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COMMERCE MENTORSHIP PROGRAM

# MIDTERM REVIEW SESSION

# MATH 104

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**Problems with Limits:**

1. Evaluate:

**a)** $\lim_{x\to 11} \frac{2x^2-21x-11}{x^2-6x-55}$

Answer: $\frac{23}{16}$

$$= \lim_{x\to 1} \frac{\cancel{(x-11)}(2x+1)}{(x-11)(x+5)}$$

$$= \lim_{x\to 11} \frac{(2x+1)}{(x+5)} = \frac{(2(11)+1)}{(11+5)} = \frac{23}{16}$$

**b)** $\lim_{x\to 0} \frac{x}{5-\sqrt{x+25}}$

Answer: $-10$

$$= \lim_{x\to 0} \frac{x(5+\sqrt{x+25})}{(5-\sqrt{x+25})(5+\sqrt{x+25})}$$

$$= \lim_{x\to 0} \frac{x(5+\sqrt{x+25})}{\cancel{25}-x-\cancel{25}}$$

$$= \lim_{x\to 0} \frac{\cancel{x}(5+\sqrt{x+25})}{-\cancel{x}} = \lim_{x\to 0} -5-\sqrt{x+25} = -5-\sqrt{0+25} = -5-5 = -10$$

**c)** $\lim_{x\to 2} \frac{2x^2-7x-4}{x+4}$

Answer: $-\frac{5}{3}$

$$= \lim_{x\to 2} \frac{2(2)^2-7(2)-4}{(2)+4}$$

$$= \lim_{x\to 2} \frac{8-14-4}{6}$$

$$= \lim_{x\to 2} -\frac{10}{6} = -\frac{5}{3}$$

**d)** $\lim_{x\to\sqrt{2}} \frac{x-2}{x(x-\sqrt{2})}$

Answer: 2

$$= \lim_{x\to\sqrt{2}} \frac{(x+\sqrt{2})\cancel{(x-\sqrt{2})}}{x\cancel{(x-\sqrt{2})}}$$

$$= \lim_{x\to\sqrt{2}} \frac{(x+\sqrt{2})}{\sqrt{2}} = \frac{(\sqrt{2}+\sqrt{2})}{\sqrt{2}} = \frac{2\cancel{\sqrt{2}}}{\cancel{\sqrt{2}}} = 2$$

2. Evaluate the following limit **<u>if</u>** it exists. If it does not exist, state so.

$$f(x) = \begin{cases} -2(x+1)+2 & x < 0 \\ e^x + 1 & x \geq 0 \end{cases}$$

$\lim_{x\to 0} f(x)$ exists if $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x)$

| $\lim_{x\to 0^-} f(x)$ | $\lim_{x\to 0^+} f(x)$ |
|---|---|
| $= \lim_{x\to 0} -2(x+1)+2$ | $= \lim_{x\to 0} e^x + 1$ |
| $= -2(0+1)+2$ | $= e^0 + 1$ |
| $= 0$ | $= 1+1 = 2$ |

$$\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$$

$\boxed{\text{The limit does not exist}}$

**Continuity Problem:**

3. Given the restriction of ***a* > 1,** find the values for ***a*** and ***b*** such that ***f(x)*** is continuous for all real numbers

$$f(x)\begin{cases}(x-2)^2+b & x>2\\ \frac{1}{2}x+a & -2\le x\le 2\\ -(x+a)^2+3 & x<-2\end{cases}$$

To be continuous, each piece has to connect to each other

Solving for a:

when $x=-2$

$$\frac{1}{2}x+a=-(x+a)^2+3$$

$$\frac{1}{2}(-2)+a=-(-2+a)^2+3$$

$$-1+a=-(a^2-4a+4)+3$$

$$-1+a=-a^2+4a-4+3$$

$$-1+a=-a^2+4a-1$$

$$a^2-4a+1-1+a=0$$

$$a^2-3a=0$$

$$a(a-3)=0$$

$$a=0,3$$

$a>1$ $\therefore$ $\boxed{a=3}$

Solving for b:

when $x=2$

$$\frac{1}{2}x+(3)=(x-2)^2+b$$

$$\frac{1}{2}(2)+3=(2-2)^2+b$$

$$1+3=b$$

$$\boxed{b=4}$$

$$\boxed{\begin{array}{l}a=3\\ b=4\end{array}}$$

**Using IVT:**

4. Prove that this equation has a root using IVT

$$f(x) = -2x^3 - 3x^2 + 4$$

The equation has a root when $f(x) = 0$

First find two points, one above and one below $f(x) = 0$

Example:

when $x = 0$ $f(0) = -2(0)^3 - 3(0)^2 + 4 = 4$

when $x = 1$ $f(1) = -2(1)^3 - 3(1)^2 + 4 = -1$

$f(1) < 0 < f(0)$

Since $f(x)$ is continuous, because of IVT we can say there exists an $x$ value between 0 and 1 such that $f(x)$ would be 0.

$\therefore$ $f(x)$ has a root.

**Problems involving Differentiation:**

5.

a) Carefully state the limit definition of the derivative of a function $y = f(x)$

$$\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}, \text{ where the limit exists}$$

b) Use the limit definition of the derivative to find *f'(4)* for the following function. No marks will be given for the use differentiation rules.

$$f(x) = x^2 + 2$$

$$f(x+h) = (x+h)^2+2$$
$$= x^2+2xh+h^2+2$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h \to 0} \frac{(x^2+2xh+h^2+2)-(x^2+2)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^2}+2xh+h^2+\cancel{2}-\cancel{x^2}-\cancel{2}}{h}$$

$$= \lim_{h \to 0} \frac{2x\cancel{h}+h^{\cancel{2}}}{\cancel{h}}$$

$$= \lim_{h \to 0} 2x+h$$

$$= 2x$$

$$f'(x) = 2x$$
$$f'(4) = 2(4) = \boxed{8}$$

6. Let $g(x) = [f(x)]^3 + 5e^{f(x)} + 2, \quad f(2) = 1, f'(2) = 3$

Find $g'(2)$

$$g(x) = [f(x)]^3 + 5e^{f(x)} + 2$$

$$g'(x) = 3f'(x)[f(x)]^2 + 5f'(x)e^{f(x)}$$

$$g'(2) = 3f'(2)[f(2)]^2 + 5f'(2)e^{f(2)}$$

$$= 3(3)(1)^2 + 5(3)e^1$$

$$\boxed{= 9 + 15e^1}$$

$15

**Lengthier Problems:**

7. Mr. Smith sells crates of watermelons. Currently he sells 70 crates of watermelons for \$20. He purchases a watermelon packager that costs \$100. Each crate \$~~[illegible]~~ to produce. His customers are price sensitive. As the price increases, less people will buy. He estimates that every 2\$ increase in price results in 4 less customers purchasing watermelons.

a) Find the linear demand function for the crates as a function of quantity (***q***)

$$\frac{\Delta p}{\Delta q} = \frac{2}{-4} = -\frac{1}{2}$$

$$p = \frac{\Delta p}{\Delta q} q + b$$

$$p = -\frac{1}{2}q + b$$

solving for b:

(70, 20)

$$20 = -\frac{1}{2}(70) + b$$

$$20 = -35 + b$$

$$b = 55$$

$$\boxed{p = -\frac{1}{2}q + 55}$$

b) Find the Revenue function as a function of quantity (***q***)

$R(q) = p \cdot q$ (price times quantity)

$$p = -\frac{1}{2}q + 55$$

$$R(q) = \left(-\frac{1}{2}q + 55\right) q$$

$$\boxed{= -\frac{1}{2}q^2 + 55q}$$

c) How much should Mr. Smith charge for each crate of watermelons to maximize his profit?

$\boxed{\text{Profit} = P(q) = R(q) - C(q)}$ (revenue - cost)

$R(q) = -\frac{1}{2}q^2 + 55q$ (from part b)

$C(q) = 15q + 100$ (10 per crate, 100 for machine)

$\therefore P(q) = -\frac{1}{2}q^2 + 55q - 15q - 100$

$= -\frac{1}{2}q^2 + 40q - 100$

$P'(q) = (-\frac{1}{2})(2)(q) + 40$

$= -q + 40$

$P'(q) = 0$

$-q + 40 = 0$

$q = 40$ (optimal quantity)

$p = -\frac{1}{2}q + 55$ (price function from a)

$p = -\frac{1}{2}(40) + 55$

$= -20 + 55$

$= 35$

$\boxed{\text{Mr. Smith should charge } \$35}$

8. Find the equation of the tangent line to $f(x) = 2x^2 + 5x + 2$ when $x = -1$

$f(x) = 2x^2 + 5x + 2$

$f'(x) = 4x + 5$

$f(-1) = 2(-1)^2 + 5(-1) + 2 = 2 - 5 + 2 = -1$

$f'(-1) = -4 + 5 = 1$

When $x = -1$ $f(x) = -1$

and the slope is $f'(x) = 1$

$\therefore$ $y = mx + b$ $(-1, -1)$ $m = 1$

solving for $b$

$-1 = 1(-1) + b$

$-1 = -1 + b$

$b = 0$

$\boxed{y = x}$

**Challenge Question:**

9. Find the equation of the line that passes through (0, -3) and is tangent to the graph of

$f(x) = x^3 + 2x^2 + 1$

Let $(a, f(a))$ be the point the tangent line intersects $f(x)$

$$\frac{\Delta y}{\Delta x} = \frac{f(a) - (-3)}{a - 0} = \frac{f(a)+3}{a} = \frac{a^3+2a^2+4}{a}$$

$f'(x) = 3x^2 + 4x$

$f'(a) = 3a^2 + 4a$

Find when $\frac{\Delta y}{\Delta x} = f'(a)$

$$\frac{a^3+2a^2+4}{a} = 3a^2+4a$$

$a^3+2a^2+4 = 3a^3+4a^2$

$2a^3+2a^2-4=0$

$a^3+a^2-2=0$

$(a-1)(a^2+2a+2)=0$

$a = 1$

$f'(1) = 3(1)^2+4(1)$

$= 7$

$y = mx+b$ $\quad (0,-3)$

$-3 = 7(0)+b$

$b = -3$

$$\boxed{y = 7x-3}$$