Lecture Activities
Maximum Likelihood Estimation (MLE)

Consider the process model

$$X_{i+1} = \alpha X_i, \ i = 1, 2, 3,$$

and data model

$$Y_i = X_i + \epsilon_i, \ i = 2, 3.$$

Assume that the conditional distribution of Y given X is

$$f_{Y|X}(y_2, \ldots, y_{N-1}|x_1, \ldots, x_N; \tau^2) = \prod_{i=2}^{N-1} \frac{1}{\tau\sqrt{2\pi}} e^{-1/2(y_i - x_i)^2/\tau^2}.$$

.

1. Show that

$$f_{Y|X}(y_2, \ldots, y_{N-1}|x_1, \ldots, x_N; \tau^2) = \prod_{i=2}^{N-1} \frac{1}{\tau\sqrt{2\pi}} e^{-1/2(y_i - \alpha^{i-1}x_1)^2/\tau^2}$$

   Solution: Use the formula $x_{i+1} = \alpha x_i$ to replace x_i with $x_i = \alpha^{i-1}x_1$

2. The maximum likelihood estimate finds the value of x_1 that maximizes the probability that the data were observed, i.e. we want to find x_1^* that satisfies

$$\max_{x_1} \prod_{i=2}^{N-1} \frac{1}{\tau\sqrt{2\pi}} e^{-1/2(y_i - \alpha^{i-1}x_1)^2/\tau^2}. \tag{1}$$

   Show that (**??**) is equivalent to

$$\max_{x_1} \left(\frac{1}{\tau\sqrt{2\pi}}\right)^{N-2} e^{-1/2\tau^2 \sum\limits_{i=2}^{N-1} (y_i - \alpha^{i-1}x_1)^2}. \tag{2}$$

   Solution: The product of exponential is the sum of its exponents.

3. Show that (**??**) is equivalent to

$$\min_{x_1} \sum_{i=2}^{N-1} (y_i - \alpha^{i-1}x_1)^2. \tag{3}$$

   Solution: The maximum of e^{-x} is the minimum value of x. The constants do not effect the maximum or minimum.

4. Given data y_2 and y_3, and states x_1, x_2, x_3, x_4 use the Lecture activities from Tuesday to prove that the maximum likelihood estimate is $x_1^* = \frac{y_2 + \alpha y_3}{\alpha(1+\alpha^2)}$.

   Solution: We are minimizing

$$\mathcal{J}(x_2, x_3; y_2, y_3) = (x_2 - y_2)^2 + (x_3 - y_3)^2,$$

   the same function as Tuesday. $x_1^* = \frac{y_2 + \alpha y_3}{\alpha(1+\alpha^2)}$.