Calculus 3 notes

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Vector Functions

Dot product:

$$a \cdot b = |a||b|cos(\theta).$$

Cross product: To find the cross product of a and b find the determinant of 3x3 matrix with unit vectors: $\mathbf{i}, \mathbf{j}, \mathbf{k}$ for the first row. Additionally:

$$|a \times b| = |a|b|sin(\theta).$$

The vector equation of a line starting at point P, (a, b, c), with parralel vector $\vec{v}$, $\langle d, e, f \rangle$, and paramater t:

$$r(t) = P + t\vec{v} = \langle a + dt, b + et, c + ft \rangle .$$

Given a constant point on the plane P_0, (x_0, y_0, z_0) and a point on the plane P, (x, y, z), the vector $\vec{P_0P}$ must be orthogonal to the normal vector of the plane N, $\langle A, B, C \rangle$. Therefore,

$$N \cdot \vec{P_0P} = 0,$$

and

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0.$$

Given vector function $r(t) = \langle f(t), g(t), h(t) \rangle$,

$$r'(t) = \langle f'(t), g'(t), h'(t) \rangle ,$$

and

$$R(t) = \langle F(t), G(t), H(t) \rangle .$$

Length of curve from a to b is:

$$\int_a^b |r'(t)|dt,$$

$$s(t) = \int_0^t |r'(x)|dx.$$

An equation $y = f(x)$ can be paramaterized as $r(t) = \langle t, f(t) \rangle$.

The unit tangent vector: $T(t) = r'(t)/|r'(t)|$. The principle unit normal vector: $N(t) = T'(t)/|T'(t)|$. The binomial vector: $B = T \times N$. The osculating plane is the one formed by T and N. Curvature:

$$\kappa(x) = |\frac{dT}{ds}| = \frac{|T'(t)|}{|r'(t)|} = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3}$$

Torsion:

$$\tau(x) = -N \cdot \frac{dB}{ds} = \frac{(r'(t) \times r''(t)) \cdot r'''(t)}{|r'(t) \cdot r''(t)|}.$$

Working backwords from an acceleration vector, $\vec{a}$, we can derive a position vector function of an object affected by gravity.

$$\vec{a} = \langle 0, -9.8 \rangle .$$

Given an initial velocity vector, $\vec{v_0}$ with speed, s, and angle, α:

$$\vec{v_0} = \langle s\cos(\alpha), s\sin(\alpha) \rangle ,$$

$$v(t) = \langle s\cos(\alpha), -9.8t + s\sin(\alpha) \rangle ,$$

and

$$r(t) = \left\langle s\cos(\alpha)t, -4.9t^2 + s\sin(\alpha)t \right\rangle ,$$

$$x = s\cos(\alpha)t, y = -4.9t^2 + s\sin(\alpha)t$$

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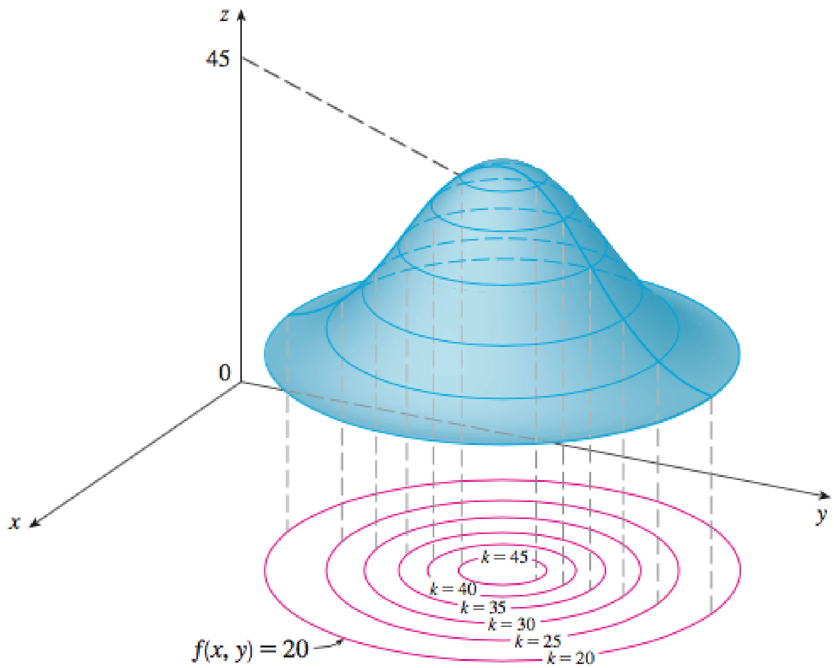
Partial Derivatives

Domain for a function of two variables can be written as:

$$D = \{(x, y) \mid x + y! = 2, xy < -1\}.$$

Similar to contour drawings on maps, level curves of a function f of two variables are the curves with equations $f(x, y) = k$ where k is a constant.



Functions of three variables are the same as functions of two variables and functions of multiple variables can be packaged into a function of one vector variable.

For multivariable limits, the notation is:

$$\lim_{(x,y)\to(a,b)} f(x,y) = L.$$

Since the limit of a multi variable function can be approached from any way, the definition of what makes a limit not exist has to be redefined. If $f(x,y) \to L_1$ as $(x,y) \to (a,b)$ along a path C_1 and $f(x,y) \to L_2$ as $(x,y) \to (a,b)$ along a path C_2 and $L_1 \neq L_2$, then $\lim_{(x,y)\to(a,b)} f(x,y)$ does not exist. The easiest paths to evaluate are often the x and y-axis. If evaluating on the x-axis (or y-axis) substitute in $y = b$ (or $x = a$) and then solve

$$\lim_{(x,y)\to(a,b)} f(x,b),$$

or

$$\lim_{(x,y)\to(a,b)} f(a,y).$$

Substituting in $y = mx$ is also an easy path to evaluate. A polynomial function of two variables is in the form:

$$f(x,y) = cx^n y^n.$$

A rational polynomial is the ratio of two polynomials. Since a polynomial is continous everywhere, a polynomial p has the property:

$$\lim_{(x,y)\to(a,b)} p(x,y) = p(a,b).$$

Similarly a rational function q has the same property provided $r(a,b) \neq 0$:

$$\lim_{(x,y)\to(a,b)} q(x,y) = \frac{p(a,b)}{r(a,b)} = q(a,b).$$

Squeeze theorem applies to multivariable functions also. Additionally, the epsilon-delta definition of a limit is useful in proving that the limit is a value. The definition states that for every $\epsilon > 0$ there exists a $\delta > 0$. δ bounds the domain such that:

$$0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta,$$

and ϵ bounds the range such that:

$$|f(x,y) - L| < \epsilon.$$

It's often easiest to start with the epsilon inequality first, and finding a relation between delta and epsilon such as $\delta = \epsilon/3$ completes the proof. The epsilon-delta definition extends to vector functions:

$$0 < |\vec{x} - \vec{a}| < \delta,$$

and

$$|f(\vec{x}) - L| < \epsilon.$$

A multivariable function is defined as continous if

$$\lim{(x,y) \to (a,b)} f(x,y) = f(a,b).$$

If two functions f and g are continuous then their composition function $h = f \circ g = f(g(x,y))$ is also continous. Fix $y = b$ so that $g(x) = f(x,b)$. The derivative of f with respect to x at a is denoted by:

$$g'(a) = f_x(a,b).$$

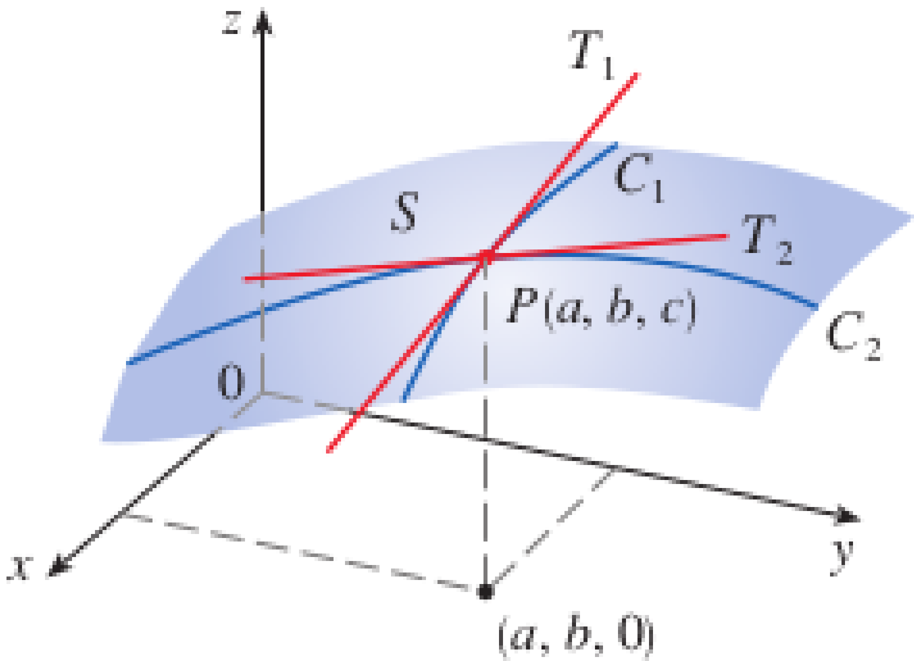
Therefore, the limit definition becomes:

$$f_x(a,b) = \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h},$$

or

$$f_y(a,b) = \lim_{h\to 0} \frac{f(a,b+h) - f(a,b)}{h}.$$

Visually:



When implicity differentiating equations with respect to x such as:

$$x^3 + y^3 + z^3 + 6xyz = 0,$$

where $z = f(x, y)$. Treat y as a constant and z as a function of x. Therefore, the derivative would be:

$$3x^2 + 3z^2\frac{\partial z}{\partial x} + 6yz + 6xy\frac{\partial z}{\partial x} = 0.$$

Partial derivatives of functions of three variables are the same. Second order partial derivatives are denoted as:

$$(f_x)_x = \frac{\partial^2 f}{\partial x^2},$$

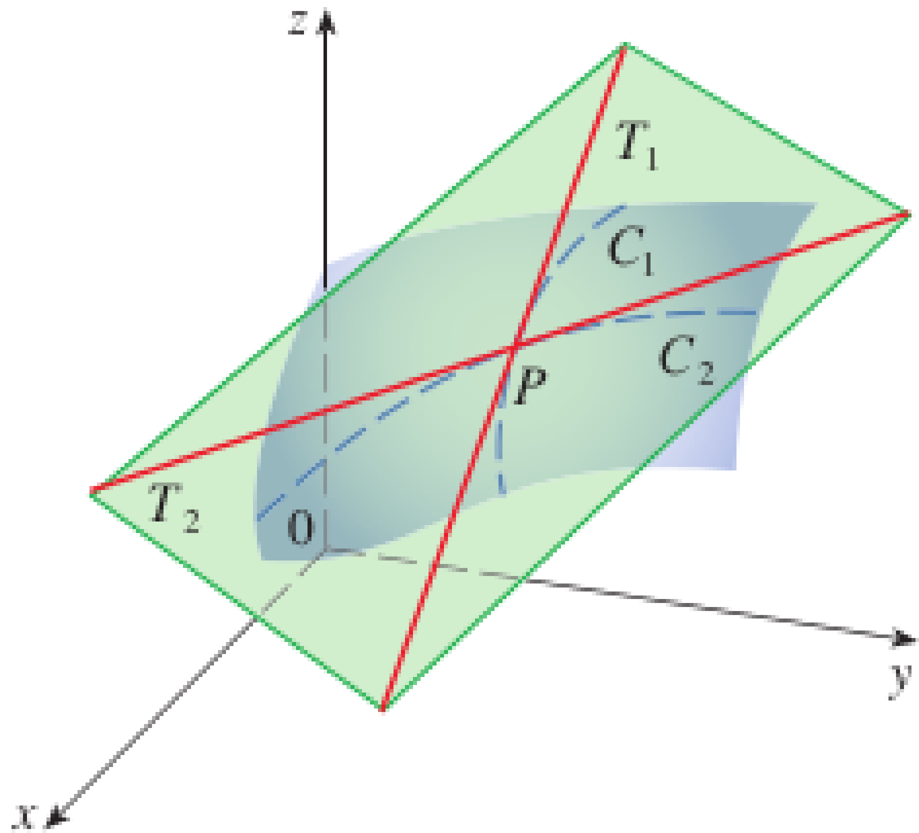
and partial derivatives of order n are denoted as:

$$\frac{\partial^n f}{\partial x^n}.$$

Clairaut's theorem states that if f is defined on a domain disk D that contains the point (a, b) then

$$f_{xy}(a, b) = f_{yx}(a, b).$$

Tangent plane contains both tangent lines from the derivative with respect to x and the derivative with respect to y.



Another equation of a plane is:

$$z - z_0 = a(x - x_0) + b(y - y_0),$$

where $a = -A/C$ and $b = -B/C$. Remember the normal vector: $N = \langle A, B, C \rangle$. Since setting $y = y_0$ or $x = x_0$ represents the tangent line in the x or y plane respectively, the equation of a tangent plane given the derivatives with respect to x and y at the point (x_0, y_0, z_0) is:

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

This equation can be used to approximate z close to (x_0, y_0). Specifically, this linear function:

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b),$$

is called the linearization for f at (a, b). A two variable function is considered differentiable at (a, b) if:

$$\Delta z = f_x(a, b)\Delta x + f_y(a, b)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y,$$

where ϵ_1 and ϵ_2 are functions of Δx and Δy such that ϵ_1 and $\epsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$. Rephrased, if the partial derivatives exist and are continous at (a,b), then the function is differentiable at (a,b).

Δz is the actual change in z, while dz is the linear approximation of the change in z:

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy.$$

One of the chain rules for multivariable functions:

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial z}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial z}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}.$$

When x and y are functions of two variables s and t,

$$\frac{\mathrm{d}z}{\mathrm{d}s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s},$$

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}.$$

Finding y' from the implicitly defined equation:

$$x^3 + y^3 = 6xy.$$

We can write this equation as:

$$F(x,y) = x^3 + y^3 - 6xy = 0.$$

Knowing that y can be written as a function of x and that $F(x,y) = 0$, we can rewrite

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x} = 0,$$

as

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{F_x}{F_y}.$$

Therefore, in the above equation

$$y' = -\frac{F_x}{F_y} = -\frac{x^2 - 2y}{y^2 - 2x}.$$

Extending this result, for the equation

$$F(x,y,z) = F(x,y,f(x,y)) = 0,$$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z},$$

and

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}.$$

The directional derivative of f at (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle a, b \rangle$:

$$D_u f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}.$$

If f is a directional function of x and y, then f has a directional derivative in the direction of the unit vector $\mathbf{u} = \langle a, b \rangle$ and

$$D_u f(x, y) = f_x(x, y)a + f_y(x, y)b = \langle f_x(x, y), f_y(x, y) \rangle \cdot \mathbf{u}.$$

$\langle f_x(x, y), f_y(x, y) \rangle$ is called the gradient vector of f and is also denoted by ∇f. Also defined by:

$$\nabla f(x, y) = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}.$$

Therefore, the directional derivative can be written as:

$$D_u f(x, y) = \nabla f(x, y) \cdot \mathbf{u}.$$

The maximum of the directional derivative $D_u f(\mathbf{x})$ is $|\nabla f(\mathbf{x})|$), and it occurs when $\mathbf{u}$ has the same direction as the gradient vector $\nabla f(\mathbf{x})$.

Given a level surface $F(x, y, z) = k$, the gradient vector ∇F is perpendicular to the surface, and therefore, the tangent plane to the surface can be represented by the point and the gradient vector of that point. Using the standard equation of a plane:

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0.$$

The normal line is the line perpendicular to the surface at point (x_0, y_0, z_0), and therefore, its symmetric equations are:

$$\frac{x - x_0}{F_x(x_0, y_0, z_0)} = \frac{y - y_0}{F_y(x_0, y_0, z_0)} = \frac{z - z_0}{F_z(x_0, y_0, z_0)}.$$

For an equation defined by $z = f(x, y)$, we can rewrite the equation as:

$$F(x, y, z) = f(x, y) - z = 0.$$

In this situation

$$\begin{aligned} F_x(x_0, y_0, z_0) &= f_x(x_0, y_0), \\ F_y(x_0, y_0, z_0) &= f_y(x_0, y_0), \\ F_z(x_0, y_0, z_0) &= -1. \end{aligned}$$

If f has a local maximum or minimum at (a, b) and the first partial derivatives exist at (a, b), then $f_x(a, b) = 0$ and $f_y(a, b) = 0$. Also, $\nabla f = \mathbf{0}$. f has a critical point if $f_x(a, b) = 0$ or $f_y(a, b)$ or either one of the partial derivatives don't exist. A critical point doesn't guarantee a maximum or minimum. The second derivatives test:

$$D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2.$$

If $D > 0$ and $f_{xx}(a,b) > 0$, then (a,b) is a local minimum, if $D > 0$ and $f_{xx}(a,b) < 0$, then (a,b) is a local maximum, and if $D < 0$ then $f(a,b)$ is a saddle point.

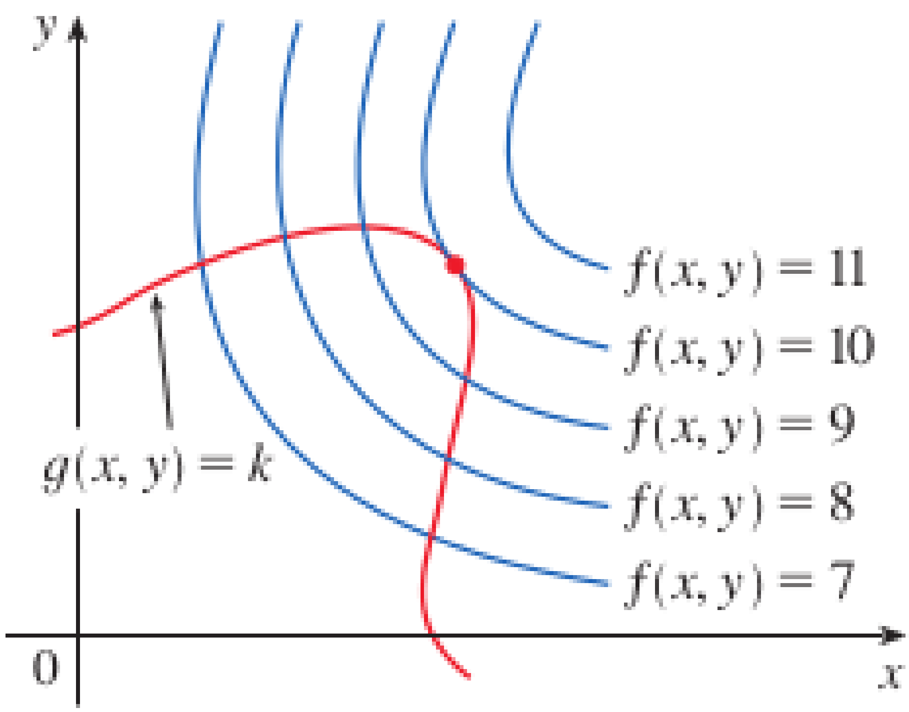
A closed set in $\mathbb{R}^2$ is one that contains its boundary points, while a bounded set in $\mathbb{R}^2$ is one contained in a disk. In order to find maximums and minimums on a closed set, find the critical points inside the set and then find the critical points on the boundary.

Use given conditions in optimization problem to simplify problem with less variables.

Lagrange's method seeks maximize $f(x,y)$ given $g(x,y) = k$. The gradient vectors at the intersection point of $f(x_0,y_0) = c$ and $g(x,y) = k$ must be multiples of each other:

$$\nabla f(x_0,y_0) = \lambda \nabla g(x_0,y_0).$$

Visually:



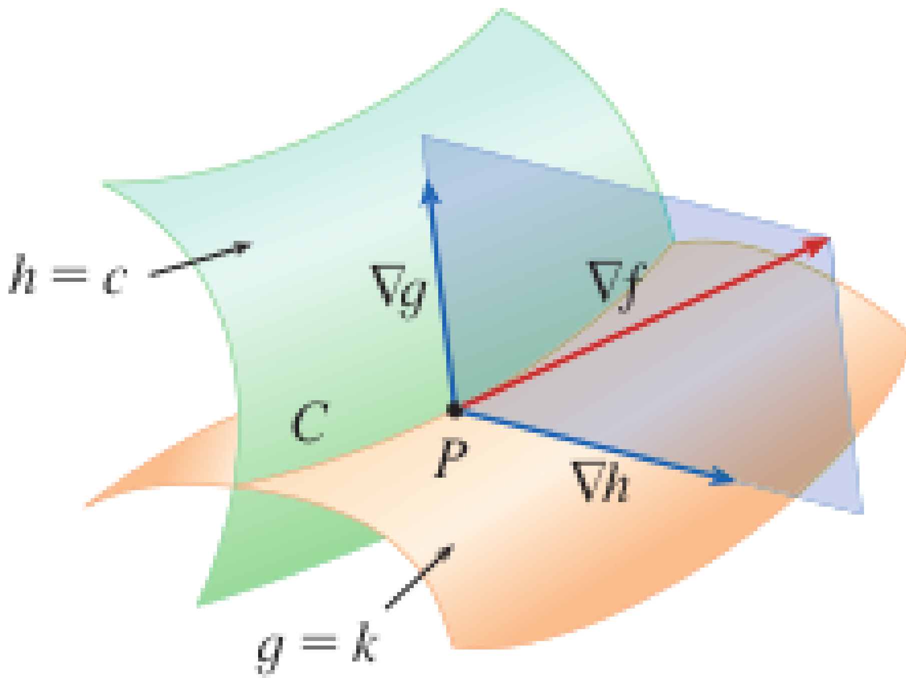
This method extends to three dimensions where $f(x,y,z)$ is maximized given $g(x,y,z) = k$. Lagrange's method also assumes that $\nabla g \neq \mathbf{0}$. If $\lambda = 0$, then $\nabla f = \mathbf{0}$ and (x_0,y_0,z_0) is a critical point. The constant λ is called a Lagrange multiplier. In terms of components:

$$\nabla f = \lambda \nabla g,$$

$$f_x = \lambda g_x, f_y = \lambda g_y, f_z = \lambda g_z.$$

Solving these equations given $g(x, y, z) = k$ solves the problem.

Lagrange's method extends to two conditions. Visually:



Therefore,

$$\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0) + \mu \nabla h(x_0, y_0, z_0).$$