A Finite Element Method for Nonlinear Elliptic Problems

Article *in* SIAM Journal on Scientific Computing · August 2013

DOI: 10.1137/120887655 · Source: arXiv

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A NONVARIATIONAL FINITE ELEMENT METHOD FOR NONLINEAR ELLIPTIC PROBLEMS

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Abstract. We present a continuous finite element method for fully nonlinear elliptic equations. The tools we use are (1) a Newton linearisation, yielding a sequence of linear PDEs in nonvariational form and (2) the discretisation proposed in [LP11] allowing us to work directly on the strong form of a linear PDE. An added benefit to making use of this discretisation method is that a *recovered (finite element) Hessian* is a biproduct of the solution process. Benchmark numerical results illustrate the convergence properties of the scheme for some test problems.

1. Introduction

Fully nonlinear PDEs arise in many areas, including differential geometry (the Monge–Ampère equation), mass transportation (the Monge–Kantorovich problem), dynamic programming (Bellmans equation) and fluid dynamics (the geostrophic equations).

It is difficult to pose numerical methods for fully nonlinear equations for three main reasons. The first more obvious one is the strong nonlinearity on the highest order derivative. The second is the fact that a fully nonlinear equation does not always admit a classical solution even if the problem data is sufficiently smooth. The third is that the problem may not admit a unique solution, but multiple, then even if one could construct a numerical approximation it is difficult to know which solution is being approximated.

Regardless of the problems, numerical simulation of fully nonlinear second order elliptic equations have been the brunt of much recent study, particularly for the case of Monge–Ampère of which [DG06, FN08b, LR05, Obe08, OP88] are selected examples.

For general fully nonlinear equations some methods have been presented. In [Böh08] the author presents a C^1 finite element method and goes to great lengths to show stability and consistency of the scheme. The basis of this argument comes from Stetter [Ste73]. The practical relevance of this approach is questionable, however, since the C^1 finite elements are complicated and computationally expensive, the minimal order of the polynomial basis that falls under the framework is 5, using the Argyris element for example.

In [FN07, FN08b, FN08a] the authors give a method in which they approximate the general second order fully nonlinear PDE by a sequence of fourth order quasilinear PDEs. These are quasilinear biharmonic equations which are discretised via mixed finite elements. In fact for the Monge–Ampère equation, which admits two solutions, of which one is convex and another concave, this method allows for the approximation of both solutions via the correct choice of a parameter. On the other hand although computationally less expensive than C^1 finite elements (an alternative to mixed methods for solving the biharmonic problem), the mixed formulation

Date: March 17, 2011.

still results in an extremely large algebraic system and the lack of maximum principle for general 4th order equations makes it hard to apply vanishing viscosity arguments to prove convergence.

In this paper, we propose an alternative to the method of vanishing moments of Feng and Neilan. We consider the following model problem

$$\mathscr{N}[u] := F(\mathrm{D}^2 u) - f = 0 \tag{1.1}$$

with homogeneous Dirichlet boundary conditions where f is prescribed and $F : \mathbb{R}^{d\times d} \to \mathbb{R}$ is a general second order operator which is uniformly elliptic (see Definition 2.2).

The method we propose consists of applying a Newton linearisation to the fully nonlinear PDE. This results in a sequence of linear nonvariational PDEs. At this point the problem falls into the framework of the nonvariational finite element method (NVFEM) proposed in [LP11]. We numerically study various problems that are specifically constructed to be well posed.

The paper is set out as follows. In §2 we introduce some notation and set out the model problem and discuss its ellipticity. In §3 we look at linearisations of the continuous problem showing they result in sequences of nonvariational PDEs. In §4 we then present a review of the nonvariational finite element method proposed in [LP11] and make use of it in order to discretise the linearised problem. In §5 we numerically demonstrate the performance of our discretisation on a class of fully nonlinear PDE, those that are elliptic and well posed without constraining our solution to a certain class of functions. we will apply the discretisation

2. Notation

2.1. **Functional set-up.** Let $\Omega \subset \mathbb{R}^d$ be an open and bounded Lipschitz domain. We denote $\mathrm{L}_2(\Omega)$ to be the space of square (Lebesgue) integrable functions on Ω together with its inner product $\langle v, w\rangle := \int_\Omega vw$ and norm $\|v\| := \|v\|_{\mathrm{L}_2(\Omega)} = \langle v, v\rangle^{1/2}$. We denote by $\langle v \,|\, w\rangle$ the action of a distribution v on the function w.

We use the convention that the derivative $\mathrm{D}u$ of a function $u : \Omega \to \mathbb{R}$ is a row vector, while the gradient of u, ∇u is the derivatives transpose (an element of $\mathbb{R}^d$, representing $\mathrm{D}u$ in the canonical basis). Hence

$$\nabla u = (\mathrm{D}u)^{\mathsf{T}}. \tag{2.1}$$

For second derivatives, we follow the common innocuous abuse of notation whereby the Hessian of u is denoted as $\mathrm{D}^2 u$ (instead of the consistent $\nabla \mathrm{D}u$) and is represented by a $d \times d$ matrix.

The standard Sobolev spaces are [Cia78, Eva98]

$$\mathrm{H}^k(\Omega) := \mathrm{W}_2^k(\Omega) = \left\{ \phi \in \mathrm{L}_2(\Omega) : \sum_{|\boldsymbol{\alpha}|\leq k} \mathrm{D}^{\boldsymbol{\alpha}}\phi \in \mathrm{L}_2(\Omega) \right\}, \tag{2.2}$$

$$\mathrm{H}_0^1(\Omega) := \text{closure of } \mathrm{C}_0^\infty(\Omega) \text{ in } \mathrm{H}^1(\Omega) \tag{2.3}$$

where $\boldsymbol{\alpha} = \{\alpha_1, ..., \alpha_d\}$ is a multi-index, $|\boldsymbol{\alpha}| = \sum_{i=1}^d \alpha_i$ and derivatives $\mathrm{D}^{\boldsymbol{\alpha}}$ are understood in a weak sense.

We consider the case when the model problem (1.1) is uniformly elliptic in the following sense.

2.2. **Definition** (ellipticity [CC95])**.** The problem (1.1) is said to be *uniformly elliptic* if for any $\boldsymbol{M} \in \operatorname{Sym}^+(\mathbb{R}^{d\times d})$, the space of symmetric positive definite $d \times d$ matrices, there exist *ellipticity constants* $\lambda, \Lambda > 0$ such that:

$$\lambda \sup_{|\boldsymbol{\xi}|=1} |\boldsymbol{N}\boldsymbol{\xi}| \leq F(\boldsymbol{M}+\boldsymbol{N}) - F(\boldsymbol{M}) \leq \Lambda \sup_{|\boldsymbol{\xi}|=1} |\boldsymbol{N}\boldsymbol{\xi}| \quad \forall\, \boldsymbol{N} \in \operatorname{Sym}^+(\mathbb{R}^{d\times d}). \tag{2.4}$$

If F is differentiable (2.4) can be obtained from conditions on the (Fréchet) derivative of F. A generic point $\boldsymbol{M} \in \operatorname{Sym}^+(\mathbb{R}^{d\times d})$ is written, as is customary, as

$$\boldsymbol{R} = \begin{bmatrix} m_{1,1} & \dots & m_{1,d} \\ \vdots & \ddots & \vdots \\ m_{d,1} & \dots & m_{d,d} \end{bmatrix} \tag{2.5}$$

and the derivative of F in a direction $\boldsymbol{L}$ is given by

$$\mathrm{D}F(\boldsymbol{M})\boldsymbol{L} = F'(\boldsymbol{M}){:}\boldsymbol{L} \tag{2.6}$$

where the *derivative matrix* $F'(\boldsymbol{M})$ is defined by

$$F'(\boldsymbol{M}) = \begin{bmatrix} \partial F(\boldsymbol{M})/\partial m_{1,1} & \dots & \partial F(\boldsymbol{M})/\partial m_{d,1} \\ \vdots & \ddots & \vdots \\ \partial F(\boldsymbol{M})/\partial m_{1,d} & \dots & \partial F(\boldsymbol{M})/\partial m_{d,d} \end{bmatrix}. \tag{2.7}$$

2.3. **Proposition** (ellipticity criterion for differentiable elliptic operators)**.** *If F is differentiable, then (2.4) is satisfied if and only if for $\boldsymbol{N} \in \operatorname{Sym}^+(\mathbb{R}^{d\times d})$ there exist a constant $\mu > 0$ such that*

$$\boldsymbol{\xi}^{\mathsf{T}} F'(\boldsymbol{M})\boldsymbol{\xi} \geq \mu \left|\boldsymbol{\xi}\right|^2 \qquad \forall\, \boldsymbol{\xi} \in \mathbb{R}^d. \tag{2.8}$$

2.4. **Assumption** (smoothness of the elliptic operator)**.** We shall assume that

$$F \in \mathrm{C}^1(\operatorname{Sym}^+(\mathbb{R}^{d\times d})), \tag{2.9}$$

and that F satisfies (2.8).

3. On the linearisation of fully nonlinear problems

In this work we will study Newton's method, although noting that fixed point methods can be used due to the relation between fully nonlinear problems and nonvariational problems as characterised in the following remark.

3.1. **Fixed point methods.**

We can rewrite (1.1) into a more familiar form using the chain rule and the fundamental theorem of calculus

$$\mathscr{N}[u] = \left[\int_0^1 F'(t\mathrm{D}^2 u)\,\mathrm{d}t\right]{:}\mathrm{D}^2 u + F(\boldsymbol{0}) - f = 0. \tag{3.1}$$

Setting

$$\boldsymbol{N}(\mathrm{D}^2 u) = \int_0^1 F'(t\mathrm{D}^2 u)\,\mathrm{d}t, \tag{3.2}$$

$$g = f - F(\boldsymbol{0}), \tag{3.3}$$

then if u solves (1.1), it also solves

$$\boldsymbol{N}(\mathrm{D}^2 u){:}\mathrm{D}^2 u = g. \tag{3.4}$$

A fixed point method would then consist in: finding a sequence $(u^n)_{n\in\mathbb{N}_0}$ such that for each $n \in \mathbb{N}_0$

$$\boldsymbol{N}(\mathrm{D}^2 u^n){:}\mathrm{D}^2 u^{n+1} = g, \tag{3.5}$$

with u^0 given.

3.2. **Newton's method.** Given an initial guess u^0, we define the Newton step for (1.1) as: For $n \in \mathbb{N}_0$ find u^{n+1} such that:

$$\mathscr{N}'[u^n]\left(u^{n+1}-u^n\right) = -\mathscr{N}[u^n]. \tag{3.6}$$

Rewriting it in terms of the nonlinear operator.

$$\begin{aligned}\mathscr{N}'[u]v &= \lim_{\epsilon\to 0} \frac{\mathscr{N}[u+\epsilon v]-\mathscr{N}[u]}{\epsilon} \\ &= \lim_{\epsilon\to 0} \frac{F(\mathrm{D}^2u+\epsilon \mathrm{D}^2 v)-F(\mathrm{D}^2u)}{\epsilon} \\ &= F'(\mathrm{D}^2u):\mathrm{D}^2v.\end{aligned} \tag{3.7}$$

Combining (3.6) and (3.7) then results in the following nonvariational sequence of linear PDEs. Given u^0 for each $n \in \mathbb{N}_0$ find u^{n+1} such that

$$F'(\mathrm{D}^2u^n):\mathrm{D}^2\left(u^{n+1}-u^n\right) = f - F(\mathrm{D}^2u^n). \tag{3.8}$$

If we attempted to rewrite (3.8) into a variational form we would introduce an advection term which would depend on second derivatives of F. This procedure could result in the problem becoming advection–dominated and unstable for conforming FEM, as was demonstrated numerically in [LP11].

4. The nonvariational finite element method

The structure of (3.5) and (3.8) motivates the use of the nonvariational finite element method (NVFEM) introduced in [LP11]. For completeness we give a brief review of the method.

This method was designed for problems of the following form. Let $\boldsymbol{A} \in \mathrm{L}_\infty(\Omega)^{d\times d}$ and for each $\boldsymbol{x} \in \Omega$ let $\boldsymbol{A}(\boldsymbol{x}) \in \mathrm{Sym}^+(\mathbb{R}^{d\times d})$, the space of bounded, symmetric, positive definite, $d \times d$ matrices. We seek $u \in \mathrm{H}^2(\Omega) \cap \mathrm{H}^1_0(\Omega)$ such that

$$\left\langle \boldsymbol{A}{:}\mathrm{D}^2u, \phi\right\rangle = \langle f, \phi\rangle \qquad \forall\, \phi \in \mathrm{H}^1(\Omega), \tag{4.1}$$

where the data $f \in \mathrm{L}_2(\Omega)$ is prescribed. To facilitate its discretisation we represent the linear model problem as a mixed method utilising a generalised Hessian.

4.1. **Remark** (generalised Hessian)**.** Given a function $v \in \mathrm{H}^2(\Omega)$ and let $\boldsymbol{n} : \partial\Omega \to \mathbb{R}^d$ be the outward pointing normal of Ω then the Hessian of v, D^2v satisfies the following identity:

$$\left\langle \mathrm{D}^2v, \phi\right\rangle = -\int_\Omega \nabla v \otimes \nabla\phi + \int_{\partial\Omega} \nabla v \otimes \boldsymbol{n}\phi \quad \forall\, \phi \in \mathrm{H}^1(\Omega). \tag{4.2}$$

If $v \in \mathrm{H}^1(\Omega) \cap \mathrm{H}^1(\partial\Omega)$ the left hand side of (4.2) need not be finite but is well defined via duality, in this case we may define a functional, $\boldsymbol{H}[v]$, such that

$$\langle \boldsymbol{H}[v], \phi\rangle := \left\langle \mathrm{D}^2v \,|\, \phi\right\rangle \quad \forall\, \phi \in \mathrm{H}^1(\Omega). \tag{4.3}$$

We consider $\boldsymbol{H}[v]$ to be the generalised Hessian of v.

The mixed formulation of the model problem (4.1) we consider is to seek the pair $(u, \boldsymbol{H}[u]) \in \mathrm{H}^1(\Omega) \cap \mathrm{H}^1(\partial\Omega) \times \mathrm{L}_2(\Omega)^{d\times d}$ such that

$$\langle \boldsymbol{H}[u], \phi\rangle + \int_\Omega \nabla u \otimes \nabla\phi - \int_{\partial\Omega} \nabla u \otimes \boldsymbol{n}\;\phi = \boldsymbol{0} \tag{4.4}$$

$$\langle \boldsymbol{A}{:}\boldsymbol{H}[u], \psi\rangle = \langle f, \psi\rangle \quad \forall\,(\phi,\psi) \in \mathrm{H}^1(\Omega) \times \mathrm{L}_2(\Omega) \tag{4.5}$$

We discretise (4.4)–(4.5) for simplicity with a standard piecewise polynomial approximation for test and trial spaces for both problem variable, u, and auxiliary variable, $\boldsymbol{H}[u]$. Formally, let $\mathscr{T}$ be a conforming, shape regular triangulation of Ω, namely, $\mathscr{T}$ is a finite family of sets such that

(1) $K \in \mathscr{T}$ implies K is an open simplex (segment for $d = 1$, triangle for $d = 2$, tetrahedron for $d = 3$),
(2) for any $K, J \in \mathscr{T}$ we have that $\overline{K} \cap \overline{J}$ is a full subsimplex (i.e., it is either $\emptyset$, a vertex, an edge, a face, or the whole of $\overline{K}$ and $\overline{J}$) of both $\overline{K}$ and $\overline{J}$ and
(3) $\bigcup_{K \in \mathscr{T}} \overline{K} = \overline{\Omega}$.

We use the convention where $h : \Omega \to \mathbb{R}$ denotes the *meshsize function* of $\mathscr{T}$, i.e.,

$$h(\mathbf{x}) := \max_{\overline{K} \ni \mathbf{x}} h_K. \tag{4.6}$$

We introduce the *finite element spaces*

$$\mathbb{V} := \left\{ \Phi \in \mathrm{H}^1(\Omega) : \ \Phi|_K \in \mathbb{P}^p \, \forall \, K \in \mathscr{T} \text{ and } \Phi \in \mathrm{C}^0(\Omega) \right\}, \tag{4.7}$$

$$\mathring{\mathbb{V}} := \mathbb{V} \cap \mathrm{H}^1_0(\Omega), \tag{4.8}$$

where $\mathbb{P}^k$ denotes the linear space of polynomials in d variables of degree no higher than a positive integer k. We consider $p \geq 1$ to be fixed and denote by $\mathring{N} := \dim \mathring{\mathbb{V}}$ and $N = \mathring{N} + \dot{N} := \dim \mathbb{V}$.

Our discrete problem then reads: Find $(u_h, \boldsymbol{H}[u_h]) \in \mathring{\mathbb{V}} \times \mathbb{V}^{d \times d}$ such that

$$\langle \boldsymbol{H}[u_h], \Phi \rangle + \int_\Omega \nabla u_h \otimes \nabla \Phi - \int_{\partial\Omega} \nabla u_h \otimes \boldsymbol{n} \, \Phi = \mathbf{0} \tag{4.9}$$

$$\langle \boldsymbol{A} {:} \boldsymbol{H}[u_h], \Psi \rangle = \langle f, \Psi \rangle \qquad \forall \, (\Phi, \Psi) \in \mathring{\mathbb{V}} \times \mathbb{V}. \tag{4.10}$$

5. Fully nonlinear PDEs

5.1. **Remark** (constraints)**.** Many fully nonlinear elliptic PDEs must be supplied with constraints in order to admit a unique solution. For example the Monge–Ampère–Dirichlet (MAD) problem is given by

$$\begin{aligned} \det \mathrm{D}^2 u &= f && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{5.1}$$

In this case

$$F'(\boldsymbol{X}) = \operatorname{Cof} \boldsymbol{X}, \tag{5.2}$$

the matrix of cofactors of $\boldsymbol{X}$. This implies that the linearisation of MAD is only well posed if we restrict the class of functions we consider to those that satisfy the following three equivalent statements

(1) There is a $\lambda > 0$ such that $\boldsymbol{\xi}^\intercal \operatorname{Cof} \mathrm{D}^2 u \boldsymbol{\xi} \geq \lambda \left|\boldsymbol{\xi}\right|^2 \quad \forall \, \boldsymbol{\xi} \in \mathbb{R}^d$

(2) There is a $\lambda > 0$ such that $\boldsymbol{\xi}^\intercal \mathrm{D}^2 u \boldsymbol{\xi} \geq \lambda \left|\boldsymbol{\xi}\right|^2 \quad \forall \, \boldsymbol{\xi} \in \mathbb{R}^d$

(3) u is strictly convex.

Due to difficulties arising from the passing of these constraints from the continuous level down to the discrete level in this work we will study fully nonlinear PDEs which have no such constraint.

5.2. **Assumption** (the linearisation must be well posed)**.** We assume that our linearisation is well posed, that is (2.8) is assumed to be true *without* constraining the class of functions we are considering. Important examples of fully nonlinear PDE fall into this category such as Bellmans equation.

5.3. **Definition** (nonlinear finite element method (NLFEM))**.** Suppose we are given a BVP of the form, finding $u \in \mathrm{H}^2(\Omega) \cap \mathrm{H}^1_0(\Omega)$ such that

$$\mathscr{N}[u] = F(\mathrm{D}^2 u) - f = 0 \qquad \text{in } \Omega, \tag{5.3}$$

which satisfies Assumption 5.2.

Upon applying Newton's method to solve problem (5.3) we obtain a sequence of functions $(u^n)_{n\in\mathbb{N}_0}$ solving the following linear equations in nonvariational form,

$$\boldsymbol{N}(\mathrm{D}^2 u^n){:}\mathrm{D}^2 u^{n+1} = g(\mathrm{D}^2 u^n) \tag{5.4}$$

where

$$\boldsymbol{N}(\boldsymbol{X}) := F'(\boldsymbol{X}), \tag{5.5}$$

$$g(\boldsymbol{X}) := f - F(\boldsymbol{X}) + F'(\boldsymbol{X}){:}\boldsymbol{X}. \tag{5.6}$$

The nonlinear finite element method to approximate (5.4) is: Given an initial guess $u_h^0 := \Pi_0 u^0$ † for each $n \in \mathbb{N}_0$ find $(u_h^{n+1}, \boldsymbol{H}[u_h^{n+1}])_{n\in\mathbb{N}_0} \in \mathring{\mathbb{V}}$ such that

$$\begin{aligned}\left\langle \boldsymbol{H}[u_h^{n+1}], \Phi\right\rangle + \int_\Omega \nabla u_h^{n+1} \otimes \nabla\Phi - \int_{\partial\Omega} \nabla u_h^{n+1} \otimes \boldsymbol{n}\ \Phi = \boldsymbol{0} \text{ and}\\ \left\langle \boldsymbol{N}(\boldsymbol{H}[u_h^n]){:}\boldsymbol{H}[u_h^{n+1}], \Psi\right\rangle = \langle g(\boldsymbol{H}[u_h^n]), \Psi\rangle \qquad \forall\ (\Phi,\Psi) \in \mathbb{V}\times\mathring{\mathbb{V}}.\end{aligned} \tag{5.7}$$

We now give an algorithm for the general method.

5.4. **The NVFEM for a class of fully nonlinear problems.**
Require: $(\mathscr{T}_0, u^0, p, N, K_{\max}, \mathrm{tol})$
Ensure: (u_h) the NVFE approximation of (5.7)

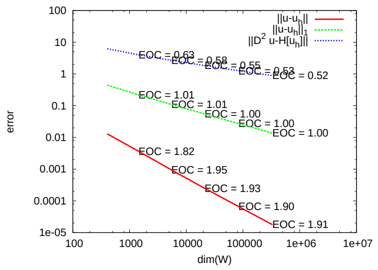
```
k = 0
while k ≤ K_max do
    V_k = FE Space(T_k, p)
    if k = 0 then
        u_h^0 = Λ^{V_0} u^0
        H[u_h^0] = Hessian Recovery(u_h^0, V_0)
    end if
    n = 0
    while n ≤ N do
        [u_h^{n+1}, H[u_h^{n+1}]] = NVFEM(V_k, N(H[u_h^n]), g(H[u_h^n]))
        if ||u_h^{n+1} − u_h^n|| ≤ tol then
            break
        end if
        n = n + 1
    end while
    T_{k+1} = Global Refine(T_k)
    k = k + 1
end while
```

5.5. **Remark** (quasilinear problems)**.** The numerical scheme given by Definition 5.3 and Algorithm 5.4 is reminiscent to that of the fixed point linearised quasilinear test problem in [LP11, §4.5] with the added complication of dealing with the finite element Hessian.

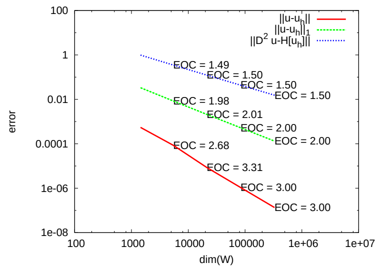
In a general case if we apply a Newton linearisation to the quasilinear problem the result is a sequence of nonvariational linear PDEs whose problem coefficients depend on the Hessian of the previous iterate as in the fully nonlinear case. This method further generalises that proposed in [LP11, §4.5] to general quasilinear PDEs using Newton's method.

†Note we have a choice for the initial finite element Hessian. We may either calculate $\mathrm{D}^2 u^0$ by hand and project onto the FE space or we may automate the procedure by computing the finite element Hessian of u_h^0 which is our preferred option.

FIGURE 1. Numerical experiments for Example 6.1. Choosing f appropriately such that $u(\boldsymbol{x}) = \sin(\pi x_1)\sin(\pi x_2)$. We run Algorithm 5.4 with an initial guess $u^0 = 0$ until $\left\|u_h^{n+1} - u_h^n\right\| \leq h$, setting $u_h := u_h^N$ the final Newton iterate of the sequence. Here we are plotting log–log error plots together with experimental convergence rates for $\mathrm{L}_2(\Omega), \mathrm{H}^1(\Omega)$ error functionals for the problem variable, u_h, and an $\mathrm{L}_2(\Omega)$ error functional for the auxiliary variable, $\boldsymbol{H}[u_h]$. Notice that there is a "superconvergence" of the auxiliary variable for both approximations.



(a) Taking $\mathbb{V}$ to be the space of piecewise linear functions on Ω $(p = 1)$.



(b) Taking $\mathbb{V}$ to be the space of piecewise quadratic functions on Ω $(p = 2)$.

6. NUMERICAL EXPERIMENTS

Each of the numerical experiments were carried out using the DOLFIN interface for FEniCS [LW10] making use of Gnuplot and ParaView for the graphics. Each of the test domains is the square $\Omega = [-1, 1]^2$ which is triangulated using a criss-cross mesh.

6.1. **Example** (a simple fully nonlinear PDE)**.** We consider the problem

$$\begin{aligned} \mathscr{N}[u] := |\Delta u| + 2\Delta u - f &= 0 \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{6.1}$$

which is specifically constructed to be uniformly elliptic. Indeed

$$F'(\mathrm{D}^2 u) = (\operatorname{sign}(\Delta u) + 2)\, \boldsymbol{I}. \tag{6.2}$$

which is uniformly positive definite.

The Newton linearisation of the problem is then: Given u^0, for $n \in \mathbb{N}_0$ find u^{n+1} such that

$$(\operatorname{sign}(\Delta u^n) + 2)\, \boldsymbol{I}{:}\mathrm{D}^2(u^{n+1} - u^n) = f - |\Delta u^n| - 2\Delta u^n. \tag{6.3}$$

and our approximation scheme is nothing but 5.7 with

$$\boldsymbol{N}(\boldsymbol{X}) = (\operatorname{sign}(\operatorname{trace} \boldsymbol{X}) + 2)\, \boldsymbol{I} \tag{6.4}$$

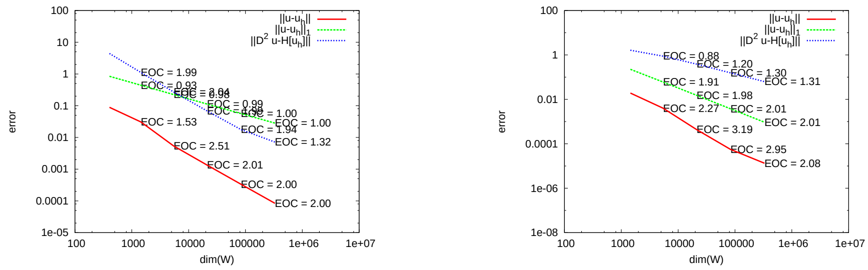
$$g(\boldsymbol{X}) = g - |\operatorname{trace} \boldsymbol{X}| - 2 \operatorname{trace} \boldsymbol{X}. \tag{6.5}$$

Figure 1 details a numerical experiment on this problem.

6.2. **Example.** The problem is for $d = 2$

$$\begin{aligned} \mathscr{N}[u] := (\partial_{11} u)^3 + (\partial_{22} u)^3 + \partial_{11} u + \partial_{22} u - f &= 0 \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{6.6}$$

Figure 2. Numerical experiments for Example 6.2. Choosing f appropriately such that $u(\boldsymbol{x}) = \exp(-10|\boldsymbol{x}|)$. We run Algorithm 5.4 with an initial guess $u^0 = 0$ until $\left\|u_h^{n+1} - u_h^n\right\| \leq h$, setting $u_h := u_h^N$ the final Newton iterate of the sequence. Here we are plotting log–log error plots together with experimental convergence rates for $\mathrm{L}_2(\Omega), \mathrm{H}^1(\Omega)$ error functionals for the problem variable, u_h, and an $\mathrm{L}_2(\Omega)$ error functional for the auxiliary variable, $\boldsymbol{H}[u_h]$. Notice that there is a "superconvergence" of the auxiliary variable for both approximations.



(a) Taking $\mathbb{V}$ to be the space of piecewise linear functions on Ω ($p = 1$).

(b) Taking $\mathbb{V}$ to be the space of piecewise quadratic functions on Ω ($p = 2$).

The problem is again uniformly elliptic since the linearisation of (6.6) is positive definite,

$$F'(\boldsymbol{X}) = \begin{bmatrix} 3\partial_{22}u^2 + 1 & 0 \\ 0 & 3\partial_{11}u^2 + 1 \end{bmatrix}. \tag{6.7}$$

The Newton linearisation of the problem is then: Given u^0 find $(u^n)_{n\in\mathbb{N}_0}$ such that

$$\boldsymbol{N}(\mathrm{D}^2 u^n):\mathrm{D}^2\left(u^{n+1} - u^n\right) = g(\mathrm{D}^2 u^n), \tag{6.8}$$

where in this case

$$\boldsymbol{N}(\mathrm{D}^2 u^n) := \begin{bmatrix} 3\partial_{22}u^2 + 1 & 0 \\ 0 & 3\partial_{11}u^2 + 1 \end{bmatrix} \tag{6.9}$$

$$g(\mathrm{D}^2 u^n) := f - (\partial_{11}u^n)^3 - (\partial_{22}u^n)^3 - \partial_{11}u^n - \partial_{22}u^n. \tag{6.10}$$

Figure 2 details a numerical experiment on this problem.

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