$\pi_i(t)$ is the probability that is in state i after Δt or (plus) the probability that is in some state k ($k \neq i$) at time t and moves to i in Δt

$$\pi_i(t+\Delta t) = \overbrace{P[x(t+\Delta t)=i \mid x(t)=i]}^{p_{ii} \to 1-\sum p_{ij} \to 1-\sum q_{ij}\Delta t} P[x(t)=i] + \sum_{\text{all } k} \underbrace{P[x(t+\Delta t)=i \mid x(t)=k]}_{p_{ki}} P[x(t)=k]$$

$$\pi_i(t+\Delta t) = \pi_i(t)\Big(1 - \sum_{j\neq i} q_{ij}(t)\Delta t\Big) + \sum_{k\neq i} q_{ki}(t)\Delta t\, \pi_k(t)$$

$$\pi_i(t) - \underbrace{\sum_{j\neq i} q_{ij}(t)\Delta t}\, \pi_i(t) + \sum_{k\neq i} q_{ki}(t)\Delta t\, \pi_k(t)$$

$$\boxed{q_{ii}(t) = -\sum_{j\neq i} q_{ij}(t)}$$

$$\underbrace{q_{ii}\Delta t\, \pi_i(t) + \sum_{k\neq i} q_{ki}(t)\Delta t\, \pi_k(t)}$$

$$\sum_{\text{all } k} q_{ki}(t)\Delta t\, \pi_k(t)$$

$$\lim_{\Delta t\to 0} \frac{\pi_i(t+\Delta t) - \pi_i(t)}{\Delta t} = \lim_{\Delta t\to 0}\left(\sum_{\text{all } k} \frac{q_{ki}(t)\pi_k \cancel{\Delta t}}{\cancel{\Delta t}}\right) \rightarrow \boxed{\frac{d\pi_i(t)}{dt} = \sum_{\text{all } k} q_{ki}(t)\pi_k(t)}$$