Data Assimilation and Climate
AIMS Rwanda, March, 2020

Lecture Activities
Discrete time stochastic process

1. Recall the following, if we have state estimates X_1, X_2, X_3, X_4 and data Y_2, Y_3:

   - Re-analysis: estimate X_1 given Y_2 and Y_3
   - Filtering: estimate X_2 given Y_2 or
     estimate X_3 given Y_2 and Y_3
   - Forecasting: estimate X_4 given Y_2 and Y_3

   Consider the Bayesian approach to assimilate data into a process model, i.e. we find the conditional pdf $f_{X|y}$ where

   $$f_{X|y} \propto f_{Y|x}(\mathbf{y}) f_X(\mathbf{x}).$$

   Now view the process and data models as time discrete stochastic processes i.e. assume we have state estimates $\mathbf{X}(0), \mathbf{X}(1), \mathbf{X}(2), \mathbf{X}(3)$ and data $\mathbf{Y}(1), \mathbf{Y}(2)$.

   (a) If we want to filter, and find an estimate of $\mathbf{X}(1)$, which conditional pdf do we find?

   Solution: $f_{X(1),y(1)}$

   (b) If we want to filter, and find an estimate of $\mathbf{X}(2)$, which conditional pdf do we find?

   Solution: $f_{X(2),y(1:2)}$

   (c) If we want to forecast, and find an estimate of $\mathbf{X}(3)$, which conditional pdf do we find?

   Solution: $f_{X(3),y(1:2)}$

2. Assume we have non-stationary stochastic process model

   $$\mathbf{X}(i) \quad = \quad \mathbf{M}_i \mathbf{X}(i-1) + \boldsymbol{\delta}_i \quad i = 1, \ldots, N$$

   (a) Let $\mathbf{X} \in \mathbb{R}^n$ contain variables X_j that represent temperature, pressure and velocity at grid points in space. If we use the process model $X_j(i+1) = \alpha_j X_j(i) + (\delta_j)_i$ for $j = 1, \ldots, n$ and $i = 1, \ldots, N$. Identify $\mathbf{M}_i$.

   Solution: For all i

   $$\mathbf{M}_i = \begin{pmatrix} \alpha_1 & 0 & \ldots & \ldots \\ 0 & \alpha_2 & 0 & \ldots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & \ldots & \alpha_n \end{pmatrix}$$

   .

   (b) If $\mathbf{X}(0) \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ and $\boldsymbol{\delta}_i \sim \mathcal{N}(\mathbf{0}, \mathbf{Q}_i)$, then $\mathbf{X}(1) \sim \mathcal{N}(\mathbf{M}_1 \boldsymbol{\mu},)$. If $\mathbf{X}(2) = \mathbf{M}_2 \mathbf{X}(1) + \boldsymbol{\delta}_1$, find mean and covariance of $\mathbf{X}(2)$.

   Solution: $\mathbf{X}(2) \sim \mathcal{N}(\mathbf{M}_2 \mathbf{M}_1 \boldsymbol{\mu}, \mathbf{M}_2(\mathbf{M}_1 \boldsymbol{\Sigma} \mathbf{M}_1^T + \mathbf{Q}_1)\mathbf{M}_2^T + \mathbf{Q}_2)$

(c) Find $\boldsymbol{\mu}_{2|0}$ and $\boldsymbol{\Sigma}_{2|0}$.

Solution: $\boldsymbol{\mu}_{2|0} = E[\mathbf{X}(2)] = \mathbf{M}_2\mathbf{M}_1\boldsymbol{\mu}$, $\boldsymbol{\Sigma}_{2|0} = \mathrm{Var}[\mathbf{X}(2)] = \mathbf{M}_2(\mathbf{M}_1\boldsymbol{\Sigma}\mathbf{M}_1^T + \mathbf{Q}_1)\mathbf{M}_2^T + \mathbf{Q}_2$