ANSWERS

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Evaluate:

1.) $\lim_{x\to 4}\left(\frac{x^2-16}{x^2-9x+20}\right)$

$$= \lim_{x\to 4}\frac{(x-4)(x+4)}{(x-4)(x-5)}$$

$$= \lim_{x\to 4}\frac{(x+4)}{(x-5)}$$

Plugging in x=4, we get

$$= -8$$

2.) $\lim_{x\to 5}\left(\frac{x-5}{\sqrt{2x-6}-2}\right)$

$$= \lim_{x\to 5}\frac{(x-5)(\sqrt{2x-6}+2)}{(\sqrt{2x-6}-2)(\sqrt{2x-6}+2)}$$

$$= \lim_{x\to 5}\frac{(x-5)(\sqrt{2x-6}+2)}{2x-6-4}$$

$$= \lim_{x\to 5}\frac{(x-5)(\sqrt{2x-6}+2)}{2(x-5)}$$

$$= \lim_{x\to 5}\frac{(\sqrt{2x-6}+2)}{2}$$

Plugging in x=5, we get

$$= 2$$

3.) $\lim_{x \to -\infty}\left(\frac{\sqrt{9x^2 - x + 1}}{4x - 5}\right)$

$$= \lim_{x \to -\infty} \frac{\sqrt{x^2(9 - \frac{1}{x} + \frac{1}{x^2})}}{x(4 - \frac{5}{x})}$$

$$= \lim_{x \to -\infty} \frac{|x|\sqrt{(9 - \frac{1}{x} + \frac{1}{x^2})}}{x(4 - \frac{5}{x})}$$

$$|x| = -x \; where \; x < 0, therefore$$

$$= \lim_{x \to -\infty} \frac{-x\sqrt{(9 - \frac{1}{x} + \frac{1}{x^2})}}{x(4 - \frac{5}{x})}$$

$$= \lim_{x \to -\infty} \frac{-\sqrt{(9 - \frac{1}{x} + \frac{1}{x^2})}}{(4 - \frac{5}{x})}$$

$$Plugging \; in \; as \; x \to \; -\infty, \frac{1}{x} \, and \, \frac{1}{x^2} \to 0$$

$$= \frac{-\sqrt{(9 - 0 + 0}}{(4 - 0)}$$

$$= \frac{-3}{4}$$

4.) $\lim_{x\to 2}\left(\frac{x-2}{\left|\sqrt{x}-\sqrt{2}\right|}\right)$

$$\lim_{x\to 2^-}\frac{x-2}{|\sqrt{x}-\sqrt{2}|}$$

$$=\lim_{x\to 2^-}\frac{x-2}{-(\sqrt{x}-\sqrt{2})}$$

$$=\lim_{x\to 2^-}-(\sqrt{x}+\sqrt{2})$$

$$=-2\sqrt{2}$$

$$\lim_{x\to 2^+}\frac{x-2}{|\sqrt{x}-\sqrt{2}|}$$

$$=\lim_{x\to 2^+}\frac{x-2}{\sqrt{x}-\sqrt{2}}$$

$$=\lim_{x\to 2^+}(\sqrt{x}+\sqrt{2})$$

$$=2\sqrt{2}$$

$$\textit{The left and right}-\textit{hand side limits are different, so the limit does not exist.}$$

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1.) Find the value of *a* for which the function f(x) is continuous for all x

$$f(x)=\begin{cases}3x+a, x\le e\\ 2alnx, x>e\end{cases}$$

$$\textit{These two functions are continuous over their own domains:}$$
$$\textit{we must make } f(x) \textit{continuous at e by setting the left and right hand limits equal.}$$

$$\lim_{x\to e^-}f(x)$$

$$=\lim_{x\to e^-}(3x+a)$$

$$=3e+a$$

$$\lim_{x\to e^+}f(x)$$

$$=\lim_{x\to e^+}2alnx$$

$$=2a$$

$$\lim_{x\to e^-}f(x)=\lim_{x\to e^+}f(x)$$

$$3e+a=2a$$

$$a=3e$$

2.) Prove that the following equation has a solution

$$2^x = x + e$$

$$f(x) = x + e - 2^x$$

$$f(-3) = -3 + e - 2^{-3} < 0$$

$$f(0) = 0 + e - 2^0 > 0$$

$f(x)$ *is continuous over the domain* $[-3,0]$.
Therefore, by IVT, there exists a c such that $-3 \leq c \leq 0$ *and* $f(c) = 0$.
Therefore, $2^x = x + e$ *has a solution.*

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1.) Find $f'(x)$, where $f(x) = e^{12x}\cos(x)$

$$f'(x) = e^{12x} * 12 * \cos(x) - \sin(x) e^{12x}$$

2.) Let $f(x) = \sqrt{x}$. Use the definition of the derivative to find $f'(4)$. No marks will be given for the use of any differentiation rules.

$$f'(4) = \lim_{x \to 4} \frac{\sqrt{x} - \sqrt{4}}{x - 4}$$

$$f'(4) = \lim_{x \to 4} \frac{(\sqrt{x} - \sqrt{4})(\sqrt{x} + \sqrt{4})}{(x - 4)(\sqrt{x} + \sqrt{4})}$$

$$f'(4) = \lim_{x \to 4} \frac{(x - 4)}{(x - 4)(\sqrt{x} + \sqrt{4})}$$

$$f'(4) = \lim_{x \to 4} \frac{1}{(\sqrt{x} + \sqrt{4})}$$

$$f'(4) = \frac{1}{4}$$

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$$\text{Evaluate } \lim_{x\to 0} \frac{\sin(x)}{-0.5x}$$

$$\lim_{x\to 0} \frac{\sin(x)}{-0.5x} = \lim_{x\to 0} \frac{\cos(x)}{-0.5}$$

$$\lim_{x\to 0} \frac{\sin(x)}{-0.5x} = \frac{1}{-0.5} = -2$$

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1. Find the equation of the tangent lines tangent to the graph of $x^2 + y^2 = 25$ at x = 2

$$2^2 + y^2 = 25$$
$$y = \pm\sqrt{21}$$

$$x^2 + y^2 = 25$$
$$2x + 2y * \frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = \frac{-x}{y}$$

NOTE: Since we have 2 values of y, we will have two different derivatives at x=2: that is, two different tangent lines at x=2.

Positive case ($y = \sqrt{21}$)

$$\frac{dy}{dx} = \frac{-2}{\sqrt{21}}$$

$$y = mx + b$$

$$y = \frac{-2}{\sqrt{21}}x + b$$

Plugging in our values of x and y…

$$\sqrt{21} = \frac{-2}{\sqrt{21}} * 2 + b$$
$$b = \sqrt{21} + \frac{4}{\sqrt{21}}$$
$$y = \frac{-2}{\sqrt{21}}x + \sqrt{21} + \frac{4}{\sqrt{21}}$$

Repeating for the negative case, we get…

$$y = \frac{2}{\sqrt{21}}x - \sqrt{21} - \frac{4}{\sqrt{21}}$$

Example: Suppose that you are a player in the MMORPG "Treestory" and you are trying to make profit selling "Work Gloves", a popular in-game item. You notice that when you try to sell your Work Gloves for $16 each, your customer demand is 20 units. For every $2 decrease in unit price, the customer demand goes up by 10 units. Find the demand function linking p and q.

$$p = mq + b \; (Note\text{: } this\ mirrors\ our\ y = mx + b\ formula)$$

$$m = \frac{\Delta p}{\Delta q} = \frac{16 - 14}{20 - 30} = \frac{-1}{5}$$

$$p = \frac{-1}{5}q + b$$

Plugging in p=16 and q=20…

$$16 = \frac{-1}{5}(20) + b$$

$$20 = b$$

$$p = \frac{-1}{5}q + 20$$

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Example: "Appleson: We Hate Children Inc." is a textbook manufacturer that prints paper textbooks for students to purchase. Every month, they rent an industrial-grade printer at $50 a month to produce economics textbooks, which cost $70 in printing and paper expenses each to produce. What are their fixed, variable, average, marginal, and total costs?

$$Fixed\ cost = 50$$
$$Variable\ cost = 70$$
$$Average\ cost = \frac{50 + 70q}{q}$$
$$Marginal\ cost = 70$$
$$Total\ cost = 50 + 70q$$

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Example: A company's cost and demand curves are given by

$$p + \sqrt{q} = 150 \ \ and\ C(q) = 2500 + 6q$$

Determine the selling price which would produce the most profit.

$$Recall: Revenue = price * quantity\ sold$$
$$Profit = Revenue - Total\ Cost$$

$$\boxed{\begin{gathered} p = 150 - \sqrt{q} \\ R(q) = q(150 - \sqrt{q}) \\ P(q) = q\left(150 - \sqrt{q}\right) - (2500 + 6q) \\ P(q) = 144q - q^{1.5} - 2500 \\ P'(q) = 144 - 1.5q^{0.5} \\ 0 = 144 - 1.5q^{0.5} \\ q = 9216 \end{gathered}}$$

Example: Suppose the price and quantity demanded of a product are related as follows:

$$p = 20 - q$$

1.) Find the price elasticity of demand when p = 8

$$\varepsilon = \frac{dq}{dp} * \frac{p}{q}$$

$$\varepsilon = -1 * \frac{8}{12} = \frac{-2}{3}$$

2.) To maximize revenues, should the company increase or decrease their price?

They should increase their price.

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Example:

Let f(x) be a differentiable function such that

$$f(1) = 10 \text{ and } -1 \leq f'(x) \leq 2 \text{ everywhere.}$$

Obtain upper and lower bounds on f(5)

$$By\ Mean\ Value\ Theorem:$$

$$-1 \leq \frac{f(b) - f(a)}{b - a} \leq 2$$

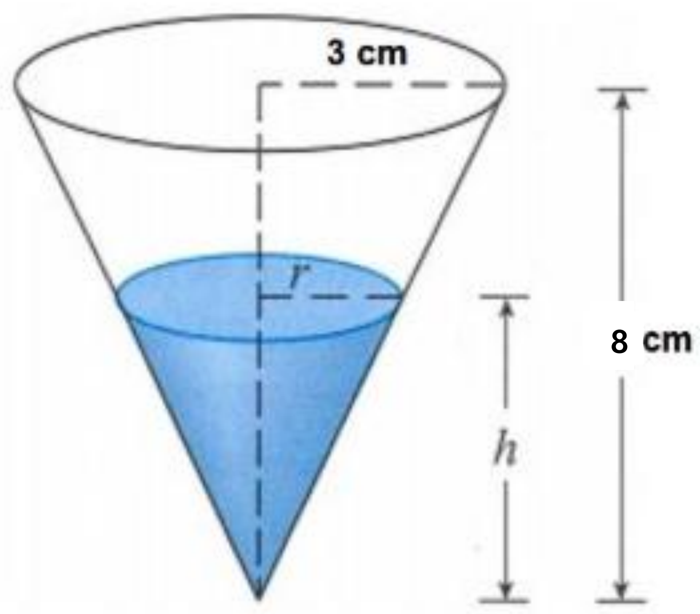
Choosing b=5 and b=1...

$$-1 \leq \frac{f(b) - 10}{5 - 1} \leq 2$$

$$-4 \leq f(b) - 10 \leq 8$$

$$6 \leq f(b) - 10 \leq 18$$

1.) Water is being poured into a cone shaped cup at a rate of of $50cm^3$ per second. If the cup has a height of 8 cm and a top radius of 3cm, how fast is the water level rising when it is 4 cm full?



$$By\ similar\ triangles,$$

$$\frac{r}{h} = \frac{3}{8}$$

$$r = \frac{3}{8}h$$

The volume of a cone is given by:

$$V = \frac{1}{3}\pi r^2 h$$

$$V = \frac{1}{3}\pi(\frac{3}{8}h)^2 h$$

$$V = \frac{1}{3}\pi(\frac{3}{8}h)^2 h$$

$$V = \frac{3\pi}{64}h^3$$

$$\frac{dV}{dt} = \frac{9\pi}{64}h^2 * \frac{dh}{dt}$$

From the question, we have dV/dt = 50 and h=4.

$$50 = \frac{9\pi}{64}4^2 * \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{200}{9\pi}$$

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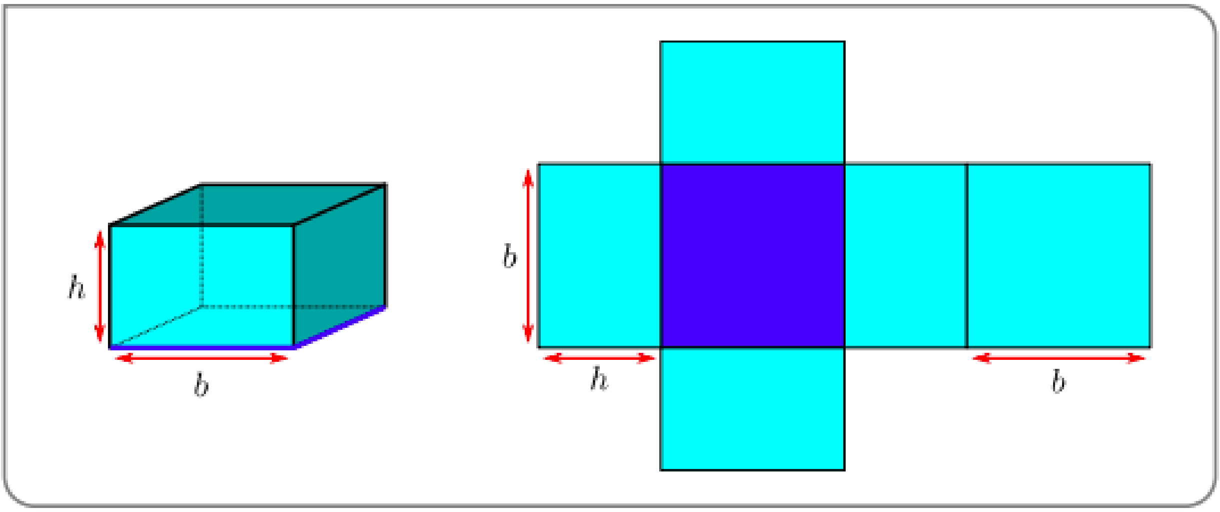
1.) A closed rectangular container with a square base is to be made from two different materials. The material for the base costs $5 per square meter, while the material for the other five sides costs $1 per square meter. Find the dimensions of the container which has the largest possible volume if the total cost of materials is $72.



By simple geometry,

$$V = b^2 h$$

$$C = 5b^2 + 4bh + b^2$$
$$72 = 5b^2 + 4bh + b^2$$

Isolating for h, we get:

$$h = \frac{72 - 6b^2}{4b} = \frac{3}{2} * \frac{12 - b^2}{b}$$

Substituting into volume, we get:

$$V = b^2 \frac{3}{2} * \frac{12 - b^2}{b}$$

$$V = 18b - \frac{3}{2}b^3$$

Differentiate both sides, then set dV/db equal to 0.

$$0 = 18 - \frac{9}{2}b^2$$
$$b^2 = 4$$

Length can't be negative, so:

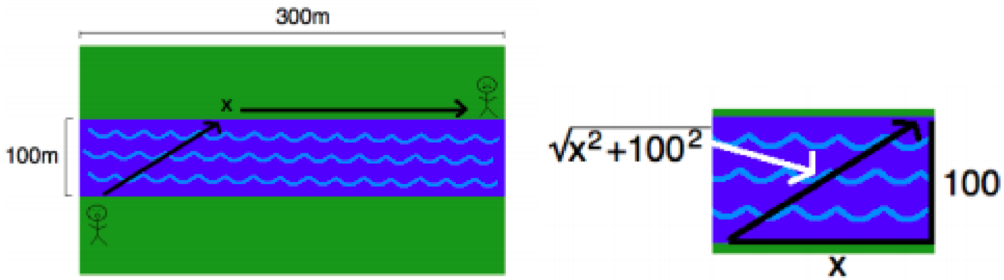
$$b = 2$$

Plugging back in:

$$h = 6$$

The dimensions of the box are 2 x 2 x 6

1.) You are standing on the bank of a river that is 100m wide, and see 12 large kegs of beer calling your name 300m up the opposite shore. You can swim at 3m/s and run at 5m/s, and you want to get to the beer as quickly as possible. To what point on the opposite shore should you swim, before running the rest of the way?



$$T = swim\ time + run\ time$$

$$Recall: time = \frac{distance}{speed}$$

$$T(x) = \frac{\sqrt{x^2 + 100^2}}{3} + \frac{300 - x}{5}$$

Taking the derivative and simplifying:

$$T'(x) = \frac{x}{3\sqrt{x^2 + 100^2}} - \frac{1}{5}$$

Set T'(x) = 0 and solve for x

$$x = 75$$

At this point, we can also check our global minima and maxima for x (x = 0 and x = 300) to see if either of these provide a value for T(x) that is lower than T(75). By doing this, we confirm that T(75) is the smallest value for T.

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1.) Sketch $f(x) = \frac{x}{x^2-4}$

The function has vertical asymptotes where the denominator = 0, at $x = \pm 2$

The function has only one intercept at (0,0)

$$\lim_{x \to \infty} \frac{x}{x^2 - 4} = 0 \to horizontal\ asymptote\ at\ y = 0$$

Evaluating left and right hand side limits at the asymptotes:

$$\lim_{x \to 2^-} f(x) = -\infty$$

$$\lim_{x \to 2^+} f(x) = \infty$$

$$\lim_{x \to -2^-} f(x) = -\infty$$

$$\lim_{x \to -2^+} f(x) = \infty$$

Now, we evaluate $f'(x)$

$$f'(x) = \frac{-(x^2 + 4)}{(x^2 - 4)^2}$$

There are no critical pts and there are singular pts at $x = \pm 2$

$f'(x)\ is\ negative\ everywhere\ except\ at\ x = \pm 2, so\ the\ function\ f(x)\ is\ decreasing\ everywhere\ except\ those\ points.$

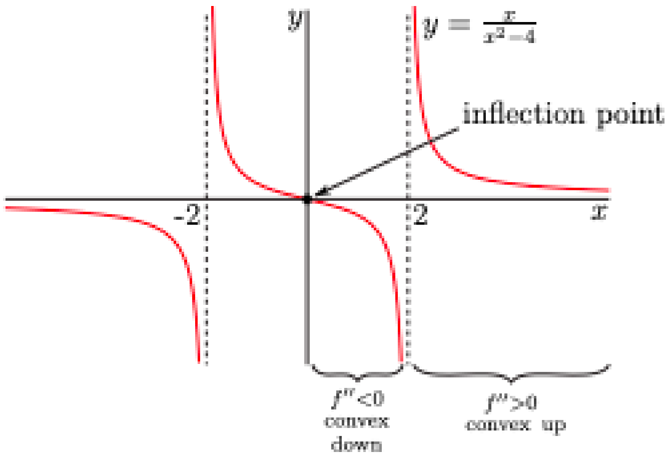
Now, we evaluate $f''(x)$

$$f''(x) = \frac{2x(x^2+12)}{(x^2-4)^3}$$

Evaluating for $f''(x) = 0, we\ find\ that\ x = 0\ is\ an\ inflection\ point.$

When x<-2 and 0<x<2, f''(x)<0, so f(x) is concave down.
When -2<x<0 and x>2, f''(x)>0, so f(x) is concave up.

This information is all you need to sketch the following graph:



2.) Sketch $f(x) = x^3 - 6x^2 + 9x - 54$

The function has no vertical asymptotes, since it exists over all real numbers.
Solving for x and y intercepts, we get: $(0,-54)$, $(6,0)$

For very large x,

$$f(x) \to \begin{cases} +\infty & \text{as } x \to +\infty \\ -\infty & \text{as } x \to -\infty \end{cases}$$

Hence, there are no horizontal asymptotes

Now, we find the first derivative

$$f'(x) = 3(x-3)(x-1)$$

- When $x < 1$, $(x-1) < 0$ and $(x-3) < 0$, so $f'(x) > 0$.
- When $1 < x < 3$, $(x-1) > 0$ and $(x-3) < 0$, so $f'(x) < 0$.
- When $3 < x$, $(x-1) > 0$ and $(x-3) > 0$, so $f'(x) > 0$.
- Summarising all this

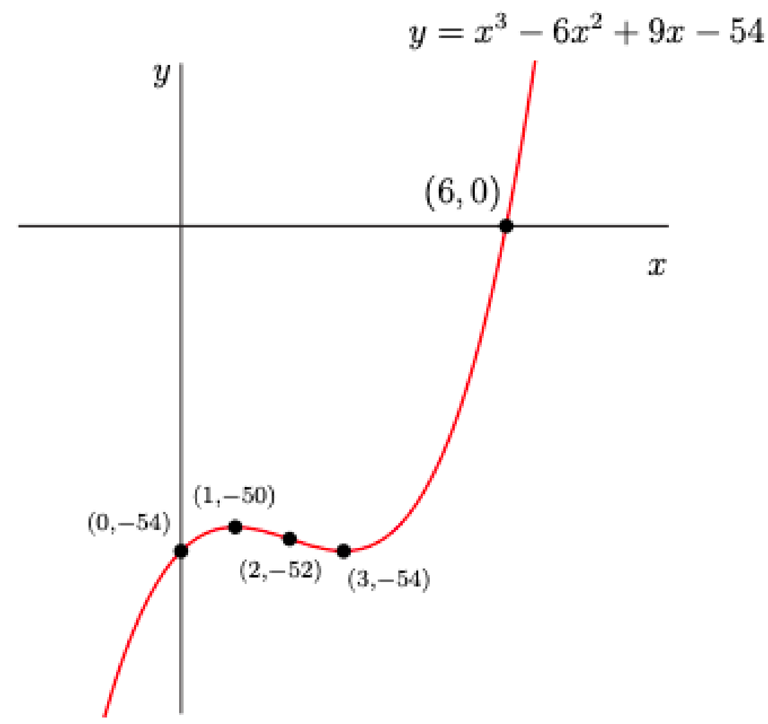
	$(-\infty, 1)$	1	(1,3)	3	$(3, \infty)$
$f'(x)$	positive	0	negative	0	positive
	increasing	maximum	decreasing	minimum	increasing

So the point $(1, f(1)) = (1, -50)$ is a local maximum. The point $(3, f(3)) = (3, -54)$ is a local minimum.

Now, we evaluate the second derivative

$$f''(x) = 6x - 12$$

- So $f''(x) = 0$ when $x = 2$. This splits the real line into the intervals $(-\infty, 2)$ and $(2, \infty)$.
- When $x < 2$, $f''(x) < 0$.
- When $x > 2$, $f''(x) > 0$.
- Thus the function is convex down for $x < 2$, then convex up for $x > 2$. Hence $(2, f(2)) = (2, -52)$ is an inflection point.



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1) Compute the Taylor polynomial of degree 3 of $f(x) = xlnx$ at $a = 1$

By our taylor polynomial formula:

$$f(x) \sim f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 + \frac{1}{3!}f'''(a)(x-a)^3$$

$$f(a) = 0, f'(a) = lna + 1 = 1, f''(a) = \frac{1}{a} = 1. f'''(a) = -\frac{1}{a^2} = -1$$

$$f(x) \sim (x-1) + \frac{1}{2}(x-1)^2 - \frac{1}{6}(x-1)^3$$

2) Estimate $|\sin(0.12) - 0.12|$ by using the linear approximation of $sinx$ at $a = 0$

$$f(x) = sinx - x$$

$$f(x) \sim f(a) + f'(a)(x-a)$$

$$f(a) = 0, f'(a) = 0$$

$$f(x) \sim 0$$

$$f(x) = \sin(0.12) - 0.12$$

$$\sin(0.12) - 0.12 \sim 0$$

Taking the absolute value of both sides:

$$|\sin(0.12) - 0.12| \sim 0$$