MATH 104/184

2018W1 Midterm 2

Review Package

Solutions

By N Morishige

commerce
undergraduate
society

[15] **1. Short Problems**. Each question is worth 3 points. Put your answer in the box provided and show your work. No credit will be given for the answer without the correct accompanying work.

(a) Find the equation of the normal line to the curve

$$y = 3y^2x + 3x^2y + 1$$

at the point $(-1, 1)$.

Answer: $y = \frac{4}{3}x + \frac{7}{3}$

By implicit differentiation,

$y' = 6yy'x + 3y^2 + 6xy + 3x^2y'$

$\Rightarrow y' = y'(6xy + 3x^2) + 3y^2 + 6xy.$

$\Rightarrow y' = \dfrac{3y^2 + 6xy}{1 - 6xy - 3x^2}$

So y' at $(-1,1)$ is $\dfrac{3-6}{1+6-3} = -\dfrac{3}{4}$.

The slope of the normal line at $(-1,1)$ is $\frac{4}{3}$.

Using point-slope form, the normal line has equation $y - 1 = \frac{4}{3}(x+1)$,

or $y = \frac{4}{3}x + \frac{7}{3}$

(b) Find $\frac{dy}{dx}$ when $\tan(xy) = y + 2$.

Answer: $\frac{y\sec^2(xy)}{1 - x\sec^2(xy)}$

By implicit diffuentiation,

$\sec^2(xy) \cdot (y + xy') = y'$

$\Rightarrow y'(1 - x\sec^2(xy)) = y\sec^2 xy$

$\Rightarrow y' = \frac{y\sec^2(xy)}{1 - x\sec^2(xy)} = \frac{y}{\cos^2(xy) - x}$

(c) Find $f'(x)$ if $f(x) = x^{\log x} + (\log x)^x$

Answer:
$\frac{2}{x}\log x\left(x^{\log x}\right) + (\log x)^x\left[\log(\log x) + \frac{1}{\log x}\right]$

We can split this into two pieces:

Let $g(x) = x^{\log x}$ and $h(x) = (\log x)^x$, so that $f(x) = g(x) + h(x)$

and $f'(x) = g'(x) + h'(x)$.

First: $g(x) = x^{\log x}$.

By log. differentiation, we have: $\log g(x) = \log\left(x^{\log x}\right) = (\log x)(\log x)$

So, $\frac{g'(x)}{g(x)} = \left[(\log x)^2\right]' = 2(\log x) \cdot \frac{1}{x}$

and $g'(x) = \left(x^{\log x}\right) \cdot 2(\log x) \cdot \frac{1}{x}$

Next: $h(x) = (\log x)^x$.

Again by log. differentiation, $\log h(x) = \log\left((\log x)^x\right) = x \log(\log x)$.

So $\frac{h'}{h} = \log(\log x) + x \frac{1}{\log x} \cdot \frac{1}{x}$ (product rule + chain rule)

and $h'(x) = (\log x)^x\left[\log(\log x) + \frac{1}{\log x}\right]$

Finally $f'(x) = g'(x) + h'(x) = \frac{2}{x}\log x \cdot x^{\log x} + (\log x)^x\left[\log(\log x) + \frac{1}{\log x}\right]$.

(d) If \$1000 is invested at an annual interest rate of 2% compounded continuously, how long will it be until it earns \$2000 interest? Would I earn more interest if I requested the bank compound every month instead?

Answer: $\log 3/(.02)$; No.

Using the formula, we solve for t, where

$$\underbrace{1000+2000}_{\substack{\text{total amount} \\ \text{original capital + interest}}} = 1000\, e^{.02t}$$

So $\frac{3000}{1000} = e^{.02t}$

$$\log 3 = .02t$$

$$t = \frac{\log 3}{.02} \text{ years (about 55 years)}$$

The more frequently you compound, the more interest you get (because you are then getting interest on the interest,) So compounding continuously earns more interest than compounding monthly.

(e) State whether the function $f(x) = x\sqrt{8-x^2}$ has an absolute maximum on the interval $x \in [0, 2\sqrt{2}]$. If so, find the coordinates of the point. Does $f(x)$ have an absolute minimum on that interval? If so, find the coordinates.

Answer: ab. max at $(2,4)$; ab. min at $(0,0)$ and $(2\sqrt{2}, 0)$

Since $f(x)$ is continuous on $[0, 2\sqrt{2}]$ it attains an absolute maximum and an absolute minimum on that interval by the Extreme Value Theorem.

First find its critical points:

$$f'(x) = \sqrt{8-x^2} + x \cdot \frac{1}{2}\frac{(-2x)}{\sqrt{8-x^2}} = \sqrt{8-x^2} + \frac{-x^2}{\sqrt{8-x^2}}$$

So $f'(x) = 0 \Rightarrow 0 = \sqrt{8-x^2} + \frac{-x^2}{\sqrt{8-x^2}}$

$\Rightarrow 8 - x^2 = x^2$

$\Rightarrow x^2 = 4 \Rightarrow x = \pm 2$. However only $x = 2$ is in the domain.

So we check $x = 2$, $x = 0$ and $x = 2\sqrt{2}$ for extremes values:

$f(0) = f(2\sqrt{2}) = 0$ and $f(2) = 4$

So $(2,4)$ is an absolute maximum and $(0,0)$ and $(2\sqrt{2}, 0)$ are absolute minima.

(f) A rectangle of length 6cm and width 8cm steadily increases in length by 4cm per minute and decreases in width by 2cm per minute for five minnutes. Which of the following is true of the area A of the rectangle during those five minutes?

1. A is always decreasing.
2. A decreases at first, then increases.
3. A is always increasing.
4. A remains constant.
5. A increases at first, then decreases.

Answer:
(5)

Related rates:

$A = \ell \cdot w$

So $\frac{dA}{dt} = \frac{d\ell}{dt} \cdot w + \ell \cdot \frac{dw}{dt}$

$= 4w - 2\ell.$

At the beginning, $w = 8$ and $\ell = 6$ so $\frac{dA}{dt} > 0$ and A is increasing.

However, after two minutes, say, $\ell(2) = 14$ and $w(2) = 4$ so $\frac{dA}{dt} < 0$.

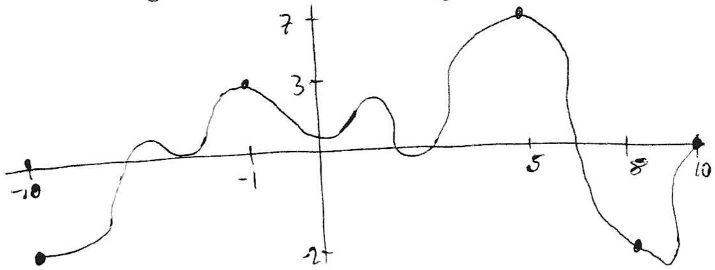
Thus (5) is the correct choice.

Long Problems. In questions 2 - ~~6~~ 5, show your work. No credit will be given for the answer without the correct accompanying work.

[14] **2.** Let $f(x)$ be a continuous and differentiable function on $[-10, 10]$, with

$$f(-10) = -2, f(-1) = 3, f(5) = 7, f(8) = -2 \quad \text{and } f(10) = 0.$$

Which of the following statements are always True? If True, briefly explain your reasoning.



(a) There is a $c \in (-10, 10)$ such that $f(c)$ is a global maximum. $\boxed{\text{TRUE}}$

EVT $\Rightarrow$ There is a global max for some $c \in [-10, 10]$

However, since $f(5) = 7$ and $f(-10) = -2$ and $f(10) = 0$

we know the global max is not at the endpoints

So, there is a global max on $(-10, 10)$.

(b) $f'(c) > 0$ for all $c \in (-1, 5)$.

Not necessarily. The slope can go up and down between those points.

(c) There is a $c \in (-10, 10)$ such that $\cancel{f(c)}$ c is a critical point. $\boxed{\text{True}}$

Rolle with $a = -10$, $b = -8$. $\Rightarrow$

$f(-10) = -2 = f(-8)$. So there is a $c \in (-10, -8)$ with $f'(c) = 0$, ie c is a critical point. Such a c is also in $(-10, 10)$.

(d) There is a $c \in (-4, 9)$ such that $f'(c) = -3$. $\boxed{\text{True}}$

MVT applied to $x = 5$ and $x = 8$

$\Rightarrow$ There is a $c \in (5,8)$ s.t. $f'(c) = \frac{f(8)-f(5)}{8-5} = \frac{-2-7}{3} = -3$.

Such a c is also in $(-4,9)$.

(e) f has a global minimum in $(-10, 10)$.

Not necessarily. EVT says there is a global minimum in $[-10, 10]$.

It could be at the endpoints; we don't know otherwise.

(f) f has a local maximum in $(-1, 5)$.

Not necessarily. f could increase monotonically between -1 and 5.

(g) There is a $c \in (-10, 10)$ such that $f''(c) = 0$. $\boxed{\text{True}}$

True. This is a two-step process.

First By IVT, we know there is an $a \in (-10,-1)$ such that $f(a) = 0$.

Also By IVT, we know there is another $b \in (5,8)$ such that $f(b) = 0$.

Now, applying Rolle to the function $f'(x)$ on $[a,b]$, we know there is a $c \in (a,b)$ with $\underset{\displaystyle (f')'}{\underset{\shortparallel}{f''(c)}} = 0$.

[10] **3.** The demand curve of a product is given by $4p + q + pq = 252$, where p is the price in dollars and q is in hundreds of units. The price elasticity of demand is $\epsilon(p) = \dfrac{p}{q}\dfrac{dq}{dp}$.

(a) Compute $\epsilon(p)$ when $p = \$3$.

- Find q when $p=3$ $\Rightarrow$ $12+q+3q=252 \Rightarrow 4q=240 \Rightarrow q=60$
- Find dq/dp implicit differentiation : $4+q'+pq'+q=0 \Rightarrow q' = -\dfrac{4+q}{1+p}$; So $\mathcal{E} = \dfrac{p}{q}\left(-\dfrac{4+q}{1+p}\right)$
- $\mathcal{E}(3) = \dfrac{p}{q}\dfrac{dq}{dp}(3) = \dfrac{3}{60}\cdot\left(-\dfrac{4+60}{1+3}\right) = -\dfrac{3}{60}\dfrac{64}{4} = -\dfrac{48}{60} = -\dfrac{4}{5} = \boxed{-.8}$

(b) If price is lowered from \$3 by 2%, what is the approximate change in demand?

$\mathcal{E} = \dfrac{\%\Delta q}{\%\Delta p}$ so $\mathcal{E}(3) = -.8 = \dfrac{\%\Delta q}{-2\%} \Rightarrow \%\Delta q = 1.6\%$ demand increases $\approx \boxed{1.6\%}$

(c) Does revenue increase or decrease if price is increased from \$3 by 2%?

When $p=3$, $\mathcal{E}(p) = -.8$ so $|\mathcal{E}|<1 \Rightarrow$ price $\uparrow$ means Revenue $\uparrow$

Alternatively, $\dfrac{dR}{dp} = q\,(1+\mathcal{E}) > 0$ when $p=3$ so price $\uparrow$ means $\boxed{\text{revenue } \uparrow}$

always positive

positive when $|\mathcal{E}|<1$

(d) At what price is revenue maximized?

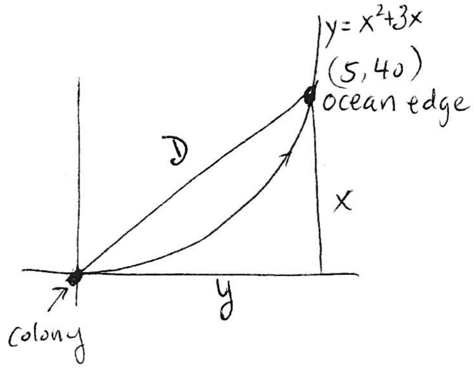
Price is maximized if $\mathcal{E} = -1$ (ie $\frac{dR}{dp} = 0$)

$\mathcal{E} = -\left(\dfrac{4+q}{1+p}\right)\dfrac{p}{q} = -1 \Rightarrow 4p+qp = q+pq \Rightarrow \underline{4p=q}$. Put this back into the

demand equation $\rightarrow$ $4p+4p+p(4p) = 252 \rightarrow 4p^2+8p-252=0 \rightarrow p^2+2p-63 = (p-7)(p+9)=0$

Since $p \geq 0$, p must be $\boxed{7}$ for revenue to be maximized. (q would be 28).

[10] **4.** An emperor penguin walks from his colony, located at the origin $(0,0)$, to the ocean edge at $(5,40)$ along a curved path described by the curve $y = x^2 + 3x$. The distance between the penguin and the colony increases at a constant rate of 1m/sec. What is the penguin's speed in the y direction at the moment he reaches the ocean edge?



Related rates: we are given $\frac{dD}{dt}$, want $\frac{dy}{dt}$.

We have two relations.

Let D be the distance of the penguin from the colony. Suppose the penguin position is (x, y)

★ By Pythagorus, $x^2 + y^2 = D^2 \Rightarrow 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2D\frac{dD}{dt}$

$\Rightarrow x\frac{dx}{dt} + y\frac{dy}{dt} = D\frac{dD}{dt}$.

We are given that $\frac{dD}{dt} = 1$, so $x\frac{dx}{dt} + y\frac{dy}{dt} = D \cdot 1$

At ocean edge, $(x, y) = (5, 40)$ and $D = \sqrt{5^2 + 40^2}$

So at ocean edge $\Rightarrow$ $5\frac{dx}{dt} + 40\frac{dy}{dt} = \sqrt{5^2+40^2}$. (†)

★ We also know the penguin path is $y = x^2 + 3x \Rightarrow \frac{dy}{dt} = 2x\frac{dx}{dt} + 3\frac{dx}{dt}$

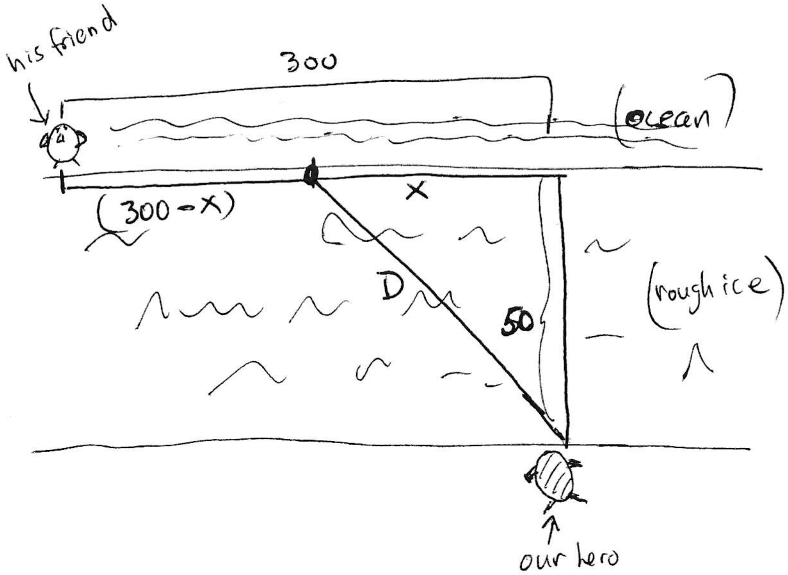
So at ocean edge, $\frac{dy}{dt} = (2 \cdot 5 + 3)\frac{dx}{dt} = 13\frac{dx}{dt}$ (††)

Combining (†) with (††),

$5\frac{1}{13}\frac{dy}{dt} + 40\frac{dy}{dt} = \sqrt{5^2+40^2} \Rightarrow \boxed{\frac{dy}{dt} = \frac{\sqrt{5^2+40^2}}{\left(40 + \frac{5}{13}\right)}}$

[10] **5.** The penguin is stopped before a 50m wide strip of rough ice that runs east-west. Just beyond the strip lies the open ocean. He sees his mate has crossed the rough ice strip and is standing 300m west of his position at the edge of the ocean. The rough ice is tough going for penguin feet, and he can only travel at .5m/sec on the rough ice. Once he reaches the water, he can swim in the open ocean along the shore at 3m/sec. He wants to get to his mate as quickly as possible. At what point on the opposite edge of the rough ice strip should he aim in order to minimize the total time to reach his mate?

Optimization.



Since speed = $\frac{\text{distance}}{\text{time}}$,

time = $\frac{\text{distance}}{\text{Speed}}$

Suppose the penguin aims for a point x meters along the far edge of the rough ice strip measured west from a directly opposite crossing.

Then the rough ice crossing will be distance $D = \sqrt{x^2+50^2}$. His rough ice speed is .5 so the time to cross the rough ice is $\frac{\sqrt{x^2+50^2}}{.5} = 2\sqrt{x^2+50^2}$.

Once he hits the ocean, he has to swim $(300-x)$m to his friend, at 3m/sec, so the swim time is $\frac{300-x}{3}$

The total time taken is the sum $\text{Time} = 2\sqrt{x^2+50^2} + \frac{300-x}{3}$.

The critical point is when $T'=0 \Rightarrow 0 = 2\frac{x}{\sqrt{x^2+50^2}} - \frac{1}{3} \Rightarrow \sqrt{x^2+50^2} = 6x$

$\Rightarrow x^2+50^2 = 36x^2 \Rightarrow 35x^2 = 50^2 \Rightarrow \boxed{x = \frac{50}{\sqrt{35}}\text{ m.}}$

If you take the 2nd derivative $T'' \Rightarrow$

$T'' = \frac{50^2}{(\sqrt{x^2+50^2})^3} > 0$ so $x = \frac{50}{\sqrt{35}}$ is a minimum ~~time~~ for time.