$

b) Use the limit definition of the derivative to find ***f'(4)*** for the following function. No marks will be given for the use of differentiation rules.

$$\boldsymbol{f(x) = x^2 + 2}$$

6. Let $\boldsymbol{g(x) = [f(x)]^3 + 5e^{f(x)} + 2}$, $\boldsymbol{f(2) = 1, f'(2) = 3}$

Find $\boldsymbol{g'(2)}$

Lengthier Problems:

7. Mr. Smith sells crates of watermelons. Currently, he sells 70 crates of watermelons for \$20 each. He purchases a watermelon packager that costs \$100. Each crate costs \$15 to produce. His customers are price sensitive. As the price increases, less people will buy. He estimates that every \$2 increase in price results in 4 less customers purchasing watermelons.

a) Find the linear demand function for the crates as a function of quantity (***q***)

b) Find the Revenue function as a function of quantity (***q***)

c) How much should Mr. Smith charge for each crate of watermelons to maximize his profit?

8. Find the equation of the tangent line to $\boldsymbol{f(x) = 2x^2 + 5x + 2}$ when $\boldsymbol{x = -1}$

Challenge Question:

9. Find the equation of the line that passes through (0, -3) and is tangent to the graph of

$$\boldsymbol{f(x) = x^3 + 2x^2 + 1}$$

Appendix

Formulas pertaining to limits:

1. $\lim_{x \to a}[p(x) + g(x)] = \lim_{x \to a} p(x) + \lim_{x \to a} g(x)$

2. $\lim_{x \to a}[p(x) - g(x)] = \lim_{x \to a} p(x) - \lim_{x \to a} g(x)$

3. For every real number k,

$\lim_{x \to a}[kp(x)] = k \lim_{x \to a} p(x)$

4. $\lim_{x \to a}[p(x)\ q(x)] = \lim_{x \to a} p(x) \times \lim_{x \to a} q(x)$

5. $\lim_{x \to a} \dfrac{p(x)}{q(x)} = \dfrac{\lim_{x \to a} p(x)}{\lim_{x \to a} q(x)}$

Formulas on differentiation

Basic Derivatives Rules

Constant Rule: $\dfrac{d}{dx}(c) = 0$

Constant Multiple Rule: $\dfrac{d}{dx}[cf(x)] = cf'(x)$

Power Rule: $\dfrac{d}{dx}(x^n) = nx^{n-1}$

Sum Rule: $\dfrac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$

Difference Rule: $\dfrac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x)$

Product Rule: $\dfrac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$

Quotient Rule: $\dfrac{d}{dx}\left[\dfrac{f(x)}{g(x)}\right] = \dfrac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$

Chain Rule: $\dfrac{d}{dx} f(g(x)) = f'(g(x))g'(x)$

Terms Relating to Application of Commerce.

Price = p
Quantity = q
Revenue = $R(q)$ (Revenue as a function of Quantity) = $p*q$ (Price times Quantity)
Cost = $C(q)$ (Cost as a function of Quantity) = Fixed Cost + Variable Cost

- Fixed Cost is a constant, its whatever you pay upfront
- Variable Cost depends on q (Quantity), as each additional item adds cost

Break-Even Point: $R(q) = C(q)$ or when **Profit = 0**
Profit/Loss = $R(q) – C(q)$ or **Revenue – Cost**
Marginal Cost = $C'(q)$ (The derivative of the cost function)
Marginal Revenue = $R'(q)$ (The derivative of the revenue function)

Marginal in this case means one more unit. Marginal Cost is the cost of producing **one more** unit, and Marginal Revenue is the revenue gained from selling **one more** unit.

"Marginal" and derivatives are related because they both measure the instant rate of change at a point.

Profit is maximized where:

- $P'(q) = 0$, or
- $MR = MC$, or
- ***Marginal Profit = 0***.