Calculus 3 notes

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Vector Functions

Dot product:

$$a \cdot b = |a||b|cos(\theta).$$

Cross product: To find the cross product of a and b find the determinant of 3x3 matrix with unit vectors: $\mathbf{i}, \mathbf{j}, \mathbf{k}$ for the first row. Additionally:

$$|a \times b| = |a|b|sin(\theta).$$

The vector equation of a line starting at point P, (a, b, c), with parralel vector $\vec{v}$, $\langle d, e, f \rangle$, and paramater t:

$$r(t) = P + t\vec{v} = \langle a + dt, b + et, c + ft \rangle .$$

Given a constant point on the plane P_0, (x_0, y_0, z_0) and a point on the plane P, (x, y, z), the vector $\vec{P_0 P}$ must be orthogonal to the normal vector of the plane N, $\langle a, b, c \rangle$. Therefore,

$$N \cdot \vec{P_0 P} = 0,$$

and

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

Given vector function $r(t) = \langle f(t), g(t), h(t) \rangle$,

$$r'(t) = \langle f'(t), g'(t), h'(t) \rangle ,$$

and

$$R(t) = \langle F(t), G(t), H(t) \rangle .$$

Length of curve from a to b is:

$$\int_a^b |r'(t)|dt,$$

$$s(t) = \int_0^t |r'(x)|dx.$$

An equation $y = f(x)$ can be paramaterized as $r(t) = \langle t, f(t) \rangle$. The unit tangent vector: $T(t) = r'(t)/|r'(t)|$. The principle unit normal vector: $N(t) = T'(t)/|T'(t)|$. The binomial vector: $B = T \times N$. The osculating plane is the one formed by T and N. Curvature:

$$\kappa(x) = |\frac{dT}{ds}| = \frac{|T'(t)|}{|r'(t)|} = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3}$$

Torsion:

$$\tau(x) = -N \cdot \frac{dB}{ds} = \frac{(r'(t) \times r''(t)) \cdot r'''(t)}{|r'(t) \cdot r''(t)|}.$$

Working backwords from an acceleration vector, $\vec{a}$, we can derive a position vector function of an object affected by gravity.

$$\vec{a} = \langle 0, -9.8 \rangle .$$

Given an initial velocity vector, $\vec{v_0}$ with speed, s, and angle, α:

$$\vec{v_0} = \langle s\cos(\alpha), s\sin(\alpha) \rangle ,$$

$$v(t) = \langle s\cos(\alpha), -9.8t + s\sin(\alpha) \rangle ,$$

and

$$r(t) = \left\langle s\cos(\alpha)t, -4.9t^2 + s\sin(\alpha)t \right\rangle ,$$

$$x = s\cos(\alpha)t, y = -4.9t^2 + s\sin(\alpha)t$$

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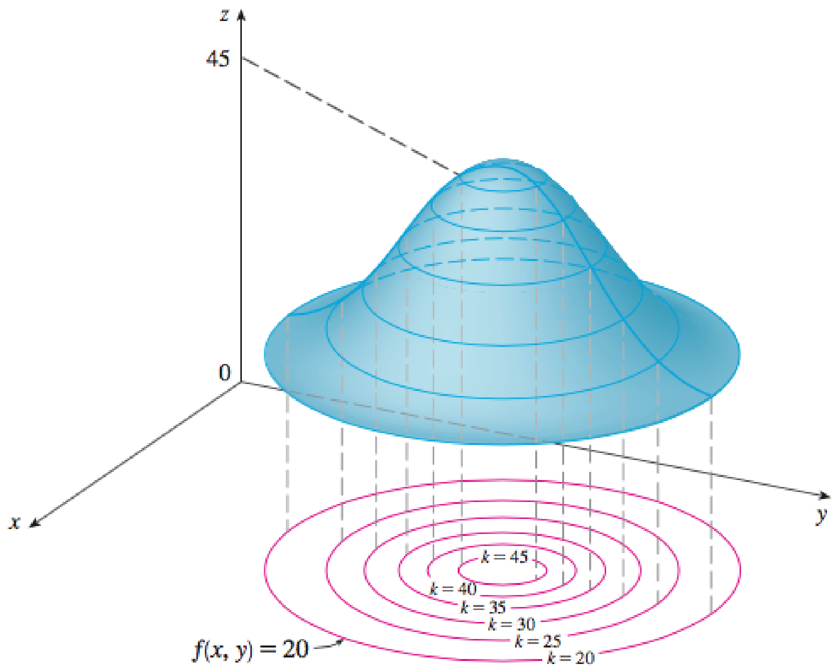
Partial Derivatives

Domain for a function of two variables can be written as:

$$D = \{(x, y) \mid x + y! = 2, xy < -1\}.$$

Similar to contour drawings on maps, level curves of a function f of two variables are the curves with equations $f(x, y) = k$ where k is a constant.



Functions of three variables are the same as functions of two variables. Functions of multiple variables can be packaged into a function of one vector variable. For multi variable limits, the notation is:

$$\lim_{(x,y)\to(a,b)} f(x,y) = L.$$

Since the limit of a multi variable function can be approached from any way, the definition of what makes a limit not exist has to be redefined. If $f(x,y) \to L_1$ as $(x,y) \to (a,b)$ along a path C_1 and $f(x,y) \to L_2$ as $(x,y) \to (a,b)$ along a path C_2 and $L_1 \neq L_2$, then $\lim_{(x,y)\to(a,b)} f(x,y)$ does not exist. The easiest paths to evaluate are often the x and y-axis. If evaluating on the x-axis (or y-axis) substitute in $y = b$ (or $x = a$) and then solve

$$\lim_{(x,y)\to(a,b)} f(x,b),$$

or

$$\lim_{(x,y)\to(a,b)} f(a,y).$$

Substituting in $y = mx$ is also an easy path to evaluate. A polynomial function of two variables is in the form:

$$f(x,y) = cx^n y^n.$$

A rational polynomial is the ratio of two polynomials. Since a polynomial is continous everywhere, a polynomial p has the property:

$$\lim_{(x,y)\to(a,b)} p(x,y) = p(a,b).$$

Similarly a rational function q has the same property provided $r(a,b) \neq 0$:

$$\lim_{(x,y)\to(a,b)} q(x,y) = \frac{p(a,b)}{r(a,b)} = q(a,b).$$

Squeeze theorem applies to multivariable functions also. Additionally, the epsilon-delta definition of a limit is useful in proving that the limit is a value. The definition states that for every $\epsilon > 0$ there exists a $\delta > 0$. δ bounds the domain such that:

$$0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta,$$

and ϵ bounds the range such that:

$$|f(x,y) - L| < \epsilon.$$

It's often easiest to start with the epsilon inequality first, and finding a relation between delta and epsilon such as $\delta = \epsilon/3$ completes the proof. The epsilon-delta definition extends to vector functions:

$$0 < |\vec{x} - \vec{a}| < \delta,$$

and

$$|f(\vec{x}) - L| < \epsilon.$$

A multivariable function is defined as continous if

$$\lim (x,y) \to (a,b) f(x,y) = f(a,b).$$

If two functions f and g are continuous then their composition function $h = f \circ g = f(g(x,y))$ is also continous. Fix $y = b$ so that $g(x) = f(x,b)$. The derivative of f with respect to x at a is denoted by:

$$g'(a) = f_x(a,b) = \frac{\partial f}{\partial x}.$$

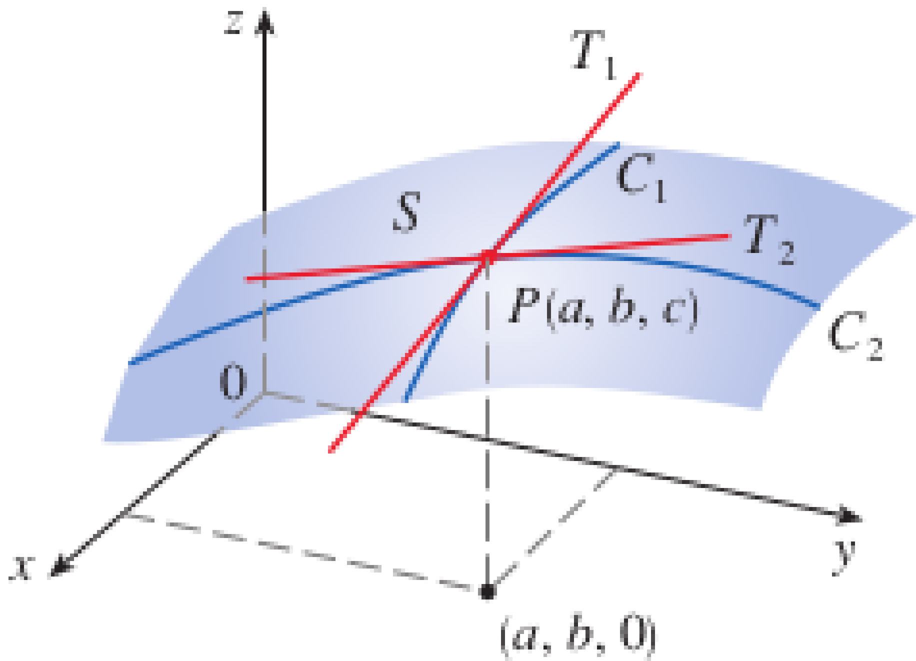
Therefore, the limit definition becomes:

$$f_x(a,b) = \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h},$$

or

$$f_y(a,b) = \lim_{h\to 0} \frac{f(a,b+h) - f(a,b)}{h}.$$

Visually:



When implicity differentiating equations with respect to x such as:

$$x^3 + y^3 + z^3 + 6xyz = 0,$$

where $z = f(x, y)$. Treat y as a constant and z as a function of x. Therefore, the derivative would be:

$$3x^2 + 3z^2\frac{\partial z}{\partial x} + 6yz + 6xy\frac{\partial z}{\partial x} = 0.$$

Partial derivatives of functions of three variables are the same. Second order partial derivatives are denoted as:

$$(f_x)_x = \frac{\partial^2 f}{\partial x^2},$$

and partial derivatives of order n are denoted as:

$$\frac{\partial^n f}{\partial x^n}.$$

Clairaut's theorem states that if f is defined on a domain disk D that contains the point (a, b) then

$$f_{xy}(a, b) = f_{yx}(a, b).$$