

# Calculus 3 notes

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## Vector Functions

Dot product:

$$a \cdot b = |a||b|\cos(\theta).$$

Cross product: To find the cross product of a and b find the determinant of 3x3 matrix with unit vectors:  $\mathbf{i}, \mathbf{j}, \mathbf{k}$  for the first row. Additionally:

$$|a \times b| = |a||b|\sin(\theta).$$

The vector equation of a line starting at point  $P$ ,  $(a, b, c)$ , with parallel vector  $\vec{v}$ ,  $\langle d, e, f \rangle$ , and parameter  $t$ :

$$r(t) = P + t\vec{v} = \langle a + dt, b + et, c + ft \rangle.$$

Given a constant point on the plane  $P_0$ ,  $(x_0, y_0, z_0)$  and a point on the plane  $P$ ,  $(x, y, z)$ , the vector  $\vec{P_0P}$  must be orthogonal to the normal vector of the plane  $N$ ,  $\langle A, B, C \rangle$ . Therefore,

$$N \cdot \vec{P_0P} = 0,$$

and

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0.$$

Given vector function  $r(t) = \langle f(t), g(t), h(t) \rangle$ ,

$$r'(t) = \langle f'(t), g'(t), h'(t) \rangle,$$

and

$$R(t) = \langle F(t), G(t), H(t) \rangle.$$

Length of curve from a to b is:

$$\int_a^b |r'(t)| dt,$$

$$s(t) = \int_0^t |r'(x)| dx.$$

An equation  $y = f(x)$  can be paramaterized as  $r(t) = \langle t, f(t) \rangle$ .

The unit tangent vector:  $T(t) = r'(t)/|r'(t)|$ . The principle unit normal vector:  $N(t) = T'(t)/|T'(t)|$ . The binomial vector:  $B = T \times N$ . The osculating plane is the one formed by  $T$  and  $N$ . Curvature:

$$\kappa(x) = \left| \frac{dT}{ds} \right| = \frac{|T'(t)|}{|r'(t)|} = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3}$$

Torsion:

$$\tau(x) = -N \cdot \frac{dB}{ds} = \frac{(r'(t) \times r''(t)) \cdot r'''(t)}{|r'(t) \cdot r''(t)|}.$$

Working backwards from an acceleration vector,  $\vec{a}$ , we can derive a position vector function of an object affected by gravity.

$$\vec{a} = \langle 0, -9.8 \rangle.$$

Given an initial velocity vector,  $\vec{v}_0$  with speed,  $s$ , and angle,  $\alpha$ :

$$\vec{v}_0 = \langle s \cos(\alpha), s \sin(\alpha) \rangle,$$

$$v(t) = \langle s \cos(\alpha), -9.8t + s \sin(\alpha) \rangle,$$

and

$$r(t) = \langle s \cos(\alpha)t, -4.9t^2 + s \sin(\alpha)t \rangle,$$

$$x = s \cos(\alpha)t, y = -4.9t^2 + s \sin(\alpha)t$$

## Partial Derivatives

Domain for a function of two variables can be written as:

$$D = \{(x, y) \mid x + y \neq 2, xy < -1\}.$$

Similar to contour drawings on maps, level curves of a function  $f$  of two variables are the curves with equations  $f(x, y) = k$  where  $k$  is a constant.



Functions of three variables are the same as functions of two variables and functions of multiple variables can be packaged into a function of one vector variable.

For multivariable limits, the notation is:

$$\lim_{(x,y) \rightarrow (a,b)} f(x,y) = L.$$

Since the limit of a multi variable function can be approached from any way, the definition of what makes a limit not exist has to be redefined. If  $f(x,y) \rightarrow L_1$  as  $(x,y) \rightarrow (a,b)$  along a path  $C_1$  and  $f(x,y) \rightarrow L_2$  as  $(x,y) \rightarrow (a,b)$  along a path  $C_2$  and  $L_1 \neq L_2$ , then  $\lim_{(x,y) \rightarrow (a,b)} f(x,y)$  does not exist. The easiest paths to evaluate are often the x and y-axis. If evaluating on the x-axis (or y-axis) substitute in  $y = b$  (or  $x = a$ ) and then solve

$$\lim_{(x,y) \rightarrow (a,b)} f(x,b),$$

or

$$\lim_{(x,y) \rightarrow (a,b)} f(a,y).$$

Substituting in  $y = mx$  is also an easy path to evaluate. A polynomial function of two variables is in the form:

$$f(x,y) = cx^ny^n.$$

A rational polynomial is the ratio of two polynomials. Since a polynomial is continuous everywhere, a polynomial  $p$  has the property:

$$\lim_{(x,y) \rightarrow (a,b)} p(x,y) = p(a,b).$$

Similarly a rational function  $q$  has the same property provided  $r(a,b) \neq 0$ :

$$\lim_{(x,y) \rightarrow (a,b)} q(x,y) = \frac{p(a,b)}{r(a,b)} = q(a,b).$$

Squeeze theorem applies to multivariable functions also. Additionally, the epsilon-delta definition of a limit is useful in proving that the limit is a value. The definition states that for every  $\epsilon > 0$  there exists a  $\delta > 0$ .  $\delta$  bounds the domain such that:

$$0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta,$$

and  $\epsilon$  bounds the range such that:

$$|f(x,y) - L| < \epsilon.$$

It's often easiest to start with the epsilon inequality first, and finding a relation between delta and epsilon such as  $\delta = \epsilon/3$  completes the proof. The epsilon-delta definition extends to vector functions:

$$0 < |\vec{x} - \vec{a}| < \delta,$$

and

$$|f(\vec{x}) - L| < \epsilon.$$

A multivariable function is defined as continuous if

$$\lim_{(x,y) \rightarrow (a,b)} f(x,y) = f(a,b).$$

If two functions  $f$  and  $g$  are continuous then their composition function  $h = f \circ g = f(g(x,y))$  is also continuous. Fix  $y = b$  so that  $g(x) = f(x,b)$ . The derivative of  $f$  with respect to  $x$  at  $a$  is denoted by:

$$g'(a) = f_x(a,b).$$

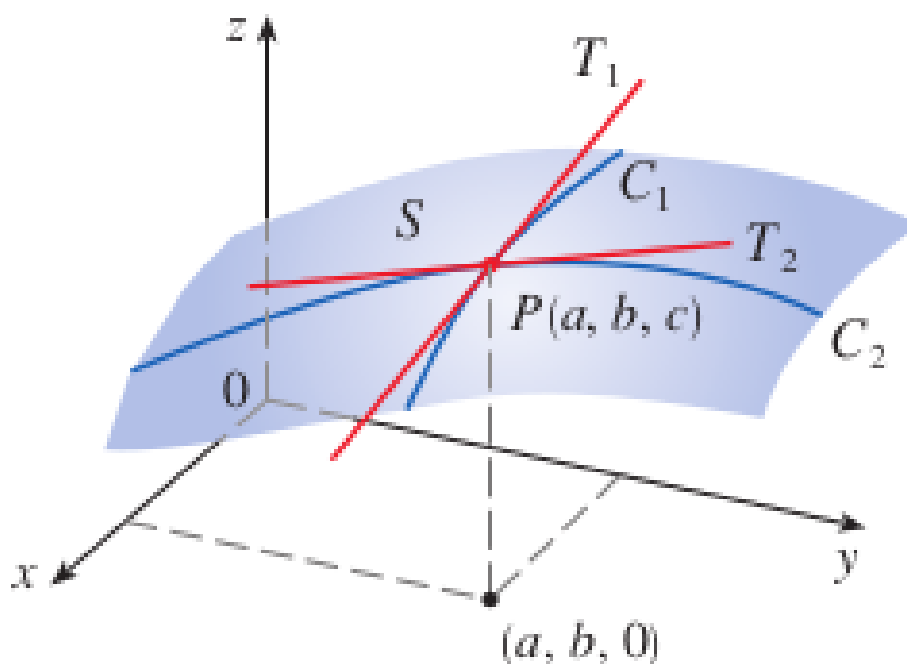
Therefore, the limit definition becomes:

$$f_x(a,b) = \lim_{h \rightarrow 0} \frac{f(a+h,b) - f(a,b)}{h},$$

or

$$f_y(a,b) = \lim_{h \rightarrow 0} \frac{f(a,b+h) - f(a,b)}{h}.$$

Visually:



When implicitly differentiating equations with respect to  $x$  such as:

$$x^3 + y^3 + z^3 + 6xyz = 0,$$

where  $z = f(x, y)$ . Treat  $y$  as a constant and  $z$  as a function of  $x$ . Therefore, the derivative would be:

$$3x^2 + 3z^2 \frac{\partial z}{\partial x} + 6yz + 6xy \frac{\partial z}{\partial x} = 0.$$

Partial derivatives of functions of three variables are the same. Second order partial derivatives are denoted as:

$$(f_x)_x = \frac{\partial^2 f}{\partial x^2},$$

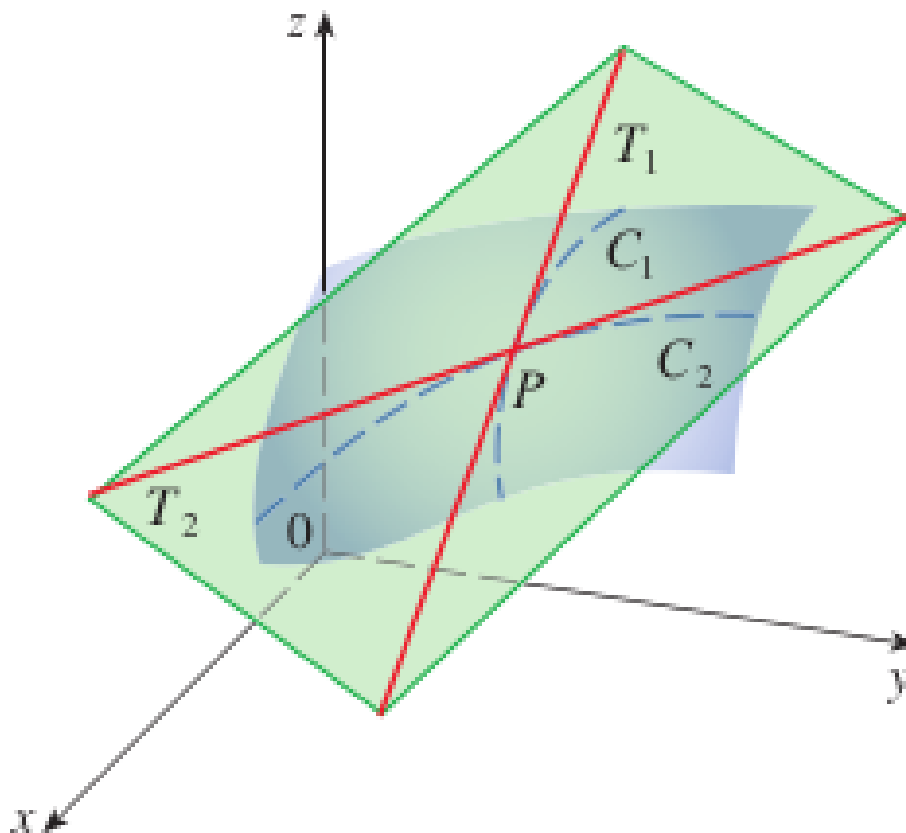
and partial derivatives of order  $n$  are denoted as:

$$\frac{\partial^n f}{\partial x^n}.$$

Clairaut's theorem states that if  $f$  is defined on a domain disk  $D$  that contains the point  $(a, b)$  then

$$f_{xy}(a, b) = f_{yx}(a, b).$$

Tangent plane contains both tangent lines from the derivative with respect to  $x$  and the derivative with respect to  $y$ .



Another equation of a plane is:

$$z - z_0 = a(x - x_0) + b(y - y_0),$$

where  $a = -A/C$  and  $b = -B/C$ . Remember the normal vector:  $N = \langle A, B, C \rangle$ . Since setting  $y = y_0$  or  $x = x_0$  represents the tangent line in the  $x$  or  $y$  plane respectively, the equation of a tangent plane given the derivatives with respect to  $x$  and  $y$  at the point  $(x_0, y_0, z_0)$  is:

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

This equation can be used to approximate  $z$  close to  $(x_0, y_0)$ . Specifically, this linear function:

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b),$$

is called the linearization for  $f$  at  $(a, b)$ . A two variable function is considered differentiable at  $(a, b)$  if:

$$\Delta z = f_x(a, b)\Delta x + f_y(a, b)\Delta y + \epsilon_1\Delta x + \epsilon_2\Delta y,$$

where  $\epsilon_1$  and  $\epsilon_2$  are functions of  $\Delta x$  and  $\Delta y$  such that  $\epsilon_1$  and  $\epsilon_2 \rightarrow 0$  as  $(\Delta x, \Delta y) \rightarrow (0, 0)$ . Rephrased, if the partial derivatives exist and are continuous at  $(a, b)$ , then the function is differentiable at  $(a, b)$ .

$\Delta z$  is the actual change in  $z$ , while  $dz$  is the linear approximation of the change in  $z$ :

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy.$$

One of the chain rules for multivariable functions:

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}.$$

When  $x$  and  $y$  are functions of two variables  $s$  and  $t$ ,

$$\frac{dz}{ds} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s},$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t}.$$

Finding  $y'$  from the implicitly defined equation:

$$x^3 + y^3 = 6xy.$$

We can write this equation as:

$$F(x, y) = x^3 + y^3 - 6xy = 0.$$

Knowing that  $y$  can be written as a function of  $x$  and that  $F(x, y) = 0$ , we can rewrite

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} \frac{dy}{dx} = 0,$$

as

$$\frac{dy}{dx} = -\frac{F_x}{F_y}.$$

Therefore, in the above equation

$$y' = -\frac{F_x}{F_y} = -\frac{x^2 - 2y}{y^2 - 2x}.$$

Extending this result, for the equation

$$F(x, y, z) = F(x, y, f(x, y)) = 0,$$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z},$$

and

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}.$$

The directional derivative of  $f$  at  $(x_0, y_0)$  in the direction of a unit vector  $\mathbf{u} = \langle a, b \rangle$ :

$$D_{\mathbf{u}}f(x_0, y_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}.$$

If  $f$  is a directional function of  $x$  and  $y$ , then  $f$  has a directional derivative in the direction of the unit vector  $\mathbf{u} = \langle a, b \rangle$  and

$$D_{\mathbf{u}}f(x, y) = f_x(x, y)a + f_y(x, y)b = \langle f_x(x, y), f_y(x, y) \rangle \cdot \mathbf{u}.$$

$\langle f_x(x, y), f_y(x, y) \rangle$  is called the gradient vector of  $f$  and is also denoted by  $\nabla f$ . Also defined by:

$$\nabla f(x, y) = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j}.$$

Therefore, the directional derivative can be written as:

$$D_{\mathbf{u}}f(x, y) = \nabla f(x, y) \cdot \mathbf{u}.$$

The maximum of the directional derivative  $D_{\mathbf{u}}f(\mathbf{x})$  is  $|\nabla f(\mathbf{x})|$ , and it occurs when  $\mathbf{u}$  has the same direction as the gradient vector  $\nabla f(\mathbf{x})$ .

Given a level surface  $F(x, y, z) = k$ , the gradient vector  $\nabla F$  is perpendicular to the surface, and therefore, the tangent plane to the surface can be represented by the point and the gradient vector of that point. Using the standard equation of a plane:

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0.$$

The normal line is the line perpendicular to the surface at point  $(x_0, y_0, z_0)$ , and therefore, its symmetric equations are:

$$\frac{x - x_0}{F_x(x_0, y_0, z_0)} = \frac{y - y_0}{F_y(x_0, y_0, z_0)} = \frac{z - z_0}{F_z(x_0, y_0, z_0)}.$$

For an equation defined by  $z = f(x, y)$ , we can rewrite the equation as:

$$F(x, y, z) = f(x, y) - z = 0.$$

In this situation

$$F_x(x_0, y_0, z_0) = f_x(x_0, y_0),$$

$$F_y(x_0, y_0, z_0) = f_y(x_0, y_0),$$

$$F_z(x_0, y_0, z_0) = -1.$$

If  $f$  has a local maximum or minimum at  $(a, b)$  and the first partial derivatives exist at  $(a, b)$ , then  $f_x(a, b) = 0$  and  $f_y(a, b) = 0$ . Also,  $\nabla f = \mathbf{0}$ .  $f$  has a critical point if  $f_x(a, b) = 0$  or  $f_y(a, b) = 0$  or either one of the partial derivatives don't exist. A critical point doesn't guarantee a maximum or minimum. The second derivatives test:

$$D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2.$$



If  $D > 0$  and  $f_{xx}(a, b) > 0$ , then  $(a, b)$  is a local minimum, if  $D > 0$  and  $f_{xx}(a, b) < 0$ , then  $(a, b)$  is a local maximum, and if  $D < 0$  then  $(a, b)$  is a saddle point.

A closed set in  $\mathbb{R}^2$  is one that contains its boundary points, while a bounded set in  $\mathbb{R}^2$  is one contained in a disk. In order to find maximums and minimums on a closed set, find the critical points inside the set and then find the critical points on the boundary.