A Decomposition of US Debt-to-GDP Movements

1 Method

This is a slightly altered version of a backward-looking exercise in Hall and Sargent's (2022) paper. Begin with the government's budget constraint (GBC), written as

$$M_t + Q_t^B B_t + Q_t^R R_t + Q_t^U U_t + P_t S_t = M_{t-1} + Q_t^B B_{t-1} + Q_t^R R_{t-1} + Q_t^U U_{t-1} \tag{1}$$

where:

- M_t: non-interest-bearing Federal Reserve (Fed) liabilities outstanding
- B_t: privately-held (issued less Fed-held) Treasuries
- R_t: interest-bearing reserves outstanding
- U_t: Fed-issued overnight reverse repurchase agreements outstanding
- Q_t^B: price of privately-held Treasuries, defined as the market value of private-held Treasuries over its par value
- Q_t^R: price of interest-bearing reserves outstanding, defined as the inverse of the interest rate on reserves (IRR)
- Q_t^U: price of Fed-issued overnight reverse repurchase agreements outstanding, defined as the inverse of the reverse repo award rate
- P_t: Aggregate price level
- S_t: Real-valued primary surplus

Define the holding period return (HPR) on Treasuries, Reserves, and Reverse Repos $i_{t-1,t}^j$ according to

$$1 + i_{t-1,t}^j = \frac{Q_t^j}{Q_{t-1}^j} \tag{2}$$

where $j \in \{B, R, U\}$.

Combine the GBC (1) and definition of the HPR (1) to write

$$\begin{aligned} & M_t + Q_t^B B_t + Q_t^R R_t + Q_t^U U_t + P_t S_t \\ & \qquad = M_{t-1} + \left(1 + i_{t-1,t}^B\right) Q_{t-1}^B B_{t-1} + \left(1 + i_{t-1,t}^R\right) Q_{t-1}^R R_{t-1} + \left(1 + i_{t-1,t}^U\right) Q_{t-1}^U U_{t-1} \end{aligned} \tag{3}$$

Divide both sides of (3) by nominal GDP $P_t Y_t$ which yields

$$\begin{aligned} & \frac{M_t}{P_t Y_t} + \frac{Q_t^B B_t}{P_t Y_t} + \frac{Q_t^R R_t}{P_t Y_t} + \frac{Q_t^U U_t}{P_t Y_t} + \frac{P_t S_t}{P_t Y_t} \\ & \qquad = \frac{M_{t-1}}{P_t Y_t} + \frac{\left(1 + i_{t-1,t}^B\right) Q_{t-1}^B B_{t-1}}{P_t Y_t} + \frac{\left(1 + i_{t-1,t}^R\right) Q_{t-1}^R R_{t-1}}{P_t Y_t} + \frac{\left(1 + i_{t-1,t}^U\right) Q_{t-1}^U U_{t-1}}{P_t Y_t} \end{aligned} \tag{4}$$

Approximate the growth in $(P_t Y_t)^{-1}$ as

$$(P_t Y_t)^{-1} = (1 - \pi_t - g_t) (P_{t-1} Y_{t-1})^{-1} \tag{5}$$

where π_t is inflation and g_t is real GDP growth, both from time $t-1$ to t.

Combining the GBC (4) with the approximation (5) gives us

$$\begin{aligned} & \frac{M_t}{P_t Y_t} + \frac{Q_t^B B_t}{P_t Y_t} + \frac{Q_t^R R_t}{P_t Y_t} + \frac{Q_t^U U_t}{P_t Y_t} + \frac{P_t S_t}{P_t Y_t} \\ & = (1 - \pi_t - g_t) \left[\frac{M_{t-1}}{P_{t-1} Y_{t-1}} + \frac{\left(1 + i_{t-1,t}^B\right) Q_{t-1}^B B_{t-1}}{P_{t-1} Y_{t-1}} + \frac{\left(1 + i_{t-1,t}^R\right) Q_{t-1}^R R_{t-1}}{P_{t-1} Y_{t-1}} + \frac{\left(1 + i_{t-1,t}^U\right) Q_{t-1}^U U_{t-1}}{P_{t-1} Y_{t-1}} \right] \end{aligned} \tag{6}$$

Which can be rearranged as

$$\underbrace{\frac{M_t + Q_t^B B_t + Q_t^R R_t + Q_t^U U_t}{P_t Y_t} - \frac{M_{t-1} + Q_{t-1}^B B_{t-1} + Q_{t-1}^R R_{t-1} + Q_{t-1}^U U_{t-1}}{P_{t-1} Y_{t-1}}}_{\text{change in debt-to-GDP}}$$

$$= \underbrace{i_{t-1,t}^B \frac{Q_{t-1}^B B_{t-1}}{P_{t-1} Y_{t-1}}}_{\text{return on Treasuries}} + \underbrace{i_{t-1,t}^R \frac{Q_{t-1}^R R_{t-1}}{P_{t-1} Y_{t-1}}}_{\text{return on reserves}} + \underbrace{i_{t-1,t}^U \frac{Q_{t-1}^U U_{t-1}}{P_{t-1} Y_{t-1}}}_{\text{return on reverse repos}} + \underbrace{-\frac{P_t S_t}{P_t Y_t}}_{\text{primary deficit}}$$

$$\underbrace{- \pi_t \frac{M_{t-1} + Q_{t-1}^B B_{t-1} + Q_{t-1}^R R_{t-1} + Q_{t-1}^U U_{t-1}}{P_{t-1} Y_{t-1}}}_{\text{inflation}} \underbrace{- g_t \frac{M_{t-1} + Q_{t-1}^B B_{t-1} + Q_{t-1}^R R_{t-1} + Q_{t-1}^U U_{t-1}}{P_{t-1} Y_{t-1}}}_{\text{real growth}}$$

$$\underbrace{- i_{t-1,t}^B (\pi_t + g_t) \frac{Q_{t-1} B_{t-1}}{P_{t-1} Y_{t-1}} - i_{t-1,t}^R (\pi_t + g_t) \frac{Q_{t-1}^R R_{t-1}}{P_{t-1} Y_{t-1}} - i_{t-1,t}^U (\pi_t + g_t) \frac{Q_{t-1}^U U_{t-1}}{P_{t-1} Y_{t-1}}}_{\text{residual}} \quad (7)$$

and where the rewritten GBC (7) can be cumulatively constructed for consecutive time periods t. The terms in underbraces are plotted in the figure.

2 Data

The data come from:

- B_t: Par value of privately-held Treasury securities. Primary source: `par_treasury_debt` from Dallas Fed's Market Value of U.S. Government Debt; fallback: FRED series `FDHBPIN` (Federal Debt Held by Private Investors).
- R_t: Interest-bearing reserves outstanding. Primary source: FRED series `WRBWFRBL` (Reserve Balances with Federal Reserve Banks, Weekly); fallback: `LDMB` (Depository Institution Deposits at Federal Reserve Banks).
- U_t: Fed-issued overnight reverse repurchase agreements outstanding. Primary source: FRED

series `WLRRAL` (Reverse Repurchase Agreements, Weekly); fallback: `LOLRPA` (Reverse Repurchase Agreements).

- M_t: Non-interest-bearing Federal Reserve liabilities, calculated as total Fed liabilities minus reserves minus reverse repos. Total Fed liabilities from FRED series `WLTLECL` (Federal Reserve Total Liabilities, Weekly); fallback: `LTOTL` (Federal Reserve Total Liabilities).

- Q_t^B: Market price of Treasury securities, calculated as the ratio of market value to par value. Market value from FRED series `MVPHGFD027MNFRBDAL` (Market Value of Privately Held Federal Debt, Dallas Fed); par value as specified above for B_t.

- Q_t^R: Price of interest-bearing reserves, calculated as $1/(1 + r_t^R/100)$ where r_t^R is the interest rate on reserves. Primary source: FRED series `IORB` (Interest Rate on Reserve Balances); fallback: `IORR` (Interest Rate on Required Reserves).

- Q_t^U: Price of reverse repos, calculated as $1/(1 + r_t^U/100)$ where r_t^U is the reverse repo award rate. Source: FRED series `RRPONTSYAWARD` (Overnight Reverse Repurchase Agreement Award Rate).

- P_t: GDP deflator (implicit price deflator for GDP). Source: Bureau of Economic Analysis NIPA Table T10109 (Real Gross Domestic Product, Implicit Prices).

- S_t: Real primary surplus, calculated as the residual term required to satisfy the government budget constraint equation (7). Not directly observed but implied by the debt dynamics and other budget constraint components.

Note: All Federal Reserve (FRED) series are accessed via the FRED API (`https://fred.stlouisfed.org/`). BEA data are accessed via the BEA API (`https://apps.bea.gov/api/`). Historical Fed balance sheet data for 1947-2011 are sourced from archived Federal Reserve statistical releases. All data are converted to quarterly frequency.

References

Hall, George J. and Thomas J. Sargent. 2022. "Three World Wars: Fiscal-Monetary Consequences." *Proceedings of the National Academy of Sciences* 119(18).