Maths for CS – Assignment 2b

Module Code 55-402612

The function f: N → N is defined recursively by the formula

$$f(n) = \begin{cases} 1 & n = 0 \\ 3f(n-1) - 1 & otherwise \end{cases}$$

a) Use the above definition to calculate f(1), f(2), f(3), f(4), and f(5).

i. f(1) = 1 = 1
ii. f(2) = 3f(2-1)-1 = 3f(1)-1 = 3(1)-1 = 3-1 = 2
iii. f(3) = 3f(3-1)-1 = 3f(2)-1 = 3(2)-1 = 6-1 = 5
iv. f(4) = 3f(4-1)-1 = 3f(3)-1 = 3(5)-1 = 15-1 = 14
v. f(5) = 3f(5-1)-1 = 3f(4)-1 = 3(14)-1 = 42-1 = 41

n	1	2	3	4	5
f(n)	1	2	5	14	41

b) Prove by induction that

$$f(n) = \frac{3^n + 1}{2}$$

for all n ∈ N.

i. Check that it works for the base case:

$$f(0) = \frac{3^0 + 1}{2} = \frac{1 + 1}{2} = \frac{2}{2} = 1$$

ii. Assume that

$$f(k) = \frac{3^k + 1}{2}$$

for some $k \geq 0$

iii. We want

$$f(k+1) = \frac{3^{k+1} + 1}{2}$$

iv. $f(k+1) = 3f\big((k+1) - 1\big) - 1$

$f(k+1) = 3f(k) - 1$

$f(k+1) = 3\left(\dfrac{3^k + 1}{2}\right) - 1$

$f(k+1) = \dfrac{3(3^k) + 3}{2} - 1$

$f(k+1) = \dfrac{3^{k+1} + 3}{2} - 1$

$f(k+1) = \dfrac{3^{k+1} + 3}{2} - \dfrac{2}{2}$

$\boldsymbol{f(k+1) = \dfrac{3^{k+1} + 1}{2}}$

c) Another function g: N → Z is defined by g(n) = 2n − 1. State (g ∘ f)(n) for n ∈ N.

n	1	2	3	4	5
f(n)	1	2	5	14	41
(g ∘ f)(n)	1	3	9	27	81

$$(g \circ f)(n) = 3^{n-1}$$

d) A further function h: Z → R is defined by h(n) = n^2. State (h ∘ g)(n) for n ∈ N.

n	1	2	3	4	5
f(n)	1	2	5	14	41
(g ∘ f)(n)	1	3	9	27	81
(h ∘ g)(n)	1	9	81	729	6561

$$(h \circ g)(n) = 9^{n-1}$$

e) Hence, show that ((h ∘ g) ∘ f) = (h ∘ (g ∘ f)).

Let n = 3
f(n) = 5
h o g(f(n)) = 9^{5-1} = 6561

g o f(n) = 81
h(g o f(n)) = 81^2 = 6561