$$p_{ij}(t+\Delta t) = \sum_{\text{all } k} p_{ik}(t)\, p_{kj}(\Delta t)$$

$$p_{ij}(t+\Delta t) = \sum_{k\neq j} p_{ik}(t)\, p_{kj}(\Delta t) + p_{ij}(t)\, p_{jj}(\Delta t)$$

$$\frac{p_{ij}(t+\Delta t) - p_{ij}(t)}{\Delta t} = \sum_{k\neq j} p_{ik}(t) \frac{p_{kj}(\Delta t)}{\Delta t} + \frac{p_{ij}(t)\, p_{jj}(\Delta t)}{\Delta t} - \frac{p_{ij}(t)}{\Delta t}$$

$$\frac{p_{ij}(t+\Delta t) - p_{ij}(t)}{\Delta t} = \sum_{k\neq j} p_{ik}(t) \frac{p_{kj}(\Delta t)}{\Delta t} + p_{ij}(t) \underbrace{\left(\frac{p_{jj}(\Delta t) - 1}{\Delta t} \right)}_{q_{jj}}$$

taking the limit $\Delta t \to 0$ we have:

$$\frac{dp_{ij}(t)}{dt} = \underbrace{\sum_{k\neq j} p_{ik}(t)\, q_{kj} + p_{ij}(t)\, q_{jj}}$$

$$\boxed{\frac{dp_{ij}(t)}{dt} = \sum_{\text{all } k} p_{ik}(t)\, q_{kj}}$$