The Finite Element Method for the Analysis of Non-Linear and Dynamic Systems

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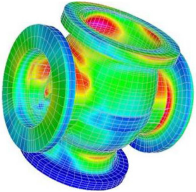
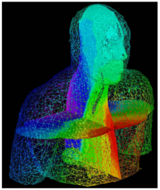
Lecture 1 - 21 September, 2017





Learning Goals

- understanding the limits of static linear finite element analysis
- understanding the difference between linear and nonlinear differential operators
- introducing the Newton-Raphson method as a fundamental tool for solving nonlinear algebraic equations
- building a big picture of most common nonlinear problems encountered in engineering

Linear Differential Operators

Is a mapping acting on elements of a space to produce elements of the same space, e.g.: functions $f : R \to R$.

Force balance:

$$E\frac{d^2u}{dx^2} = L(u) = -P$$

Differential operator:

$$L(\cdot) = E\frac{d^2(\cdot)}{dx^2}$$

The differential operator is linear:

$$L\left(au_1 + bu_2\right) = aL\left(u_1\right) + bL\left(u_2\right)$$

The superposition principle holds.

Linear differential equation:

$$L\left(u\right) = b, \text{ with } L\left(u\right) = \sum_{i=0}^{n} a_{2i}\frac{d^{2i}u}{dx^{2i}}$$

$$\begin{cases} \frac{d^j u}{dx^j}|_{0,L}, j \in \{0, ..., n-1\} \text{ essential BCs (primal quantities)} \\ \frac{d^j u}{dx^j}|_{0,L}, j \in \{n, ..., 2n-1\} \text{ natural BCs (dual quantities)} \end{cases}$$

$$\downarrow$$

Linear algebraic equation:

$$\underset{(n\times n_u)\times(n\times n_u)}{\mathbf{K}_u} \underset{(n\times n_u)\times 1}{\begin{bmatrix} \mathbf{u} \\ \vdots \\ \frac{d^{(n-1)}\mathbf{u}}{dx^{(n-1)}} \end{bmatrix}} = \underset{(n\times n_u)\times 1}{\mathbf{F}_u}$$

Linear differential equation:

$$L(u) = b, \text{ with } L(u) = \sum_{i=0}^{n} a_{2i} \frac{d^{2i}u}{dx^{2i}}$$

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Nonlinear Differential Operator

Is a mapping acting on elements of a space to produce elements of the same space, e.g.: functions $f : R \to R$.

Force balance (in presence of large strains):

$$L(u) = E\frac{d^2u}{dx^2} + E\frac{du}{dx}\frac{d^2u}{dx^2} = -P$$

Differential operator:

$$L(\cdot) = E\frac{d^2(\cdot)}{dx^2} + E\frac{d(\cdot)}{dx}\frac{d^2(\cdot)}{dx^2}$$

The differential operator L is nonlinear:

$$L(au_1 + bu_2) = E\frac{d^2(au_1 + bu_2)}{dx^2} + E\frac{d(au_1 + bu_2)}{dx}\frac{d^2(au_1 + bu_2)}{dx^2} \neq aL(u_1) + bL(u_2)$$

... the superposition principle does no longer apply.

The **principle of virtual displacements** facilitates the derivation of discretized equations:

$$\mathbf{R}_u(\mathbf{u}) = \mathbf{F}_u$$

$$\int_\Omega \{\delta\epsilon\}^T \{\boldsymbol{\sigma}\} d\Omega = \int_\Omega \delta\mathbf{u}^T \mathbf{P}_u^{vol} d\Omega + \int_\Gamma \delta\mathbf{u}^T \mathbf{P}_u^{sur} \cdot \overrightarrow{d\Gamma}$$

- σ : stress state generated by volume $\mathbf{P}_u^{vol}$ and surface $\mathbf{P}_u^{sur}$ loads.
- $\delta\mathbf{u}$ and $\delta\epsilon$: compatible variations of displacement $\mathbf{u}$ and strain ϵ fields.

Nonlinear Static FEA

The **principle of virtual displacements** facilitates the derivation of discretized equations:

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Nonlinear differential equation:

$$L\left(u\right)=b$$

$$\begin{cases} \frac{d^j u}{dx^j}\big|_{0,L}, j \in \{0, ..., n-1\} \text{ essential BCs (primal quantities)} \\ \frac{d^j u}{dx^j}\big|_{0,L}, j \in \{n, ..., 2n-1\} \text{ natural BCs (dual quantities)} \end{cases}$$

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Nonlinear algebraic equation:

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The solution of the governing BVs turns into a minimization problem that can be solved with standard optimization tools:

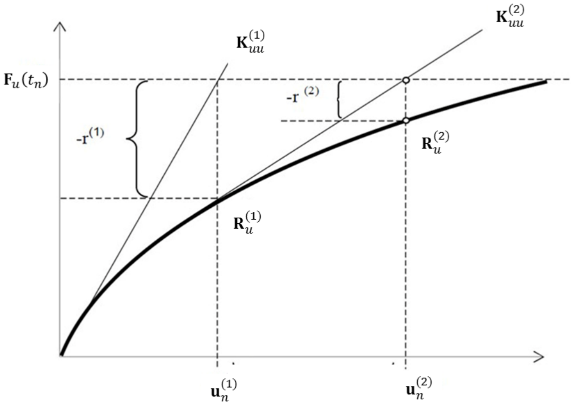
$$\hat{\mathbf{u}}_n = \arg\underset{u_n}{min}\left(\mathbf{F}_u\left(t_n\right) - \mathbf{R}_u\left(\mathbf{u}_n\right)\right)$$

Key idea:

$$\mathbf{R}_u\left(\mathbf{u} + \Delta\mathbf{u}\right) \approx \mathbf{R}_u\left(\mathbf{u}\right) + \mathbf{K}_u \Delta\mathbf{u}$$

where $\mathbf{K}_u = \nabla_{\mathbf{u}} \mathbf{R}_u$ is the so called tangent stiffness matrix.

Graphical representation of the Newton-Raphson algorithm for the monodimensional case:



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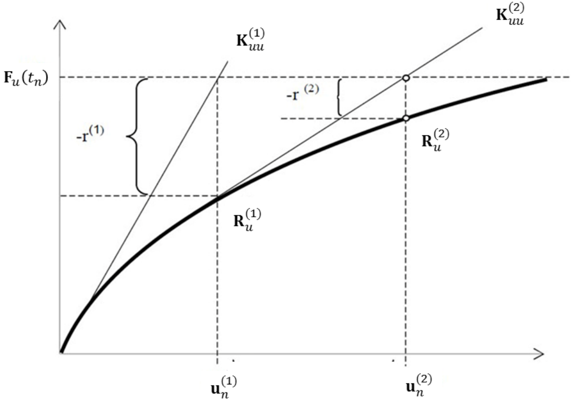
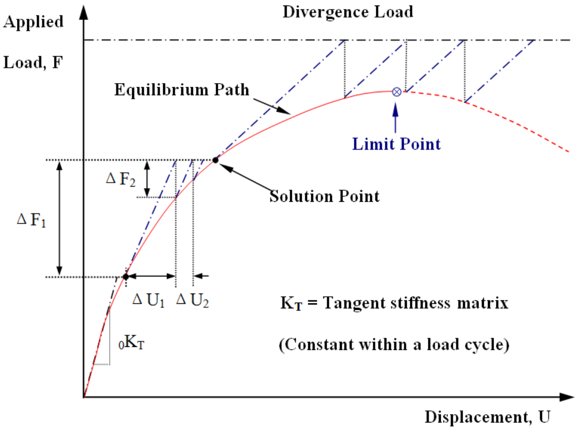
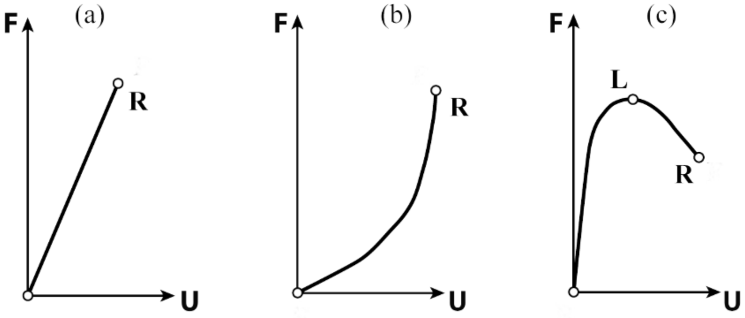


... evaluations of tangent stiffness matrices may be reduced.

Graphical representation of the Newton-Raphson algorithm for the monodimensional case:

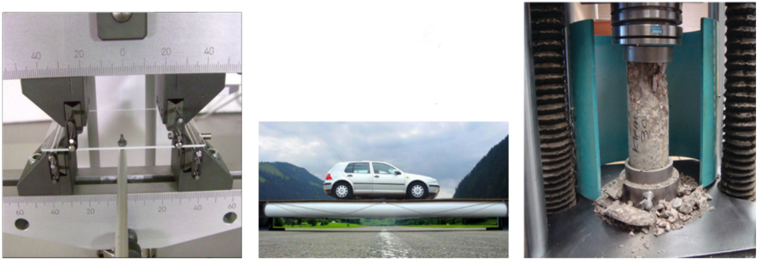




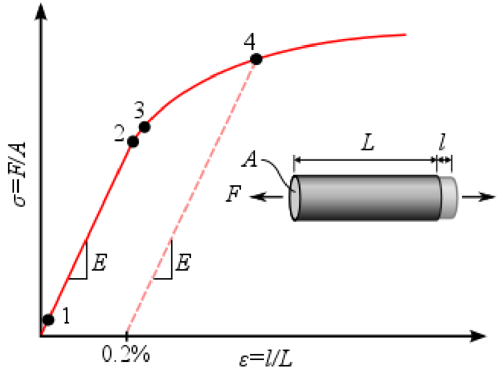
(a) elastic (brittle behavior materials); (b) increase of stiffness as load increases (pneumatic structures); (c) commonly observed behavior (concrete or steel) ©C.A. Felippa: A tour of Nonlinear Analysis.

Material Nonlinearities

(a) (b) (c)



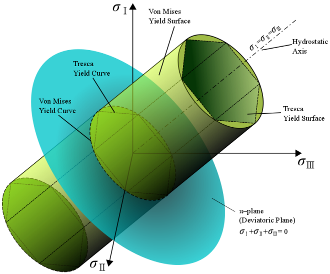
(a) elastic (brittle behavior materials); (b) increase of stiffness as load increases (pneumatic structures); (c) commonly observed behavior (concrete or steel) ©C.A. Felippa: A tour of Nonlinear Analysis.



Hysteresis loop ©Wikipedia

$$\epsilon = \epsilon_e + \epsilon_p$$

ϵ_e : elastic strain, ϵ_p : plastic strain.



Von Mises's and Tresca's yielding surfaces ©Wikipedia

$$\text{if } f(\sigma, \sigma_y) < 0 \rightarrow \dot{\sigma} = E\dot{\epsilon}$$

$$\text{if } f(\sigma, \sigma_y) = 0 \rightarrow \begin{cases} \dot{\sigma} = E(\dot{\epsilon} - \dot{\epsilon}_p), \text{ if } f(\sigma, \sigma_y(\alpha)) = 0 \\ \dot{\epsilon}_p = \dot{\lambda}\nabla f \\ \dot{\alpha} = \dot{\lambda} p \\ \dot{f} = 0 \end{cases}$$

Large Strain

Force balance:

$$\frac{d\sigma}{dx} + P = 0$$

Constitutive model:

$$\sigma = E\epsilon$$

Strain measure:

$$\epsilon = \frac{du}{dx} \rightarrow \text{linear}$$

Force balance:

$$\frac{d\sigma}{dx} + P = 0$$

Constitutive model:

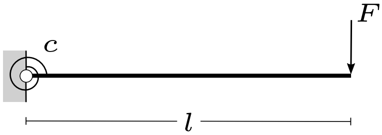
$$\sigma = E\epsilon$$

Strain measure:

$$\epsilon = \frac{du}{dx} + \frac{1}{2}\left(\frac{du}{dx}\right)^2 \rightarrow \text{Nonlinear!}$$

Large displacements of a rigid beam

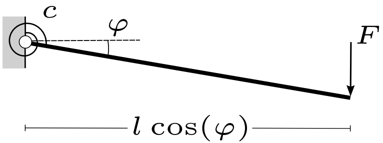
Small displacements



Equilibrium equation:

$$Fl = c\varphi$$

Large displacements

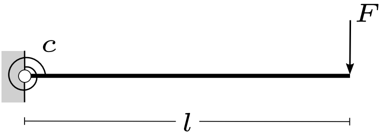


Equilibrium equation:

$$Fl\cos(\varphi) = c\varphi$$

Example taken from: "Nonlinear Finite Element Methods" by P. Wriggers, Springer, 2008

Small displacements



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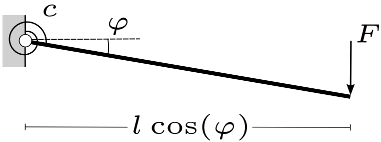
Large displacements



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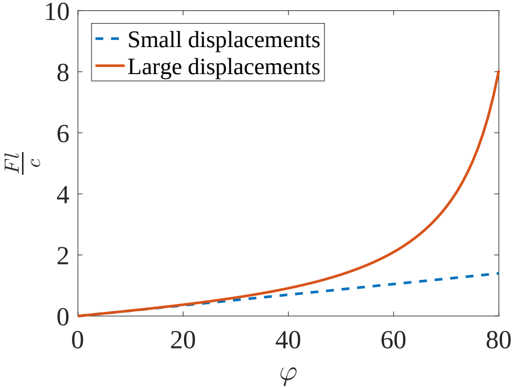
Large displacements of a rigid beam



Force versus rotation

Examples

- Example #1: Forming of a metal profile
- Example #2: Large deformation of a race-car tyre
- Example #3: Buckling of a truss structure

Nonlinear differential equation:

$$L\left(u, \dot{u}, \ddot{u}\right) = b$$

$$\begin{cases} \frac{d^j u}{dx^j}|_{0,L}, j \in \{0, ..., n-1\} \text{ essential BCs} \\ \frac{d^j u}{dx^j}|_{0,L}, j \in \{n, ..., 2n-1\} \text{ natural BCs} \end{cases}$$

$$\frac{d^j u}{dx^j}|_{t=0}, j \in \{0, ..., n-1\} \text{ Initial Conditions (ICs) on essential BCs}$$

$\downarrow$

Nonlinear algebraic equation:

$$\underset{(n\times n_u)\times(n\times n_u)}{\mathbf{M}_u} \underset{(n\times n_u)\times 1}{\begin{bmatrix} \ddot{\mathbf{u}} \\ \vdots \\ \frac{d^{(n-1)}\ddot{\mathbf{u}}}{dx^{(n-1)}} \end{bmatrix}} + \underset{(n\times n_u)\times 1}{\mathbf{R}_u} \left(\underset{(n\times n_u)\times 1}{\begin{bmatrix} \mathbf{u} \\ \vdots \\ \frac{d^{(n-1)}\mathbf{u}}{dx^{(n-1)}} \end{bmatrix}}, \underset{(n\times n_u)\times 1}{\begin{bmatrix} \dot{\mathbf{u}} \\ \vdots \\ \frac{d^{(n-1)}\dot{\mathbf{u}}}{dx^{(n-1)}} \end{bmatrix}} \right) = \underset{(n\times n_u)\times 1}{\mathbf{F}_u}$$

Nonlinear Dynamic FEA

Nonlinear differential equation:

$$L\left(u, \dot{u}, \ddot{u}\right) = b$$

$$\begin{cases} \frac{d^j u}{dx^j}|_{0,L}, j \in \{0, ..., n-1\} \text{ essential BCs} \\ \frac{d^j u}{dx^j}|_{0,L}, j \in \{n, ..., 2n-1\} \text{ natural BCs} \end{cases}$$

$$\frac{d^j u}{dx^j}|_{t=0}, j \in \{0, ..., n-1\} \text{ Initial Conditions (ICs) on essential BCs}$$

$\downarrow$

Nonlinear algebraic equation:

$$\underset{(n\times n_u)\times(n\times n_u)}{\mathbf{M}_u} \underset{(n\times n_u)\times 1}{\begin{bmatrix} \ddot{\mathbf{u}} \\ \vdots \\ \frac{d^{(n-1)}\ddot{\mathbf{u}}}{dx^{(n-1)}} \end{bmatrix}} + \underset{(n\times n_u)\times 1}{\mathbf{R}_u} \left(\underset{(n\times n_u)\times 1}{\begin{bmatrix} \mathbf{u} \\ \vdots \\ \frac{d^{(n-1)}\mathbf{u}}{dx^{(n-1)}} \end{bmatrix}}, \underset{(n\times n_u)\times 1}{\begin{bmatrix} \dot{\mathbf{u}} \\ \vdots \\ \frac{d^{(n-1)}\dot{\mathbf{u}}}{dx^{(n-1)}} \end{bmatrix}} \right) = \underset{(n\times n_u)\times 1}{\mathbf{F}_u}$$

Nonlinear Dynamic FEA

Also in this case, the solution of the governing BVs turns into a minimization problem that can be solved with standard optimization tools:

$$\{\hat{\mathbf{u}}_n, \dot{\hat{\mathbf{u}}}_n, \ddot{\hat{\mathbf{u}}}_n\} = \underset{u_n, \dot{u}_n, \ddot{u}_n}{\arg min} \left(\mathbf{F}_u\left(t_n\right) - \mathbf{R}_u\left(\mathbf{u}_n, \dot{\mathbf{u}}_n\right) - \mathbf{M}_u \ddot{\mathbf{u}}_n\right)$$

In detail:

- A minimization problem must be solved at each load step.
- The previous time step solution is taken as starting point.
- Apparently the number of unknowns increased ...

- Example #1: Shake table test of a concrete frame
- Example #2: Crash test of a car

Rail buckling driven by restrained thermal expansion ©Wikipedia

$$\epsilon = \epsilon_e + \epsilon_T$$

ϵ_e : elastic strain, ϵ_T : thermal strain.

$$\begin{aligned}
\epsilon_{xx} &= \frac{1}{E}\left(\sigma_{xx} - \nu\left(\sigma_{yy} + \sigma_{zz}\right)\right) + \alpha\left(T - T_0\right) \\
\epsilon_{yy} &= \frac{1}{E}\left(\sigma_{yy} - \nu\left(\sigma_{zz} + \sigma_{xx}\right)\right) + \alpha\left(T - T_0\right) \\
\epsilon_{zz} &= \frac{1}{E}\left(\sigma_{zz} - \nu\left(\sigma_{xx} + \sigma_{yy}\right)\right) + \alpha\left(T - T_0\right) \\
\epsilon_{xy} &= \frac{\sigma_{xy}}{2G} \\
\epsilon_{xz} &= \frac{\sigma_{xz}}{2G} \\
\epsilon_{yz} &= \frac{\sigma_{yz}}{2G}
\end{aligned}$$

α : coef. of thermal expansion; T : abs. temperature; T_0 : ref. temperature.

Linear Thermoelasticity

$$\epsilon = \epsilon_e + \epsilon_T$$

ϵ_e : elastic strain, ϵ_T : thermal strain.

$$\epsilon_{xx} = \frac{1}{E}\left(\sigma_{xx} - \nu\left(\sigma_{yy} + \sigma_{zz}\right)\right) + \alpha\left(T - T_0\right)$$

$$\epsilon_{yy} = \frac{1}{E}\left(\sigma_{yy} - \nu\left(\sigma_{zz} + \sigma_{xx}\right)\right) + \alpha\left(T - T_0\right)$$

$$\epsilon_{zz} = \frac{1}{E}\left(\sigma_{zz} - \nu\left(\sigma_{xx} + \sigma_{yy}\right)\right) + \alpha\left(T - T_0\right)$$

$$\epsilon_{xy} = \frac{\sigma_{xy}}{2G}$$

$$\epsilon_{xz} = \frac{\sigma_{xz}}{2G}$$

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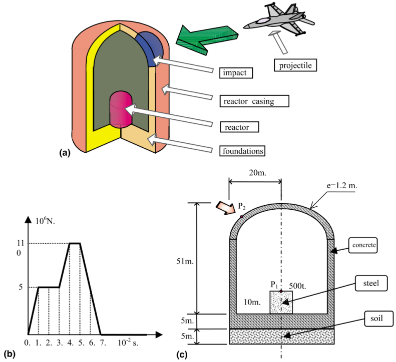
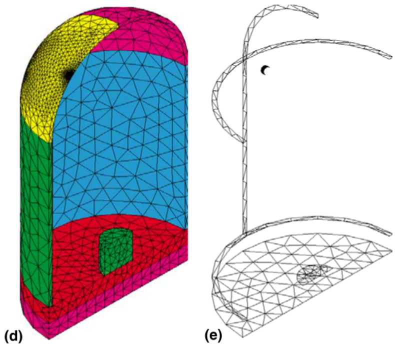
α : coef. of thermal expansion; T : abs. temperature; T_0 : ref. temperature.

- Example #1: Buckling of a railway
- Example #2: Spring forging



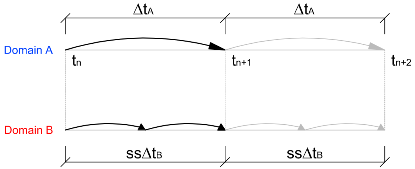
a) Impact on the containment vessel of a nuclear reactor casing; b) normal load applied at the point of impact; c) geometry of the structure.

Combescure, A., Gravouil, A. (2002). A numerical scheme to couple subdomains with different time-steps for predominantly linear transient analysis. Computer Methods in Applied Mechanics and Engineering, 191(1112), 11291157.



d) mesh of the structure; e) mesh of the interfaces.

Combescure, A., Gravouil, A. (2002). A numerical scheme to couple subdomains with different time-steps for predominantly linear transient analysis. Computer Methods in Applied Mechanics and Engineering, 191(1112), 11291157.



Task sequence of the partitioned algorithm.

$$\begin{cases} \mathbf{M}_u^A \ddot{\mathbf{u}}_{n+1}^A + \mathbf{R}_u^A \left(\mathbf{u}_{n+1}^A, \dot{\mathbf{u}}_{n+1}^A\right) = \mathbf{F}_u^A \left(t_{n+1}\right) + \mathbf{L}^{A^T} \boldsymbol{\Lambda}_{\mathbf{n+1}} \\ \mathbf{M}_u^B \ddot{\mathbf{u}}_{n+\frac{j}{ss}}^B + \mathbf{R}_u^B \left(\mathbf{u}_{n+\frac{j}{ss}}^B, \dot{\mathbf{u}}_{n+\frac{j}{ss}}^B\right) = \mathbf{F}_u^B \left(t_{n+\frac{j}{ss}}\right) + \mathbf{L}^{B^T} \boldsymbol{\Lambda}_{\mathbf{n+\frac{j}{ss}}} \end{cases}$$

$$\mathbf{L}^A \dot{\mathbf{u}}_{n+\frac{j}{ss}}^A + \mathbf{L}^B \dot{\mathbf{u}}_{n+\frac{j}{ss}}^B = \mathbf{0}$$

Combescure, A., Gravouil, A. (2002). A numerical scheme to couple subdomains with different time-steps for predominantly linear transient analysis. Computer Methods in Applied Mechanics and Engineering, 191(1112), 11291157.

- Example #1: Bullet impact
- Example #1: Fluid-structure interaction