MATH 104/184

2018W1 Midterm 2

Review Package

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[15] **1. Short Problems**. Each question is worth 3 points. Put your answer in the box provided and show your work. No credit will be given for the answer without the correct accompanying work.

(a) Find the equation of the normal line to the curve

$$y = 3y^2x + 3x^2y + 1$$

at the point $(-1, 1)$.

Answer:

(b) Find $\dfrac{dy}{dx}$ when $\tan(xy) = y + 2$.

Answer:

(c) Find $f'(x)$ if $f(x) = x^{\log x} + (\log x)^x$

Answer:

(d) If \$1000 is invested at an annual interest rate of 2% compounded continuously, how long will it be until it earns \$2000 interest? Would I earn more interest if I requested the bank compound every month instead?

Answer:

(e) State whether the function $f(x) = x\sqrt{8 - x^2}$ has an absolute maximum on the interval $x \in [0, 2\sqrt{2}]$. If so, find the coordinates of the point. Does $f(x)$ have an absolute minimum on that interval? If so, find the coordinates.

Answer:

(f) A rectangle of length 6cm and width 8cm steadily increases in length by 4cm per minute and decreases in width by 2cm per minute for five minnutes. Which of the following is true of the area A of the rectangle during those five minutes?

1. A is always decreasing.
2. A decreases at first, then increases.
3. A is always increasing.
4. A remains constant.
5. A increases at first, then decreases.

Answer:

Long Problems. In questions 2 - 5, show your work. No credit will be given for the answer without the correct accompanying work.

[14] **2.** Let $f(x)$ be a continuous and differentiable function on $[-10, 10]$, with

$$f(-10) = -2, f(-1) = 3, f(5) = 7, f(8) = -2 \quad \text{and } f(10) = 0.$$

Assume its derivative $f'(x)$ is also continuous and differentiable on $[-10, 10]$. Which of the following statements are always True? If True, briefly explain your reasoning.

(a) There is a $c \in (-10, 10)$ such that $f(c)$ is a global maximum.

(b) $f'(c) > 0$ for all $c \in (-1, 5)$.

(c) There is a $c \in (-10, 10)$ such that c is a critical point.

(d) There is a $c \in (-4, 9)$ such that $f'(c) = -3$.

(e) f has a global minimum in $(-10, 10)$.

(f) f has a local maximum in $(-1, 5)$.

(g) There is a $c \in (-10, 10)$ such that $f''(c) = 0$.

[10] **3.** The demand curve of a product is given by $4p + q + pq = 252$, where p is the price in dollars and q is in hundreds of units. The price elasticity of demand is $\epsilon(p) = \frac{p}{q}\frac{dq}{dp}$.

(a) Compute $\epsilon(p)$ when $p = \$3$.

(b) If price is lowered from \$3 by 2%, what is the approximate change in demand?

(c) Does revenue increase or decrease if price is increased from \$3 by 2%?

(d) At what price is revenue maximized?

[10] **4.** An emperor penguin walks from his colony, located at the origin $(0, 0)$, to the ocean edge at $(5, 40)$ along a curved path described by the curve $y = x^2 + 3x$. The distance between the penguin and the colony increases at a constant rate of 1m/sec. What is the penguin's speed in the y direction at the moment he reaches the ocean edge?

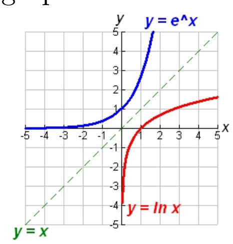
[10] **5.** The penguin is stopped before a 50m wide strip of rough ice that runs east-west. Just beyond the strip lies the open ocean. He sees his mate has crossed the rough ice strip and is standing 300m west of his position at the edge of the ocean. The rough ice is tough going for penguin feet, and he can only travel at .5m/sec on the rough ice. Once he reaches the water, he can swim in the open ocean along the shore at 3m/sec. He wants to get to his mate as quickly as possible. At what point on the opposite edge of the rough ice strip should he aim in order to minimize the total time to reach his mate?

1. Cheat Sheet

1.1. Preliminaries.

- exp and log

  negative exponents $\to$ reciprocal
  fractional exponent $\to$ roots
  $b^x b^y = b^{x+y}$
  $\log(xy) = \log x + \log y$
  $(b^x)^y = b^{xy}$
  $\log(e^x) = x = e^{\log x}$ (where defined)
  $e^0 = 1, \log 1 = 0$
  graph:

- inverse functions, horizontal line test

1.2. Business problem:

- **quantity** q (number of widgets)
- **price** p (price per widget)
- **Revenue** R (money you receive when you sell q widgets for price p)
- **Cost** $C(q)$ (what it costs to make widgets)
  - **Fixed cost** F (costs that don't depend on the number of widgets)
  - **Variable cost** $V(q)$ (costs that depend on the number of widgets)
- **Break-Even Points** q such that $C(q) = R(q)$
- **Profit** P (the financial gain; the difference between amount earned and spent)
- **Demand** (equation that expresses the relation between quantity and price) - assumed linear for the MT1

In math:

$$R = pq$$
$$C(q) = F + V(q)$$
$$P = R - C$$
$$m = \frac{\Delta p}{\Delta q}$$
$$p - P_0 = m(q - Q_0)$$

given the data point (P_0, Q_0)

1.3. **Limits.**

Limits may not exist for many reasons. If they do, define:

- **Left-hand limit:** $\lim_{x \to a^-} f(x) = K$
  $f(x)$ gets closer to K as x approaches a from the left, i.e. $x < a$
- **Right-hand limit:** $\lim_{x \to a^+} f(x) = L$
  $f(x)$ gets closer to L as x approaches a from the right, i.e. $x > a$
- **Limit:** $\lim_{x \to a} f(x) = L$
  $f(x)$ gets closer to L as x approaches a both from the right or the left

Some strategies

- If left limit and right limit exist but are not equal, the limit DNE.
- The limit does not have to be the value of the function.
- Rational function (polynomial divided by polynomial) - if you get 0/0, then factor
- Rationalize denominator
- Limit laws, e.g. the sum of limits is the limit of sums, etc
- If you know the function is continuous at the point, then we can 'plug in'. Continuous functions include polynomials, exp, sin, cos, rational functions where the denominator is not 0. Also sums, products, compositions of such functions.
- Some of the ways functions can go bad: dividing by zero, roots of negatives, log of $x \leq 0$, going outside the defined domain.
- If a rational function $\to \pm\infty$, decide sign by checking signs of numerator and denominator

1.4. **Continuity.**

- A function $f(x)$ is **continuous** at a if $\lim_{x \to a} f(x) = f(a)$

1.5. **Intermediate Value Theorem.**

Memorize this theorem!

- Let f be continuous on $[a, b]$. Then for any Y between $f(a)$ and $f(b)$, we can find a $c \in [a, b]$ with $f(c) = Y$.
  This just means f takes every value (y-coord) in between its endpoint values

1.6. **Derivatives.**

- Rate of change: think slope. Secants give average rate of change; tangents are instantaneous rates of change.

Memorize this definition!

- Let $a \in \mathbf{R}$ and $f(x)$ be defined on an open interval containing a. The **derivative** of $f(x)$ at x=a is defined by

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

  if the limit exists.

1.7. **Differentiation.**

$$\frac{d}{dx}x^n = nx^{n-1}$$
$$\frac{d}{dx}\Big(f(x) + g(x)\Big) = f'(x) + g'(x)$$
$$\frac{d}{dx}\Big(cf(x)\Big) = cf'(x)$$
$$\frac{d}{dx}\Big(f(x)g(x)\Big) = f'(x)g(x) + f(x)g'(x)$$
$$\frac{d}{dx}\Big(\frac{f(x)}{g(x)}\Big) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$$
$$\frac{d}{dx}e^x = e^x$$

- Finding equations of tangent lines

2. Cheat Sheet

2.1. **Trig, exponential, log.**

- trig derivatives
- exponential and logarithmic functions
- growth/decay models

$$\frac{d}{dx}\sin x = \cos x \qquad \frac{d}{dx}\tan x = \sec^2 x \qquad \frac{d}{dx}\cot x = -\csc^2 x$$
$$\frac{d}{dx}\cos x = -\sin x \qquad \frac{d}{dx}\sec x = \sec x \tan x \qquad \frac{d}{dx}\csc x = -\csc x \cot x$$
$$\frac{d}{dx}e^x = e^x \qquad \frac{d}{dx}\log x = \frac{1}{x}$$

2.2. **Chain Rule.** (p. 157)

- Key: Figure out which is the outside function and which is the inside function.
- Then: Derive the outside, and multiply by the inside derivative.
- Can iterate recipe to multiple nested functions.
- **Do not confuse with the product rule!**

$$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$$

2.3. **Implicit Differentiation.** (p. 173)

- Use when you can't isolate y as a function of x.
- Take $\frac{d}{dx}$ of both sides of equation and use chain rule. A lot.
- Remember $\frac{d}{dx}x = 1$, $\frac{d}{dx}y = \frac{dy}{dx}$.
- Gather all y' terms on one side and solve for it.
- Use $\frac{dy}{dx}$ as the slope to tangent line as before, or (-1/slope) of normal line

2.4. **Logarithmic Differentiation.** (p. 172)

- Take log then differentiate. Then multiply by the original function.

$$\frac{d}{dx}\log f(x) = \frac{f'(x)}{f(x)} \Longrightarrow f'(x) = f(x)\left(\frac{d}{dx}\log f(x)\right)$$

  Useful because log turns products into sums, quotients into differences, and powers into products.
- Used for relative rates of change (e.g. elasticity)

2.5. **Compound Interest.** (Handout)

A future value
P beginning value (principal)
r interest rate (as decimal or fraction)
n compounding periods per year
t years invested

- compound interest $A = P\left(1 + \frac{r}{n}\right)^{nt}$
- continuously compounded interest $A = Pe^{rt}$

2.6. **Business terminology and notation.** (Handout)

- **quantity** q (number of widgets)
- **price** p (price per widget)
- **Revenue** R (money you receive when you sell q widgets for price p)
- **Cost** $C(q)$ (what it costs to make widgets)
  - **Fixed cost** F (costs that don't depend on the number of widgets)
  - **Variable cost** $V(q)$ (costs that depend on the number of widgets)
- **Break-Even Points** (q where costs equal revenue $C(q) = R(q)$)
- **Profit** P (financial gain; difference between amount earned R and spent C)
- **Demand** (equation that expresses the relation between quantity and price)
- **Marginal Cost** MC (additional cost to make another widget)
- **Marginal Revenue** MR (additional revenue for making another widget)
- **Marginal Profit** MP (additional profit for making another widget. Equivalently, difference between marginal revenue and marginal cost. Profit is maximized when $MP = 0$.)
- **Elasticity of Demand** ϵ (percent change in quantity demanded divided by percent change in price). $\epsilon < 0$ by law of demand

  If $|\epsilon| > 1$, widget is *price elastic* (wide choice of widgets), revenue decreasing as price increases at that price
  If $|\epsilon| < 1$, widget is *price inelastic* (widget is necessary staple), revenue increasing as price increases at that price
  If $|\epsilon| = 1$, widget is *price unit elastic.*

In math:

$$R = pq$$
$$C(q) = F + V(q)$$
$$P = R - C$$
$$\epsilon = \frac{p}{q}\frac{dq}{dp}$$
$$\frac{dR}{dp} = q(1 + \epsilon)$$

'Marginal' (`means take the derivative of that function with respect to` q.)

$$MC = \frac{dC}{dq}$$
$$MR = \frac{dR}{dq}$$
$$MP = \frac{dP}{dq} = MR - MC = \frac{dR}{dq} - \frac{dC}{dq}$$

Maximum profit when $MP = 0$, or equivalently, $MR = MC$.

2.7. **Theorems.**

- Key: Understand what the theorem means. In plain English.
- Memorize these theorems! Verbatim.
- Which theorem to use? *(What are we trying to show?)*
- Check hypothesis satisfied
- What does the conclusion of the theorem imply for my problem?

2.7.1. *Rolle's Theorem.* (Baby Mean Value) (Theorem 2.13.1)

- Assume:
  - a and b real numbers, $a < b$
  - f continuous on $[a, b]$
  - f differentiable on (a, b)
  - $f(a) = f(b)$.
- Then:
  - There is a c, with $a < c < b$, such that $f'(c) = 0$.

  This just means if f has the same value at its endpoints, it must have a horizontal tangent somewhere in between

2.7.2. *Mean Value Theorem.* (Theorem 2.13.4)

- Assume:
  - a and b real numbers, $a < b$
  - f continuous on $[a, b]$
  - f differentiable on (a, b)
- Then:
  - There is a c, with $a < c < b$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

  This just means there's a tangent to f with the same slope as that of the secant line between the endpoints. I.e., there's some point where the instantaneous rate of change equals the average rate of change.

2.7.3. *Extreme Value Theorem.* (Theorem 3.5.11)

- Assume:
  - a and b real numbers, $a < b$
  - f continuous on $[a, b]$
- Then:
  - f attains a max and min on $[a, b]$.

2.8. **Related Rates.**

- Key: Organize your head
- Draw a picture.
- Label everything. *(Pick good names.)*
- What's constant? What's changing?
- What are we looking for? *(Differentiate what? With respect to what? At what specific instant?)*
- Write equations that relate {things you know} $\leftrightarrow$ {things you want to know}
- Differentiate relations using chain rule. Solve. *(Only plug in numbers for variables at the very end. Constants can use numbers from the beginning.)*

2.9. **Min/Max/Crit.**

- Absolute Min/Max (Defn 3.5.3)
  Let $f(x)$ be defined on a closed interval $[a, b]$ and $a \leq c \leq b$.
  - If $f(c) \geq f(x)$ for every x in $[a, b]$, then $f(c)$ is an absolute maximum.
  - If $f(c) \leq f(x)$ for every x in $[a, b]$, then $f(c)$ is an absolute minimum.
- Local Min/Max (Defn 3.5.3)
  Let $f(x)$ be defined on a closed interval $[a, b]$ and $a \leq e < c < f \leq b$.
  - If $f(c) \geq f(x)$ for every x in the open interval $(e, f) \subset [a, b]$, then $f(c)$ is a local maximum.
  - If $f(c) \leq f(x)$ for every x in the open interval $(e, f) \subset [a, b]$, then $f(c)$ is a local minimum.
- Critical Points (Defn 3.5.6)
  Let $f(x)$ be defined on a closed interval $[a, b]$ and $a \leq c \leq b$. If $f'(c)$ exists and is zero, we call $x = c$ a critical point.
- Check (by plugging in) for possible global extremes at (Theorem 3.5.12):
  - $f'(x) = 0$ (crit points)
  - $f'(x)$ DNE
  - endpoints of domain
- Recognizing local min/max at a critical point $f'(c) = 0$
  - f' changes sign at $x = c$: f' neg $\Rightarrow f$ dec; f' pos $\Rightarrow f$ inc;
  - f'' is $+$ (min) or $-$ (max)

2.10. **Optimization.**

- Know your problem:
  Recognize it's an optimization problem; e.g. '-est' words (find the biggest, cheapest, shortest, minimize, maximize)
- Gather info:
  Draw a picture. Pick good labels. Write down given infos. Know what's constant, what's variable. Figure out domains of variables.
- What are you optimizing??
  Write down the thing you're optimizing as a function, e.g. $f(x, w, \theta, \sin\alpha, p, R, e^t)$, of everything else.
- Reduce to one variable:
  Use constraints (e.g. from the given infos, the geometry, fixed constants, given infos) to express the thing you're optimizing as a function of one variable $f(x)$
- Find your extreme candidates:
  Find the criticals, where $f' = 0$, f' DNE, endpoints of domain. These are candidates for extremes
- Test your candidates:
  Try first, second derivative tests, or plug in to check for extremes
- Reflect on your answer:
  Does it make sense? Correct units (area should be m^2, not m/s)? No negative quantities sold or other physical impossibilities?