MATH 104/184

2017W1 Midterm 1

Review Package

Solutions

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[36] **1. Short Problems**. Each question is worth 3 points. Put your answer in the box provided and show your work. No credit will be given for the answer without the correct accompanying work.

(a) Find the equation to the tangent line to $f(x) = x\cos x$ at $x = \pi/2$.

The slope of the tangent line is given by the derivative:

$f'(x) = \cos x - x\sin x$ (product rule).

$f'(\pi/2) = \cos \pi/2 - \pi/2 \sin \pi/2 = \boxed{-\pi/2}$ is the slope at $x = \pi/2$.

When $x = \pi/2$, $f(\pi/2) = \pi/2 \cos \pi/2 = 0$ so the $\boxed{\text{point is } (\pi/2, 0).}$

The equation of the tangent line is then:

$(y-0) = -\frac{\pi}{2}(x - \pi/2)$

Answer:

$y = -\frac{\pi}{2}\left(x - \frac{\pi}{2}\right)$

(b) Evaluate $\lim_{x\to 1}\left(\frac{1}{x-1}-\frac{1}{x^2-x}\right)$.

The expression approaches $\frac{1}{0}-\frac{1}{0}$ which does not make sense

So we'll try putting it a common denominator

$$\lim_{x\to 1}\left(\frac{1}{x-1}-\frac{1}{x^2-x}\right)=\lim_{x\to 1}\frac{x-1}{x^2-x}=\lim_{x\to 1}\frac{x-1}{x(x-1)}=\lim_{x\to 1}\frac{1}{x}=1.$$

Answer:
1

(c) Evaluate $\lim_{x\to -2} \dfrac{2x^2+x-6}{2x+4}$.

The expression is like $\frac{0}{0}$ which does not make sense, so we'll try factoring:

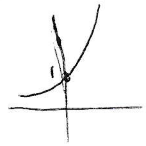
$$\lim_{x\to -2} \frac{2x^2+x-6}{2x+4} = \lim_{x\to -2} \frac{(x+2)(2x-3)}{2(x+2)} = \lim_{x\to -2} \frac{2x-3}{2} = -\frac{7}{2}$$

Answer: $-7/2$

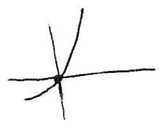
(d) Evaluate $\lim_{h \to 0^-} \frac{1}{e^{2h} - 1}$.

The expression is like $\frac{1}{0}$ or $\pm\infty$. But which sign is it?

We know e^{2h} has graph:



So $e^{2h}-1$ has graph:



So $e^{2h}-1$ is <u>negative</u> as we approach 0 from below.

So $\lim_{h \to 0^-} \frac{1}{e^{2h}-1} = -\infty$

Answer:

$-\infty$

(e) Let $f(x) = \dfrac{2x+6}{x+1}$.

Find the equations of all tangent lines to f which are parallel to the line $x + y = 2$.

The line $x+y=2$ or $y=-x+2$ has slope -1.

To find the slope of a tangent line to f, we take its derivative

$$f'(x) = \frac{2(x+1) - 1(2x+6)}{(x+1)^2} \quad \text{(quotient rule)}$$

$$= \frac{-4}{(x+1)^2}$$

We want to find tangent lines with slope -1:

$$-1 = \frac{-4}{(x+1)^2}$$

$$(x+1)^2 = 4$$

$$x+1 = \pm 2$$

So: $x = 1$ or -3.

When $x=1$, $f(x) = \frac{2+6}{1+1} = 4$ so the tangent line at $(1,4)$ has equation:

$y-4 = -(x-1)$ or $y = -x+5$

When $x=-3$, $f(x) = 0$ so the tangent line at $(-3,0)$ has equation

$y = -(x+3)$

Answer: $y=-x+5$ and $y=-x-3$

(f) Let $f(x)$ be the same function as above, $f(x) = \dfrac{2x+6}{x+1}$.

Find the equations of all tangent lines to f which are perpendicular to the line $x+y=2$.

The slope to the line $x+y=2$ is -1 so the perpendicular ~~has~~ has slope $-\left(\frac{1}{-1}\right) = +1$.

We found in e) that $f'(x) = \dfrac{-4}{(x+1)^2}$

When does the tangent line have slope 1?

$$1 = \frac{-4}{(x+1)^2}$$

$$(x+1)^2 = -4$$

But this can never happen. So there are no such lines.

Answer:
No such tangent lines

(g) If $f(x)$ is a function satisfying $f(0) = 1$ and $f'(0) = 4$, find the equation of the tangent line to the graph of $g(x) = f(x)e^x$ at $x = 0$.

$g'(x) = f'(x)e^x + f(x)e^x$ (product rule)

at $x=0$

$g'(0) = f'(0)e^0 + f(0)e^0$

$= 4 \cdot 1 + 1 \cdot 1 = 5$ is the slope

Also, when $x=0$,

$g(0) = f(0)e^0 = 1 \cdot 1 = 1$.

So the equation of the tangent line is

$y - 1 = 5(x - 0)$

Answer: $y = 5x + 1$

(h) Let $f(x)$ be defined by

$$f(x) = \begin{cases} \dfrac{|x|^2}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Is $f(x)$ continuous at $x = 0$? Is it differentiable at $x = 0$?

When $x \geq 0$, $|x|^2 = x^2$. Also when $x < 0$, $|x|^2 = x^2$.

So $\frac{|x|^2}{x} = \frac{x^2}{x} = x$ for all x.

So we can write $f(x) = \begin{cases} \frac{|x|^2}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ as $f(x) = \begin{cases} x & x \neq 0 \\ 0 & x = 0 \end{cases}$

which is the same as $f(x) = x$ for all real x.

This is continuous and differentiable everywhere, including at $x = 0$.

Answer: Yes, Yes.

(i) Find numbers a and b that makes

$$f(x) = \begin{cases} e^x + a & \text{if } x > 2 \\ bx^2 + 1 & \text{if } 1 \leq x \leq 2 \\ 3x^3 - b & \text{if } x < 1 \end{cases}$$

$f(x)$ continuous for all real numbers. With these values, is $f(x)$ differentiable at $x = 1$?

On each piece, $f(x)$ is continuous since its a sum of polynomials and exponentials

So we only need to check at $x=1$ and $x=2$.

At $x=1$, $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (3x^3-b) = 3-b$ and $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (bx^2+1) = b+1$

We must have $3-b = b+1$ or $\boxed{b=1}$ for f to be continuous at $x=1$.

At $x=2$, $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} 1x^2+1 = 5$ and $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} e^x+a = e^2+a$

So we must have $5 = e^2+a$ or $\boxed{a = 5-e^2}$ for f to be continuous at $x=2$.

For the derivative to exist, we need $\lim_{x\to 1} \frac{f(x)-f(1)}{x-1}$ to exist.

The left limit is $\lim_{x\to 1^-} \frac{(3x^3-1)-(3\cdot 1^3-1)}{x-1} = \lim_{x\to 1^-} \frac{3x^3-3}{x-1} = \lim_{x\to 1^-} 3(x^2+x+1) = 9$

The right limit is $\lim_{x\to 1^+} \frac{(x^2+1)-(1^2+1)}{x-1} = \lim_{x\to 1^+} \frac{x^2-1}{x-1} = \lim_{x\to 1^+} (x+1) = 2$.

Since $9 \neq 2$, the limit does not exist and $f(x)$ is not differentiable at $x=1$.

Answer:

$a = 5-e^2$, $b = 1$; No.

(j) Evaluate $\lim_{x\to 3} \dfrac{x-3}{\sqrt{5x+1}-4}$.

This approaches $\frac{0}{0}$ so we'll try rationalizing the denominator

$$\lim_{x\to 3} \frac{x-3}{\sqrt{5x+1}-4} = \lim_{x\to 3} \frac{(x-3)(\sqrt{5x+1}+4)}{(\sqrt{5x+1}-4)(\sqrt{5x+1}+4)} = \lim_{x\to 3} \frac{(x-3)(\sqrt{5x+1}+4)}{(5x+1-16)}$$

$$= \lim_{x\to 3} \frac{(x-3)(\sqrt{5x+1}+4)}{5(x-3)} = \lim_{x\to 3} \frac{\sqrt{5x+1}+4}{5} = \frac{8}{5}$$

Answer: 8/5

(k) Evaluate $\lim_{x\to 0} \frac{\sin 2x}{\sqrt{x^2+2}}$.

this approaches $\frac{0}{2} = 0$.

We can just "plug in" since the function is continuous at $x = 0$.

Answer:

0

(1) Evaluate $\lim\limits_{x\to 2} \dfrac{x^4 - 16}{x^2 - 4}$.

This approaches $\frac{0}{0}$ so we'll try factorizing

$$\lim_{x\to 2} \frac{x^4-16}{x^2-4} = \lim_{x\to 2} \frac{(x^2-4)(x^2+4)}{x^2-4} = \lim_{x\to 2} (x^2+4) = 8.$$

Answer: 8.

Long Problems. In questions 2 - 6, show your work. No credit will be given for the answer without the correct accompanying work.

[10] **2.**

(a) Carefully state what it means for a function $f(x)$ to be continuous at $x = a$.[3pts]

$f(x)$ is continuous at $x=a$ if $f(a)$ exists, $\lim_{x \to a^-} f(x)$ exists, $\lim_{x \to a^+} f(x)$ exists

and $f(a) = \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$.

(b) Carefully state the definition of the derivative $f'(a)$ of a function $f(x)$ at $x = a$.[3pts]

If $f(x)$ is defined on an open interval containing a, Then

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$$\left(\text{OR} \quad \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \right)$$

(c) Let $f(x)$ be defined as

$$f(x) = \begin{cases} -3x^2 + |x| & \text{if } x < 0 \\ x^3 - x & \text{if } x \geq 0 \end{cases}$$

Is $f(x)$ continuous at $x = 0$? If so, what is $f(0)$? Is it differentiable at $x = 0$? If so, what is $f'(0)$?

Justify your answers. [4pts]

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (-3\cdot 0^2 + |0|) = 0$ and $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (x^3 - x) = 0$

So $\boxed{f(x) \text{ is continuous at } x=0 \text{ and } f(0) = 0}$

For the derivative $f'(0)$, the left limit is

$$\lim_{x\to 0^-} \frac{-3x^2 + |x| - (0+0)}{x - 0} = \lim_{x\to 0^-} \frac{-3x^2 - x}{x} = \quad \text{Since } -|x| = x \text{ if } x < 0$$

$$= \lim_{x\to 0^-} (-3x - 1) = -1$$

and the right limit is

$$\lim_{x\to 0^+} \frac{x^3 - x - 0}{x - 0} = \lim_{x\to 0^+} (x^2 - 1) = -1$$

So $\lim_{x\to 0^+} \frac{f(x) - f(a)}{x - a} = \lim_{x\to 0^-} \frac{f(x) - f(a)}{x - a}$ and $\lim_{x\to 0} \frac{f(x) - f(a)}{x - a}$ exists and $= -1$.

Since f is also defined on an open interval containing 0,

$\boxed{f \text{ is differentiable at } x=0}$. and $\boxed{f'(0) = -1}$

[10] **3**. Use the definition of the derivative as a limit to find $f'(x)$ for the following functions. No marks will be given for the use of differentiation rules.

(a) $f(x) = \dfrac{1}{\sqrt{x+1}}$. [5pts]

$$f'(x) = \lim_{h \to 0} \frac{\frac{1}{\sqrt{x+h+1}} - \frac{1}{\sqrt{x+1}}}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{x+1} - \sqrt{x+h+1}}{h \cdot \sqrt{x+h+1}\sqrt{x+1}}$$

$$= \lim_{h \to 0} \frac{(x+1)-(x+h+1)}{h\sqrt{x+h+1}\sqrt{x+1}\left(\sqrt{x+1}+\sqrt{x+h+1}\right)}$$

$$= \lim_{h \to 0} \frac{-h}{h\sqrt{x+h+1}\sqrt{x+1}\left(\sqrt{x+1}+\sqrt{x+h+1}\right)}$$

$$= \frac{-1}{2(x+1)\left(\sqrt{x+1}\right)}$$

$$= \boxed{\frac{-1}{2}(x+1)^{-3/2}}$$

Use the definition of the derivative as a limit to find $f'(x)$ for the following functions. No marks will be given for the use of differentiation rules.

(b) $f(x) = \dfrac{x}{3x+2}$. [5pts]

$$f'(x) = \lim_{h\to 0} \frac{\frac{x+h}{3(x+h)+2} - \frac{x}{3x+2}}{h}$$

$$= \lim_{h\to 0} \frac{(3x+2)(x+h) - x(3(x+h)+2)}{h\,(3(x+h)+2)(3x+2)}$$

$$= \lim_{h\to 0} \frac{(3x+2)h - 3xh}{h\,(3(x+h)+2)(3x+2)}$$

$$= \lim_{h\to 0} \frac{2h}{h(3(x+h)+2)(3x+2)}$$

$$= \lim_{h\to 0} \frac{2}{(3(x+h)+2)(3x+2)}$$

$$= \boxed{\frac{2}{(3x+2)^2}}$$

[8] **4.** True or False. Justify your answer or give a counterexample.

(a) If $f(x)$ and $g(x)$ are differentiable at $x = a$, then $\dfrac{f(x)}{g(x)}$ is also differentiable at $x = a$. [2pts]

False take any $g(x)$ such that $g(a) = 0$

ie. $f(x) = 1$, $g(x) = x - a$. then $\dfrac{f}{g}$ is not even continuous at $x = a$. much less differentiable

[8] **4.** True or False. Justify your answer or give a counterexample.

(b) If $\lim_{x\to a^+} f(x)$ and $\lim_{x\to a^-} f(x)$ exist, then $\lim_{x\to a}(f(x)+x)$ exists. [2pts]

False $\lim_{x\to a} f(x)$ only exists if $\lim_{x\to a^+} f(x)$ and $\lim_{x\to a^-} f(x)$ exist and are equal. E.g.

$$f(x) = \begin{cases} e^x & x \geq a \\ -e^x & x \leq a \end{cases}$$

Since $e^x + x \neq -e^x + x$ for any x

(Since $e^x > 0$ and $-e^x < 0$)

the $\lim_{x\to a}(f(x)+x)$ does not exist.

[8] **4.** True or False. Justify your answer or give a counterexample.

(c) Let $f(x)$ be continuous on $[0,1]$ and $f(0) = 0$, $f(1) = -1$. Then there is a $c \in [0,1]$ such that $f(x) = -1/2$. [2pts]

Yes, this is

True by I.V.T.

[8] **4.** True or False. Justify your answer or give a counterexample.

(d) Let $f(x)$ be continuous on $[0,1]$ and $f(0) = 0$, $f(1) = -1$. Then for all $c \in [0,1]$, $\lim_{x \to c} \frac{f(x) - f(c)}{x - c} > 1$. [2pts]

This is <u>false</u> for many reasons.

let $f(x) = -x$

then $f'(x) = -1$ for all $c \in [0,1]$.

~~thus~~

[8] **5.**

(a) Let $f(x)$ be defined by

$$f(x) = \begin{cases} \cos x & \text{if } x < c \\ 2x + 7 & \text{if } x \geq c \end{cases}$$

for some number c. Show that there is a number c that makes $f(x)$ continuous on all real numbers. [4pts]

Suppose we define $g(x) = 2x+7 - \cos x$.

This is continuous on all real numbers.

Since $-1 \leq \cos x \leq 1$, for very large x, say $x = 100$, $g(x) > 0$. Similarly for very small x, say $x = -100$ $g(x) < 0$. Then, by IVT, there is a $c \in [-100, 100]$ such that $g(c) = 0$.

That c makes $f(x)$ continuous for all real numbers.

(b) Show that $f(x) = 2^x - 7x$ has a root in between 5 and 6. [4pts]

$f(x)$ is continuous on $[5,6]$.

$f(5) = 32 - 35 = -3 < 0$

$f(6) = 64 - 42 = 22 > 0$

So by IVT there is a $c \in [5,6]$ such that

$f(c) = 0$.

That is, f has a root in between 5 and 6.

[8] **6.**

(a) Show that there are at least two solutions to $x^4 - 2x^3 + 3x^2 - 1 = 0$ in $[-1, 1]$. [4pts]

We want to apply IVT twice.

The polynomial $x^4 - 2x^3 + 3x^2 - 1 = f(x)$ is continuous on all real numbers.

$f(-1) = 5 > 0.$

$f(0) = -1 < 0.$

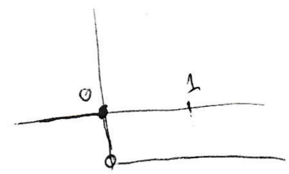
So by IVT, $f(x)$ has a root in $[-1, 0]$.

Also $f(1) = 3 > 0$ and again by IVT, $f(x)$ has a root in $[0, 1]$.

(b) Give an example of a function defined on all real numbers such that $f(0) = 0$, $f(1) = -1$ and $f(x) \neq 1/2$ for any $x \in [0, 1]$. [4pts]

This would be similar to IVT which we know is true. So in order to find such a function, it must not satisfy the hypothesis of continuity.

For example $f(x) = \begin{cases} 0 & x \leq 0 \\ -1 & x > 0 \end{cases}$ will do



[12] **7.** The CMP sells 200 review packs a term at \$60 each. They pay \$200 per term to rent their office and each pack costs \$10 to print. They discover that rival Perp001 are charging \$59.99 for review packs. So, they estimate that for every \$20 cheaper they can sell their packs, their sales will increase by 100 per term.

(a) Find the linear demand equation using p for price and q for for quantity

The data point is $(q_0, p_0) = (200, 60)$.

The slope is $\frac{-20}{100} = -\frac{1}{5}$.

So the linear demand equation is

$$(p - 60) = -\frac{1}{5}(q - 200)$$

$$\boxed{p = -\frac{1}{5}q + 100}$$

(b) What should they charge to maximize profit?

$$R = pq = -\frac{1}{5}q^2 + 100q$$

$$C = 200 + 10q$$

$$\text{Profit} = R - C = -\frac{1}{5}q^2 + 100q - 200 - 10q = -\frac{1}{5}q^2 + 90q - 200$$

$$= -\frac{1}{5}(q^2 - 450q) - 200$$

$$= -\frac{1}{5}(q - 225)^2 - 200 + \frac{(225)^2}{5}$$

Profit is maximized when $q = 225$

So $p = -\frac{1}{5}225 + 100 = \boxed{\$55}$

What would the break even point be?

$BE(q) \Rightarrow$ where $R-C=0$

$$0 = -\frac{1}{5}(q-225)^2 - 200 + \frac{(225)^2}{5}$$

$$\Rightarrow (q-225)^2 = (225)^2 - 1000$$

$$q = \pm 5\sqrt{1985} + 225.$$

[12] **8**. The following year, the CMP notice their sales are still 200 review packs per term, even though they are only charging \$45 per pack. They find that sneaky rival Perp001 have slashed their pack price to \$40. In order to stay competitive, they cut a deal with a local printer to reduce printing costs to \$5 per pack and move into a broom closet office that costs \$100 per term.
Now they estimate that for every \$10 more they charge for their packs, their sales will decrease by 20 every term.

(a) Find the linear demand equation using p for price and q for for quantity

The data point is $(200, 45)$

The slope is $\frac{+10}{-20} = -\frac{1}{2}$

So the linear demand equation is

$$(p-45) = -\frac{1}{2}(q-200)$$

or $\boxed{p = -\frac{1}{2}q + 145}$

(b) What should they charge to maximize profit?

$$\text{Cost} = 100 + 5q$$

$$R = pq = -\frac{1}{2}q^2 + 145q$$

$$\text{Profit} = R - C = -\frac{1}{2}q^2 + 145q - 100 - 5q = -\frac{1}{2}q^2 + 140q - 100$$

$$= -\frac{1}{2}(q^2 + 280q) - 100 = -\frac{1}{2}(q-140)^2 - 100 + \frac{(140)^2}{2}$$

So profit is maximized when $q = 140$

$\Rightarrow$ $p = -\frac{1}{2}\cdot 140 + 145 = \boxed{\$75}$

What would the break even point be?

$$0 = -\frac{1}{2}(q-140)^2 - 100 + \frac{(140)^2}{2}$$

$$(q-140)^2 = (140)^2 - 200$$

$$q = \pm 10\sqrt{194} + 140.$$