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A new frequency weighted Fourier-based method for model order reduction ⋆

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Abstract

This paper presents a new, analytically driven frequency weighted model order reduction method. The method is grounded on the Fourier-based decomposition of a high-order state space model. The method is designed for discrete-time systems, but it can be easily applied to continuous-time ones. The main advantage of the proposed algorithm is a class of quadratic time complexity as compared to the cubic one for the classical frequency weighted method, the major feature resulting from the application of analytical methods for calculation of factorizations for controllability and observability Gramians. The simulation experiment confirms the effectiveness of the proposed method both in terms of high modeling accuracy and low computational cost.

Key words: Identification and model reduction, Frequency weighted, Discrete Fourier transform, Gramians, Large scale complex systems

1 Introduction

Model order reduction is an effective method which has been applied in various theoretical and practical developments involving controls, circuit design, complex system modeling and analysis, approximation of fractional-order systems, and many others. The goal of the reduction is to generate a low-order model, which accurately describes complex system dynamics in a given adequacy range. It is worth mentioning that the reduction of high order systems is a time consuming process, in general. In this paper, we consider two model order reduction methods, that is Fourier model reduction (FMR) and frequency weighted (FW) method.

The FMR method (Willcox and Megretski, 2005) uses discrete-time Fourier coefficients to develop a model with a reduced number of state variables. Fourier coefficients can be efficiently calculated in an analytical way, which enables to obtain a very accurate approximation to the system dynamics. Usually, the obtained model contains only a fraction of state variables as compared to the original system. Furthermore, the obtained model, usually called the intermediate one, can be reduced for the second time by another model order reduction technique, e.g. balanced truncation approximation (BTA) method, see e.g. Antoulas (2005); Moore (1981). As a result we have obtained the combined FMR-BTA method (Stanisławski et al., 2017). The BTA method is based on the Gramians' factorizations in order to determine transformation matrices which reduce the model. The method minimizes an approximation error at the whole frequency range, which sometimes is not necessary in case of a given adequacy range of the model. In that case it is more desirable to reduce model errors in a certain frequency interval rather than in the whole frequency range (Rydel and Stanisławski, 2015). The FW methods are an extended versions of BTA and they are based on a direct application of input/output weighting functions in the frequency domain. This leads to a better model accuracy in selected frequency ranges. The first FW method has been proposed by Enns (1984). However, despite of the simplicity of the method and successful employment in many applications, this method cannot guarantee the stability of the reduced model in case of two-sided weighting. Several modifications to the Enns method have been offered in the literature to cope with the stability problem (Campbell et al., 2000; Ghafoor and Sreeram, 2007; Imran et al., 2014a; Lin and Chiu, 1992; Sahlan and Sreeram, 2009; Sreeram et al., 1995; Sreeram and Sahlan, 2012; Varga and Anderson, 2003; Wan Muda et al., 2011; Wang et al., 1999). All those methods provide comparable frequency response errors

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and yield computable *a priori* error bounds. Proper selections of weighting functions enable a significant reduction of approximation errors for a given frequency range (Rydel and Stanisławski, 2015). Properties of weighting functions are determined on the basis of frequency response of the model and a range of model adequacy. However, all those methods are of time complexity $\mathcal{O}(n^3)$.

In this paper we present a new model order reduction method, which is a combination of the FMR and FW methods and is called the FMR-FW method. The introduced algorithm is realized in two steps 1) the FMR method is applied to the original system in order to obtain the intermediate model and 2) the FW method is applied to the intermediate model generating the final reduced order model. The main advantage of the method are simple *analytical* formulae for selection of the controllability and observability Gramians' factorizations, which leads to reduction of computational complexity of the reduction algorithm.

This paper is organized as follows. Fundamentals of the FMR and FW methods are recalled in Section 2. The main result in terms of simple, analytical formulae for determination of the controllability and observability Gramians' factorizations is presented in Section 3. Consequently, this Section also contains a time complexity analysis for the introduced algorithm and presents some simplifications of the introduced algorithm both for the one-sided weighting and for SISO system cases. A numerical example of Section 4 confirms the effectiveness of the introduced methodology both in terms of modeling accuracy and low time complexity. Conclusions of Section 5 complete the paper.

2 Preliminaries

Consider a discrete-time LTI MIMO state-space system $G = \{A, B, C, D\}$ as follows

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) + Du(k) \qquad k = 0, 1, ... \end{aligned} \tag{1}$$

where $x(k) \in \Re^n$ is the state vector, $u(k) \in \Re^{n_u}$ and $y(k) \in \Re^{n_y}$ are the input and output vectors, respectively, $A \in \Re^{n\times n}$, $B \in \Re^{n\times n_u}$, $C \in \Re^{n_y\times n}$ and $D \in \Re^{n_y\times n_u}$.

2.1 Fourier method

It is well known that the system of Eqn. (1) can be described in the transfer function matrix form

$$G(z) = C(Iz - A)^{-1}B + D \tag{2}$$

or using the Fourier decomposition

$$G(z) = \sum_{k=0}^{\infty} g_k z^{-k} \tag{3}$$

with $g_k \in \Re^{n_y\times n_u}$, $k = 0, 1, ...$, being the impulse response matrix components, calculated as

$$g_k = \begin{cases} D & k = 0 \\ CA^{k-1}B & k = 1, 2, ... \end{cases} \tag{4}$$

Alternatively, g_k can be calculated in a recursive way

$$g_k = CH_{k-1},\ H_0 = B,\ H_k = AH_{k-1},\ k = 1, 2, ... \tag{5}$$

A finite-length approximation of Eqn. (3) can be obtained by bounding the summation process

$$\hat{G}(z) = \sum_{k=0}^{m} g_k z^{-k} \tag{6}$$

where g_k, $k = 0, ..., m$, are as in Eqn. (4) or (5). An approximation error for the model (6) is analyzed in Willcox and Megretski (2005). The model based on Fourier series expansion (FMR) can be presented in the state space form $\hat{G} = \{\hat{A}, \hat{B}, \hat{C}, \hat{D}\}$ as

$$\begin{aligned} \hat{x}(k+1) &= \hat{A}\hat{x}(k) + \hat{B}u(k) \\ \hat{y}(k) &= \hat{C}\hat{x}(k) + \hat{D}u(k) \end{aligned} \tag{7}$$

where $\hat{x}(k) \in \Re^{n_f}$, $\hat{A} \in \Re^{n_f\times n_f}$, $\hat{B} \in \Re^{n_f\times n_u}$, $\hat{C} \in \Re^{n_y\times n_f}$, $\hat{D} \in \Re^{n_y\times n_u}$, $n_f = mn_u$, and the underlying matrices are as follows

$$\left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & \hat{D} \end{array}\right] = \left[\begin{array}{cccccc|c} 0 & 0 & 0 & \dots & 0 & 0 & I \\ I & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & I & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & I & 0 & 0 \\ \hline g_1 & g_2 & g_3 & \cdots & g_{m-1} & g_m & g_0 \end{array}\right] \tag{8}$$

with $I \in \Re^{n_u\times n_u}$ and $0 \in \Re^{n_u\times n_u}$ being the identity and zero matrices, respectively, and the components $g_k \in \Re^{n_y\times n_u}$, $k = 0, ..., m$, calculated as in Eqn. (4) or (5).
It is well known that the number m of the Fourier elements can seriously affect the approximation error of the intermediate model and it has to be sufficiently large in order to produce a model of satisfactorily low approximation error. The outlined FMR method generates an intermediate model of order $n_f = mn_u$, which will be used in the next part of the paper.

2.2 Frequency weighted method

The FW method belongs to the class of the SVD-based methods and relies on the concept of balanced model realization (Antoulas, 2005; Moore, 1981), with application of weighting functions. The aim of the reduction algorithm is determination of a dominant part of the model, which can be obtained through calculation of the balancing (or transformation) matrix T and its inverse. The transformation matrix is not unique and there exist several algorithms to solve the inverse problem (Antoulas, 2005; Varga, 1991; Varga and Anderson, 2003). However, almost all of them are based on determination of the Cholesky factorizations of the controllabilty and observability Gramians, which is the first, most time consuming step in the whole algorithm. All subsequent steps of the FW algorithm are identical to those for the BTA method (Antoulas, 2005; Moore, 1981).

In the paper, we apply the FW method to the intermediate system given by Eqn. (8), which can be transformed and partitioned as follows

$$T\hat{A}T^{-1} = \begin{bmatrix} A_r & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, T\hat{B} = \begin{bmatrix} B_r \\ B_2 \end{bmatrix}, \hat{C}T^{-1} = \begin{bmatrix} C_r & C_2 \end{bmatrix}$$

where the submatrices A_r, B_r, C_r describe the reduced-order state-space system $G_r = \{A_r, B_r, C_r, \hat{D}\}$ of order n_r, such that $n_r < n_f << n$.

The FW method has been developed for stable models with stable input and output weighting functions with minimal realizations $W_i = \{A_i, B_i, C_i, D_i\}$ and $W_o = \{A_o, B_o, C_o, D_o\}$ of orders n_i and n_o, respectively. Accounting that no pole-zero cancellations occur during forming of $\hat{G}W_i$ and $W_o\hat{G}$, the augmented systems are given as follows (Antoulas, 2005; Imran et al., 2014a; Wang et al., 1999)

$$\hat{G}W_i = \left[\begin{array}{c|c} \tilde{A}_i & \tilde{B}_i \\ \hline \tilde{C}_i & \tilde{D}_i \end{array}\right] = \left[\begin{array}{cc|c} \hat{A} & \hat{B}C_i & \hat{B}D_i \\ 0 & A_i & B_i \\ \hline \hat{C} & \hat{D}C_i & \hat{D}D_i \end{array}\right]$$

$$W_o\hat{G} = \left[\begin{array}{c|c} \tilde{A}_o & \tilde{B}_o \\ \hline \tilde{C}_o & \tilde{D}_o \end{array}\right] = \left[\begin{array}{cc|c} \hat{A} & 0 & \hat{B} \\ B_o\hat{C} & A_o & B_o\hat{D} \\ \hline D_o\hat{C} & C_o & D_o\hat{D} \end{array}\right]$$

It is well known that the controllability and observability Gramians are computed based on the input weight $(\hat{G}W_i)$, and the output weight $(W_o\hat{G})$, respectively.

The frequency weighted controllability and observability Gramians

$$\tilde{P} = \begin{bmatrix} \tilde{P}_{11} & \tilde{P}_{12} \\ \tilde{P}_{12}^T & \tilde{P}_{22} \end{bmatrix} \quad \tilde{Q} = \begin{bmatrix} \tilde{Q}_{11} & \tilde{Q}_{12} \\ \tilde{Q}_{12}^T & \tilde{Q}_{22} \end{bmatrix} \tag{9}$$

where $\tilde{P}_{11} \in \Re^{n_f \times n_f}$ and $\tilde{Q}_{11} \in \Re^{n_f \times n_f}$, are solutions to the discrete-time Lyapunov equations

$$\tilde{A}_i\tilde{P}\tilde{A}_i^T - \tilde{P} = -\tilde{B}_i\tilde{B}_i^T \qquad \tilde{A}_o^T\tilde{Q}\tilde{A}_o - \tilde{Q} = -\tilde{C}_o^T\tilde{C}_o \tag{10}$$

where $\tilde{P} \in \Re^{(n_f+n_i)\times(n_f+n_i)}$ and $\tilde{Q} \in \Re^{(n_f+n_o)\times(n_f+n_o)}$.

It is important that the application of the weighting functions affects the order of the solutions $\tilde{P}$ and $\tilde{Q}$ of the Lyapunov equations. However, to calculate the matrix T and its inverse, the matrices P and Q of order n_f are required. Enns (1984) proposed the first solution of the problem through the assumptions $P = P_{en} = \tilde{P}_{11}$ and $Q = Q_{en} = \tilde{Q}_{11}$. The stability of the reduced model is not guaranteed in the case of two-sided weighting, unless either of the weighting functions is an all-pass filter $W_i = I$ or $W_o = I$. Lin and Chiu (1992) have solved the above problem, however, their approach is applicable only for strictly proper weighting filters and requires no pole-zero cancellations possibly occurring in the system and weighting functions. The improvements to that technique have been presented by Sreeram et al. (1995) and Sreeram and Sahlan (2012). To circumvent the stability issue some other methods were developed, e.g. Campbell et al. (2000); Wang et al. (1999), Varga and Anderson (2003), Imran et al. (2014a,b), the partial fraction expansion technique (Ghafoor and Sreeram, 2007; Sahlan and Sreeram, 2009; Wan Muda et al., 2011) and others.

3 Main Results

Consider the input/output weighting functions in terms of the FIR filters described as

$$W_i(z) = \sum_{k=0}^{m_i} w_k^i z^{-k} \qquad W_o(z) = \sum_{k=0}^{m_o} w_k^o z^{-k} \tag{11}$$

where m_i and m_o denote the numbers of Fourier coefficients for the input and output filters, respectively. For simplicity and without loss of generality, we assume that $W_i(z)$ and $W_o(z)$ are MIMO square filters with dimensions $n_u \times n_u$ and $n_y \times n_y$, respectively, so that $w_k^i \in \Re^{n_u \times n_u}$, $k = 0, ..., m_i$, and $w_k^o \in \Re^{n_y \times n_y}$, $k = 0, ..., m_o$. The orders of $W_i(z)$ and $W_o(z)$ are $n_i = m_i n_u$ and $n_o = m_o n_y$, respectively.
Here we present the main result of the paper, which is a new, simple method for determination of the controllability and observability Gramian factorizations for the FMR-based model given by Eqn. (7) (intermediate model), with the input/output weighting filters as in Eqn. (11), leading to a new FMR-FW reduction method.

Theorem 1 *Consider the FMR model as in Eqn. (7) cascaded with input/output weighting filters as in Eqn. (11) and let $\tilde{P}$ and $\tilde{Q}$ be the controllability and observability Gramians, respectively, satisfying the discrete-time Layapunov equations* (10). *Then*

1) factorization of the controllability Gramian $\tilde{S} \in \Re^{(m+m_i)n_u \times (m+m_i)n_u}$ *such that* $\tilde{P} = \tilde{S}^T\tilde{S}$ *is as follows*

$$\tilde{S} = \begin{bmatrix} w_0^i & 0 & 0 & \dots & 0 & 0 & I & 0 & \dots & 0 \\ w_1^i & w_0^i & 0 & \dots & 0 & 0 & 0 & I & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ w_{m_i-1}^i & w_{m_i-2}^i & w_{m_i-3}^i & \dots & 0 & 0 & 0 & 0 & \dots & I \\ w_{m_i}^i & w_{m_i-1}^i & w_{m_i-2}^i & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & w_{m_i}^i & w_{m_i-1}^i & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & w_0^i & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & w_1^i & w_0^i & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & w_{m_i}^i & w_{m_i-1}^i & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & w_{m_i}^i & 0 & 0 & \dots & 0 \end{bmatrix} \tag{12}$$

with $I \in \Re^{n_u \times n_u}$ *and* $0 \in \Re^{n_u \times n_u}$ *being the identity and zero matrices, respectively, and*

2) factorization of the observability Gramian $\tilde{R} \in \Re^{(m+m_o)n_y \times (mn_u + m_o n_y)}$ *such that* $\tilde{Q} = \tilde{R}^T\tilde{R}$ *is as follows*

$$\tilde{R} = \begin{bmatrix} r_m & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ r_{m-1} & r_m & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \\ r_1 & r_2 & \dots & r_m & 0 & 0 & \dots & 0 \\ r_{m_o-1,1} & r_{m_o-1,2} & \dots & r_{m_o-1,m} & w_{m_o}^o & 0 & \dots & 0 \\ r_{m_o-2,1} & r_{m_o-2,2} & \dots & r_{m_o-2,m} & w_{m_o-1}^o & w_{m_o}^o & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ r_{0,1} & r_{0,2} & \dots & r_{0,m} & w_1^o & w_2^o & \dots & w_{m_o}^o \end{bmatrix} \tag{13}$$

with

$$r_l = \sum_{k=l}^{l+m_o} w_{m_o+l-k}^o g_k \qquad r_{j,l} = \sum_{k=0}^{j} w_{j-k}^o g_{l+k} \tag{14}$$

for $l = 1, ..., m$, $j = 0, ..., m_o - 1$ *and* $g_k = 0 \in \Re^{n_y \times n_u}$ *for* $k > m$.

Proof. The controllability Gramian obtained from factorization (12) can be presented in form of Eqn. (9) with the submatrices $\tilde{P}_{11}$, $\tilde{P}_{12}$, $\tilde{P}_{22}$ as follows

$$\tilde{P}_{11} = \begin{bmatrix} p_0^{11} & p_1^{11} & \dots & p_{m-1}^{11} \\ {p_1^{11}}^T & p_0^{11} & \dots & p_{m-2}^{11} \\ \vdots & \vdots & \ddots & \vdots \\ {p_{m-1}^{11}}^T & {p_{m-2}^{11}}^T & \dots & p_0^{11} \end{bmatrix}$$

$$\tilde{P}_{12} = \begin{bmatrix} p_0^{12} & p_1^{12} & \dots & p_{m_i-1}^{12} \\ 0 & p_0^{12} & \dots & p_{m_i-2}^{12} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & p_0^{12} \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}$$

$$\tilde{P}_{22} = I \in \Re^{n_i \times n_i}$$

where the parameter matrices

$$p_j^{11} = \sum_{k=0}^{m_i-j} {w_{k+j}^i}^T w_k^i \qquad j = 0, ..., m-1 \tag{15}$$

$$p_j^{12} = {w_j^i}^T \qquad j = 0, ..., m_i - 1 \tag{16}$$

The left hand side of the discrete-time Lyapunov equation (10) for the controllability Gramian is as follows

$$\tilde{A}_i\tilde{P}\tilde{A}_i^T - \tilde{P} = \begin{bmatrix} l_0^1 - p_0^{11} & \dots & l_{m-1}^1 - p_{m-1}^{11} & l_0^2 - p_0^{12} & \dots & l_{m_i-1}^2 - p_{m_i-1}^{12} \\ \vdots & \vdots & \ddots & \vdots & & \\ {l_{m-1}^1}^T - {p_{m-1}^{11}}^T & \dots & 0 & 0 & \dots & 0 \\ {l_0^2}^T - {p_0^{12}}^T & \dots & 0 & -I & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & & \\ {l_{m_i-1}^2}^T - {p_{m_i-1}^{12}}^T & \dots & 0 & 0 & \dots & 0 \end{bmatrix} \tag{17}$$

where

$$l_j^1 = \begin{cases} \sum_{k=1}^{m_i} w_k^i {w_k^i}^T & j = 0 \\ \sum_{k=0}^{m_i-j} w_{k+j}^i {p_k^{12}}^T & j = 1, ..., m_i \\ 0 & j = m_i + 1, ..., m-1 \end{cases} \tag{18}$$

$$l_j^2 = \begin{cases} 0 & j = 0 \\ w_j^i & j = 1, ..., m_i - 1 \end{cases} \tag{19}$$

with $0 \in \Re^{n_u \times n_u}$ and $I \in \Re^{n_u \times n_u}$. Now, inserting Eqns. (15) and (16) into (17) we arrive at $\tilde{A}_i\tilde{P}\tilde{A}_i^T - \tilde{P} = -\tilde{B}_i\tilde{B}_i^T$.

This completes the proof for the part 1.
Proof for the observability part 2 follows the lines of that for the controllability part, with the left hand side of the discrete-time Lyapunov equation

$$\tilde{A}_o^T \tilde{Q} \tilde{A}_o - \tilde{Q} =$$
$$-\begin{bmatrix} g_1^T w_0^{o\,T} w_0^o g_1 & \ldots & g_1^T w_0^{o\,T} w_0^o g_m & g_1^T w_0^{o\,T} w_1^o & \ldots & g_1^T w_0^{o\,T} w_{m_o}^o \\ g_2^T w_0^{o\,T} w_0^o g_1 & \ldots & g_2^T w_0^{o\,T} w_0^o g_m & g_2^T w_0^{o\,T} w_1^o & \ldots & g_2^T w_0^{o\,T} w_{m_o}^o \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ g_m^T w_0^{o\,T} w_0^o g_1 & \ldots & g_m^T w_0^{o\,T} w_0^o g_m & g_m^T w_0^{o\,T} w_1^o & \ldots & g_m^T w_0^{o\,T} w_{m_o}^o \\ w_1^{o\,T} w_0^o g_1 & \ldots & w_1^{o\,T} w_0^o g_m & w_1^{o\,T} w_1^o & \ldots & w_1^{o\,T} w_{m_o}^o \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ w_{m_o}^{o\;T} w_0^o g_1 & \ldots & w_{m_o}^{o\;T} w_0^o g_m & w_{m_o}^{o\;T} w_1^o & \ldots & w_{m_o}^{o\;T} w_{m_o}^o \end{bmatrix} \quad (20)$$

being equal to $-C_o^T C_o$. This completes the proof for the observability part 2. □

The results of Theorem 1 can be easily used in various frequency weighted methods in order to compute the frequency weighted controllability and observability Gramians given by Eqn. (10). All the subsequent steps depend on the selected reduction technique (see Subsection 2.2). In case of use of the Enns technique, decompositions of the controllability and observability Gramians such that $P_{en} = S_{en}^T S_{en}$ and $Q_{en} = R_{en}^T R_{en}$, can be obtained by simple elimination of the last $m_i n_u$ and $m_o n_y$ columns of the $\tilde{S}$ and $\tilde{R}$ matrices presented in Theorem 1, respectively, so that $\tilde{S}_{en} \in \Re^{(m+m_i)n_u \times mn_u}$ and $\tilde{R}_{en} \in \Re^{(m+m_o)n_y \times mn_u}$.

3.1 Time complexity

The construction process for the decomposition of the controllability Gramian $\tilde{S}$ by use of Theorem 1 does not require any computational operations and the whole process only involves inserting of the Fourier coefficient matrices of the input weighting filter $W_i(z)$ into the $\tilde{S}$ matrix. On the other hand, calculation of the decomposition of the observability Gramian $\tilde{R}$ by use of Theorem 1 requires computational operations and involves 1) calculation do the Fourier coefficient matrices of the original system g_k, $k = 1, ..., m$ (see Eqns. (7) and (8)), and 2) determination of the elements r_l and $r_{j,l}$, $l = 1, ..., m$, $j = 0, ..., m_o - 1$ (see Eqn. (14)). Finally, the time complexity is as follows

$$\mathcal{O}(mn^2) \leq 2m(nn_y n_u + n^2 n_u) + 2mn_y^2 n_u m_o(1 + m_o) \quad (21)$$

Thus, the calculation of the factorizations for the controllability and observability Gramians is a class of quadratic time complexity $\mathcal{O}(n^2)$, but it may remarkably depend on the length m of the intermediate model. Luckily, for (essentially) large-scale systems there is typically $m << n$, compare Willcox and Megretski (2005). Note that solutions of the Lyapunov equations in the classical FW methods are a class of cubic time complexity $\mathcal{O}(n^3)$ (Antoulas, 2005).

3.2 One-sided weighting case

Using one-sided weighting in the FMR-FW method can further reduce the time complexity of the reduction algorithm. Employing the output weighting filter only, that is $W_i(z) = I$, the factorization of observability Gramian is still given by Eqn. (13) and the factorization of controllability Gramian is simplified to the identity matrix $\tilde{S} = I \in \Re^{n_f \times n_f}$. Alternatively, using input weights only, that is $W_o(z) = I$, the factorization of the controllability Gramian is still given by Eqn. (12), but factorization of the observability Gramian is as follows

$$\tilde{R} = \begin{bmatrix} g_m & 0 & 0 & \ldots & 0 \\ g_{m-1} & g_m & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ g_1 & g_2 & g_3 & \cdots & g_m \end{bmatrix} \quad (22)$$

Remark 1 *Note that using input weighting only can significantly reduce time complexity in calculation of the Gramian factorizations. In this case, in order to determine the $\tilde{S}$ and $\tilde{R}$ matrices (Eqns.* (12) *and*(22)*) we have to calculate the Fourier decomposition only. In case of use of output weighting only, since $\tilde{R}$ is calculated from Eqn.* (13)*, the time complexity remains as in Eqn.* (21)*. Therefore, using input weighting only, that is $W_i := W_i W_o$, is recommended for reduction of LTI SISO systems.*

The presented method can be easily applied to reduction of continuous-time systems. Note that the reduction algorithm is still realized in the z-domain. Therefore, the reduction process is realized in three main steps 1) discretization of a continuous-time system, 2) reduction process using the presented methodology and 3) conversion of the reduced-order model back to the continuous-time domain (see exemplary Willcox and Megretski (2005)).

4 Numerical example

We demonstrate the effectiveness of the introduced methodology on the test example for a (moderate) large-scale dynamical system presented in Penzl (2006). Originally, it is a SISO continuous-time system of order 1006 which is discretized using the Tustin method, with the sampling period 0.002 [s]. The presented FMR-FW method is now compared with the methods of BTA (Antoulas, 2005), FMR (Willcox and Megretski, 2005) and FW (Enns, 1984). A frequency weighting function

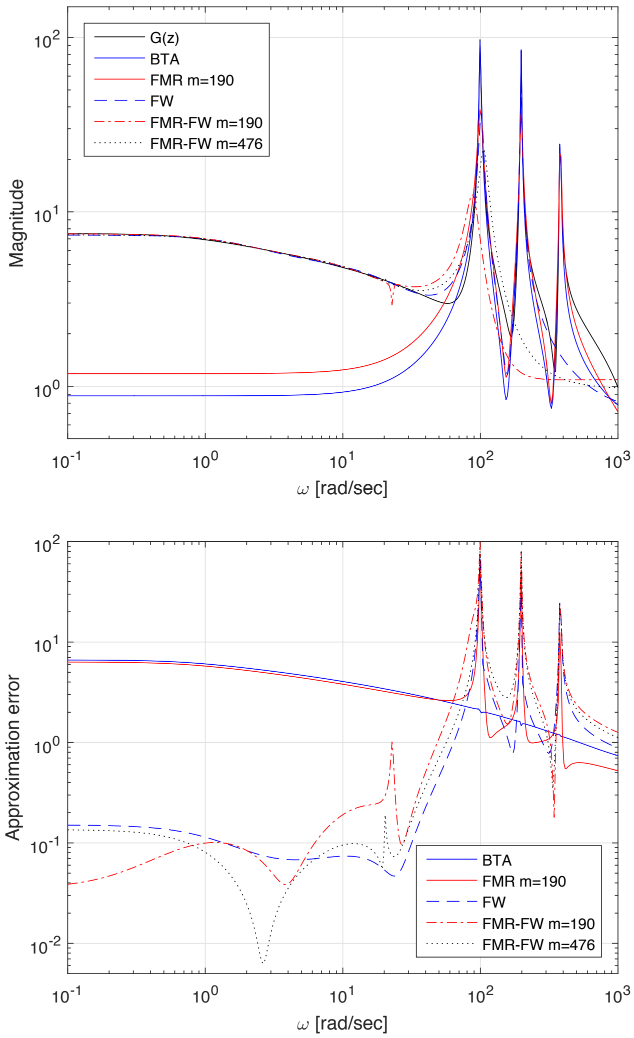

Fig. 1. Frequency responses and approximation errors for the reduced models

is applied as a one-sided input weight in the form of a low-pass FIR filter with cut-off frequency 20 [rad/s] and implementation length $m_i = 100$. The approximation order of the reduced models it assumed to be $n_r = 6$ and the model adequacy range is $\omega \in [0.1, 25]$ [rad/s]. We use two different numbers of coefficients in the Fourier (intermediate) model (8) that is $m = 190$ and $m = 476$ in the FMR-FW method.

Frequency domain characteristics such as magnitudes and approximation errors for the particular reduced order models are presented in Fig. 1. Table 1 contains both frequency and time domain approximation errors for the analyzed models in terms of 1) steady state error of the approximation (DCE), 2) mean square approximation error for the frequency characteristics in the frequency range $\omega \in [0.1, 25]$ [rad/s] (MSE_ω) and 3) mean square approximation error for the step response for $t \in [0, 50]$ (MSE_t). Moreover, Table 1 presents calculation times

Table 1
Approximation errors of reduced-order models

	DCE	MSE_ω	$\mathcal{H}_\infty$	MSE_t	time [s]
BTA	6.63	30.58	6.63	43.58	1.013
FMR $m = 476$	6.59	30.15	29.13	42.98	0.201
FMR $m = 190$	6.33	27.60	60.24	39.72	0.0622
FW	0.151	0.01172	67.61	0.03213	1.48
FMR-FW $m = 190$	0.0355	0.02350	99.51	0.03160	0.0716
FMR-FW $m = 476$	0.136	8.885e-3	82.26	0.03775	0.213

consumed by the reduction process for the considered reduction algorithms.

The results presented in Fig. 1 and Table 1 show that the FW and the two different FMR-FW methods give very similar results in terms of approximation errors for both frequency and time domains. This shows the value of the introduced FMR-FW methodology. It is clearly visible that the application of the weighting function in the form of the low-pass filter in the FW and FMR-FW methods significantly improves the approximation accuracy in the frequency range $\omega \in [0.1, 25]$ [rad/s] as well as for the steady state.

It is important to note that, unsurprisingly, FMR-FW with $m = 476$ gives better results in the frequency domain than for $m = 190$, but the reduction process in the former case is some 3 times more time consuming. On the other hand, it is rather surprising that the model with $m = 190$ provides better time-domain performance than that for $m = 476$ due to a higher steady-state approximation accuracy. Moreover, the FMR-FW method with $m = 190$ is some 20 times faster than the FW method.

It can be seen from Table 1 that the introduced FMR-FW method is significantly faster than the FW one. Moreover, taking into account that computational complexity of the FMR-FW algorithm is a class of the quadratic time instead of the cubic one for the classical FW algorithm, the superiority of the FMR-FW method will be greater for more complex systems. In our specific numerical example, the order n is relatively low, so that lower gains in time complexity can appear for higher ratios m to n. In contrast, for higher ratios n to m, as in e.g. Willcox and Megretski (2005), where $n = 13928$ and $m = 200$, the applicability of our methodology is additionally substantiated. Still, it should be emphasized that, for large-scale systems, m is always lower than n so that time complexity gains can always be obtained here. In general, selection of m may not necessarily depend on n, but rather on dynamic properties of a large-scale system, in particular poles and zeros.

5 Conclusion

This paper presents a new frequency weighted Fourier-based model order reduction method, called FMR-FW. Since controllability and observability Gramian factorizations are given by simple *analytical* formulae instead of solving Lyapunov equations, the proposed method is computationally more efficient and faster than the other SVD-based reduction methods. The computational complexity of FMR-FW is a class of the quadratic time instead of the cubic one for the classical FW method. The introduced FMR-FW method is especially useful for reduction of large-scale systems. A simulation example shows that a low-order model obtained by employing the introduced methodology enables to obtain similar results as in case of the classical FW method, but the model-order reduction process is significantly faster.
The intriguing open issue of 'optimal' selection of the number m of Fourier coefficients is left for future research.

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