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I. VOIGT NOTATION

For the stress tensor, it is simple:

$$\begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} = \begin{pmatrix} \sigma_{1} & \sigma_{6} & \sigma_{5} \\ \sigma_{6} & \sigma_{2} & \sigma_{4} \\ \sigma_{5} & \sigma_{4} & \sigma_{3} \end{pmatrix} \tag{1}$$

For the strain tensor, we have to consider coefficients.

$$\begin{pmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{21} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{31} & \epsilon_{32} & \epsilon_{33} \end{pmatrix} = \begin{pmatrix} \epsilon_{1} & \epsilon_{6}/2 & \epsilon_{5}/2 \\ \epsilon_{6}/2 & \epsilon_{2} & \epsilon_{4}/2 \\ \epsilon_{5}/2 & \epsilon_{4}/2 & \epsilon_{3} \end{pmatrix}, \tag{2}$$

i.e.,

$$\epsilon_1 = \epsilon_{11} \tag{3}$$

$$\epsilon_2 = \epsilon_{22} \tag{4}$$

$$\epsilon_3 = \epsilon_{33} \tag{5}$$

$$\epsilon_4 = 2\epsilon_{23} = \gamma_{yz} \tag{6}$$

$$\epsilon_5 = 2\epsilon_{13} = \gamma_{xz} \tag{7}$$

$$\epsilon_6 = 2\epsilon_{12} = \gamma_{xy} \tag{8}$$

That is, the shear components are "engineering" shear strains. This definition is convenient due to various reasons. For example, *with this notation*, the following is satisfied;

$$\sigma_i = \sum_{j=1}^{6} B_{ij}\epsilon_j, \tag{9}$$

where B_{ij} are the stress-strain coefficients (Wallace [1] (Sec. 1.2)).

Note: If the reference state is under zero stress $B_{ij} = C_{ij}$.

References:

- Nye [2] (Sec. VIII.2, p. 99)

- Grimvall [3] (Sec. 3.2, p. 28)
- https://en.wikipedia.org/wiki/Voigt_notation

II. CUBIC

In the standard orientation, the stress-strain relation can be given by

$$\begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{pmatrix} = \begin{pmatrix} B_{11} & B_{12} & B_{12} & 0 & 0 & 0 \\ B_{12} & B_{11} & B_{12} & 0 & 0 & 0 \\ B_{12} & B_{12} & B_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & B_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & B_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & B_{44} \end{pmatrix} \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \end{pmatrix} \tag{10}$$

This can be reduced as

$$\begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{pmatrix} = \begin{pmatrix} B_{11}\epsilon_1 + B_{12}\epsilon_2 + B_{12}\epsilon_3 \\ B_{12}\epsilon_1 + B_{11}\epsilon_2 + B_{12}\epsilon_3 \\ B_{12}\epsilon_1 + B_{12}\epsilon_2 + B_{12}\epsilon_3 \\ B_{44}\epsilon_4 \\ B_{44}\epsilon_5 \\ B_{44}\epsilon_6 \end{pmatrix} = \begin{pmatrix} \epsilon_1 & \epsilon_2 + \epsilon_3 & 0 \\ \epsilon_2 & \epsilon_1 + \epsilon_3 & 0 \\ \epsilon_3 & \epsilon_1 + \epsilon_2 & 0 \\ 0 & 0 & \epsilon_4 \\ 0 & 0 & \epsilon_5 \\ 0 & 0 & \epsilon_6 \end{pmatrix} \begin{pmatrix} B_{11} \\ B_{12} \\ B_{44} \end{pmatrix} \tag{11}$$

If we write this using tensor strains,

$$\begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{pmatrix} = \begin{pmatrix} \epsilon_{11} & \epsilon_{22} + \epsilon_{33} & 0 \\ \epsilon_{22} & \epsilon_{11} + \epsilon_{33} & 0 \\ \epsilon_{33} & \epsilon_{11} + \epsilon_{22} & 0 \\ 0 & 0 & 2\epsilon_{23} \\ 0 & 0 & 2\epsilon_{13} \\ 0 & 0 & 2\epsilon_{12} \end{pmatrix} \begin{pmatrix} B_{11} \\ B_{12} \\ B_{44} \end{pmatrix} \tag{12}$$

III. HEXAGONAL

In the standard orientation, the stress-strain relation can be given by

$$\begin{pmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\\ \sigma_4\\ \sigma_5\\ \sigma_6\end{pmatrix} = \begin{pmatrix} B_{11} & B_{12} & B_{13} & 0 & 0 & 0\\ B_{12} & B_{11} & B_{13} & 0 & 0 & 0\\ B_{13} & B_{13} & B_{33} & 0 & 0 & 0\\ 0 & 0 & 0 & B_{44} & 0 & 0\\ 0 & 0 & 0 & 0 & B_{44} & 0\\ 0 & 0 & 0 & 0 & 0 & B_{66}\end{pmatrix}\begin{pmatrix}\epsilon_1\\ \epsilon_2\\ \epsilon_3\\ \epsilon_4\\ \epsilon_5\\ \epsilon_6\end{pmatrix} \tag{13}$$

where

$$B_{66} = \frac{B_{11} - B_{12}}{2}. \tag{14}$$

This can be reduced to

$$\begin{pmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\\ \sigma_4\\ \sigma_5\\ \sigma_6\end{pmatrix} = \begin{pmatrix} B_{11}\epsilon_1 + B_{12}\epsilon_2 + B_{13}\epsilon_3\\ B_{12}\epsilon_1 + B_{11}\epsilon_2 + B_{13}\epsilon_3\\ B_{13}\epsilon_1 + B_{13}\epsilon_2 + B_{33}\epsilon_3\\ B_{44}\epsilon_4\\ B_{44}\epsilon_5\\ [(B_{11} - B_{12})/2]\epsilon_6\end{pmatrix} = \begin{pmatrix} \epsilon_1 & 0 & \epsilon_2 & \epsilon_3 & 0\\ \epsilon_2 & 0 & \epsilon_1 & \epsilon_3 & 0\\ 0 & \epsilon_3 & 0 & \epsilon_1 + \epsilon_2 & 0\\ 0 & 0 & 0 & 0 & \epsilon_4\\ 0 & 0 & 0 & 0 & \epsilon_5\\ \epsilon_6/2 & 0 & -\epsilon_6/2 & 0 & 0\end{pmatrix}\begin{pmatrix}B_{11}\\ B_{33}\\ B_{12}\\ B_{13}\\ B_{44}\end{pmatrix} \tag{15}$$

If we write this using tensor strains,

$$\begin{pmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\\ \sigma_4\\ \sigma_5\\ \sigma_6\end{pmatrix} = \begin{pmatrix} \epsilon_{11} & 0 & \epsilon_{22} & \epsilon_{33} & 0\\ \epsilon_{22} & 0 & \epsilon_{11} & \epsilon_{33} & 0\\ 0 & \epsilon_{33} & 0 & \epsilon_{11} + \epsilon_{22} & 0\\ 0 & 0 & 0 & 0 & 2\epsilon_{23}\\ 0 & 0 & 0 & 0 & 2\epsilon_{13}\\ \epsilon_{12} & 0 & -\epsilon_{12} & 0 & 0\end{pmatrix}\begin{pmatrix}B_{11}\\ B_{33}\\ B_{12}\\ B_{13}\\ B_{44}\end{pmatrix} \tag{16}$$

[1] D. C. Wallace, *Thermodynamics of Crystals* (Dover Publications, New York, 1998) p. 484.

[2] J. F. Nye, *Physical properties of crystals: their representation by tensors and matrices* (Oxford university press, 1985).

[3] G. Grimvall, *Thermophysical Properties of Materials* (Elsevier, 1999).