1.2 Practice problems

Describe the asymptotics of the following expressions as $x \to \infty$, $x \to -\infty$ and $x \to 0$. (i.e. find simplified expressions for the following as $x \to \infty$, $x \to -\infty$, $x \to 0$)

1. $\frac{x^4+e^x-\cos(x)}{2}$

As $x \to \infty$, e^x dominates in the numerator. $\boxed{\frac{x^4+e^x-\cos(x)}{2} \sim \frac{e^x}{2}}$

As $x \to -\infty$, x^4 dominates in the numerator (e^x approaches 0). $\boxed{\frac{x^4+e^x-\cos(x)}{2} \sim \frac{x^4}{2}}$

As $x \to 0$, e^x and $-\cos(x)$ are both significant. $\boxed{\frac{x^4+e^x-\cos(x)}{2} \sim \frac{e^x-\cos(x)}{2}}$

Remark: In all cases, we cannot neglect 2 in the denominator. This is because eliminating it would multiply the function's output by 2. When simplifying expressions asymptotically, make sure all neglected terms have no measurable effect on the original expression as $x \to$ a certain value.

2. $2^x + 100000\cos(30000x)$

For all x, $100000\cos(30000x)$ is bound between -100000 and 100000.

As $x \to \infty$, 2^x dominates. $\boxed{2^x + 100000\cos(30000x) \sim 2^x}$

As $x \to -\infty$, $100000\cos(30000x)$ dominates. $\boxed{2^x + 100000\cos(30000x) \sim 100000\cos(30000x)}$

As $x \to 0$, $100000\cos(30000x)$ dominates (2^x approaches 1, $100000\cos(30000x)$ approaches 100000). $\boxed{2^x + 100000\cos(30000x) \sim 100000\cos(30000x)}$

3. $\frac{x^2+\sqrt{|x|}-2}{x^2+1}$

As $x \to \infty$, x^2 dominates in the numerator and x^2 dominates in the denominator.

$$\boxed{\frac{x^2+\sqrt{|x|}-2}{x^2+1} \sim \frac{x^2}{x^2} = 1}$$

As $x \to -\infty$ x^2 dominates in the numerator and x^2 dominates in the denominator

$$\boxed{\frac{x^2+\sqrt{|x|}-2}{x^2+1} \sim \frac{x^2}{x^2} = 1}$$

As $x \to 0$, -2 dominates in the numerator and $+1$ dominates in the denominator

$$\boxed{\frac{x^2+\sqrt{x}-2}{x^2+1} \sim \frac{-2}{1} = -2}$$

2.4 Practice problems

Evaluate the following limits, or show that they do not exist.

1. $\lim_{x\to 0} x^5 - 32x^3 + 1$

$\lim_{x\to 0} (x^5 - 32x^3 + 1) = (0)^5 - 32(0)^3 + 1$

$\lim_{x\to 0} (x^5 - 32x^3 + 1) = 1$

2. $\lim_{x\to -\frac{1}{2}} \dfrac{-|\frac{1}{3x^2}| + \frac{1}{x^2-1}}{2x+1}$

Since $\frac{1}{3x^2} > 0$ for all $x \neq 0$, remove the abs. value. $\lim_{x\to -\frac{1}{2}} \dfrac{-|\frac{1}{3x^2}| + \frac{1}{x^2-1}}{2x+1} = \lim_{x\to -\frac{1}{2}} \dfrac{-\frac{1}{3x^2} + \frac{1}{x^2-1}}{2x+1}$

$\lim_{x\to -\frac{1}{2}} \dfrac{-\frac{1}{3x^2} + \frac{1}{x^2-1}}{2x+1} = \lim_{x\to -\frac{1}{2}} \left(\dfrac{-(x^2-1)}{3x^2(x^2-1)} + \dfrac{3x^2}{3x^2(x^2-1)}\right)/(2x+1)$

$\lim_{x\to -\frac{1}{2}} \dfrac{-\frac{1}{3x^2} + \frac{1}{x^2-1}}{2x+1} = \lim_{x\to -\frac{1}{2}} \left(\dfrac{2x^2+1}{3x^2(x^2-1)(2x+1)}\right)$ Can't be simplified.

Let's check left- and right-handed limits

$\lim_{x\to -\frac{1}{2}^-} \left(\dfrac{2x^2+1}{3x^2(x^2-1)(2x+1)}\right) = \dfrac{\text{pos. number}}{(\frac{3}{4})(-\frac{3}{4})(\text{very small neg. \#})} = +\infty$

$\lim_{x\to -\frac{1}{2}^+} \left(\dfrac{2x^2+1}{3x^2(x^2-1)(2x+1)}\right) = \dfrac{\text{pos. number}}{(\frac{3}{4})(-\frac{3}{9})(\text{very small pos. \#})} = -\infty$

Since the left- and right-handed limits aren't equal, $\boxed{\lim_{x\to -\frac{1}{2}} \dfrac{-|\frac{1}{3x^2}| + \frac{1}{x^2-1}}{2x+1} = DNE}$

3. $\lim_{x\to -1} \sqrt{\dfrac{x^3 + x^2 + x + 1}{3x^2 + 3x}}$

For all $x < 0$, $\dfrac{x^3+x^2+x+1}{3x^2+3x}$ is negative and the expression $\sqrt{\dfrac{x^3+x^2+x+1}{3x^2+3x}}$ is undefined. Proof:

$\sqrt{\dfrac{x^3+x^2+x+1}{3x^2+3x}} = \sqrt{\dfrac{(x^2+1)(x+1)}{3x(x+1)}} = \sqrt{\dfrac{x^2+1}{3x}}$ (this expression is equal to $\sqrt{\dfrac{x^3+x^2+x+1}{3x^2+3x}}$ at all x except $x=-1$)

x^2+1 is always positive, $3x$ is always negative for negative x inputs.

So $\boxed{\lim_{x\to -1} \sqrt{\dfrac{x^3+x^2+x+1}{3x^2+3x}} = DNE}$

4. $\lim_{j\to 1} \dfrac{3 - 3j}{2 - \sqrt{5 - j}}$

$\lim_{j\to 1} \dfrac{3-3j}{2-\sqrt{5-j}} = \lim_{j\to 1} \dfrac{(3-3j)(2+\sqrt{5-j})}{(2-\sqrt{5-j})(2+\sqrt{5-j})}$

$= \lim_{j\to 1} \dfrac{6+3\sqrt{5-j}-6j-3j\sqrt{5-j}}{4-(5-j)}$

$= \lim_{j\to 1} \dfrac{6-6j+3\sqrt{5-j}-3j\sqrt{5-j}}{-1+j}$

$= \lim_{j\to 1} \dfrac{6(1-j)+3\sqrt{5-j}(1-j)}{(-1+j)}$

$= \lim_{j\to 1} \dfrac{-6(1-j)-3\sqrt{5-j}(1-j)}{(1-j)}$

$= \lim_{j\to 1} \dfrac{(-6-3\sqrt{5-j})(1-j)}{(1-j)}$

$= \lim_{j\to 1} (-6-3\sqrt{5-j})$

$= -6-3\sqrt{5-1}$

$= -6-3(2)$

$\boxed{= -12}$

5. $\lim_{x\to 2} (x-2)^3 \cos[(\frac{x^3+2}{x^2+3})^3 + 999]$

This looks too complicated to try and simplify. Let's try to apply the squeeze theorem. Let $\cos[(\frac{x^3+2}{x^2+3})^3 + 999]$ be denoted as $\cos(\ldots)$

$-1 \leq \cos(\ldots) \leq 1$ for all $x \in \mathbb{R}$

$-(x-2)^3 \leq (x-2)^3\cos(\ldots) \leq (x-2)^3$

If $\lim_{x\to 2} -(x-2)^3 = \lim_{x\to 2} (x-2)^2$, $\lim_{x\to 2} (x-2)^2 \cos(\ldots)$ equals those limits.

$\lim_{x\to 2} -(x-2)^3 = -(2-2)^3 = 0$

$\lim_{x\to 2} (x-2)^3 = (2-2)^3 = 0$

Hence by squeeze theorem,

$\boxed{\lim_{x\to 2} (x-2)^2 \cos[(\frac{x^3+2}{x^2+3})^3 + 999] = 0}$

Evaluate the following limits at infinity, or show that they do not exist.

1. $\lim_{x\to-\infty} \dfrac{5x^2+4x-1}{3x^3+2}$

$$\lim_{x\to-\infty} \frac{5x^2+4x-1}{3x^3+2} = \lim_{x\to-\infty} \frac{x^3\left(\frac{5}{x}+\frac{4}{x^2}-\frac{1}{x^3}\right)}{x^3\left(3+\frac{2}{x^3}\right)}$$

$$\lim_{x\to-\infty} \frac{5x^2+4x-1}{3x^3+2} = \lim_{x\to-\infty} \frac{\frac{5}{x}+\frac{4}{x^2}-\frac{1}{x^3}}{3+\frac{2}{x^3}}$$

All terms approach 0 except the constant 3 in the denominator.

So, $\boxed{\lim_{x\to-\infty} \dfrac{5x^2+4x-1}{3x^3+2} = 0}$.

Alternatively, use an asymptotic argument. As $x\to-\infty$

$$\frac{5x^2+4x-1}{3x^3+2} \sim \frac{5x^2}{3x^3} = 0 \text{ because } |3x^3| \gg |5x^2|$$

Therefore $\lim_{x\to-\infty} \dfrac{5x^2+4x-1}{3x^3+2} = 0$.

2. $\lim_{x\to\infty} \sqrt{x^2+6x}-x$

$$\lim_{x\to\infty} \sqrt{x^2+6x}-x = \lim_{x\to\infty} \frac{(\sqrt{x^2+6x}-x)(\sqrt{x^2+6x}+x)}{\sqrt{x^2+6x}+x}$$

$$= \lim_{x\to\infty} \frac{(x^2+6x)-x^2}{\sqrt{x^2+6x}+x}$$

$$= \lim_{x\to\infty} \frac{6x}{\sqrt{x^2+6x}+x} = \lim_{x\to\infty} \frac{6x}{\sqrt{x^2(1+\frac{6}{x})}+x}$$

$$= \lim_{x\to\infty} \frac{6x}{\sqrt{x^2}\sqrt{1+\frac{6}{x}}+x} = \lim_{x\to\infty} \frac{6x}{x\sqrt{1+\frac{6}{x}}+x}$$

$$= \lim_{x\to\infty} \frac{6}{\sqrt{1+\frac{6}{x}}+1} = \frac{6}{\sqrt{1}+1}$$

$$\boxed{=3}$$

3. $\lim_{x\to-\infty} \dfrac{\sin(x)}{x^2+3x+2}$

As $x\to-\infty$ $-1\le \sin(x)\le 1$, while $x^2+3x+2 \sim x^2$, $x^2\to+\infty$, so essentially, the expression becomes a number in the range $[-1,1]$ divided by something much bigger. So $\boxed{\lim_{x\to-\infty} \dfrac{\sin(x)}{x^2+3x+2} = 0}$.

4. $\lim_{x\to-\infty} \dfrac{x+1}{|\sqrt{x^2}|}$

The absolute value sign is meaningless since $\sqrt{x^2}\ge 0$ for all $x\in\mathbb{R}$. $\lim_{x\to-\infty} \dfrac{x+1}{|\sqrt{x^2}|} = \lim_{x\to-\infty} \dfrac{x+1}{\sqrt{x^2}}$

$$\lim_{x\to-\infty} \frac{x+1}{\sqrt{x^2}} = \lim_{x\to-\infty} \frac{x+1}{-x}$$

$$= \lim_{x\to-\infty} \frac{x(1+\frac{1}{x})}{x(-1)}$$

$$= \lim_{x\to-\infty} \frac{1+\frac{1}{x}}{-1} = \frac{1}{-1}$$

$$\boxed{=-1}$$

Remark: $\sqrt{x^2}=|x|$ (graph them!). Whenever we deal with negative x, we can remove $|x|$ by writing $-x$ in its place instead.

5. $\lim_{x\to0^+} \dfrac{x^2+\frac{1}{x}}{x-1}$

with $x\to0^+$, $\dfrac{x^2+\frac{1}{x}}{x-1} \sim \dfrac{\frac{1}{x}}{-1}$. Thus, as $x\to0^+$, the numerator $x^2+\frac{1}{x}$ will "blow up" to $+\infty$ while the denominator is -1. So $\boxed{\lim_{x\to0^+} \dfrac{x^2+\frac{1}{x}}{x-1} = -\infty}$

3 Continuity

3.1 Conditions for continuity

For a function $f(x)$ to be continuous at $x = a$, three conditions must be satisfied;

1. $f(a)$ exists.
2. $\lim_{x\to a} f(x)$ exists; that is, both one-sided limits $\lim_{x\to a^+} f(x)$ and $\lim_{x\to a^-} f(x)$ are equal.
3. $f(a) = \lim_{x\to a} f(x)$.

Intuitively, a function that is continuous at a point is **connected** at that point; the part of the function to the left of that point is connected to the part of the function to the right of that point.

3.2 Continuity of common functions

The following functions are continuous everywhere on intervals on which they are defined.

- Polynomials, constants, exponentials, $\cos(x)$, $\sin(x)$. These are also defined for all $x \in \mathbb{R}$
- Non-polynomial power functions, rational functions
- $\tan(x)$, logarithms

By arithmetic of continuity, the sum, difference, product and quotient (if the denominator $\neq 0$) of functions continuous at $x = a$ returns a function also continuous at $x = a$.

3.3 Practice problems

1. Where is the function $f(x) = \frac{sin(x)+3}{x^2-2x+1}$ contiunous?

We have a sum of two functions continuous everywhere in the numerator, and a polynomial, a type of function always continuous everywhere in the denominator, hence $f(x)$ should be continuous everywhere by arithmetic of continuity, unless there are x at which the denominator is 0. Let's check if there are such x.

$x^2-2x+1=0$
$x^2-x-x+1=0$
$x(x-1)-1(x-1)=0$
$(x-1)^2=0$
$x-1=0$
$x=1$, $\boxed{\text{so, } f(x) \text{ is continuous everywhere except at } x=1 \text{ where } f(x) \text{ is undefined}}$

2. Find the values b, c such that $f(x)$ is continuous everywhere.

$$f(x) = \begin{cases} 6 - cx & \text{if } x < 2c \\ x^2 & \text{if } x > 2c \\ b & \text{if } x = 2c \end{cases}$$

Since $f(x)$ is always defined by polynomials (that are continuous everywhere) on each subdomain, we need the function to be continuous at x where the function "switches" subfunctions, at $x = 2c$.

If $f(2c) = \lim_{x\to 2c^-} f(x) = \lim_{x\to 2c^+} f(x)$, $f(x)$ will be continuous everywhere. To first solve for c, let's equate the one-sided limits together.

$\lim_{x\to 2c^-}(6-cx) = \lim_{x\to 2c^+}(x^2)$

$6-2c^2=4c^2$

$6=6c^2$

$c^2=1$

$c=\pm 1$.

Use either one-sided limit to solve for b.

$f(2c) = \lim_{x\to 2c^+} f(x)$

$b = 4c^2$

$b = 4(\pm 1)^2$

$\boxed{b=4}$

So $\boxed{b=4,\ c=\pm 1}$

4.4 Practice problems

1. Using the definition of the derivative, find $f'(a)$ given $f(x) = 2x^2 + 1$

Use either definition of the derivative.

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$
$$f'(a) = \lim_{x \to a} \frac{(2x^2+1) - (2a^2+1)}{x - a}$$
$$f'(a) = \lim_{x \to a} \frac{2x^2 - 2a^2}{x - a}$$
$$f'(a) = \lim_{x \to a} \frac{2(x^2 - a^2)}{x - a}$$
$$f'(a) = \lim_{x \to a} \frac{2(x+a)(x-a)}{(x-a)}$$
$$f'(a) = \lim_{x \to a} 2(x+a)$$
$$f'(a) = 2(2a)$$
$$\boxed{f'(a) = 4a}$$

OR

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$
$$f'(a) = \lim_{h \to 0} \frac{2(a+h)^2 + 1 - (2a^2+1)}{h}$$
$$f'(a) = \lim_{h \to 0} \frac{2a^2 + 4ah + 2h^2 + 1 - 2a^2 - 1}{h}$$
$$f'(a) = \lim_{h \to 0} \frac{4ah + 2h^2}{h}$$
$$f'(a) = \lim_{h \to 0} 4a + 2h \Rightarrow \boxed{f'(a) = 4a}$$

2. Use the definition of the derivative to compute the derivative of the function $f(x) = \frac{1}{x^2+3}$

We must use $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ since we want the derivative as a function.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
$$f'(x) = \lim_{h \to 0} \frac{\frac{1}{(x+h)^2+3} - \frac{1}{x^2+3}}{h}$$
$$f'(x) = \lim_{h \to 0} \left(\frac{(x^2+3)}{((x+h)^2+3)(x^2+3)} - \frac{(x+h)^2+3}{((x+h)^2+3)(x^2+3)} \right) \cdot \frac{1}{h}$$
$$f'(x) = \lim_{h \to 0} \left(\frac{x^2+3-x^2-2xh-h^2-3}{((x+h)^2+3)(x^2+3)} \right) \cdot \frac{1}{h}$$
$$f'(x) = \lim_{h \to 0} \frac{-2xh - h^2}{((x+h)^2+3)(x^2+3)} \cdot \frac{1}{h}$$
$$f'(x) = \lim_{h \to 0} \frac{-2x - h}{((x+h)^2+3)(x^2+3)}$$
$$f'(x) = \frac{-2x}{(x^2+3)(x^2+3)}$$
$$\boxed{f'(x) = \frac{-2x}{(x^2+3)^2}}$$

3. Is the following function differentiable at $x = 0$? Justify your answer using the definition of the derivative.

$$f(x) = \begin{cases} x\cos(x) & \text{if } x \geq 0 \\ \sqrt{x^2 + x^4} & \text{if } x < 0 \end{cases}$$

If $f(x)$ is differentiable at $x=0$, $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ with $a = 0$ must exist (or equivalently, $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$). Take the left- and right-handed limits.

$$\lim_{x \to a^-} \frac{f(x) - f(a)}{x - a} = \lim_{x \to 0^-} \frac{f(x) - f(0^-)}{x - 0^-}$$
$$= \lim_{x \to 0^-} \frac{\sqrt{x^2+x^4} - \sqrt{(0)^2 + (0)^4}}{x - 0^-}$$
$$= \lim_{x \to 0^-} \frac{\sqrt{x^2+x^4}}{x}$$
$$= \lim_{x \to 0^-} \frac{\sqrt{x^2(1+x^2)}}{x}$$
$$= \lim_{x \to 0^-} \frac{-x(1+x^2)}{x}$$
$$= \lim_{x \to 0^-} -(1+x^2)$$
$$= -1.$$

Right-handed limit:

$$\lim_{x \to a^+} \frac{f(x) - f(a)}{x - a} = \lim_{x \to 0^+} \frac{f(x) - f(a)}{x - a}$$
$$= \lim_{x \to 0^+} \frac{x\cos(x) - (0^+)\cos(0^+)}{x - 0^+}$$
$$= \lim_{x \to 0^+} \frac{x\cos(x)}{x}$$
$$= \lim_{x \to 0^+} \cos(x)$$
$$= 1$$

Since the left- and right-sided limit definitions of the derivative are NOT equal, $\boxed{f(x) \text{ is NOT differentiable at } x = 0}$.

4. Let $f(x) = x^2 + 3x + 1$, with an "artificially" restricted domain $x \in [-1000, 1000]$. On what interval does the derivative exist?

$f(x)$ is a polynomial and polynomials are differentiable (i.e. continuous and smooth) at all x when the domain is unrestricted. Therefore, on $[-1000, 1000]$, $f(x)$ is $\boxed{\text{differentiable everywhere on } (-1000, 1000)}$. It isn't differentiable on the endpoints of $[-1000, 1000]$ since we can't take the limit definition of the derivative from the left at $x = -1000$, and the right at $x = 1000$.

5.2 Practice problems

1. Let $f(x) = \frac{h(g(x))}{xm(x)}$, where $h(x)$, $g(x)$ and $m(x)$ are functions. Differentiate $f(x)$, expressing your answer only in terms of x, $h(x)$, $g(x)$, $m(x)$ and their derivatives.

By quotient rule on the outermost level,

$$\boxed{f'(x) = \frac{\overbrace{h'(g(x))g'(x)}^{\text{Derivative of numerator}}\cdot \overbrace{xm(x)}^{\text{Denominator}} - \overbrace{h(g(x))}^{\text{Numerator}}\cdot\overbrace{(m(x) + xm'(x))}^{\text{Derivative of numerator}}}{\underbrace{x^2(m(x))^2}_{\text{Denominator, squared}}}}$$

2. What is the slope of the graph $y = (\frac{\cos(x)}{x^2+2})^2$ at $x = 1$?

Slope of the graph at $x=1$ is $y'|_{x=1}$ (the derivative at $x=1$)

$$\boxed{y'|_{x=1} = 0}$$

$$\frac{dy}{dx} = y' = 2\left(\frac{\cos(x)}{x^2+2}\right)\cdot\left(\frac{-\sin(x)(x^2+2) - \cos(x)(2x)}{(x^2+2)^2}\right)$$

$$y'|_{x=1} = 2\left(\frac{\cos(1)}{1^2+2}\right)\cdot\left(\frac{-\sin(1)(1^2+2) - \cos(1)(2(1))}{(1^2+2)^2}\right)$$

$$y'|_{x=1} = 2\left(\frac{0}{3}\right)\cdot\left(\frac{(-1)(3) - (0)(2)}{(3)^2}\right)$$

3. $f(x) = e^{e^{e^x}}$. Find $f'(x)$.

We apply the chain rule thrice.

$$f'(x) = e^{(e^{e^x})}\cdot\frac{d}{dx}(e^{e^x})$$

$$f'(x) = e^{e^{e^x}}\cdot\left(e^{(e^x)}\cdot\frac{d}{dx}e^x\right)$$

$$f'(x) = e^{e^{e^x}}\cdot e^{e^x}\cdot e^x$$

$$\boxed{f'(x) = e^{(e^{e^x} + e^x + x)}}$$

4. Use the product rule to differentiate $g(x) = (x^2 + 2\log x)(8\sqrt{x})$

$$g'(x) = \frac{d}{dx}(x^2 + 2\log x)(8\sqrt{x}) + (x^2 + 2\log x)\frac{d}{dx}(8\sqrt{x})$$

$$g'(x) = \left(2x + \frac{2}{x}\right)(8\sqrt{x}) + (x^2 + 2\log x)\left(\frac{8}{2\sqrt{x}}\right)$$

$$g'(x) = 16x^{(\frac{3}{2})} + \frac{16\sqrt{x}}{x} + \frac{8x^2}{2\sqrt{x}} + \frac{16\log(x)}{2\sqrt{x}}$$

$$\boxed{g'(x) = 16(\sqrt{x})^3 + \frac{16}{\sqrt{x}} + 4(\sqrt{x})^3 + \frac{8\log(x)}{\sqrt{x}}}$$

6.3 Practice problems

1. Given $f(x) = x^3 + 2x$, find the equation for the tangent to $f(x)$ at $x = 3$

1. Find slope of $f(x)$ at $x=3$
$f'(x) = 3x^2 + 2$
$f'(3) = 3(3)^2 + 2$
$f'(3) = 29$

2. Find a point on the tangent line.
$f(3) = (3)^3 + 2(3)$
$f(3) = 33$
point: $(3, 33)$

3. Solve for b in $y = mx + b$.
$33 = (29)(3) + b$
$b = -54$

$\boxed{y = 29x - 54}$

2. Use the linear approximation to approximate $\sqrt{5}$

Linear approx: $f(x) \approx f'(a)(x-a) + f(a)$
We need a function that outputs $\sqrt{5}$, whose nearby values are similar. We can use $f(x) = \sqrt{x}$. We need a known point on $f(x)$ near the point $(5, \sqrt{5})$. Let's use $(4, \sqrt{4}) \Rightarrow (4, 2)$. Let's solve for $f'(4)$. $f'(x) = \frac{1}{2\sqrt{x}}$, $f'(4) = \frac{1}{4}$.

$f(x) \approx \frac{1}{4}(x-4) + 2$, for x near $a = 4$

$f(5) \approx \frac{1}{4}(5-4) + 2$

$\boxed{f(5) \approx \frac{9}{4}}$

3. Use the linear approximation to approximate $\log(3)$

Linear approx: $f(x) \approx f'(a)(x-a) + f(a)$
We choose $f(x) = \log(x)$, and approximate $f(3) = \log(3)$. A nearby x-value that gives a known output is $a = e$.

$f'(x) = \frac{1}{x}$, $f'(e) = \frac{1}{e}$

$f(e) = \log(e) = 1$

$f(x) \approx \frac{1}{e}(x-e) + 1$, for x near $a = e$

$f(3) \approx \frac{1}{e}(3-e) + 1$

$f(3) \approx \frac{3}{e} - 1 + 1$

$\boxed{f(3) \approx \frac{3}{e}}$

7.3 Practice problems

1. Find the derivative of the curve $xy^2 + x^2y = 2$ at the point $(1, 1)$.

$xy^2 + x^2y = 2$, where y is a function of x.

$\frac{d}{dx}(xy^2 + x^2y) = \frac{d}{dx}(2)$

Differentiate the LHS by product rule and chain rule.

$((1)(y^2) + (x)(2y \cdot y')) + ((2x)(y) + (x^2)(y')) = 0$

$y^2 + 2xy \cdot y' + 2xy + x^2y' = 0$

$2xy \cdot y' + x^2y' = -2xy - y^2$

$y'(2xy + x^2) = -2xy - y^2$

$y' = \frac{-2xy - y^2}{2xy + x^2}$

$y'|_{x=1,y=1} = \frac{-2(1)(1) - (1)^2}{2(1)(1) + (1)^2}$

$y'|_{x=1,y=1} = \frac{-2 - (1)^2}{2 + 1}$

$\boxed{y'|_{x=1,y=1} = -1}$

2. Find the tangent to the curve $x^2 + y^2 = 5$ at the point $(-1, 2)$.

Remark: Since $x^2 + y^2 = 5$ represents a circle centered at $(0,0)$ with radius $\sqrt{5}$, and because our point $(-1,2)$ is in the 2nd quadrant, we can rearrange for $y = \sqrt{5 - x^2}$ (the upper half of $x^2 + y^2 = 5$), differentiate and plug in $x = -1$, or we can differentiate implicitly.

$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(5)$

$2x + 2y \cdot y' = 0$

$y' = \frac{-2x}{2y}$

$y'|_{x=-1,y=2} = \frac{-2(-1)}{2(2)}$

$\boxed{y'|_{x=-1,y=2} = \frac{1}{2}}$

3. Let $y = x^{\log x}$. Find $\frac{dy}{dx}$ in terms of x.

Use logarithmic differentiation.

$\log(y) = \log(x^{\log(x)})$

$\log(y) = \log(x)\log(x)$

$\frac{d}{dx}(\log(y)) = \frac{d}{dx}(\log(x)\log(x))$

$\frac{1}{y} \cdot y' = \frac{1}{x}\log(x) + \log(x)\frac{1}{x}$

$y' = y\left[\frac{2}{x}\log(x)\right]$

$\boxed{y' = x^{\log(x)}\left[\frac{2}{x}\log(x)\right]}$

4. Differentiate $g(x) = \frac{5x^2(\cos x)(\log x)}{(x+1)(x+2)}$, expressing the derivative in terms of x only.

Use logarithmic differentiation to simplify the differentiation process.

$\log(g(x)) = \log\left(\frac{5x^2\cos(x)\log(x)}{(x+1)(x+2)}\right)$

$\log(g(x)) = \log(5x^2\cos(x)\log(x)) - \log((x+1)(x+2))$

$\log(g(x)) = \log(5x^2) + \log(\cos(x)) + \log(\log(x)) - \log(x+1) - \log(x+2)$

$\frac{d}{dx}\log(g(x)) = \frac{d}{dx}\left[\log(5x^2) + \log(\cos(x)) + \log(\log(x)) - \log(x+1) - \log(x+2)\right]$

$\frac{1}{g(x)}g'(x) = \frac{10x}{5x^2} - \frac{\sin(x)}{\cos(x)} + \frac{1}{x\log(x)} - \frac{1}{x+1} - \frac{1}{x+2}$

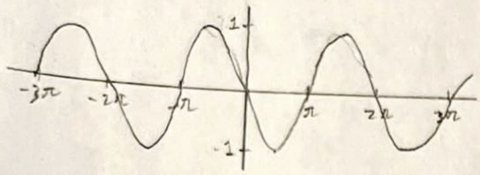
$\boxed{g'(x) = \left[\frac{5x^2\cos(x)\log(x)}{(x+1)(x+2)}\right]\left(\frac{2}{x} - \tan(x) + \frac{1}{x\log(x)} - \frac{1}{x+1} - \frac{1}{x+2}\right)}$

8.5 Practice problems

1. Consider the function $f(x) = \sin(x)$. On which intervals is this function concave down? Give a general interval with endpoints in terms of n, $n \in Z$

$f(x)$ is concave down on all intervals where $f''(x)$ is negative.

$f'(x) = \cos(x)$, $f''(x) = -\sin(x)$



rough sketch of $f''(x)$ about $x=0$

The intervals are $(0, \pi)$, $(2\pi, 3\pi)$, $(4\pi, 5\pi)$, and so on. Thus, the general interval is

$$\boxed{(2n\pi, (2n+1)\pi)}$$

2. Does $\log x$, defined on $0 < x < \infty$, have any global maxima or minima? Explain with reference to the value of its derivative.

Since $\log(x)$ is defined on $0 < x < \infty$, and differentiable and continuous there, any global extrema will be at critical points where $\frac{d}{dx}\log(x) = 0$.

$\frac{d}{dx} = \frac{1}{x}$, for $0 < x < \infty$. The derivative is always positive on $0 < x < \infty$, so there are no critical points and $\boxed{\text{no global maxima or minima}}$

3. Find the local extrema of $f(x) = x^3 - 2x$. Are these points also global extrema?

Since $f(x)$ is continuous and differentiable everywhere, local extrema (if they exist) will be at critical points, $x = \pm\sqrt{\frac{2}{3}}$

$f(x) = x^3 - 2x$

$f'(x) = 3x^2 - 2$

$0 = 3x^2 - 2$

$x^2 = \frac{2}{3}$

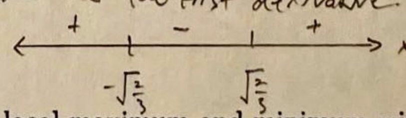
$f''(x) = 6x$

$f''(\sqrt{\frac{2}{3}}) = 6\sqrt{\frac{2}{3}}$

$f''(-\sqrt{\frac{2}{3}}) = -6\sqrt{\frac{2}{3}}$

$\boxed{\text{So there is a local min at } \left(\sqrt{\tfrac{2}{3}}, \left(\sqrt{\tfrac{2}{3}}\right)^3 - 2\sqrt{\tfrac{2}{3}}\right) \text{ and a local max at } \left(-\sqrt{\tfrac{2}{3}}, \left(-\sqrt{\tfrac{2}{3}}\right)^3 + 2\sqrt{\tfrac{2}{3}}\right).}$

To see whether these are global max, examine the first derivative.

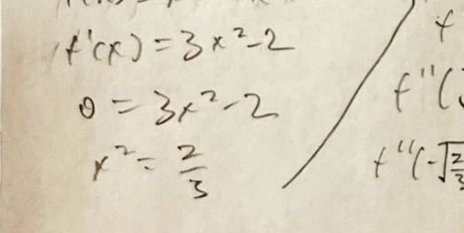


Since the function is decreasing for all x before the local min and increasing for all x after the local max,

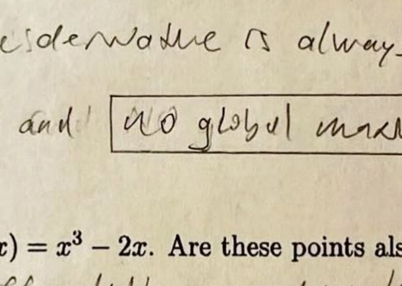
$\boxed{\text{These points are NOT global extrema.}}$

4. Let $f(x) = \cos(x)$. On (-2,4), find its local maximum and minimum with reference to the sign of $f''(x)$ at their x-values.

Rough sketch of $\cos(x)$



$-2 > -\pi$, $4 < \frac{3\pi}{2}$, so let's draw a sketch of $\cos(x)$ with the region we want to look at.



$\boxed{\begin{array}{l}\text{local max: } (0, 1) \\ \text{local min: } (\pi, -1)\end{array}}$

To confirm these local extrema, let's make sure they are critical points and that $f''(x) < 0$ for the local max, and $f''(x) > 0$ for the local min.

$f'(x) = -\sin(x)$. $f'(x) = 0$ when $x = \pi n, n \in Z$.

$\boxed{f''(x) = -\cos(x), \ f''(0) = -1, \ f''(\pi) = 1}$

9.2 Practice problems

1. Let $y = \sqrt{x+1}$.

(a) What are the intercepts and asymptotes of this curve?

Y is defined on $1 \le X < \infty$,

Finding y-intercept(s): $y = \sqrt{0+1}$

$y = 1$

$\boxed{\text{y-intercept: } (0,1)}$

Finding x-intercept(s): $0 = \sqrt{x+1}$

$0 = x+1$

$x = -1$

$\boxed{\text{x-intercept: } (-1,0)}$

There are no "blowups", so no vertical asymptotes.

$\lim_{x\to\infty}(y) = \infty$, so no horizontal asymptotes either.

$\lim_{x\to\infty}(y) = L$ where L is finite, in the case of a horizontal asymptote.

(b) On what intervals does the curve increase/decrease? Where are the local and global extrema?

$y = \sqrt{x+1}$, $\frac{dy}{dx} = \frac{1}{2\sqrt{x+1}}$

No critical points as the derivative is always positive,

$\boxed{\text{So y is increasing on } -1 < x < \infty}$

Local extrema will be at any critical points since y has no singular points.

There are $\boxed{\text{no local extrema}}$ since there are no critical points.

Since y is always increasing, there can only possibly be a global minimum, located at the very smallest x, if it exists, since y is defined on an interval with a closed left endpoint, a global minimum exists; it is at $(-1, \sqrt{-1+1}) = \boxed{(-1,0)}$

(c) Differentiate twice to obtain $f''(x)$. What are the intervals of concavity? Where are the inflection points?

$f(x) =$

$f(x) = y = \sqrt{x+1}$

$f'(x) = \frac{1}{2\sqrt{x+1}}$

$f''(x) = \frac{(0)2\sqrt{x+1} - (1)\frac{2}{2\sqrt{x+1}}}{(2\sqrt{x+1})^2}$

(by quotient rule)

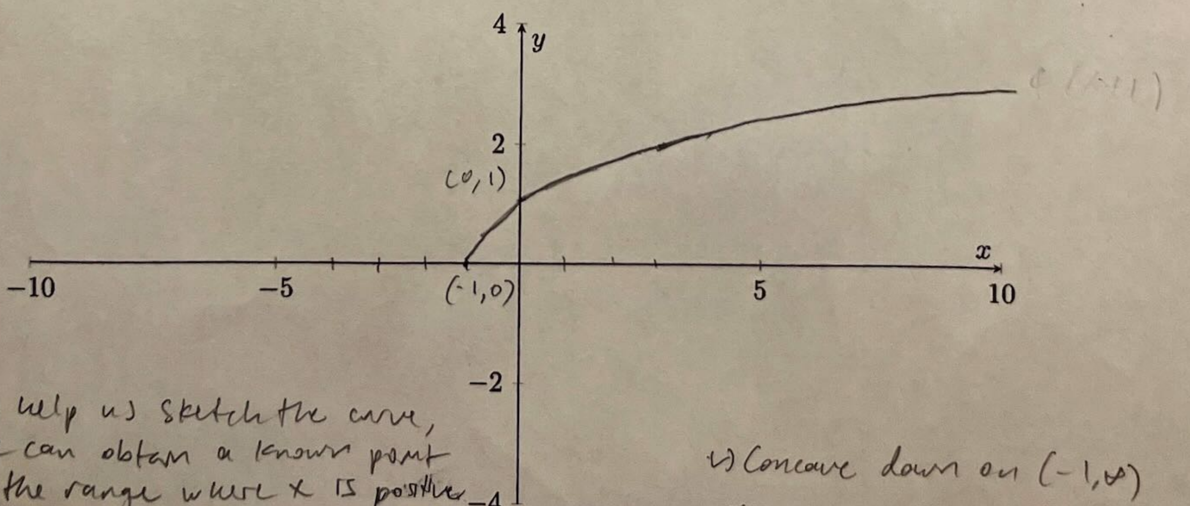
$f''(x) = -\frac{1}{\sqrt{x+1}} \times \frac{1}{(2\sqrt{x+1})^2}$

$f''(x) = -\frac{1}{4(x+1)\sqrt{x+1}}$

$f''(x) < 0$ for $-1 < x < \infty$ since $(x+1)\sqrt{x+1} > 0$ for all $x > -1$.

$\boxed{\text{Thus } f(x) \text{ is concave down on } (-1,\infty) \text{ and there are no points of inflection}}$

(d) Sketch $y = \sqrt{x+1}$. Indicate any intercepts, asymptotes, critical points and singular points.



To help us sketch the curve, we can obtain a known point in the range where x is positive e.g.: $x = 3$, $y = 2$

↳ Concave down on $(-1,\infty)$

↳ Increasing on $(-1,\infty)$

↳ y-int: $(0,1)$

↳ x-int: $(-1,0)$

2. Let $y = \frac{x+2}{2x^2+2}$. P.S. Part C can't be fully solved without graphing technology. I am sorry for any confusion caused!

(a) What are the intercepts and asymptotes of this curve?

Finding y-int: $y = \frac{0+2}{2(0)^2+2}$

$y = 1$

$\boxed{\text{y-int: } (0,1)}$

Finding x-int: $0 = \frac{x+2}{2x^2+2}$

$0 = x+2$

$x = -2$

$\boxed{\text{x-int: } (-2,0)}$

Since it's a rational function, vertical asymptotes (if they exist) should appear at x where the denominator is 0

$2x^2+2=0$

$x^2 = -\frac{2}{2}$

Since there are no solutions to this equation, there are no vertical asymptotes.

HA's should appear at y values of $\lim_{x\to-\infty} y$ and $\lim_{x\to\infty} y$.

$\lim_{x\to\infty} \frac{x+2}{2x^2+2} = 0$, $\lim_{x\to-\infty} \frac{x+2}{2x^2+2} = 0$,

$\boxed{\text{So there is a horizontal asymptote at } y=0}$

(b) On what intervals does the curve increase/decrease? Where are the local and global extrema?

$y' = \frac{(1)(2x^2+2)-(x+2)(4x)}{(2x^2+2)^2}$

$y' = \frac{-2x^2-8x+2}{(2x^2+2)^2}$

Find critical points to eventually find increasing and decreasing intervals of f(x).

$0 = \frac{-2x^2-8x+2}{(2x^2+2)^2}$

$0 = -2x^2-8x+2$

$0 = -x^2-4x+1$

$0 = x^2+4x-1$

using quadratic formula,

$x = -2+\sqrt{5}, -2-\sqrt{5}$.

− + −

$-2-\sqrt{5}$ $-2+\sqrt{5}$ x

Number line for f'(x)

f(x) increases on $(-2-\sqrt{5}, -2+\sqrt{5})$

f(x) decreases on $(-\infty, -2-\sqrt{5}) \cup (-2+\sqrt{5}, \infty)$

$\boxed{\text{Local min: } \left(-2-\sqrt{5}, \frac{-\sqrt{5}}{2(-2-\sqrt{5})^2+2}\right)}$

$\boxed{\text{local max: } \left(-2+\sqrt{5}, \frac{\sqrt{5}}{2(-2+\sqrt{5})^2+2}\right)}$

$\boxed{\text{Local max and min are also global max and min}}$ (examine the derivative's sign, asymptote, x-int)

(c) Differentiate twice to obtain $f''(x)$. What are the intervals of concavity? Where are the inflection points?

$y' = \frac{-2x^2-8x+2}{(2x^2+2)^2}$

$y' = \frac{2(-x^2-4x+1)}{(2(x^2+1))^2}$

$y' = \frac{-x^2-4x+1}{2(x^2+1)^2}$

$y'' = \frac{1}{2} \cdot \frac{d}{dx}\left(\frac{-x^2-4x+1}{(x^2+1)^2}\right)$

$y'' = -\frac{1}{2} \cdot \frac{d}{dx}\left(\frac{x^2+4x-1}{(x^2+1)^2}\right)$

$y'' = -\frac{1}{2} \cdot \left(\frac{(2x+4)(x^2+1)^2-(x^2+4x-1)(2(x^2+1)\cdot 2x)}{((x^2+1)^2)^2}\right)$

$y'' = -\frac{1}{2} \cdot \left(\frac{(2x+4)(x^2+1)-(x^2+4x-1)(4x)}{(x^2+1)^3}\right)$

$y'' = \frac{(x+2)(x^2+1)-(2x)(x^2+4x-1)}{-(x^2+1)^3}$

$y'' = \frac{x^3+6x^2-3x-2}{(x^2+1)^3}$

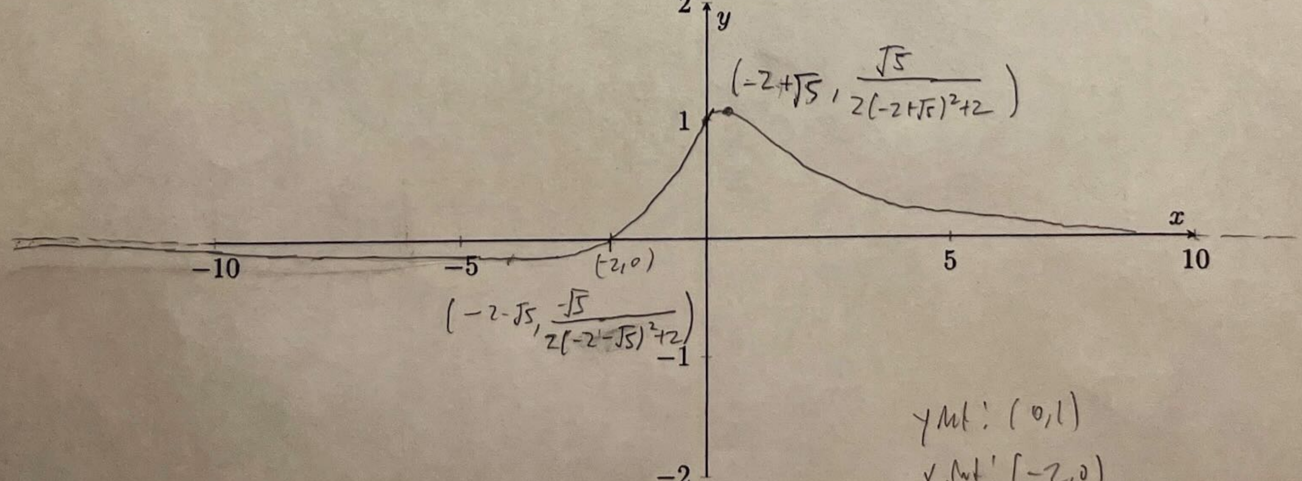
Solving graphically, the intervals for

Concave up: $(-\infty, -0.387...) \cup (0.806..., \infty)$

Concave down: $(-0.387..., 0.806...)$

$\boxed{\text{Inflection pts: } (-0.387..., y|_{x=-0.387...}),\ (0.806..., y|_{x=0.806...})}$

(d) Sketch $y = \frac{x+2}{2x^2+2}$. Indicate any intercepts, asymptotes, critical points and singular points.



y-int: (0,1)

x-int: (−2,0)

Increasing on $(-2-\sqrt{5}, -2+\sqrt{5})$

Decreasing on $(-\infty, -2-\sqrt{5}) \cup (-2+\sqrt{5}, \infty)$

Horizontal asymptote at $y=0$