MATH 104

2020W1 midterm review session

Answer Key

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1. **Short Problems.** Put your answer in the box provided and show your work. No credit will be given for an answer without the accompanying work.

(a) Evaluate $$\lim_{x\to 7}\left(\frac{x^2-4x-21}{3x^2-17x-28}\right)$$

$$\lim_{x\to 7}\left(\frac{(x-7)(x+3)}{(3x+4)(x-7)}\right)$$

$$= \lim_{x\to 7}\left(\frac{x+3}{3x+4}\right)$$

$$= \frac{(7)+3}{3(7)+4}$$

$$\boxed{= \frac{2}{5}}$$

Answer: 2/5

(b) Evaluate $\lim_{x\to -3}\left(\frac{\sqrt{2x+22}-4}{x+3}\right)$

$$\lim_{x\to -3}\left(\frac{\sqrt{2x+22}-4}{x+3}\cdot\frac{(\sqrt{2x+22}+4)}{(\sqrt{2x+22}+4)}\right)$$

$$=\lim_{x\to -3}\left(\frac{2x+22-16}{(x+3)(\sqrt{2x+22}+4)}\right)$$

$$=\lim_{x\to -3}\left(\frac{2(x+3)}{(x+3)(\sqrt{2x+22}+4)}\right)$$

$$=\lim_{x\to -3}\left(\frac{2}{\sqrt{2x+22}+4}\right)$$

$$=\frac{2}{\sqrt{2(-3)+22}+4}$$

$$\boxed{=\frac{1}{4}}$$

Answer: 1/4

(c) Evaluate

$$\lim_{x\to 1}\left(\frac{1}{x^2-1}-\frac{2}{x^4-1}\right)$$

$$\lim_{x\to 1}\left(\frac{1}{x^2-1}-\frac{2}{x^4-1}\right)$$

$$=\lim_{x\to 1}\left(\frac{x^2+1}{x^2+1}\cdot\frac{1}{x^2-1}-\frac{2}{x^4-1}\right)$$

$$=\lim_{x\to 1}\left(\frac{x^2+1-2}{x^4-1}\right)$$

$$=\lim_{x\to 1}\left(\frac{x^2-1}{(x^2-1)(x^2+1)}\right)$$

$$=\lim_{x\to 1}\left(\frac{1}{x^2+1}\right)$$

$$=\frac{1}{(1)^2+1}$$

$$\boxed{=\frac{1}{2}}$$

Answer: 1/2

Long Problems. No credit will be given for the answer without the correct accompanying work.

2. Find the values of a and b such that $f(x)$ is continuous for all real numbers.

$$f(x)\begin{cases} e^x + a & \text{if } x > 2 \\ bx^2 + 1 & \text{if } 1 \le x \le 2 \\ 3x^3 - b & \text{if } x < 1 \end{cases}$$

Each piece is continuous over $(-\infty, 1), (1,2), (2, \infty)$

check $x=1$

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x)$$

$$\lim_{x \to 1^-} \left(3x^3 - b\right) = \lim_{x \to 1^+} \left(bx^2 + 1\right)$$

$$3 - b = b + 1$$

$$b = 1$$

check $x=2$

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x)$$

$$\lim_{x \to 2^-} \left(bx^2 + 1\right) = \lim_{x \to 2^+} \left(e^x + a\right)$$

$$(1)(2)^2 + 1 = e^2 + a$$

$$a = 5 - e^2$$

Answer:
a = 5 – e^2
b = 1

3.

Prove that the equation below has a root.

$$f(x) = 4x^4 - 16x^3 + 2x^2 - x + 9$$

1. f is polynomial, and $\therefore$ continuous.

$f(1) = 4 - 16 + 2 - 1 + 9$ $\quad$ $f(0) = 0 - 0 + 0 - 0 + 9$

$f(1) = -2$ $\quad$ $f(0) = 9$

$\therefore$ $f(1) < 0$ $\quad$ $\therefore$ $f(0) > 0$

$$f(1) < 0 < f(0)$$

Since f is continuous, by IVT

there exists a $c \in [0,1]$

such that $f(c) = 0$.

That is, $f(x)$ has a root.

(a) Carefully state the limit definition of the derivative of the function $y = f(x)$

The derivative of $f(x)$ is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}, \quad \text{where the limit exists}$$

$$= \lim_{h \to 0}$$

(b) Use the limit definition of the derivative to find $f\,'(9)$ for the following function. No marks will be given for the use of differentiation rules.

$$f(x) = -6\sqrt{x}$$

$$f'(x) = \lim_{h\to 0} \frac{-6\sqrt{x+h} - (-6\sqrt{x})}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{6\sqrt{x} - 6\sqrt{x+h}}{h}$$

$$f'(x) = \lim_{h\to 0} 6\cdot\frac{(\sqrt{x} - \sqrt{x+h})}{h}\,\frac{(\sqrt{x} + \sqrt{x+h})}{(\sqrt{x} + \sqrt{x+h})}$$

$$f'(x) = \lim_{h\to 0} 6\cdot\frac{x-(x+h)}{h(\sqrt{x} + \sqrt{x+h})}$$

$$f'(x) = \lim_{h\to 0} 6\cdot\left(-\frac{h}{h(\sqrt{x} + \sqrt{x+h})}\right)$$

$$f'(9) = \lim_{h\to 0} 6\left(-\frac{1}{\sqrt{9} + \sqrt{9+h}}\right)$$

$$= 6\left(-\frac{1}{3+3}\right)$$

$$= -1$$

Answer:
-1

5. Let

$$h(x) = e^{3f(x)} + [f(x)]^2 \quad f(1) = 2 \quad f'(1) = 5$$

Find $h'(1)$

$$h(x) = e^{3f(x)} + [f(x)]^2$$

$$h'(x) = e^{3f(x)} \cdot 3f'(x) + 2f(x) \cdot f'(x)$$

$$h'(1) = e^{3f(1)} \cdot 3f'(1) + 2f(1) \cdot f'(1)$$

$$h'(1) = e^{3(2)} \cdot 3(5) + 2(2)(5)$$

$$\boxed{h'(1) = 15 \cdot e^{6} + 20}$$

Answer:

15*e^6 + 20

6. Find the equations of the lines parallel to $y = 2x + 3x^{-1}$ and tangent to the graph of

$$2y - 2x = 2019^{2020}$$

$2y = 2x + 2019^{2020}$

$y = x + \frac{1}{2} \cdot 2019^{2020}$

This line is parallel to our tangent line, so our tangent line's slope is 1.

$y = 2x + 3x^{-1}$

$\frac{dy}{dx} = 2 - 3x^{-2}$

$1 = 2 - 3x^{-2}$

$-1 = -3x^{-2}$

$x = \pm\sqrt{3}$

$y = 2(\sqrt{3}) + 3(\sqrt{3})^{-1}$

$y = 3\sqrt{3}$

$y = mx + b$

$m = 1 \quad x = \sqrt{3} \quad y = 3\sqrt{3}$

$3\sqrt{3} = \sqrt{3} + b$

$2\sqrt{3} = b$

$\boxed{y = x + 2\sqrt{3}}$

$y = 2(-\sqrt{3}) + 3(-\sqrt{3})^{-1}$

$y = -3\sqrt{3}$

$y = mx + b$

$m = 1 \quad x = -\sqrt{3} \quad y = -3\sqrt{3}$

$-3\sqrt{3} = -\sqrt{3} + b$

$-2\sqrt{3} = b$

$\boxed{y = x - 2\sqrt{3}}$

Answer:
y = x - 2sqrt(3)
y = x - 2sqrt(3)

7. Professor Fenceman, an economics professor at the University of Building Connections sells 200 of his award-winning review packages at \$110 each. He rents an industrial-grade printer at \$50 per term and each review pack costs \$70 to produce. Dr. Fenceman’s students are price-sensitive, and some will decide not to buy the review package if the price is too high. He estimates that for every \$20 increase in his review package price, 20 students will choose not to purchase it.

(a) Find the linear demand function for the review package as a function of q (quantity).

$$p = mq + b$$

$$m = \frac{\Delta p}{\Delta q} = \frac{20}{-20} = -1$$

$$110 = -200 + b$$

$$(q, p) \to (200, 110)$$

$$310 = b$$

$$\boxed{p = -q + 310}$$

Answer:
p = 310 - q

(b) Find the Revenue function as a function of q (quantity)

$$\text{Revenue} = p \cdot q$$

$$\text{Revenue} = 310q - q^2$$

Answer:
R(q) = 310q – q^2

(c) What should Dr. Fenceman charge to maximize profit

$$\text{Profit} = \text{Revenue} - \text{Cost}$$

$$\text{Profit} = 310q - q^2 - (50 + 70q)$$

$$\text{Profit} = 240q - q^2 - 50$$

$$\text{Profit} = -(q^2 - 240q) - 50$$

$$\text{Profit} = -(q^2 - 240q + 14400 - 14400) - 50$$

$$\text{Profit} = -(q - 120)^2 + 14350$$

Profit maxed at $q = 120$

$$p = -120 + 310$$

$$\boxed{p = 190}$$

Fence man should charge \$190

Appendix: Formulae

$$\lim_{x\to a}\left[f\left(x\right)+g\left(x\right)\right]=\lim_{x\to a}f\left(x\right)+\lim_{x\to a}g\left(x\right).$$

$$\lim_{x\to a}kf\left(x\right)=k\lim_{x\to a}f\left(x\right).$$

$$\lim_{x\to a}\left[f\left(x\right)+g\left(x\right)\right]=\lim_{x\to a}f\left(x\right)+\lim_{x\to a}g\left(x\right).$$

$$\lim_{x\to a}\frac{f\left(x\right)}{g\left(x\right)}=\frac{\lim_{x\to a}f\left(x\right)}{\lim_{x\to a}g\left(x\right)},\quad \text{if}\quad \lim_{x\to a}g\left(x\right)\neq 0.$$

$$\lim_{x\to a}\left[f\left(x\right)\right]^{p}=\left[\lim_{x\to a}f\left(x\right)\right]^{p},$$

$$\lim_{x\to a}b^{f(x)}=b^{\lim_{x\to a}f(x)},$$

Constant Rule: $\frac{d}{dx}(c) = 0$

Constant Multiple Rule: $\frac{d}{dx}\left[cf(x)\right] = cf'(x)$

Power Rule: $\frac{d}{dx}(x^n) = nx^{n-1}$

Sum Rule: $\frac{d}{dx}\left[f(x) + g(x)\right] = f'(x) + g'(x)$

Difference Rule: $\frac{d}{dx}\left[f(x) - g(x)\right] = f'(x) - g'(x)$

Product Rule: $\frac{d}{dx}\left[f(x)g(x)\right] = f(x)\,g'(x) + g(x)\,f'(x)$

Quotient Rule: $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\,f'(x) - f(x)g'(x)}{\left[g(x)\right]^2}$

Chain Rule: $\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(\cos x) = -\sin x \qquad \frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

Business Terminology and notation

Quantity **q**

Price **p**

Revenue = **pq** (Refers to the money earned by selling **q** items at **p** price)

Costs = **C(q)** (Refers to the total cost of producing **q** items)

- Fixed Cost (Stays constant regardless of production
- Variable Cost (Varies depending on the **q** value)

Break-Even Point (**Where R(q) = C(q) -> Profit = 0**)

Profit(Loss) = **Revenue - Cost**

Marginal Cost **= C' (q)** (The cost to produce an additional unit)

Marginal Revenue = **R' (q)** (The additional revenue gained from selling an additional unit)

"Marginal" means that you should take the derivative of that function.

Profit is maximized where:

- P' (q) = 0, or
- MR = MC, or
- Marginal Profit = 0.