Inverse methods and imaging

Jodi Mead
Department of Mathematics

Thanks to: Diego Domenzain
John Bradford

BOISE STATE UNIVERSITY

Inverse Problems

Consider solving problems of the form:

$$\mathbf{G}\mathbf{m} = \mathbf{d},$$

- $\mathbf{G} \in R^{m \times n}$ - mathematical model
- $\mathbf{d} \in R^{m}$ - data collected by Scientists or Engineers
- $\mathbf{m} \in R^{n}$- unknown model characteristics we need to find.

$\mathbf{G}$ is rectangular, ill-conditioned and the solution $\mathbf{m} = \mathbf{G}^{-1}\mathbf{d}$ is not possible. In other words the mathematical model $\mathbf{G}\mathbf{m}$ cannot be determined by the data $\mathbf{d}$,

Least squares

$$\mathbf{m}_{ls} = argmin_{\mathbf{m}} \|\mathbf{G}\mathbf{m} - \mathbf{d}\|_2^2$$

If $\mathbf{G}$ has linearly indenpendent columns

$$\mathbf{m}_{ls} = (\mathbf{G}^T\mathbf{G})^{-1}\mathbf{G}^T\mathbf{d}$$

Typically we can't compute this but even if we could, do we want the mathematical model to match the data exactly?

Regularization

$$\mathbf{m}_{\mathbf{L}_p} = argmin_{\mathbf{m}} \left\{ \|\mathbf{Gm} - \mathbf{d}\|_2^2 + \lambda ||\mathbf{L}_p(\mathbf{m} - \mathbf{m}_0)||_2^2 \right\}$$

$\mathbf{m}_0$ - initial estimate of $\mathbf{m}$
$\mathbf{L}_p$ - typically represents the first $(p = 1)$ or second derivative $(p = 2)$
λ - regularization parameter

Gives estimates

$$\mathbf{m}_{\mathbf{L}_p} = \mathbf{m}_0 + (\mathbf{G}^T\mathbf{G} + \lambda^2 \mathbf{L}_p^T \mathbf{L}_p)^{-1} \mathbf{G}^T \mathbf{d}$$

and the mathematical model with additional information in $\lambda \mathbf{L}_p$ can be resolved by the data.

Choice of λ

Methods:
L-curve, Generalized Cross Validation (GCV) and Morozov's Discrepancy Principle, UPRE, χ^2 method[1].

- λ large $\rightarrow$ constraint: $||\mathbf{L}_p(\mathbf{m} - \mathbf{m}_0)||_2^2 \approx 0$

$$\mathbf{m}_{\mathbf{L}_p} = argmin_{\mathbf{m}} \left\{ ||\mathbf{G}\mathbf{m} - \mathbf{d}||_2^2 + \lambda||\mathbf{L}_p(\mathbf{m} - \mathbf{m}_0)||_2^2 \right\}$$

- λ small $\rightarrow$ problem may stay ill-conditioned

$$\mathbf{m}_{\mathbf{L}_p} = \mathbf{m}_0 + (\mathbf{G}^T\mathbf{G} + \lambda^2\mathbf{L}_p^T\mathbf{L}_p)^{-1}\mathbf{G}^T\mathbf{d}$$

[1] Mead et al, 2008, 2009, 2010, 2016, 2019

Choice of $\mathbf{L}_p$[2]

$$\mathbf{m}_{\mathbf{L}_p} = argmin_{\mathbf{m}} \left\{ \|\mathbf{Gm} - \mathbf{d}\|_2^2 + \lambda ||\mathbf{L}_p(\mathbf{m} - \mathbf{m}_0)||_2^2 \right\}$$

$\mathbf{L}_0 = \mathbf{I}$ - requires good initial estimate $\mathbf{m}_0$, otherwise may not regularize the problem.

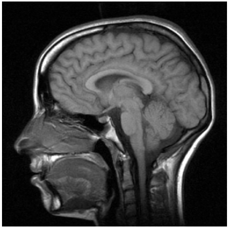
$\mathbf{L}_1$ - requires $\mathbf{L}_1 \mathbf{m}_0$, i.e. zero first derivative estimate, less information than $\mathbf{m}_0$.

$\mathbf{L}_2$ - requires $\mathbf{L}_2 \mathbf{m}_0$, leaves more degrees of freedom than first derivative, so that data has more opportunity to inform parameter estimates.

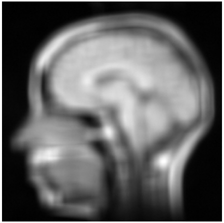
[2] Hetrick and M. 2018; Domenzain, Bradford and M. 2019

Total Variation minimization

$$\hat{\mathbf{m}} \in \underset{\mathbf{m}}{argmin} \ \frac{\lambda}{2}\|\mathbf{d} - \mathbf{G}\mathbf{m}\|_2^2 + \|\nabla \mathbf{m}\|_1$$



Exact - $\mathbf{m}$



Blurred - $\mathbf{d}$

Choice of λ

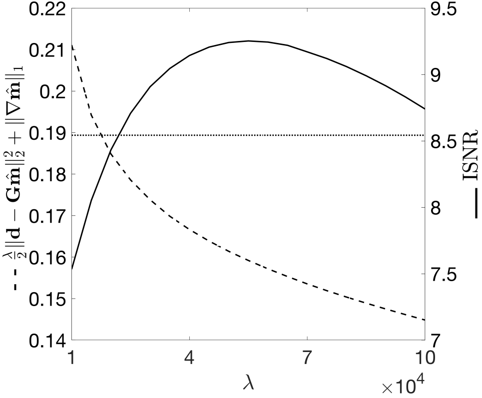
$\lambda = 500$

$\lambda = 50000$

$\lambda = 500000$

χ^2 test for λ

$$\frac{\lambda}{2}\|\mathbf{d}-\mathbf{G}\hat{\mathbf{m}}\|_2^2 + \|\nabla\hat{\mathbf{m}}\|_1 \sim \chi_m^2$$



Joint Inversion as Regularization

$$\mathbf{m}_{12} = argmin_{\mathbf{m}} \left\{ ||\mathbf{G}_1\mathbf{m} - \mathbf{d}_1||_2^2 + ||\mathbf{G}_2\mathbf{m} - \mathbf{d}_2||_2^2 \right\}$$

Objective function can be written

$$\left\| \begin{bmatrix} \mathbf{G}_1 \\ \mathbf{G}_2 \end{bmatrix} [\mathbf{m}] - \begin{bmatrix} \mathbf{d}_1 \\ \mathbf{d}_2 \end{bmatrix} \right\|_2^2 \equiv ||\mathbf{G}_{12}\mathbf{m} - \mathbf{d}_{12}||_2^2$$

Goal: Improve condition number

$$\kappa(\mathbf{G}_{12}) < \kappa(\mathbf{G}_1), \kappa(\mathbf{G}_2)$$

where $\kappa(\mathbf{G}) = \frac{\sigma_{max}(\mathbf{G})}{\sigma_{min}(\mathbf{G})}$

Singular Value Decomposition (SVD)

$$\mathbf{G}_{12} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T \rightarrow \mathbf{m}_{12} = \sum_{i=1}^{n} \frac{\mathbf{U}_{\cdot,i}^T \mathbf{d}_{12}}{\sigma_i} \mathbf{V}_{\cdot,i}$$

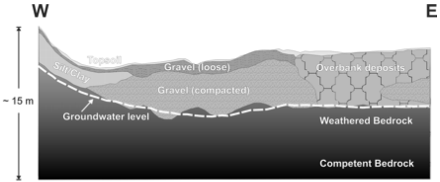
Truncated SVD (with decomposition on appropriate matrix)

$$\mathbf{m}_1 = \sum_{i=1}^{k_1} \frac{\mathbf{U}_{\cdot,i}^T \mathbf{d}_1}{\sigma_i} \mathbf{V}_{\cdot,i}, \;\; \mathbf{m}_2 = \sum_{i=1}^{k_2} \frac{\mathbf{U}_{\cdot,i}^T \mathbf{d}_2}{\sigma_i} \mathbf{V}_{\cdot,i}, \;\; \mathbf{m}_{12} = \sum_{i=1}^{l} \frac{\mathbf{U}_{\cdot,i}^T \mathbf{d}_{12}}{\sigma_i} \mathbf{V}_{\cdot,i}$$

Goal: Keep as many singular values as possible

$$l > k_1, k_2$$

Near subsurface imaging

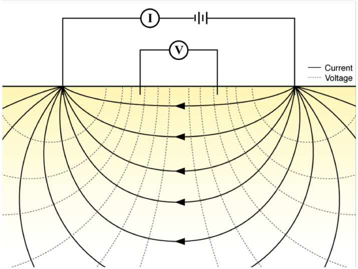


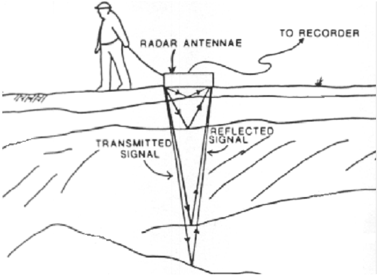
- Landfill Investigation
- Mapping and monitoring of groundwater pollution
- Determination of depth to bedrock
- Locating sinkholes, cave systems, faults and mine shafts
- Landslide assessments
- Buried foundation mapping

Boise Hydrogeophysical Research Site

Field laboratory near intersection of Warm Springs Ave and HWY 21

Near subsurface imaging with electromagnetic waves





Electrical Resistivity Tomography **Ground Penetrating Radar**

Ground Penetrating Radar (GPR)

$$\text{Damped Wave:} \quad \epsilon \frac{\partial^2 E}{\partial t^2} + \sigma \frac{\partial E}{\partial t} = \frac{1}{\mu} \nabla^2 E + f$$

- Radar signals f transmitted into the ground and energy that is reflected back to the surface is recorded.
- If there's a contrast in properties between adjacent material properties (permittivity ϵ, permeability μ and conductivity σ) a proportion of the electromagnetic pulse will be reflected back.
- Subsurface structures are imaged by measuring the amplitude and travel time of this reflected energy.

Electrical Resistivity Tomography (ERT)

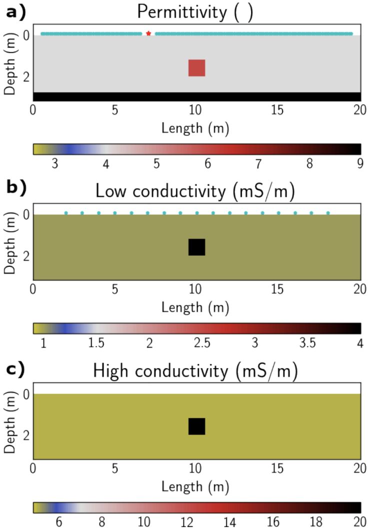
$$\text{Diffusion:} \quad -\nabla \cdot \sigma \nabla \phi = \nabla \cdot J_s$$

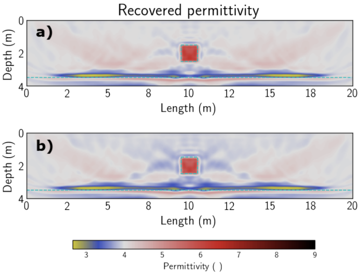
- Current is passed through the ground via outer electrodes J_s and potential difference ϕ is measured between an inner pair of electrodes.
- Only responds to variability in electrical resistivity σ exhibited by earth materials.
- ERT data must be inverted to produce detailed electrical structures of the cross-sections below the survey lines.

Forward vs Inverse modeling

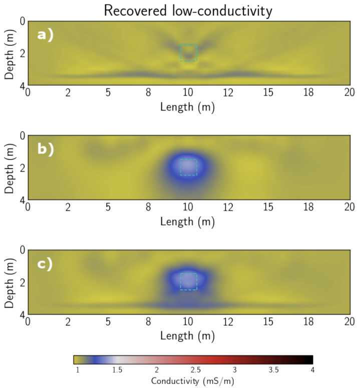
	Forward	Inverse
$\epsilon \frac{\partial^2 E}{\partial t^2} + \sigma \frac{\partial E}{\partial t} = \frac{1}{\mu} \nabla^2 E + f$	Given ϵ, μ, σ Solve for E	Given E Solve for ϵ, μ, σ
$-\nabla \cdot \sigma \nabla \phi = \nabla \cdot J_s$	Given σ solve for ϕ	Given ϕ solve for σ

Simulated subsurface

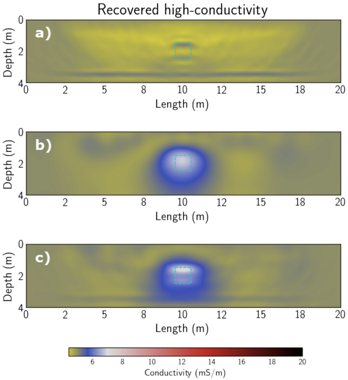




a) GPR **b)** GPR and ERT



a) GPR **b)** ERT **c)** GPR and ERT



a) GPR **b)** ERT **c)** GPR and ERT