

Applied Linear Algebra Notes

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1

1.1

Elementary Row Operations

1. Interchange two rows.
2. Multiply the elements of a row by a nonzero constant.
3. Add a multiple of the elements of one row to the corresponding elements of another.

Gauss-Jordan elimination is a systematic way to eliminate variables to arrive at a solution matrix. The equation is solved when the matrix is diagonalized. Create zeros in each column until finished. Finally, normalize the final element ((3,3) for a 3x3 matrix, etc.). Interchange rows if a diagonal element is zero. Final matrix is called the reduced echelon form. The formula can be used for multiple constants with the same coefficient matrix. To do this, just stack the constants in each column:

$$\begin{aligned}x - y + 3z &= b_1 \\ 2x - y + 4z &= b_2 \\ -x + 2y - 4z &= b_3\end{aligned}$$

for

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 11 \\ -11 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 4 \end{bmatrix}$$

would produce the augmented matrix

$$\begin{bmatrix} 1 & -1 & 3 & 8 & 0 & 3 \\ 2 & -1 & 4 & 11 & 1 & 3 \\ -1 & 2 & -4 & -11 & 2 & 4 \end{bmatrix}$$

which simplifies to

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & -2 \\ 0 & 1 & 0 & -1 & 3 & 1 \\ 0 & 0 & 1 & 2 & 1 & 2 \end{bmatrix}.$$

1.2

A matrix is in reduced echelon form if:

1. Any rows consisting entirely of zeros are grouped at the bottom of the matrix.
2. The first nonzero element of each other row is 1. This element is called a leading 1.
3. The leading 1 of each row after the first is positioned to the right of the leading 1 of the previous row.
4. All other elements in a column that contain a leading 1 are zero.

The reduced echelon of a matrix is unique. To express multiple solutions, write the leading variables in each equation (variables with the lowest subscript: x_1 , etc.) in terms of the remaining free variables. If last row has all zeros apart from constant, there are no solutions for the system. A system of linear equations is said to be homogeneous if all the constants are zero. A homogeneous system of linear equations always has a solution of all zeros. A homogenous system of linear equations that has more variables than equations (more rows than columns) has many solutions.

1.7

Curve fitting a polynomial involves estimating variables, which can be done by solving a system of linear equations. Kirchhoff's Laws state:

1. All the current flowing into a junction must flow out of it
2. The sum of the IR terms (I denotes current, R resistance) in any direction around a closed path is equal to the total voltage in the path in that direction.

Using these laws a system of linear equations can be formed to determine the current flowing through certain points. The same procedure can be used with traffic.

1.3

\mathbb{R}^n is closed under addition and scalar multiplication if the result is also in \mathbb{R}^n . To determine if \mathbf{r} is a linear combination of \mathbf{u} , \mathbf{v} , and \mathbf{w} , determine if this system has a solution:

$$c_1 \mathbf{u}_1 + c_2 \mathbf{v}_1 + c_3 \mathbf{w}_1 = \mathbf{r}_1$$

$$c_1 \mathbf{u}_2 + c_2 \mathbf{v}_2 + c_3 \mathbf{w}_2 = \mathbf{r}_2$$

$$c_1 \mathbf{u}_3 + c_2 \mathbf{v}_3 + c_3 \mathbf{w}_3 = \mathbf{r}_3.$$

The norm of a vector \mathbf{r} is $\|\mathbf{r}\|$, and it denotes the magnitude of a vector. In \mathbb{R}^n , $\|\mathbf{r}\| = \sqrt{\mathbf{r}^n}$.

1.4

Subspaces are subsets of \mathbb{R}^n that have all the same properties (subset must be closed under addition and multiplication). An example for demonstrating a subset in \mathbb{R}^3 with the form $(a, 2a, 3a)$ is closed for addition:

$$\begin{aligned}(a, 2a, 3a) + (b, 2b, 3b) &= (a + b, 2a + 2b, 3a + 3b) \\ &= (a + b, 2(a + b), 3(a + b)),\end{aligned}$$

and multiplication:

$$\begin{aligned}k(a, 2a, 3a) &= (ka, 2ka, 3ka) \\ &= (ka, 2(ka), 3(ka)).\end{aligned}$$

When every vector of a vector space can be written as a linear combination of a set of vectors, the vectors span the space. For example, the subspace in \mathbb{R}^3 defined by $(a, b, a + b)$ is spanned by the vectors $(1, 0, 1)$ and $(0, 1, 1)$ because $(a, b, a + b) = a(1, 0, 1) + b(0, 1, 1)$. Generally, vectors that make up a line in \mathbb{R}^n are subspaces. The set of solutions to any homogenous system of linear equations is a subspace.

1.5

The basis is the finite set of vectors that span a whole space. The basis must also be linearly independent. The vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ are linearly independent if the identity $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_n\mathbf{v}_n = \mathbf{0}$ is only true for $c_1 = 0, c_2 = 0, \dots, c_n = 0$. If the vectors are not linearly independent, they are linearly dependent. Vectors can span the space but not be linearly independent. For example, the vectors $(0, 1), (1, 0)$, and $(1, 1)$ span \mathbb{R}^2 , but the set of the vectors is not a basis for \mathbb{R}^2 because they are not linearly independent. Additionally, \mathbb{R}^2 is not a subspace of \mathbb{R}^3 . All subspaces must pass through the origin because they must contain the $\mathbf{0}$ vector.

2

2.1

Matrix notation:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

The main diagonal consists of a_{11}, a_{22} , and a_{33} . Addition involves adding the corresponding elements of each matrix. Both matrices must be the same size. Scalar multiplication involves multiplying every element. Matrix multiplication ($C = AB$) requires that the number of columns in matrix A , is the same as the

number of rows in matrix B . The product is equal to the product of the rows of A and the columns of B :

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}.$$

For example,

$$\begin{aligned} AB &= \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 5 & 0 & 1 \\ 3 & -2 & 6 \end{bmatrix} \\ &= \begin{bmatrix} \begin{bmatrix} 1 & 3 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \end{bmatrix} & \begin{bmatrix} 1 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ -2 \end{bmatrix} & \begin{bmatrix} 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 6 \end{bmatrix} \\ \begin{bmatrix} 2 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \end{bmatrix} & \begin{bmatrix} 2 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -2 \end{bmatrix} & \begin{bmatrix} 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 6 \end{bmatrix} \end{bmatrix} \\ &= \begin{bmatrix} (1)(5) + (3)(3) & (1)(0) + (3)(-2) & (1)(1) + (3)(6) \\ (2)(5) + (0)(3) & (2)(0) + (0)(-2) & (2)(1) + (0)(6) \end{bmatrix} \\ &= \begin{bmatrix} 14 & -6 & 19 \\ 10 & 0 & 2 \end{bmatrix}. \end{aligned}$$

Matrix multiplication is not commutative, and BA does not exist. If A is an $m \times r$ matrix and B is an $r \times n$ matrix, then AB is an $m \times n$ matrix. Three classes of matrices are notable: The zero matrix O which contains all zeros, the diagonal matrix which contains zeros everywhere except the main diagonal, and the identity matrix I which is a diagonal matrix with ones on the main diagonal. If B_1 is column one of matrix B , then $AB = [AB_1, AB_2, \dots, AB_n]$. Matrices can be partitioned into blocks, and these blocks can be multiplied.

2.2

For matrix multiplication,

1. $A(BC) = (AB)C$
2. $A(B + C) = AB + AC$
3. $(A + B)C = AC + BC$

The product of a chain of matrices will have the same number of rows as the first matrix and the same number of columns as the last matrix. For matrices, $AB = AC$ doesn't imply $B = C$, and $PQ = O$ doesn't imply $P = O$ or $Q = O$. If A is an $n \times n$ square matrix, and r and s are nonnegative integers, then

1. $A^r A^s = A^{r+s}$
2. $(A^r)^s = A^{rs}$
3. $A^0 = I_n$

Consider a homogenous system of linear equations $AX = 0$. Let X_1 and X_2 be solutions of the system. Then $AX_1 = 0$ and $AX_2 = 0$, so $AX_1 + AX_2 = A(X_1 + X_2) = 0$, and $X_1 + X_2$ is also a solution. Furthermore, if c is any scalar, then $AX_1 = 0$ and $c(AX_1) = A(cX_1) = 0$, so cX_1 is also a solution.

The corresponding homogeneous system of a matrix is one in which the right-hand side of the equations is zero, and the solution of the corresponding nonhomogeneous system is the solution of the homogeneous system plus a particular solution of the nonhomogeneous system.

2.3

The transpose of a matrix A , denoted A^T , is the matrix whose columns are the rows of the given matrix A . Let A and B be matrices and c be a scalar. There are several properties of the transpose:

1. $(A + B)^T = A^T + B^T$
2. $(cA)^T = cA^T$
3. $(AB)^T = B^T A^T$
4. $(A^T)^T = A$

A symmetrical matrix is a matrix that is equal to its transpose. The trace of A , denoted $tr(A)$, is the sum of the elements on the main diagonal. There are several properties of the trace:

1. $tr(A + B) = tr(A) + tr(B)$
2. $tr(AB) = tr(BA)$
3. $tr(cA) = ctr(A)$
4. $tr(A^T) = tr(A)$

Matrices can also contain complex numbers, and the conjugate of a matrix A is denoted \bar{A} . Furthermore, the conjugate transpose of a matrix A is denoted A^* , and a square matrix C is called Hermitian if $C = C^*$. There are several properties of the conjugate transpose:

1. $(A + B)^* = A^* + B^*$
2. $(zA)^* = \bar{z}A^*$
3. $(AB)^* = B^* A^*$
4. $(A^*)^* = A$