commerce undergraduate society

COMMERCE MENTORSHIP PROGRAM

FINAL REVIEW SESSION

MATH 104

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Section 1: Pre-Midterm Material (Practice for this can be found in the Midterm Review Package)

<u>Limits</u>:

The <u>limit</u> of a function is a value that the function approaches as the input approaches a value.

$\lim\limits_{x \to a} f(x) = c$ is an example of what a limit equation looks like.

- To interpret this equation, as ***x*** approaches ***a***, the **limit** of ***f(x)*** approaches ***c.***

For this limit to exist at ***a***, f(x) must approach the same value from both sides of ***a.***

$$\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x)$$

- a^+ means we are approaching a from the right (positive) side
- a^- means we are approaching a from the left (negative) side

To evaluate a limit, you can try evaluating ***f(x)*** when ***x = a***. If ***f(a)*** gives you a valid numerical answer, then no further work is needed. Sometimes you may end up with an invalid answer such as a number divided by 0. This is an **indeterminate form**, that can be solved with algebraic manipulation. We will go through them in the examples.

One more thing, when we say

- ***"***As ***x*** approaches ***a***, the **limit** of ***f(x)*** approaches ***c."***

This is **<u>not</u>** the same thing as proving

- *"When **x = a, f(x) = c.**"*

f(x) does **not have to** equal ***c*** to approach it, hence the existence of indeterminate forms.

<u>Continuity:</u>

A function ***f(x)*** is continuous at ***x = a*** when

1) ***f(a)*** is defined	***f(x)*** is a function, and ***a*** exists in the domain of ***f(x)***
2) $\lim_{x \to a} f(x)$ exists	As ***x*** approaches ***a***, the limit of ***f(x)* exists**
3) $\lim_{x \to a} f(x) = f(a)$	The limit of ***f(x)*** as ***x*** approaches ***a*** is the same as ***f(a)***

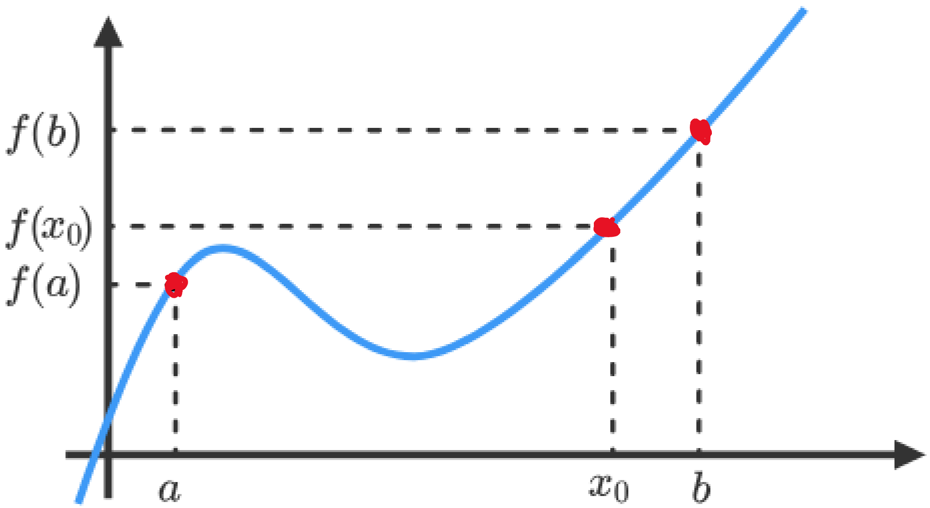
When we say continuous, we mean that the graph of the function is **unbroken**. A common explanation is that you can draw any two points on the graph without lifting your pen from your paper.

Some types of polynomial functions such as ***sin(x)*** or ***cos(x)*** are continuous throughout their whole domains. Proving this is outside the scope of MATH 104, so you can just say they are continuous without providing proofs.

Intermediate Value Theorem (IVT):

Formal Definition: If ***f*** is a continuous function whose domain is in the interval [***a***,***b***], then there exists at least one ***c*** within [***a***,***b***] such that ***f(a)*** < ***f(c)*** < ***f(b)***.

This theorem explains a simple concept. If we have two points, (***a, f(a)***) and (***b, f(b)***) on a continuous function, then the function must go through every ***y-value*** between ***f(a)*** and ***f(b).*** This means you can find any ***y-value*** between ***f(a)*** and ***f(b)***, with at least one corresponding ***x-value*** between ***a*** and ***b***.



In this graph, we can see that ***f(x0)*** is a value between ***f(a)*** and ***f(b)***, and it exists at ***x0***, which is between ***a*** and ***b***.

IVT tells us, without using the graph, that there is at least one ***x*** value between ***a*** and ***b*** where ***f(x) = f(x0).***

<u>Derivatives:</u>

The derivative is the **instantaneous rate of change** of a function at a specific point.

The limit definition of a derivative of ***f(x)*** is

$$\boldsymbol{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

f(x) is differentiable, where this limit **exists**.

f'(a), aka the derivative of ***f(x)*** at ***x = a*** can be described by

$$\boldsymbol{f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}}$$

Also note: The derivative at a point (in this case, ***a***) is equivalent to the slope of the tangent line of ***f(x)*** at ***x = a***.

You can also use differentiation rules to find the derivatives of functions depending on what the question allows you to do.

One notable rule is the power rule because of its widespread use. Other rules can be found in the appendix of this package.

$\boldsymbol{\frac{d}{dx}(x^n) = nx^{n-1}}$, Power rule.

Appendix

Formulas pertaining to limits:

1. $\lim\limits_{x \to a}[p(x) + g(x)] = \lim\limits_{x \to a} p(x) + \lim\limits_{x \to a} g(x)$

2. $\lim\limits_{x \to a}[p(x) - g(x)] = \lim\limits_{x \to a} p(x) - \lim\limits_{x \to a} g(x)$

3. For every real number k,

$\lim\limits_{x \to a}[kp(x)] = k \lim\limits_{x \to a} p(x)$

4. $\lim\limits_{x \to a}[p(x)\ q(x)] = \lim\limits_{x \to a} p(x) \times \lim\limits_{x \to a} q(x)$

5. $\lim\limits_{x \to a} \dfrac{p(x)}{q(x)} = \dfrac{\lim\limits_{x \to a} p(x)}{\lim\limits_{x \to a} q(x)}$

Formulas on differentiation

Basic Derivatives Rules

Constant Rule: $\dfrac{d}{dx}(c) = 0$

Constant Multiple Rule: $\dfrac{d}{dx}[cf(x)] = cf'(x)$

Power Rule: $\dfrac{d}{dx}(x^n) = nx^{n-1}$

Sum Rule: $\dfrac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$

Difference Rule: $\dfrac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x)$

Product Rule: $\dfrac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$

Quotient Rule: $\dfrac{d}{dx}\left[\dfrac{f(x)}{g(x)}\right] = \dfrac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$

Chain Rule: $\dfrac{d}{dx}f(g(x)) = f'(g(x))g'(x)$

Terms Relating to Application of Commerce.

Price = ***p***

Quantity = ***q***

Revenue = ***R(q)*** (Revenue as a function of Quantity) = ***p*q*** (Price times Quantity)

Cost = ***C(q)*** (Cost as a function of Quantity) = Fixed Cost + Variable Cost

- Fixed Cost is a constant, its whatever you pay upfront
- Variable Cost depends on ***q*** (Quantity), as each additional item adds cost

Break-Even Point: ***R(q) = C(q)*** or when **Profit = 0**

Profit Function ***P(q)*** = ***R(q) – C(q)*** or **Revenue – Cost**

Marginal Cost = ***C'(q)*** (The derivative of the cost function)

Marginal Revenue = ***R'(q)*** (The derivative of the revenue function)

Marginal in this case means one more unit. Marginal Cost is the cost of producing **one more** unit, and Marginal Revenue is the revenue gained from selling **one more** unit.

"Marginal" and derivatives are related because they both measure the instant rate of change at a point.

Profit is maximized where:

- ***P'(q) = 0,*** or
- ***MR = MC***, or
- ***Marginal Profit = 0***.

Section 2: Post-Midterm Material

Implicit Differentiation:

Implicit differentiation is a method to calculate derivatives for implicit equations, where you may not easily isolate ***y*** or ***f(x)***. An example of an implicit equation is $\boldsymbol{x^2 + y^2 = 1}$. This equation describes a relationship x and y, without expressing *y* as a function of *x*.

Suppose we had to find the derivative $\boldsymbol{\frac{dy}{dx}}$ for

$$\boldsymbol{x^2 + y^2 = 1}$$

You can begin by differentiating each element of the equation with respect to *x*

$$\boldsymbol{\frac{d}{dx}x^2 + \frac{d}{dx}y^2 = \frac{d}{dx}1}$$

Using simple differentiation and chain rule

$$\boldsymbol{2x + 2y\frac{dy}{dx} = 0}$$

From here you can solve for $\boldsymbol{\frac{dy}{dx}}$

$$\boldsymbol{2y\frac{dy}{dx} = -2x}$$

$$\boldsymbol{\frac{dy}{dx} = \frac{-2x}{2y} = \frac{-x}{y}}$$

1. Find $\frac{dy}{dx}$ for the equation: $\mathbf{2x^3 + 4y^2 = 5}$

2. Given $\mathbf{x^2 + y^2 = 16}$, find the equation of the tangent **lines**, at $\mathbf{x = 3}$

L'Hopital's Rule:

L'Hopital's rule provides a method to find the limits of some indeterminate functions. The rule is as follows:

$$\lim_{x\to c}\frac{\boldsymbol{f(x)}}{\boldsymbol{g(x)}} = \lim_{x\to c}\frac{\boldsymbol{f'(x)}}{\boldsymbol{g'(x)}}$$

There are three conditions for this rule to function

1. $\boldsymbol{f(x)}$ *and* $\boldsymbol{g(x)}$ are differentiable over an open interval, with point *c* being the only exception
2. $\lim_{x\to c}\frac{f(x)}{g(x)}$ evaluates to an indeterminate form $\frac{0}{0}$ *or* $\frac{\infty}{\infty}$
3. $\lim_{x\to c}\frac{f'(x)}{g'(x)}$ exists

Logarithmic Differentiation:

The idea behind logarithmic differentiation is to take the logarithm of both sides of an equation, to solve for their derivatives using implicit differentiation. Here is a general example:

$\boldsymbol{f(x) = g(x)}$, say we want to find ***f'(x)***

$\ln \boldsymbol{f(x)} = \ln \boldsymbol{g(x)}$, first take the natural logarithm of both sides

$\frac{f'(x)}{f(x)} = \frac{g'(x)}{g(x)}$, then you find the derivative of both sides with respect to x

$\boldsymbol{f'(x) = f(x)}\frac{g'(x)}{g(x)}$, you can then isolate ***f'(x)*** to solve for it

1. Find the limit: $\lim_{x \to 1} \frac{x^3 - 3x^2 + 3x - 1}{x^3 + x^2 - x - 1}$

2. Differentiate $y = 5x^{2x}$

Price Elasticity of Demand

Price elasticity of demand is a measure of the change in quantity purchased of a product, as a result of a change in price.

The formula for elasticity is:

$$\boldsymbol{e_{(p)} = \frac{dq}{dp} * \frac{p}{q}}$$

To use this equation, find the derivative of the demand function $\boldsymbol{\frac{dq}{dp}}$, then find the point ***(p, q)*** on the demand curve that you are trying to find the elasticity for.

If $\left|\boldsymbol{e_{(p)}}\right| > \mathbf{1}$, the good is elastic and a decrease in price will increase revenue

If $\left|\boldsymbol{e_{(p)}}\right| = \mathbf{1}$, the good is unit elastic and revenue is maximized

If $\left|\boldsymbol{e_{(p)}}\right| < \mathbf{1}$, the good is inelastic and an increase in price will increase revenue

1. Given the demand function $q = 100 - 10p$ is given for water bottles.

i) Find the elasticity of price when $p = 7$

ii) To maximize revenues, does the price of the bottles need to be increased or decreased?

<u>Rolle's Theorem and Mean Value Theorem:</u>

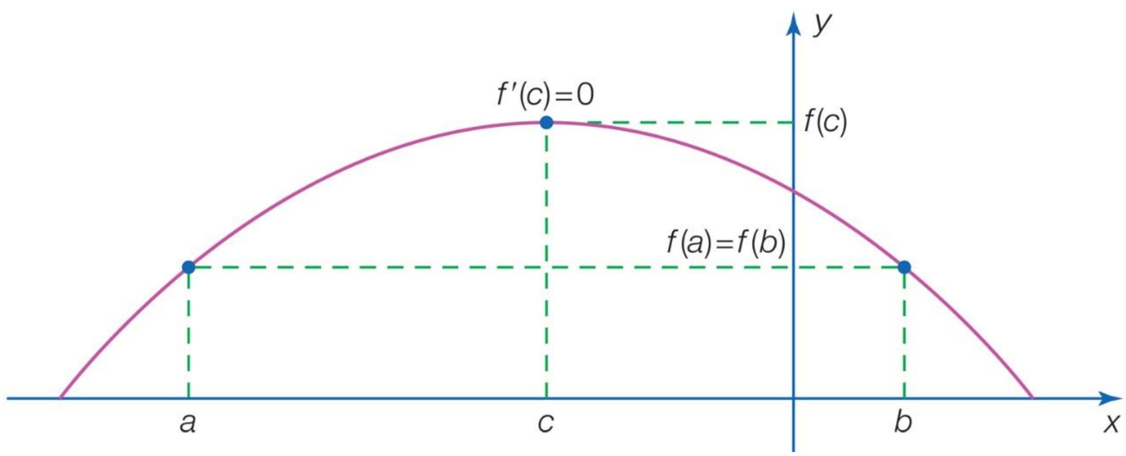
Rolle's Theorem is a fundamental proof of the Mean Value Theorem. It essentially states:

Suppose ***f(x)*** is

1. Continuous on the closed interval **[a, b]**
2. Differentiable on the open interval **(a, b)**
3. ***f(a) = f(b)***

Then there is a point ***x = c*** between ***a*** and ***b,*** where ***f'(c) = 0***

In simpler terms, if you can draw a horizontal line connecting point ***a*** and point ***b***, there is at least one point ***c*** where the slope of ***f(x)*** is 0.



This leads into the **Mean Value Theorem**, draws on a similar concept. **MVT** states:

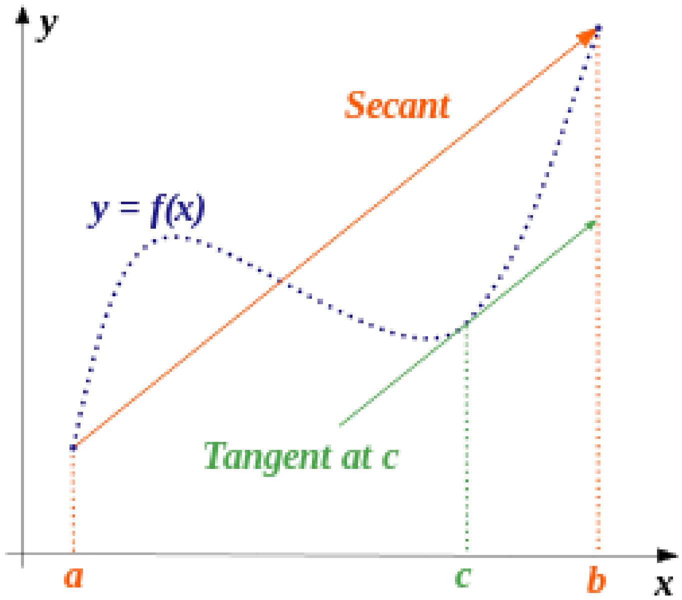
Suppose ***f(x)*** is

1. Continuous on the closed interval **[a, b]**
2. Differentiable on the open interval **(a, b)**

Then there is a point ***x* = *c*** between ***a*** and ***b*,** where

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Basically, if you can draw a line connecting point **[*a*, *f(a)*]** and point [***b***, ***f(b)]*** there is at least one point ***c*** where the slope of ***f(x)*** is the slope of the line that connects the two points.



1. Given the function $\boldsymbol{f(x) = x^2 - x - 6}$ over the interval **[-2 , 3]**, find the point c such that satisfied Rolle's Theorem.

2. Suppose that ***f(x)*** is a function that is differentiable anywhere. Given that $\boldsymbol{f(3) = 6}$ and $\boldsymbol{-3 \leq f'(x) \leq 5}$, find the upper and lower bounds of $\boldsymbol{f(1)}$

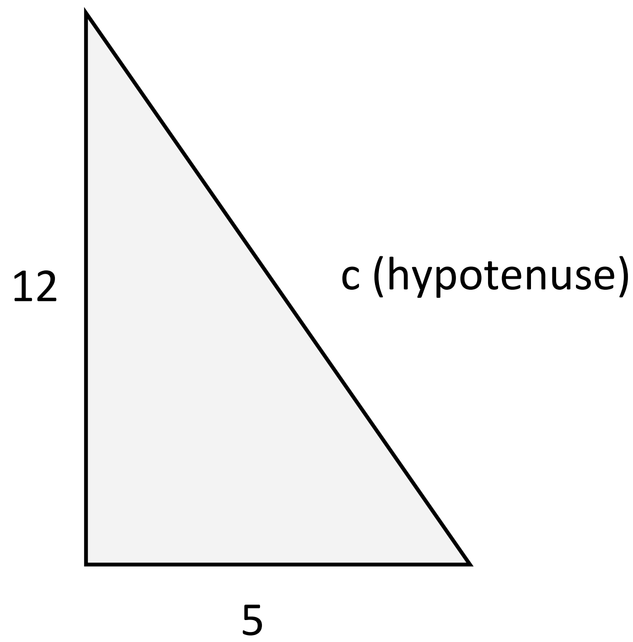
Related Rates:

Related rates problems involve being given the rate of change of one variable and using it to find the rate of change of another.

Let's set up a problem to visualize this idea.

Picture right angle triangle that is expanding. The base of the triangle is currently 5 centimeters, and the height is 12 centimeters. The base of the triangle is increasing at a rate of 4 centimeters per second. The height of the triangle is increasing at a rate of 0.5 centimeters per second. What is the rate of change of the hypotenuse?

First let's draw the triangle.



It helps to draw out these problems to have a more intuitive understanding of what the question is asking.

Now that we have a diagram, let's look at the information given to us.

We are told that when the base of the triangle Is 5 cm, the change in base is 4 cm per second.

$$\boldsymbol{b = 5,\ \ \frac{db}{dt} = 4}$$

The height of the triangle is 12 cm, and the change in height is 0.5 cm per second.

$$\boldsymbol{h = 12,\ \ \frac{dh}{dt} = 0.5}$$

Our missing variables here are $\boldsymbol{c}$ (hypotenuse), and $\frac{dc}{dt}$ (change in hypotenuse).

We can solve for the hypotenuse first. What equation relates all these variables together? Pythagorean theorem does, giving us:

$$c^2 = b^2 + h^2$$

Solving for $\boldsymbol{c}$

$$c^2 = 5^2 + 12^2 = 169$$

$$c = 13$$

Now that we have the hypotenuse, we can find the change in the hypotenuse through implicit differentiation. If we differentiate our Pythagorean theorem in respect to time ($\boldsymbol{t}$), we can get:

$$2c * \frac{dc}{dt} = 2b * \frac{db}{dt} + 2h * \frac{dh}{dt}$$

We can then isolate $\frac{dc}{dt}$, and solve for it

$$\frac{dc}{dt} = \frac{\left(2b * \frac{db}{dt} + 2h * \frac{dh}{dt}\right)}{2c} = \frac{\left(b * \frac{db}{dt} + h * \frac{dh}{dt}\right)}{c}$$

$$\frac{dc}{dt} = \frac{\left(2b * \frac{db}{dt} + 2h * \frac{dh}{dt}\right)}{2c} = \frac{\left(b * \frac{db}{dt} + h * \frac{dh}{dt}\right)}{c}$$

$$\frac{dc}{dt} = \frac{(4 * 5 + 0.5 * 12)}{13} = 2$$

The rate of change of the hypotenuse is **2 centimeters per second**.

None of the individual steps of this question are particularly difficult, however, combining all the small concepts into an answer is tricky. Here are some guidelines to solving these questions.

- Read the question carefully, and write down what information is given, and what you are looking for.
- Draw a diagram to gain a better understanding of what you are solving.
- Assign variables, for example in the question above, the base was ***b***, the height was ***h***, hypotenuse was ***c*** and time was ***t.***
- Find an equation that relates the variables. In this case it was Pythagorean theorem. A lot of these questions will use simple geometry formulas, such as the volume of a cylinder or cube.
- Check if your numbers make intuitive sense. For example, you can't have a negative radius, but you are allowed to have a negative change in radius.

1. A mixture of gold pellets and water are flowing out of a pipe into a cylindrical storm water tank at a rate of $\mathbf{15m^3}$ per second. The tank has a radius of 10 meters. As the tank fills, how fast is the height of the water mixture rising?

2. A spotlight is on the ground 25 feet away from a wall. In front of it, is a 5.5 (5 and one half, not 5'5") feet tall person, casting a shadow on the wall. The person walks towards the wall at 2.5 feet per second. At what rate is the height of the shadow changing, when the person is 10 feet away from the wall?

Optimization:

One of the main applications of differential calculus is trying to find the maximum or minimum values of a function, with the purpose of optimization.

Global Maximum and Minimum:

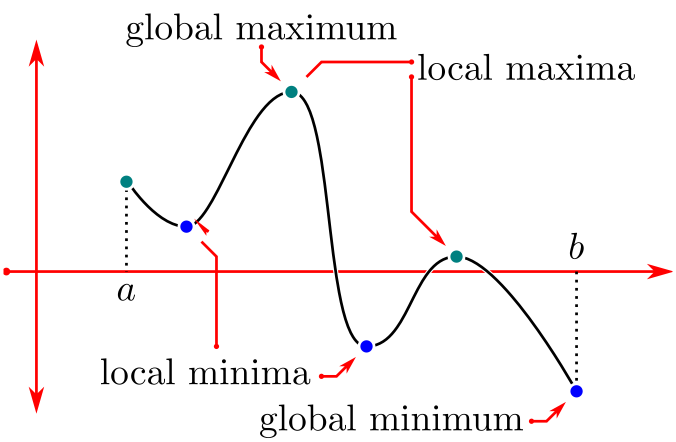
The global maximum of a function is the largest y-value on the entire domain of the function. In a similar manner, the global minimum of a function is the smallest y-value on the entire domain of the function.

Local Maximum and Minimum:

The local maximum of a function is a point on a function that has a larger y-value than all the other points near it. The local minimum of a function is a point that has a smaller y-value than the other points near it.

Fermat's Theorem states that if ***f(x)*** has a local extreme at a point, and that point is differentiable, then the derivative at that point is **0**.

Essentially, this theorem implies that you can find the local extremes by finding the points at which the derivative of the function is 0.

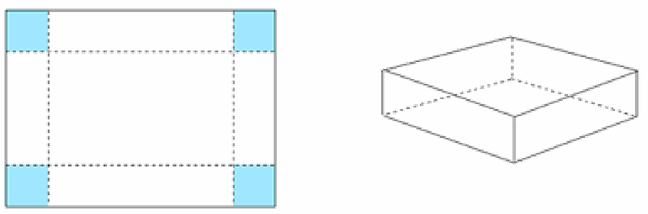


Critical points and inflection points:

Critical points of a function are points at which the first derivative $f'(x) = 0$. These are points at which the functions stop increasing or decreasing, giving them the tendency to be maximums and minimums.

Inflection points are points at which the second derivative $f''(x) = 0$. At these points, the curvature of the function changes.

1. Suppose we have a rectangular piece of cardboard that is 18 by 20 cm. We are going to cut the corners of the cardboard and fold it to make a box. What is the optimal height of the box to maximize volume?



2. You are building a fence around a rectangular plot of land that must have an area of 300 square feet. Suppose your neighbour is willing to pay the half cost of one side of this fence. What are the dimensions of the fence that would minimize cost (Assume the fence material costs the same for the length and width)?

Curve Sketching:

Sketching the graphs of complicated functions can help us understand their properties and visualize changes. The skills we learned in taking derivatives can help us with this.

Here's a list of things you can find to help sketch the graph:

1. Find the domain, note areas where the x values do not exist. Note any points of discontinuity.

2. Find intercepts. Search for any points at which $y = 0$ or $x = 0$.

3. Vertical asymptotes. For values of x that are discontinuous, you can find the limit of the function as it approaches both sides of the asymptote. Functions can never cross vertical asymptotes.

4. Horizontal asymptotes. You can find these by finding the limit of the function as x approaches $\pm\infty$. If the limit of the function is also $\pm\infty$ then there is no horizontal asymptote. Graphs may or may not cross horizontal asymptotes.

5. First Derivative. Whenever $f'(x) > 0$, $f(x)$ is increasing. When $f'(x) < 0$, $f(x)$ is decreasing.

6. Critical points, points when $f'(x) = 0$. These help you find the functions local and global maximums and minimums. These can also help you figure out when the graph stops increasing/decreasing.

7. Second Derivatives. Whenever $f''(x) > 0$, $f(x)$ is concave up. Whenever $f''(x) < 0$, $f(x)$ is concave down.

8. Inflection points, when $f''(x) = 0$. These are points at which the concavity of the function changes.

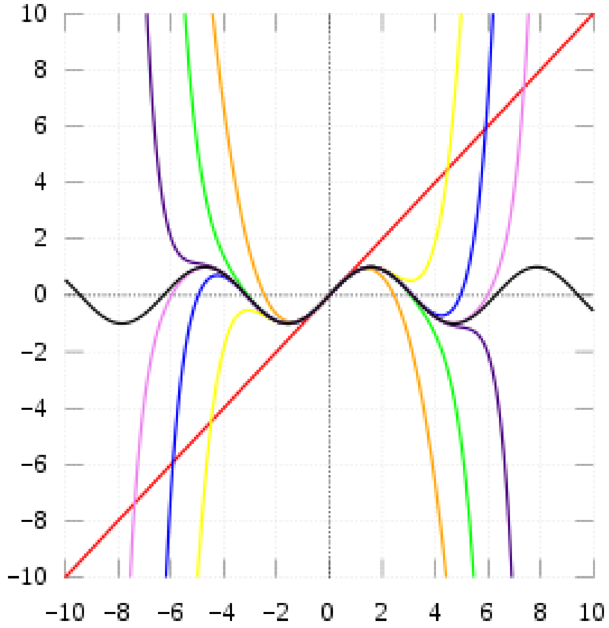
1. Sketch the graph of $\boldsymbol{f(x) = x^3 - 1}$

2. Challenge question: Sketch the graph of $\boldsymbol{f(x) = x + \frac{1}{x}}$

Taylor Polynomials:

A Taylor polynomial is a function that approximates a different function, near a given point. This is useful because it turns complex, non-polynomial functions such as $\boldsymbol{e^x}$ ***or*** $\mathbf{sin}\,(\boldsymbol{x})$ into simpler polynomials.

Here is a graph that depicts Taylor polynomial functions.



The black line is the original function $\mathbf{sin}\,(\boldsymbol{x})$.

The red line is a linear (degree 1) Taylor approximation of $\mathbf{sin}(\boldsymbol{x})\ \boldsymbol{at}\ \boldsymbol{x} = \mathbf{0}$. As you can see, this simple line is a good estimate of $\mathbf{sin}(\boldsymbol{x})$ looks like at $\boldsymbol{x} = \mathbf{0}$.

You can increase the degree of the Taylor approximation infinitely, and each time you do, your polynomial will become more and more accurate to the original function.

So how do we perform these approximations?

Linear Approximation (1st Degree Taylor Polynomial):

If you are given the point ***a***, at which to approximate ***f(x)***, your linear approximation will be:

$$\boldsymbol{f(x) \approx f(a) + \frac{f'(a)}{1!}(x-a)}$$

This approximation is referred to as linear because it only gives you a linear function.

Quadratic Approximation (2nd Degree Taylor Polynomial):

The formula for quadratic approximation is:

$$\boldsymbol{f(x) \approx f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2}$$

As you can see, it is similar to the first-degree polynomial but with a term added at the end.

Nth Degree Taylor Polynomial:

Taylor Polynomials can be calculated to an infinite degree. As you may have noticed from the previous two examples, each time you increase the degree of the polynomial you add a term at the end.

For an ***nth*** degree Taylor Polynomial, take the previous ***(n-1)th*** degree polynomial and add this.

$$\boldsymbol{\frac{f^{(n)}(a)}{n!}(x-a)^n}$$

1. Compute a third-degree Taylor polynomial for $\boldsymbol{f(x) = xe^x}$ at $\boldsymbol{a = 1}$

2. Compute a linear approximation of $\boldsymbol{f(x) = \tan(x)}$ at $\boldsymbol{a = \frac{\pi}{4}}$