Using Riemann Sums to Approximate π

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Demonstration of a slow, but creative new method of approximating π by computing the Riemann sum over a semicircle.

1 Introduction

Let $n \in \mathbb{Z}$. Then $\pi \approx \frac{8}{n^2} \sum_{i=1}^{n} \sqrt{i(n-i)}$.

This method was originally described in a paper I wrote on November 30, 2012. I discovered this method during my first semester of calculus at IUK.

2 Proof

A semicircle can be represented by the function $f(x) = \sqrt{r^2 - x^2}$. The area of a semicircle is $A = \frac{\pi}{2} r^2$, which can also be represented as an integral:

$$\frac{\pi}{2} r^2 = \int_{-r}^{r} \sqrt{r^2 - x^2}$$

Now we write this definite integral in terms of a Riemann sum, where $n \in \mathbb{Z}^+$.

$$\begin{aligned} \int_{-r}^{r} \sqrt{r^2 - x^2} &= \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)(x_i - x_{i-1}) \\ &= \lim_{n \to \infty} \sum_{i=1}^{n} \frac{2r}{n} \sqrt{r^2 - \left(\frac{2i}{n} - r \right)^2} \end{aligned}$$

It then follows that:

$$\frac{\pi}{2} r^2 = \lim_{n \to \infty} \sum_{i=1}^{n} \frac{2r}{n} \sqrt{r^2 - \left(\frac{2i}{n} - r \right)^2}$$

Let $r = 1$. Since $n > 0$,

$$
\begin{aligned}
\frac{\pi}{2} &= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n} \sqrt{1 - \left(\frac{2i}{n} - 1\right)^2} \\
\implies \pi &= \frac{4}{n} \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 - \left(\frac{2i}{n} - 1\right)^2} \\
&= \frac{4}{n} \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 - \left(\frac{4i^2}{n^2} - \frac{4i}{n} + 1\right)} \\
&= \frac{4}{n} \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{\frac{4i}{n} - \frac{4i^2}{n^2}} \\
&= \frac{4}{n} \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{\frac{4in - 4i^2}{n^2}} \\
&= \frac{4}{n} \lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n} \sqrt{i(n-i)} \\
&= \frac{8}{n^2} \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{i(n-i)}
\end{aligned}
$$

So to approximate the value of π, we take a rectangular width of $\frac{1}{n}$ and compute the sum $\frac{8}{n^2} \sum_{i=1}^{n} \sqrt{i(n-i)}$, as illustrated by the C code in the following section.

3 Example

riesum.c

```
#include <stdio.h>
#include <math.h>

int main(int argc, char *argv[]) {
 int n;
 printf("Enter a positive integer. n = ");
 for (scanf("%d",&n); n <= 0; scanf("%d",&n)) {
  while (fgetc(stdin) != '\n');
  printf("Enter a valid integer. n = ");
 }

 double sum = 0;

 for (int i = 1; i <= n; i++) {
  sum += sqrt(i*(n-i))/n;
 }

 // Note that we kept one n in the denominator of the sum to avoid
 // an integer overflow.
 sum *= 8/(double)n;

 printf(π ≈ %f\n",sum);

 return 0;
}
```

To compile, run: `gcc riesum.c -o riesum -lm`