A Formula for the Computation of Compound Interest With Payments

Let the accumulated value be defined by:

$$A_0 = P$$

$$\begin{aligned} A_n &= A_{n-1} + r(A_{n-1} + d) \\ &= A_{n-1}(1+r) + rd \end{aligned}$$

Here, r is the rate of interest, and d is the additional change made per cycle, i.e. payments to a loan, or deposits into a bank account.

Let the function $F : \mathbb{Z} \mapsto \mathbb{R}$ be defined as

$$\mathrm{F}(x) = \begin{cases} 0 & \text{if } x < 0 \\ \sum_{k=0}^{x} (1+r)^k & \text{if } x \geq 0 \end{cases}$$

Lemma 1. *If x is an integer greater than or equal to -1, then* $\mathrm{F}(x)(1+r) + 1 = \mathrm{F}(x+1)$
(Proof)

Let $x \in \mathbb{Z}$.
Case 1. $x = -1$:

$$\begin{aligned} \mathrm{F}(x)(1+r) + 1 &= \mathrm{F}(-1)(1+r) + 1, \\ &= 0 + 1, \\ &= 1, \\ &= \mathrm{F}(0), \\ &= \mathrm{F}(x+1). \end{aligned}$$

Case 2. $x \geq 0$:

$$\begin{aligned} \mathrm{F}(x)(1+r) + 1 &= \left(\sum_{k=0}^{x} (1+r)^k\right)(1+r) + 1 \qquad \text{(by definition of F)}, \\ &= \sum_{k=0}^{x} (1+r)^{k+1} + 1, \\ &= \sum_{k=1}^{x+1} (1+r)^k + 1, \\ &= \sum_{k=1}^{x+1} (1+r)^k + (1+r)^0, \\ &= \sum_{k=0}^{x+1} (1+r)^k, \\ &= \mathrm{F}(x+1). \end{aligned}$$

QED

Theorem. $A_n = P(1+r)^n + rd\,\mathrm{F}(n-1)$
(Proof by Induction)

Base Case:

$$\begin{aligned} A_0 &= P && \text{(by definition)},\\ &= P(1) + rd(0) \\ &= P(1+r)^0 + rd\,\mathrm{F}(0-1). \end{aligned}$$

Induction Step:
Suppose $A_k = P(1+r)^k + rd\,\mathrm{F}(k-1)$, for some $k \in \mathbb{Z}^{\text{nonneg}}$.
Then,

$$\begin{aligned} A_{k+1} &= A_k(1+r) + rd && \text{(by definition)},\\ &= \left[P(1+r)^k + rd\,\mathrm{F}(k-1)\right](1+r) + rd && \text{(by supposition)},\\ &= P(1+r)^k(1+r) + rd\,\mathrm{F}(k-1)(1+r) + rd, \\ &= P(1+r)^{k+1} + rd\Big[\mathrm{F}(k-1)(1+r) + 1\Big], \\ &= P(1+r)^{k+1} + rd\,\mathrm{F}(k) && \text{(by Lemma 1)}. \end{aligned}$$

QED