COMMERCE MENTORSHIP PROGRAM

MIDTERM REVIEW

MATH 100

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Contents

1 Asymptotics

Asymptotics are used to describe the behavior of a function as the input variable x approaches a certain value. In practice, these values typically are $\infty, -\infty$ and 0. Below lies a list of the asymptotics of several types of common functions that can be generalized.

1.1 Asymptotics of common functions

Polynomials (with $b > a \geq 0$, $c \in \mathcal{R}$):

- as $x \to \infty$, $x^b + x^a + c \sim x^b$
- as $x \to -\infty$, $x^b + x^a + c \sim x^b$
- as $x \to 0$, $x^b + x^a + c \sim c$
- as $x \to 0$, $x^b + x^a \sim x^a$

Exponential functions (with $c > 1$):

- as $x \to \infty$, $c^x \to \infty$
- as $x \to -\infty$, $c^x \to 0$
- as $x \to 0$, $c^x \to 1$

Exponential functions (with $0 < c < 1$):

- as $x \to \infty$, $c^x \to 0$
- as $x \to -\infty$, $c^x \to \infty$
- as $x \to 0$, $c^x \to 1$

Logarithms (with $b > 1$):

- as $x \to \infty$, $\log_b(x) \to \infty$
- as $x \to 0$, $\log_b(x) \to -\infty$

Similar to when solving for limits (which we'll get to shortly), examining more complicated functions asymptotically is a skill that is developed through practice. However, here are some useful things to remember when analyzing how expressions interact with each other:

- when $x \to \infty$, $-1 \leq sin(x), cos(x) \leq 1$
- as $x \to \infty$, $x^x \geq c^x \geq x^b \geq \log(x)$ for $c > 1$, $b > 0$

There are often terms in numerators, denominators and operators, such as absolute values and logarithms, that do not have any effect on the function when x is of extreme magnitude, due to the presence of other, more significant terms in the numerator/denominator/operator. Use the generalizations and tips mentioned above to solve!

1.2 Practice problems

Describe the asymptotics of the following expressions as $x \to \infty$, $x \to -\infty$ and $x \to 0$.

1. $\frac{x^4+e^x-cos(x)}{2}$

2. $2^x + 100000cos(30000x)$

3. $\frac{x^2+\sqrt{|x|}-2}{x^2+1}$

2 Limits

2.1 The limit of a function

The concept of limits extends nicely from our discussion of asymptotics. In both cases, we are interested in a function's behavior as x approaches some finite/infinite value. A limit that exists is denoted as

$$\lim_{x\to a} f(x) = L$$

where L is finite. This equation is read as "the limit of $f(x)$ as x approaches a equals L", and L can be visualized as the value $f(x)$ gets infinitely close to (or in some cases, such as constant functions, takes on) as $x \to a$. Note that the limit above is a "two-sided limit"; for the equation to hold true, both one-sided limits (see below) $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ must tend to L.

2.2 One-sided limits

When examining one-sided limits, we are only interested in the value of the function as x approaches a from either the left of a (i.e. $\lim_{x\to a^-} f(x)$), or the right (i.e. $\lim_{x\to a^+} f(x)$).

2.3 Evaluating limits

The most simple method to evaluate limits is plugging in a. This works for many well-behaved continuous functions, such as polynomials and exponentials. However, exam questions often require us to solve more complicated limits involving **indeterminate forms**, where it is initially unclear what the limit evaluates to. These forms include $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0 \times \infty$, $\infty - \infty$, 0^0, 1^∞, ∞^0, etc. In these cases, we must manipulate our function to an evaluable form, through tricks like:

- Factoring and cancelling an expression from both the numerator and the denominator
- Multiplying the numerator and denominator by the conjugate of the denominator
- Evaluating one-sided limits
- Combining multiple fractions into one fraction, through a common denomator
- Using trigonometric identities
- Simplifying an absolute value; $|f(x)| = -f(x)$ when $x < 0$ and $|f(x)| = f(x)$ when $x > 0$
- Factoring out the highest power of x from the denominator and cancelling it. Or equivalently, using asymptotics.

A good way of thinking about evaluating limits is to think about an **line of reasoning** that could be used; limit problems often tend to require some unstructured, creative thinking to be solved!

2.3.1 Squeeze theorem

A specialized argument sometimes used to solve limits is the squeeze theorem. Essentially, this theorem states that if a function is defined on all $[a, b]$ except possibly c, $c \in [a, b]$, and bound between two other functions on the same interval, its limit as $x \to c$ is equal to the limit of either binding functions, if **the binding two functions have equivalent limits**. It is often used when the function includes sin or cos multiplied by a non-linear polynomial (i.e. x^n with $n > 0$, $n \in \mathbb{Z}$). Please refer to practice problem 5 for an example.

2.4 Practice problems

Evaluate the following limits, or show that they do not exist.

1. $\lim_{x\to 0} x^5 - 32x^3 + 1$

2. $\lim_{x\to -\frac{1}{2}} \frac{-|\frac{1}{3x^2}| + \frac{1}{x^2-1}}{2x+1}$

3. $\lim_{x\to -1} \sqrt{\frac{x^3+x^2+x+1}{3x^2+3x}}$

4. $\lim_{j\to 1} \frac{3-3j}{2-\sqrt{5-j}}$

5. $\lim_{x\to 2} (x-2)^3 \cos[(\frac{x^3+2}{x^2+3})^3 + 999]$

Evaluate the following limits at infinity, or show that they do not exist.

1. $\lim\limits_{x \to -\infty} \dfrac{5x^2 + 4x - 1}{3x^3 + 2}$

2. $\lim\limits_{x \to \infty} \sqrt{x^2 + 6x} - x$

3. $\lim\limits_{x \to -\infty} \dfrac{sin(x)}{x^2 + 3x + 2}$

4. $\lim\limits_{x \to -\infty} \dfrac{x + 1}{|\sqrt{x^2}|}$

5. $\lim\limits_{x \to 0^+} \dfrac{x^2 + \frac{1}{x}}{x - 1}$

3 Continuity

3.1 Conditions for continuity

For a function $f(x)$ to be continuous at $x = a$, three conditions must be satisfied;

1. $f(a)$ exists.
2. $\lim_{x \to a} f(x)$ exists; that is, both one-sided limits $\lim_{x \to a^+} f(x)$ and $\lim_{x \to a^-} f(x)$ are equal.
3. $f(a) = \lim_{x \to a} f(x)$.

Intuitively, a function that is continuous at a point is **connected** at that point; the part of the function to the left of that point is connected to the part of the function to the right of that point.

3.2 Continuity of common functions

The following functions are continuous everywhere on intervals on which they are defined.

- Polynomials, constants, exponentials, $\cos(x)$, $\sin(x)$. These are also defined for all $x \in \mathbb{R}$
- Non-polynomial power functions, rational functions
- $\tan(x)$, logarithms

By arithmetic of continuity, the sum, difference, product and quotient (if the denominator $\neq 0$) of functions continuous at $x = a$ returns a function also continuous at $x = a$.

3.3 Practice problems

1. Where is the function $f(x) = \frac{sin(x)+3}{x^2-2x+1}$ contiunous?

2. Find the values b, c such that $f(x)$ is continuous everywhere.

$$f(x) = \begin{cases} 6 - cx & \text{if } x < 2c \\ x^2 & \text{if } x > 2c \\ b & \text{if } x = 2c \end{cases}$$

4 The Definition of the Derivative

The idea of the derivative originates from the slope formula for a linear function, where slope $m = \frac{y_1 - y_0}{x_1 - x_0}$. The derivative extends this formula to compute the slope at a **differentiable point** $(a, f(a))$ on a non-linear function $f(x)$, by bringing point $(b, f(b))$ infinitely close to $(a, f(a))$.

4.1 Deriving the definition of the derivative

We first apply the slope formula to our non-linear function $f(x)$. This gives us the slope of the secant line passing through $(a, f(a))$ and $(b, f(b))$, $m_{sec} = \frac{f(b)-f(a)}{b-a}$. Then, expressing b as $a + h$, where $h \in \mathbb{R}$ such that $b = a + h$, we substitute in $a + h$, simplify, and obtain $\frac{f(a+h)-f(a)}{h}$.

To complete the last step and obtain the definition of the derivative, we take the limit as $h \to 0$;

Definition of the derivative: $\lim\limits_{h\to 0} \dfrac{f(a+h) - f(a)}{h} = f'(a)$

That's it! Note that this expression is used to calculate the derivative $f'(a)$ at a point $(a, f(a))$. Alternatively, if we replace a, a number, with the placeholder x, the resulting expression can be used to obtain the derivative as a function, $f'(x)$.

Derivative at a point $(f'(a))$: The slope of a curve at a defined point $(a, f(a))$. Equivalently, it can be defined as the slope of the tangent line to that point.

Derivative as a function $(f'(x))$: The function that outputs the slope of the curve at any differentiable point on the original function.

4.2 An alternative expression for the definition of the derivative

Let's go back to our original expression for the definition of the derivative at a point. If we set $h = x - a$ and substitute we can get an alternative expression for the definition of the derivative.

Alternate Definition: $\lim\limits_{x\to a} \dfrac{f(x) - f(a)}{x - a} = f'(a)$

Arguably, this expression better shows the intuition behind the derivative at a point, as it illustrates the RHS of the secant line slope formula while "squeezing" one point to another. However, its disadvantage is that we cannot use it to calculate the derivative as a function.

4.3 Differentiability

For a point $(a, f(a))$ to be differentiable, $\lim\limits_{h\to 0} \dfrac{f(a+h) - f(a)}{h}$ (or equivalently, $\lim\limits_{x\to a} \dfrac{f(x) - f(a)}{x - a}$) must return a finite value. Implicitly, this means that two conditions must be satisfied.

1. $f(x)$ is defined and continuous at $x = a$. i.e. $f(a) = \lim\limits_{x\to a} f(x) = L$, where L is a finite value.

2. $f(x)$ must be "well-behaved" at a. informally, this can be thought of as the absence of things like sharp corners or oscillations at $x = a$.

4.4 Practice problems

1. Using the definition of the derivative, find $f'(a)$ given $f(x) = 2x^2 + 1$

2. Use the definition of the derivative to compute the derivative of the function $f(x) = \frac{1}{x^2+3}$

3. Is the following function differentiable at $x = 0$? Justify your answer using the definition of the derivative.

$$f(x) = \begin{cases} x\cos(x) & \text{if } x \geq 0 \\ \sqrt{x^2 + x^4} & \text{if } x < 0 \end{cases}$$

4. Let $f(x) = x^2 + 3x + 1$, with an "artificially" restricted domain $x \in [-1000, 1000]$. On what interval does the derivative exist?

5 Derivative Rules

Differentiation: The process of obtaining the derivative, either at a point or as a function

It is important to understand the logic behind the definition of the derivative, but if you start using it to differentiate increasingly complicated functions, you will quickly realize that it is quite inefficient at differentiating! Luckily, we have many "derivative rules" in our toolkit to more quickly differentiate. They all (1, 2, 3, 4, 5, 6) originate from the definition of the derivative.

5.1 A differentiation toolkit

Note: $\frac{d}{dx}f(x) = f'(x) = \frac{df}{dx}$. Also, $\log(x) = \ln(x) = \log_e(x)$, for this course.

Constant rule: $\frac{d}{dx}c = 0$	$\frac{d}{dx}\log_c(x) = \frac{1}{x\log(c)}$
Power rule: $\frac{d}{dx}[x^n] = nx^{n-1}, n \neq 1$	$\frac{d}{dx}[c^x] = c^x \log(c), c > 0$
Constant multiple rule: $\frac{d}{dx}[cf(x)] = cf'(x)$	$\frac{d}{dx}\sec(x) = \sec(x)\tan(x)$
$\frac{d}{dx}\sin(x) = \cos(x)$	$\frac{d}{dx}\csc(x) = -\csc(x)\cot(x)$
$\frac{d}{dx}\cos(x) = -\sin(x)$	$\frac{d}{dx}\cot(x) = -\csc^2(x)$
$\frac{d}{dx}\tan(x) = \sec^2(x) = \frac{1}{\cos^2(x)}$	$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$
$\frac{d}{dx}[e^x] = e^x$	$\frac{d}{dx}\cos^{-1}(x) = -\frac{1}{\sqrt{1-x^2}}$
$\frac{d}{dx}\log(x) = \frac{1}{x}$	$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}$

Sum and difference rule: $$\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$$

Product rule: $$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$$

Quotient rule: $$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$$

Chain rule: $$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x)$$

5.2 Practice problems

1. Let $f(x) = \frac{h(g(x))}{xm(x)}$, where $h(x)$, $g(x)$ and $m(x)$ are functions. Differentiate $f(x)$, expressing your answer only in terms of x, $h(x)$, $g(x)$, $m(x)$ and their derivatives.

2. What is the slope of the graph $y = (\frac{cos(x)}{x^2+2})^2$ at $x = 1$?

3. $f(x) = e^{e^{e^x}}$. Find $f'(x)$.

4. Use the product rule to differentiate $g(x) = (x^2 + 2\log x)(8\sqrt{x})$

6 Tangent Lines and the Linear Approximation

6.1 Tangent lines

The derivative at $x = a$ can be interpreted as the slope of the tangent line to the point $(a, f(a))$, and a common exam problem is to be asked to find the equation $y = mx + b$ of the tangent line at that point. Luckily, this usually is done easily through the two following steps:

1. Given $f(x)$, differentiate to obtain $f'(x)$ and plug in $x = a$ to find $m = f'(a)$. This is the slope of the tangent line to $x = a$, and now we have the equation $y = f'(a)x + b$
2. To solve for b, find a point (x, y) on the tangent line. Usually, the easiest of such points to find is $(a, f(a))$, which lies on both the given function and the tangent line; we have a, and we can find $f(a)$ given $f(x)$. Substitute y and x for a and $f(a)$, then rearrange for b.
$$f(a) = f'(a)a + b \Leftrightarrow b = f(a) - af'(a)$$
3. Compute b and plug its value into $y = f'(a)x + b$

We can go a step further, replacing b in our previous equation $y = f'(a)x + b$ with $f(a) - af'(a)$ and simplifying. This gives us the following useful, time-saving formula:

Tangent line equation: $y = f'(a)(x - a) + f(a)$.

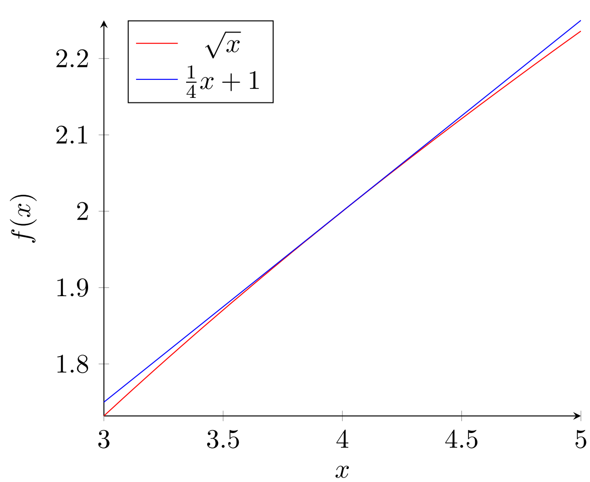
6.2 The linear approximation

A subtle use of tangent lines is that they can be used to approximate values that a function takes on near some differentiable $x = a$; many common functions like $\sqrt{x}$ output ”nice”, well-known numbers at some input values (e.g. $x = 1, 4, 9$, etc.) but not at others. If we want to know the function’s approximate value at an input near one of these, we can obtain a linear approximating function by plugging in the ”nice” inputs value into the RHS of the tangent line equation;

Linear approximation: $f(x) \approx f'(a)(x - a) + f(a)$, for some x close to a

Note that before plugging in x to approximate $f(x)$, we must obtain a, $f(a)$ and $f'(a)$.

The graph below shows the linear approximation to $f(x) = \sqrt{x}$ for x near $x = 4$.



6.3 Practice problems

1. Given $f(x) = x^3 + 2x$, find the equation for the tangent to $f(x)$ at $x = 3$

2. Use the linear approximation to approximate $\sqrt{5}$

3. Use the linear approximation to approximate $\log(3)$

7 Implicit and Logarithmic Differentiation

7.1 Implicit differentiation

Up until this point, we've differentiated explicit functions; that is, functions (typically denoted $f(x)$ or y) equated to its equivalent expression that is in terms of the independent variable (usually denoted x).

But sometimes, we are not given an explicit function. Instead we may be given an equation with y and x, where it is impossible to rearrange for y to be alone. For example, an equation like

$$y = y^3 + xy + x^3$$

In this situation, we can use the fact that the derivative of the LHS equals the derivative of the RHS, differentiate both sides with respect to x and rearrange to get y'. This way of differentiating, where we treat one variable (y) as a function of another (x) is called **implicit differentiation**, and is made possible by the chain rule.

Along with the equation with x and y, Implicit differentiation problems often provide a point (x, y) so that students are tasked with finding the tangent to that point (see practice problem 1).

7.2 Logarithmic differentiation

An application of implicit differentiation is **logarithmic differentiation**, where we take the logarithm of both sides and algebraically manipulate, before differentiating implicitly and rearranging to find the derivative.

This can be very useful to differentiate functions made up of many factors or a complicated ratio, such as $y = f(x) = (x^2+1)(\sin x - x)(x^3+1)$ or $g(x) = \frac{5x^2(\cos x)(\log x)}{(x+1)(x+2)}$ Although these functions are hypothetically differentiable by iterations of product rule and/or quotient rule, it is much more efficient to use logarithmic differentiation. For example,

$$\begin{aligned}
y &= (x^2+1)(\sin x - x)(x^3+1) \\
\log y &= \log\left[(x^2+1)(\sin x - x)(x^3+1)\right] \\
\log y &= \log(x^2+1) + \log(\sin x - x) + \log(x^3+1) \\
\frac{d}{dx}\log y &= \frac{d}{dx}\left[\log(x^2+1) + \log(\sin x - x) + \log(x^3+1)\right] \\
\frac{y'}{y} &= \frac{2x}{x^2+1} + \frac{\cos x - 1}{\sin x - x} + \frac{3x^2}{x^3+1} \\
y' &= y\left[\frac{2x}{x^2+1} + \frac{\cos x - 1}{\sin x - x} + \frac{3x^2}{x^3+1}\right] \\
y' &= \left[(x^2+1)(\sin x - x)(x^3+1)\right]\left[\frac{2x}{x^2+1} + \frac{\cos x - 1}{\sin x - x} + \frac{3x^2}{x^3+1}\right]
\end{aligned}$$

Another case that calls for logarithmic differentiation is when the function involves logarithms and powers but cannot be differentiated directly through our known derivative rules (e.g. $f(x) = x^x$).

7.3 Practice problems

1. Find the derivative of the curve $xy^2 + x^2y = 2$ at the point $(1, 1)$.

2. Find the tangent to the curve $x^2 + y^2 = 5$ at the point $(-1, 2)$.

3. Let $y = x^{\log x}$. Find $\frac{dy}{dx}$ in terms of x.

4. Differentiate $g(x) = \frac{5x^2(\cos x)(\log x)}{(x+1)(x+2)}$, expressing the derivative in terms of x only.

8 Curve Sketching Part 1: Concavity and Extrema

Before we begin sketching curves, we must first learn some additional terminology.

8.1 Concavity

1. We say that a function $f(x)$ is **concave up** at $x = c$ if $f''(c) > 0$. The tangent line to $(c, f(c))$ will lie below the curve near that point, and the function will be decreasing at a decreasing rate, increasing at an increasing rate or have a slope of zero at $x = c$.

2. We say that a function $f(x)$ is **concave down** at $x = c$ if $f''(c) < 0$. The tangent line to $(c, f(c))$ will lie above the curve near that point, and the function will be increasing at a decreasing rate, decreasing at an increasing rate or have a slope of zero at $x = c$.

3. A function changes concavity at points called **inflection points**, where $f''(x) = 0$ and $f''(x)$ changes sign.

This image shows some good illustrations of concavity.

8.2 Global maxima and minima (i.e. global extrema)

Let I be an interval $[a, b]$ or (a, b), and let a function $f(x)$ be defined for all $x \in I$. Also, let $c \in I$. Then,

1. $f(x)$ has a **global maximum** on the interval I at $x = c$ if $f(c) \geq f(x)$ for all $x \in I$.

2. $f(x)$ has a **global minimum** on the interval I at $x = c$ if $f(c) \leq f(x)$ for all $x \in I$.

If the interval I is open (i.e. $[a, b]$), then global extrema can possibly exist at any $a \leq x \leq b$. If the interval I is closed (i.e. (a, b)), then global extrema can possibly exist at any $a < x < b$

8.2.1 Finding global maxima and minima

If $f(x)$ is defined, continuous on I and I is closed, global extremum are guaranteed to exist at some $x = c$, $c \in I$. They can be found at three possible $x \in I$:

1. At the endpoints of I, If the interval I is closed. Compute $f(a)$ and $f(b)$.

2. At **critical points**, where $f'(x) = 0$. Compute $f(x)$ at critical points. Often (but not always), if a global maximum exists at $x = c$ where $f'(c) = 0$, $f(x)$ increases as $x \to c^-$, and decreases as x becomes greater than c. Similarly, in the case of a global minima at $f(c)$ where $f'(c) = 0$, $f(x)$ often decreases as $x \to c^-$, and increases as x becomes greater than c.

3. At **singular points**, where $f'(x) = DNE$. Compute $f(x)$ at singular points.

After obtaining all these values, compare them to find where $f(x)$ has global extrema. Note that a function can be a maximum of one global maximum **value** and one global minimum **value**, but a it may have multiple global extrema. An example of a function with multiple global extrema is $2 \sin x$, with restricted domain $[0, 5\pi]$. This function has 3 global extrema, at x-values $x = \frac{\pi}{2} + 2\pi n, 0 \leq n \leq 2, n \in \mathcal{Z}$. However, it only has one global maximum value, 2.

8.3 Local maxima and minima (i.e. local extrema)

Again, Let I be an interval $[a, b]$ or (a, b), and let a function $f(x)$ be defined for all $x \in I$. Also, let $c \in I$. Then,

1. $f(x)$ has a **local maximum** on the interval I at $x = c$ if $f(c) \geq f(x)$ for all $x \in I$ that are near c. Precisely, if $f(c) \geq f(x)$ on $[c-\delta, c+\delta]$, where $\delta > 0$ and $[c-\delta, c+\delta] \in I$.

2. $f(x)$ has a **local maximum** on the interval I at $x = c$ if $f(c) \leq f(x)$ for all $x \in I$ that are near c. Precisely, if $f(c) \leq f(x)$ on $[c-\delta, c+\delta]$, where $\delta > 0$ and $[c-\delta, c+\delta] \in I$.

Note that for local maxima and minima, it is not necessary that $f(c) \geq f(x)$ or $f(c) \leq f(x)$ for all $x \in I$. In other words, local extrema are not necessarily global extrema. Also, local maxima and minima cannot exist at $x = a$ or $x = b$, since local extrema must be sandwiched such that $c - \delta < c < c + \delta$ where $[c - \delta, c + \delta] \in I$.

8.3.1 Finding local maxima and minima

If $f(x)$ is defined and continuous on I, where I does not have to be closed, We can find local extrema at the same places we can find global extrema, **except at endpoints of** I;

1. At critical points, where $f'(x) = 0$. Compute $f(x)$ at critical points. Often (but not always), if a local maximum exists at $x = c$ where $f'(c) = 0$, $f(x)$ increases as $x \to c^-$, and decreases as x becomes greater than c. Similarly, in the case of a local minimum at $f(c)$ where $f'(c) = 0$, $f(x)$ often decreases as $x \to c^-$, and increases as x becomes greater than c.

2. At singular points, where $f'(x) = DNE$. Compute $f(x)$ at singular points

After finding these points, we must examine the behavior of $f(x)$ around them.

- If $f(c)$ is a critical point, obtain $f''(c)$, whose sign represents concavity. If $f''(c) < 0$, we have a local maximum at $(c, f(c))$. If $f''(c) > 0$, we have a local minimum at $(c, f(c))$

- If $f''(c) = 0$ at a critical point, $(c, f(c))$ either can be both a local maximum or minimum (e.g. any point on a constant function $f(x) = q, q \in R$), or it is not a local extrema at all (e.g. the point $(0, 0)$ on $f(x) = x^3$).

- If $f(c)$ is a singular point, examine the derivative around $x = c$. If the derivative is positive to the left of c and negative to the right of c, $(c, f(c))$ is a local maximum. If the derivative is negative to the left of c and positive to the right of c, $(c, f(c))$ is a local minimum.

Note that a global extremum at $x = c$ that isn't located on the endpoints of I also qualify as a local extremum.

8.4 Intuitive descriptions of global and local extrema

As we can see from the last two pages, defining global and local extrema results in quite a bit of headache-inducing working! The key to understanding these concepts is to remember them intuitively;

- A global maximum/minimum point is a point where a function takes on its highest/lowest value **on the interval it is defined**.

- A local maximum/minimum point is a point where a function takes on its highest/lowest value within a smaller interval (i.e. a point **between**, NOT on, a smaller interval's endpoints) that lies inside the larger interval on which the function is defined.

8.5 Practice Problems

1. Consider the function $f(x) = sin(x)$. On which intervals is this function concave down? Give a general interval with endpoints in terms of n, $n \in Z$

2. Does $\log x$, defined on $0 < x < \infty$, have any global maxima or minima? Explain with reference to the value of its derivative.

3. Find the local extrema of $f(x) = x^3 - 2x$. Are these points also global extrema?

4. Let $f(x) = \cos(x)$. On (-2,4), find its local maximum and minimum with reference to the sign of $f''(x)$ at their x-values.

9 Curve Sketching Part 2: Methodology

Now that we've learned about global and local extrema, we can apply our cumulative knowledge to systematically sketch curves. When sketching curves, we typically want to include a few features on our graph;

9.1 Features to include

(a) Features found using $f(x)$

1. Axis labels (based on the usually-provided domain of the function).
2. x- and y-intercepts.
3. Vertical and horizontal asymptotes, represented by a dotted line. Vertical asymptotes are located $x = a$ where $f(x)$ blows up to $\pm\infty$ while $x \to a$. Horizontal asymptotes are located at y which correspond to the finite limits $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$, if they exist.
4. "hole" and "jump" discontinuities

(b) Features found using $f'(x)$

1. Critical points ($f'(x) = 0$) and singular points ($f'(x) = DNE$).
2. Domains where $f(x)$ is increasing ($f'(x) > 0$) and decreasing ($f'(x) < 0$). Check for the sign of $f'(x)$ on both sides of critical and singular points.

(c) Features found using $f''(x)$

1. Inflection points where $f"(x) = 0$.
2. Domains where $f(x)$ is concave up ($f''(x) > 0$) , and concave down ($f''(x) < 0$). Check for the sign of $f''(x)$ on both sides of inflection points, as well as points where the second derivative does not exist.

9.2 Practice Problems

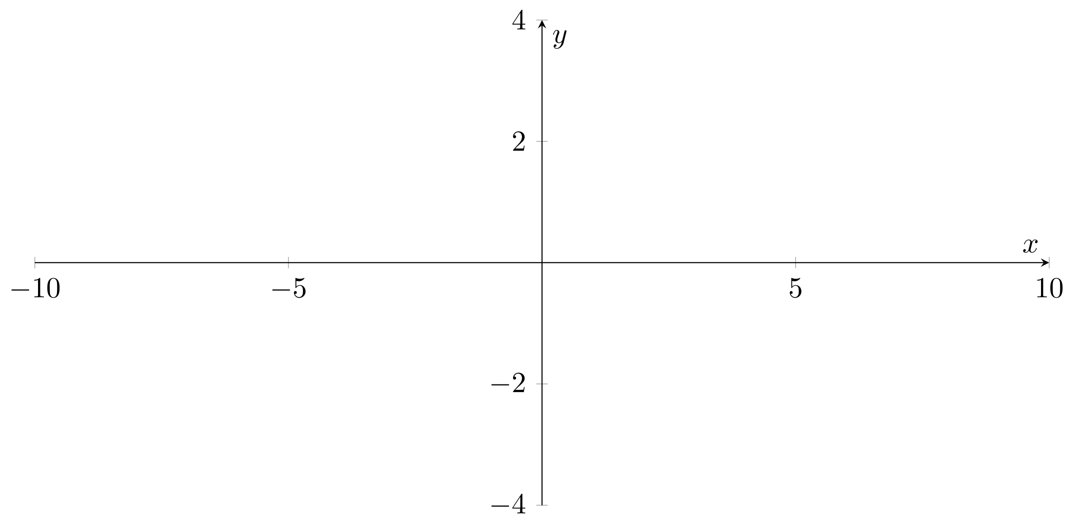
1. Let $y = \sqrt{x+1}$.

   (a) What are the intercepts and asymptotes of this curve?

   (b) On what intervals does the curve increase/decrease? Where are the local and global extrema?

   (c) Differentiate twice to obtain $f''(x)$. What are the intervals of concavity? Where are the inflection points?

   (d) Sketch $y = \sqrt{x+1}$. Indicate any intercepts, asymptotes, critical points and singular points.



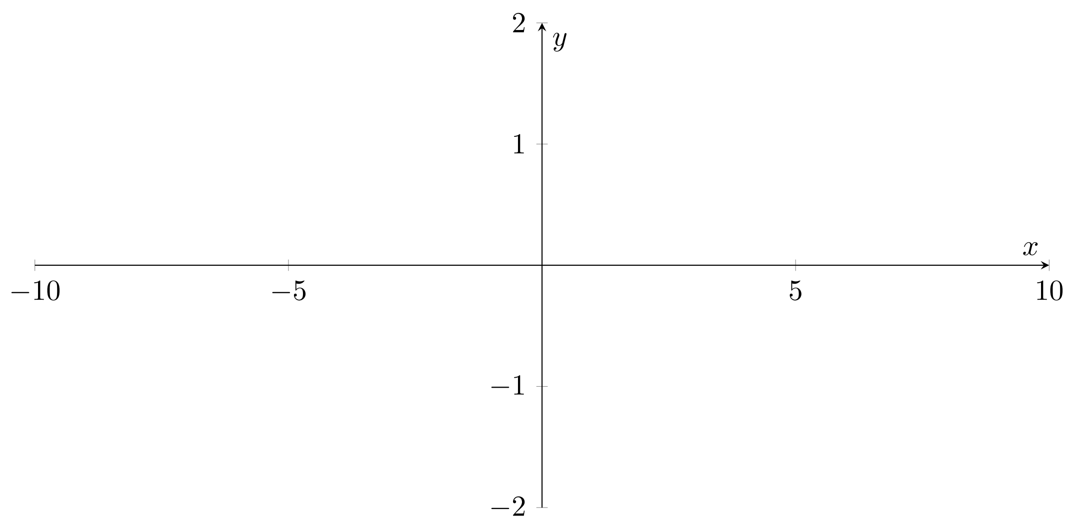
2. Let $y = \frac{x+2}{2x^2+2}$.

   (a) What are the intercepts and asymptotes of this curve?

   (b) On what intervals does the curve increase/decrease? Where are the local and global extrema?

   (c) Differentiate twice to obtain $f''(x)$. What are the intervals of concavity? Where are the inflection points?

   (d) Sketch $y = \frac{x+2}{2x^2+2}$. Indicate any intercepts, asymptotes, critical points and singular points.



10 Skippable extras

10.0.1 Additional resources

There are many great additional resources available for math help. Here are some of my favorites:

1. Piazza
2. CLP-1 Differential Calculus Textbook, a UBC-tailored Calculus I textbook written in part by current MATH 100C professor Elyse Yeager.
3. Math Learning Centre, a fantastic resource especially for help with the written assignments.
4. Sauder Math Coaching
5. Past MATH 100 final exams
6. Instructor office hours

10.0.2 Tips for the final exam

As someone who took both MATH 100 and MATH 101, here's some tips I would have given myself for the final if I could go back in time. They are listed from most important to least important.

1. Attempt all the written assignment problems individually before consulting with your group. Yes, they are very challenging and time-consuming, but I encountered very similar problems to those found on my written assignments on both the MATH 100 and MATH 101 finals. They are also excellent for developing mathematical communication skills.
2. Ask and **answer** questions on Piazza. I greatly improved my comprehension of the material by answering other classmates' questions whenver possible.
3. Use past finals to identify misunderstandings.

10.0.3 About this document

This document was created using Overleaf, an online LaTeX editor. Its functionality is best when viewed in PDF format.

Its contents were determined based on the MATH 100 Calendar, past CMP review packages, the CLP-1 textbook, Prof. Lior Silberman's MATH 100 Worksheets, and my own notes from taking MATH 100 last year. The practice problems are either from past final exams, or they were created by myself with inspiration from the CLP-1 textbook and past CMP Review Packages.

I wish you the best of luck for your upcoming Test 2 and for the rest of the term!

-Ivan Chow