

$\bar{X}$  is an unbiased estimator

$$E[\bar{X}] = E\left[\frac{1}{n} \sum_{i=1}^n x_i\right]$$

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$$E[\bar{X}] = \frac{1}{n} \cdot n \mu = \mu$$

$$E[\bar{X}] = \mu$$

$S^2$  is a unbiased estimator

$$E[S^2] = E\left[\frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})^2\right]$$

$$E[S^2] = \frac{1}{n-1} E\left[\sum_{i=1}^n (x_i - \bar{x})^2\right]$$

$$= \frac{1}{n-1} E\left[\sum_{i=1}^n (x_i^2 - 2\bar{x}x_i + \bar{x}^2)\right]$$

$$= \frac{1}{n-1} E\left[\sum_{i=1}^n x_i^2 - 2\bar{x} \sum_{i=1}^n x_i + \sum_{i=1}^n \bar{x}^2\right]$$

$$= \frac{1}{n-1} E\left[\sum_{i=1}^n x_i^2 - n\bar{x}^2\right]$$

$$= \frac{1}{n-1} \left( \sum_{i=1}^n E[x_i^2] - E[n\bar{x}^2] \right)$$

$$\frac{1}{n-1} (n(\sigma^2 + \mu^2) - n E[\bar{x}^2])$$

$$E[S^2] = \sigma^2$$