Quantum Finite Element Method with Model Order Reduction*

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Abstract. The finite element method is used to approximately solve boundary value problems for differential equations. The method discretises the parameter space and finds an approximate solution by solving a large system of linear equations.

Keywords: Quantum computing · Finite Element Method · Model Order Reduction

1 Related Works

A quantum algorithm for the finite element method has been introduced by Montanaro and Pallister [1]. Some important notes from their paper:

1. QLE algorithm is indeed applicable to the general FEM, and can achieve substantial speedups over the classical algorithm. However, the quantum speedup obtained is only at most polynomial, if the spatial dimension is fixed and the solution satisfies certain smoothness properties
2. There are two potential sources of error in producing the solution:
   (a) The discretisation process which converts the problem to a system of linear equations. exponentially.
   (b) Any inaccuracies in solving the system of equations itself and computing the desired function of the solution. (The larger the system of equations produced, the smaller the first type of error is.)

   The QLE algorithm can work with an exponentially larger set of equations in a comparable time to the classical algorithm, so this source of error can be reduced. However, the scaling with accuracy of the QLE algorithm's extraction of a solution from the system of linear equations is substantially worse than the classical algorithm. These two effects can come close to cancelling each other out.
3. The inability of the quantum algorithm to deliver exponential speedups (in some cases) is not a limitation of the algorithm itself, but rather that any quantum algorithm for the FEM will face similar constraints.

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(a) Any algorithm which needs to distinguish between two states which are distance ϵ apart must have runtime $\Omega(1/\sqrt{\epsilon})$.
(b) The FEM solving subroutine of any quantum algorithm can likely be replaced with an equivalent classical subroutine with at most a polynomial slowdown (in fixed spatial dimension and when the solution is smooth).
(c) There can be no more than a quadratic speedup if the input to the problem is arbitrary and accessed via queries to a black box or "oracle".

The work [2] and [3] shows that using Euler's method the quantum algorithm can in principle achieve an expotential improvement for approximately computing properties of the solution to the system, if the system of equations is provided implicitly. This approach requires also specifying how the equations are produced and how the property of interest is computed. If the equations are generated by a discretization procedure such as FDM, similar qualitative conclusions to those [1] derived for the FEM seem likely to hold

2 Motivation

Using a Model Order Reduction method would decrease the overall accuracy ϵ but could potentially speed up the computational time opposed to the one found in [1] with no preconditioning

$$\widetilde{\mathcal{O}}\left(\|u\|\,\|u\|_2^2/\epsilon^3 + \|u\|_1\,|u|_2/\epsilon^2\right)$$

and with optimal preconditioning

$$\widetilde{\mathcal{O}}\left(\|u\|_1/\epsilon\right)$$

The projection matrices for the Reduced Order Model (ROM) generation has to be recreated for each time which also increases the runtime but in a superposition using a quantum algorithm it could be possible to create them in parallel improving the overall computational time further. The combination of FDM and MOR could also potentially achieve an exponential speedup as it was shown in the work of [2] and [3].

3 Model Order Reduction Methods[4]

Possible methods for model order reductions:

3.1 Projection based MOR

Projection based MOR methods approximate the high dimensional state space vector $x(t) \in \mathbb{R}^n$ with the help of a vector $z(t) \in \mathbb{R}^r$ of reduced dimension $r \ll n$.

Krylov method `https://www.mw.tum.de/en/rt/research/model-order-reduction/krylov-subspace-methods/`

POD method

3.2 Truncation based MOR

Balanced Truncation

Model Truncation

3.3 Specialized eigensolution methods

The Dominant Pole Algorithm (DPA)

3.4 Other References

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   https://kiwi.oden.utexas.edu/papers/Fourier-model-reduction-Willcox-Megretski.pdf
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