2. Kinematic Variables

In relativistic heavy-ion collisions and in many other high-energy reaction processes, it is convenient to use kinematic variables which have simple properties under a change of the frame of reference. The light-cone variables x_+ and x_-, the rapidity variable y, and the pseudorapidity variable η are kinematic variables which have simple properties under a Lorentz transformation. They are commonly used. The Feynman scaling variable, x_F, which is related to the light-cone variables x_+ and x_-, is also often used. It is worthwhile to discuss these variables in detail to establish the proper language for relativistic reactions.

§2.1 Notation and Conventions

In this book, we shall follow the notation of Bjorken and Drell, *Relativistic Quantum Mechanics*, (McGraw-Hill Book Company, N.Y. 1964). We use the *natural units* $c = \hbar = 1$. The space-time coordinates of a point x are denoted by a *contravariant vector* with components x^μ:

$$x^\mu = (x^0, x^1, x^2, x^3) = (t, \boldsymbol{x}) = (t, x, y, z)\,. \tag{2.1}$$

The momentum vector p is similarly defined by a contravariant vector with components p^μ:

$$p^\mu = (p^0, p^1, p^2, p^3) = (E, \boldsymbol{p}) = (E, \boldsymbol{p}_T, p_z) = (E, p_x, p_y, p_z). \tag{2.2}$$

We shall adopt the space-time *metric tensor* $g_{\mu\nu}$ in the form

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{2.3}$$

The *covariant vector* x_μ is related to the contravariant vector x^μ through the metric tensor $g_{\mu\nu}$ by

$$x_\mu \equiv (x_0, x_1, x_2, x_3) \equiv g_{\mu\nu} x^\nu = (t, -x, -y, -z)\,, \tag{2.4}$$

where we use the notation that a repeated index implies a summation with respect to that index, unless indicated otherwise. Conversely, the contravariant vector x^μ is related to the corresponding covariant vector x_ν by

$$x^\mu \equiv g^{\mu\nu} x_\nu \,, \tag{2.5}$$

where the metric tensor $g^{\mu\nu}$ is

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} . \tag{2.6}$$

The *scalar product of two vectors* a and b is defined as

$$a \cdot b \equiv a^\mu b_\mu = g_{\mu\nu} a^\mu b^\nu = a^0 b^0 - \boldsymbol{a} \cdot \boldsymbol{b} \,.$$

The *four-momentum operator* p^μ in coordinate representation is

$$p^\mu = i\partial^\mu = (i\partial^0, i\partial^1, i\partial^2, i\partial^3) \tag{2.7a}$$

$$= (i\partial_0, -i\partial_1, -i\partial_2, -i\partial_3) = (i\frac{\partial}{\partial x^0}, -i\frac{\partial}{\partial x^1}, -i\frac{\partial}{\partial x^2}, -i\frac{\partial}{\partial x^3}) . \tag{2.7b}$$

The covariant operator ∂_i is the usual *gradient operator* ∇ defined by

$$\nabla = (\nabla_x, \nabla_y, \nabla_z) = (\partial_1, \partial_2, \partial_3) = (\partial/\partial x^1, \partial/\partial x^2, \partial/\partial x^3) \,. \tag{2.8}$$

⊕⟦ Supplement 2.1

The placement of the indices in the tensor notation of contravariant and covariant vectors is confusing for beginning students when they are dealing with relativistic kinematics. In the "common" notation for vectors, only subscript indices are used and contravariant and covariant vectors are not distinguished. This is permissible in the Euclidean space, where $g_{\mu\nu} = \delta_{\mu\nu}$ and there is no significant distinction between a contravariant vector and a covariant vector. However, in Minkowski space with the metric tensor (2.3), they are different types of vectors as they have different properties under a coordinate transformation. In the tensor notation, the contravariant vectors have superscript indices but the covariant vectors have subscript indices. The tensor notation makes a clear distinction between these two types of vectors. The easiest way to understand the distinction between them is to remember that the suffixes 'contra-' and 'co-' refer to a comparison with the gradient vector. 'Contravariant' corresponds to the nomenclature 'contragradient', and 'covariant' corresponds to the term 'cogradient' [1]. A quantity which transforms like a gradient vector is 'cogradient' and is therefore a covariant vector with a subscript index. A quantity which transforms like the coordinates is 'contragradient' and is a contravariant vector with a superscript index.

Accordingly, the coordinate x^μ and the momentum p^μ are contravariant four-vectors and they have superscript indices. The covariant vectors x_μ and p_μ, with subscript indices derived from these vectors, are different vectors because the signs of their space components are changed as in Eq. (2.4).

On the other hand, because superscript indices are clumsy to use, the common notation for a vector uses subscript indices to refer to the components of any vector. It does not distinguish a contravariant vector from a covariant vector. Unfortunately, the most commonly used vectors, such as the coordinate vector and the momentum vector, are contravariant vectors. They have superscript indices in the tensor notation but subscript indices in the common notation. It is unavoidable that one uses both notations for these quantities. Therefore, when these quantities are written out, it is necessary to make a mental note as to which notation is adopted.

It is worth recommending that one adheres to the tensor notation as much as possible, to insure that one gets the correct signs in an algebraic manipulation with vector quantities. As an example, we can follow the line of reasoning which gives the coordinate representation of the momentum operator (2.7). There, the signs in front of the ∂ operators in Eqs. (2.7a) and (2.7b) are often a source of confusion for beginning students. We begin by recalling that the coordinate vector x^μ and the momentum vector p^μ are contravariant vectors. The gradient operator ∂_μ is a covariant vector, and it is the derivative with respect to the μ component of x. As x is a contravariant vector, ∂_μ is the derivative with respect to x^μ:

$$\partial_\mu = \frac{\partial}{\partial x^\mu} .$$

All other vectors such as p_μ, x_μ, and ∂^μ are derived in terms of the basic vectors x^μ, p^μ, and ∂_μ by using the metric tensors $g_{\mu\nu}$ and $g^{\mu\nu}$ as in Eqs. (2.4) and (2.5).

Being a contravariant vector, the momentum vector has a coordinate representation which is naturally another contravariant vector $i\partial^\mu$. If we want to write out the operator $i\partial^\mu$ explicitly in terms of the derivatives of the coordinates, it is necessary to express ∂^μ in terms of the covariant operator ∂_μ by using the metric tensor $g^{\mu\nu}$. Thus, we have

$$\begin{aligned} p^\mu &= i\partial^\mu \\ &= ig^{\mu\nu}\partial_\nu . \end{aligned}$$

Writing the covariant operator ∂_ν explicitly as the derivative with respect to the contravariant vector x^μ, we have

$$\begin{aligned} p^\mu &= ig^{\mu\nu}\frac{\partial}{\partial x^\nu} \\ &= (i\frac{\partial}{\partial x^0}, -i\frac{\partial}{\partial x^1}, -i\frac{\partial}{\partial x^2}, -i\frac{\partial}{\partial x^3}) \end{aligned}$$

which is the expression in Eq. (2.7b).]] $\oplus$

§2.2 Light-Cone Variables

In many high-energy reaction processes, a detected particle can be identified as originating from one of the colliding particles. For example, in the reaction

$$b + a \to c + X ,$$

where c is a detected particle, c may sometimes be considered as fragmenting from the incident beam particle b or from the target particle a.

A reaction in which the detected particle c is described as originating from the beam particle b is called a *projectile fragmentation* reaction. The region of the momentum of c in which this type of reaction is dominant is referred to as the projectile fragmentation region. It lies in the forward direction with respect to the beam axis. Similarly, a reaction in which the detected particle c can be described as originating from the target particle a is called a *target fragmentation* reaction. The kinematic region in which this type of reaction is dominant is the target fragmentation region, which is near the region of momentum where the target particle is initially at rest.

In a reaction, kinematic quantities along the direction of the incident beam, which we shall identify as the longitudinal axis, have properties quite different from those along the transverse directions perpendicular to the beam axis. We shall designate the longitudinal axis as the z-axis. For convenience, we use the same symbol to represent a particle and its four-momentum. For example, $c = (c_0, \mathbf{c}_T, c_z)$, where c_0 is the energy of the particle c, c_z is its longitudinal momentum and $\mathbf{c}_T$ is its two-dimensional transverse momentum in the plane perpendicular to the longitudinal axis.

Two linear combinations of c_0 and c_z have special properties under a Lorentz transformation in the z-direction. The quantity

$$c_+ = c_0 + c_z \, , \tag{2.9a}$$

is called the *forward light-cone momentum* of c, while the quantity

$$c_- = c_0 - c_z \, , \tag{2.9b}$$

is called the *backward light-cone momentum* of c. For an energetic particle traveling in the forward direction (along the beam direction), its forward light-cone momentum c_+ is large, while its backward light-cone momentum c_- is small. Conversely, for a particle traveling in the backward direction opposite to that of the beam, its backward light-cone momentum is large, while its forward light-cone momentum is small.

The forward light-cone momentum of any particle in one frame is related to the forward light-cone momentum of the same particle in another boosted Lorentz frame by a constant factor. Therefore, if one considers a daughter particle c as fragmenting from a parent particle b, then the ratio of the forward light-cone momentum of c relative to that of b is independent of the Lorentz frame (see Exercise 2.1). It is convenient to introduce the *forward light-cone variable* x_+ of c relative to b, defined as the ratio of the forward light-cone momentum of the daughter particle c relative to the forward light-cone momentum of the parent particle b:

$$x_+ = \frac{c_0 + c_z}{b_0 + b_z} . \qquad (2.10)$$

The forward light-cone variable x_+ is always positive. Because a daughter particle cannot possess a forward light-cone momentum greater than the forward light-cone momentum of its parent particle, the upper limit of the forward light-cone variable x_+ is 1. It is a Lorentz-invariant quantity, independent of the Lorentz frame (see Exercise 2.1).

We have been motivated to introduce the light-cone variable x_+ to specify the relationship between the forward light-cone momentum of a daughter particle and the forward light-cone momentum of its parent particle. However, because of its Lorentz invariant property, x_+ is sometimes used to specify the relationship between the momentum of a particle c relative to another reference particle b, whether the particle b is the parent particle of c or not. In this instance, the forward light-cone momentum of the reference particle b provides the scale by which the momentum of particle c is measured.

•⟦ Exercise 2.1
Show that x_+ is a Lorentz-invariant quantity.

◊Solution:
Under a Lorentz transformation, one goes from a frame F to a moving frame F' by boosting the frame F with a velocity β in the z-direction. The four-momentum $c = (c_0, \mathbf{c}_T, c_z)$ in the frame F is transformed into the four-momentum $c' = (c_0', \mathbf{c}_T', c_z')$ in the frame F'. The momenta in the two different frames F and F' are related by the Lorentz transformation

$$c_0' = \gamma(c_0 - \beta c_z) , \qquad (1)$$

$$c_z' = \gamma(c_z - \beta c_0) , \qquad (2)$$

and

$$\mathbf{c}_T' = \mathbf{c}_T ,$$

where

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} .$$

Therefore, from Eqs. (1) and (2), the forward light-cone momentum $c_0 + c_z$ in the frame F is related to that in the frame F' by the factor $\gamma(1-\beta)$:

$$c_0' + c_z' = \gamma(1-\beta)(c_0 + c_z) . \qquad (3)$$

Similarly, for another particle b, the forward light-cone momentum $b_0 + b_z$ in frame F is related to its forward light-cone momentum in frame F' by

$$b'_0 + b'_z = \gamma(1-\beta)(b_0 + b_z)\,. \tag{4}$$

If one considers particle c to be the daughter particle and particle b to be the parent particle (or reference particle), it is useful to introduce the forward light-cone variable x_+, defined as the ratio of the forward light-cone momentum of c relative to the forward light-cone momentum of b. In the frame F, the light-cone variable x_+ is

$$x_+ = \frac{c_0 + c_z}{b_0 + b_z}\,. \tag{5}$$

In the frame F', the light-cone variable x'_+ is

$$x'_+ = \frac{c'_0 + c'_z}{b'_0 + b'_z}\,. \tag{6}$$

From Eqs. (3), (4), (5) and (6), we see that

$$x_+ = x'_+\,.$$

Therefore, the light-cone variable x_+ is independent of the frame of reference.]•

In exactly the same way, one finds that under the change of Lorentz frame, the backward light-cone momentum of one particle in one frame is related to its backward light-cone momentum in another frame by a constant factor. The factor is $\gamma(1+\beta)$ for the backward light-cone momenta, whereas for the forward light-cone momenta, the factor is $\gamma(1-\beta)$ (see Eqs. (3) and (4) of Exercise 2.1). Therefore, if one considers a daughter particle c as fragmenting from a parent target particle a, it is convenient to introduce the *backward light-cone variable* x_- defined as

$$x_- = \frac{c_0 - c_z}{a_0 - a_z}\,, \tag{2.11}$$

which is a Lorentz-invariant quantity, independent of the Lorentz frame of reference. It is a positive quantity and its upper limit is 1.

In many problems, one deals exclusively with the forward region and the backward light-cone momentum is not considered. In these cases, the forward light-cone variable x_+ is often simply referred to as the *light-cone variable* and is denoted by x. There is no ambiguity that it would be confused with the backward light-cone variable x_-. At very high energies when the energy and the longitudinal momentum are approximately the same, the light-cone variable x is just the longitudinal momentum fraction of the daughter particle c relative to the parent particle b. For this reason, the variable x is sometimes called the *longitudinal momentum fraction*, or simply the *momentum fraction* of c relative to b.

The daughter particle c may be a particle detected as a free particle in a detector. In this case, the particle is not subject to any interaction, and the components of its four-momentum obey the relation appropriate for a free particle:

$$c^2 = c_0^2 - \mathbf{c}^2 = m_c^2\,, \tag{2.12}$$

where m_c is the rest mass of c.

In the space of the momentum components of c, Eq. (2.12) can be considered as expressing m_c^2 as a function of the variables c_0 and $\mathbf{c}$, which is represented graphically by a hyperboloid of the variables c_0 and $\mathbf{c}$ characterized by the mass of the particle. Equation (2.12) is called the *mass-shell condition*. A free particle is said to be *on the mass shell* because its energy and momentum obey a simple relation with reference to the rest mass of the particle. The four-momentum c now has only three degrees of freedom, and it can be represented, for example, by $(x_+, \mathbf{c}_T)$ when the forward momentum of the parent particle b is known.

In some problems, the beam particle b is considered to be a composite system consisting of a constituent particle c and the other parts X. Sometimes, a reaction between the beam particle and the target particle is described as the interaction between the beam constituent c with the target particle. The particle c is not a free particle, and it is still subject to interactions with the other parts X. The four-momentum of c will not obey the mass-shell relation (2.12). Its four degrees of freedom can be specified by the Lorentz-invariant quantities $(x_+, c^2, \mathbf{c}_T)$, when the forward light-cone momentum of b is known. The particle c is said to be *off the mass shell*, or simply off shell. There is a simple transformation which gives $(c_0, \mathbf{c}_T, c_z)$ in terms of $(x_+, c^2, \mathbf{c}_T)$ (see Exercise 2.2). Similarly, when the particle c under consideration is a constituent of the target particle a, its four degrees of freedom can be specified by the Lorentz-invariant quantities $(x_-, c^2, \mathbf{c}_T)$ when the backward light-cone momentum of a is known.

In a typical high-energy collision process, the distribution of the transverse momentum $dN/d\mathbf{c}_T$ of the produced particles has a peak at zero transverse momentum and has a width of the order of a few hundred MeV/c. This width is small compared to the longitudinal momentum of the reaction. The momentum of the produced particles is therefore 'limited' in the transverse direction. It is often convenient to separate out the transverse degree of freedom from the longitudinal degree of freedom. For these problems, it is useful to write Eq. (2.12) in the form

$$c_0^2 - c_z^2 = m_c^2 + \boldsymbol{c}_T^2 = m_{c_T}^2 \,, \tag{2.13}$$

where m_{c_T} is called the *transverse mass* of particle c. It contains contributions from both the rest mass and the transverse momentum of the particle.

•⟦ Exercise 2.2
The variables $(c_0, \boldsymbol{c}_T, c_z)$ depend on the frame of reference. In many problems, one makes a change of variables from $(c_0, \boldsymbol{c}_T, c_z)$ to the Lorentz-invariant variables $(x_+, c^2, \boldsymbol{c}_T)$. Express the momentum components c_0 and c_z in terms of the forward light-cone variable x_+ and the Lorentz-invariant quantity c^2 when the particle c is produced from a reaction in which the beam particle is b. Also, find the relation between the differential elements $dc_0 dc_z d\boldsymbol{c}_T$ and $dx_+ dc^2 d\boldsymbol{c}_T$.

◇Solution:
From Eq. (2.10), we have

$$c_0 + c_z = x_+(b_0 + b_z) \,.$$

We also have

$$c_0 - c_z = (c_0^2 - c_z^2)/(c_0 + c_z) = (c^2 + \boldsymbol{c}_T^2)/(c_0 + c_z) \,.$$

By adding and subtracting these two equations, we obtain

$$c_0 = \frac{1}{2}\left[x_+(b_0 + b_z) + \frac{c^2 + \boldsymbol{c}_T^2}{x_+(b_0 + b_z)}\right] , \tag{1}$$

and

$$c_z = \frac{1}{2}\left[x_+(b_0 + b_z) - \frac{c^2 + \boldsymbol{c}_T^2}{x_+(b_0 + b_z)}\right] , \tag{2}$$

which expresses c_0 and c_z in terms of x_+ and c^2. From these equations, we find

$$dc_0 = \frac{1}{2}\left[dx_+\left\{(b_0 + b_z) - \frac{c^2 + \boldsymbol{c}_T^2}{x_+^2(b_0 + b_z)}\right\} + \frac{dc^2}{x_+(b_0 + b_z)}\right] , \tag{3}$$

and

$$dc_z = \frac{1}{2}\left[dx_+\left\{(b_0 + b_z) + \frac{c^2 + \boldsymbol{c}_T^2}{x_+^2(b_0 + b_z)}\right\} - \frac{dc^2}{x_+(b_0 + b_z)}\right] . \tag{4}$$

The relation between the differential elements $dc_0 dc_z$ and $dc^2 dx_+$ can be found by using the well-known Jacobian determinant formula. An intuitively simple way to get this result is by viewing the differential element $dc_0 dc_z$ as a surface area element. It has the meaning of the magnitude of the cross product $dc_0 \times dc_z$. Similarly, $dx_+ dc^2$ is a surface element in the space with coordinates x_+ and c^2, and it has the meaning of the magnitude of the cross product $dx_+ \times dc^2$ in that space. Hence, using Eqs. (3-4), we carry out the cross product and we have

$$dc_0 dc_z = | dc_0 \times dc_z | = \frac{1}{2} \frac{| dx_+ \times dc^2 |}{x_+} = \frac{1}{2} \frac{dx_+}{x_+} dc^2 . \qquad (5)$$

Therefore, the invariant volume element d^4c can be expressed in terms of dx_+, dc^2 and $d\boldsymbol{c}_T$ by

$$d^4c = dc_0 dc_z d\boldsymbol{c}_T = \frac{dx_+}{2x_+} dc^2 d\boldsymbol{c}_T . \qquad (6)$$

The relations obtained here are useful when one makes a change of variables from $(c_0, \mathbf{c}_T, c_z)$ to $(x_+, c^2, \mathbf{c}_T)$, whether the particle c is on the mass shell or not. If the particle c is on the mass-shell, the mass-shell condition can be represented by a delta function $\delta(c^2 - m^2)$ constraining the variable c^2 to the value m^2. ⟧•

•⟦Exercise 2.3
The Feynman scaling variable x_F for a detected particle c is defined as

$$x_F = \frac{c_z^*}{c_z^*(max)} , \qquad (1)$$

where the asterisks stand for quantities in the center-of-mass system. Obtain the relationship between the Feynman scaling variable x_F and the forward light-cone momentum fraction x_+.

◇Solution:
Consider the reaction $b + a \to c + X$. We shall work in the center-of-mass system, and start by expressing the denominator of Eq. (1) in terms of rest masses and the center-of-mass collision energy. This denominator is the maximum value of c_z^*. Clearly, c_z^* attains its maximum value when the undetected collection of particles 'X' in the reaction $b + a \to c + X$ consists of a single particle with a rest mass m_X, which corresponds to the minimum value of the rest mass of X allowed by the conservation laws (of baryon numbers, charge, etc). In this situation, the magnitude of the three-momentum of c is equal to the magnitude of the three-momentum of X. We have, therefore, from the energy conservation condition

$$\sqrt{(c_z^*(max))^2 + m_c^2} + \sqrt{(c_z^*(max))^2 + m_X^2} = \sqrt{s} ,$$

where $\sqrt{s}$ is the center-of-mass energy of the collision process. This equation can be solved for $c_z^*(max)$ in terms of s, and we obtain

$$c_z^*(max) = \frac{\lambda(s, m_c^2, m_X^2)}{2\sqrt{s}} , \qquad (2)$$

where the function λ is defined by

$$\lambda^2(s, m_1^2, m_2^2) = s^2 + m_1^4 + m_2^4 - 2(sm_1^2 + sm_2^2 + m_1^2 m_2^2) . \qquad (3)$$

From the definition of the Feynman scaling variable, the longitudinal momentum of c in the center-of-mass system is

$$c_z^* = x_F \frac{\lambda(s, m_c^2, m_X^2)}{2\sqrt{s}} \,. \tag{4}$$

We shall use (4) in the definition of x_+, so as to relate x_+ with x_F.

The energy of the particle c in the center-of-mass system is (see Eqs. (2.13) and (4))

$$c_0^* = \sqrt{(c_z^*)^2 + m_{cT}^2} = \sqrt{\frac{x_F^2 \lambda^2(s, m_c^2, m_X^2)}{4s} + \mathbf{c}_T^2 + m_c^2} \,. \tag{5}$$

Furthermore, we need to know the energy and the longitudinal momentum of the parent beam particle in the center-of-mass system to obtain the forward light-cone variable. In this system, the magnitude of the three-momentum of b is equal to the magnitude of the three-momentum of a. Therefore, from the definition of the center-of-mass energy $\sqrt{s}$, we have

$$\sqrt{(b_z^*)^2 + m_b^2} + \sqrt{(b_z^*)^2 + m_a^2} = \sqrt{s} \,.$$

We then have

$$b_z^* = \frac{\lambda(s, m_a^2, m_b^2)}{2\sqrt{s}} \,, \tag{6}$$

and

$$b_0^* = \frac{s + m_b^2 - m_a^2}{2\sqrt{s}} \,. \tag{7}$$

The forward light-cone momentum x_+ is defined by (2.10) as

$$x_+ = \frac{c_0^* + c_z^*}{b_0^* + b_z^*} \,, \tag{8}$$

where c_0^*, c_z^*, b_0^*, and b_z^* are available from Eqs. (4)-(7) to express x_+ as a function of the Feynman scaling variable x_F.

To obtain the inverse relation, we use Eqs. (1), and (2) and Eq. (2) of Exercise 2.2, to get a relation that expresses x_F as a function of the light-cone variable,

$$x_F = \frac{1}{2}\left[x_+(b_0^* + b_z^*) - \frac{m_{cT}^2}{x_+(b_0^* + b_z^*)}\right] \frac{2\sqrt{s}}{\lambda(s, m_c^2, m_X^2)} \,, \tag{9}$$

where b_0^* and b_z^* are given by Eqs. (6) and (7) to express x_F as functions of x_+.

It is interesting to examine the case when the rest masses are small compared to the center-of-mass energy $\sqrt{s}$. Then, because $c_z^*(max) \approx \sqrt{s}/2$, and

$$b_0^* + b_z^* \approx \sqrt{s}(1 - m_a^2/s),$$

we have

$$x_+ \approx \frac{1}{2}\left\{x_F\left(1 - \frac{m_c^2 + m_X^2}{s}\right) + \sqrt{x_F^2\left(1 - \frac{m_c^2 + m_X^2}{s}\right)^2 + \frac{4m_{cT}^2}{s}}\right\} \frac{1}{1 - m_a^2/s} \,,$$

and

$$x_F \approx \left\{ x_+(1-\frac{m_a^2}{s}) - \frac{m_{cT}^2}{x_+ s(1-m_a^2/s)} \right\} \frac{1}{1-(m_c^2+m_X^2)/s} .$$

One notices that while the forward light-cone variable is always positive, the Feynman scaling variable can be zero and negative. In the case of very high energies and $x_+ \gg 0$ or $x_F \gg 0$, the light-cone variable x_+ coincides with the Feynman scaling variable x_F. On the other hand, when x_+ or x_F are small, or when x_F is negative, the light-cone variable x_+ differs substantially from the Feynman scaling variable x_F.]•

§2.3 Rapidity Variable

Another useful variable used commonly to describe the kinematic condition of a particle is the *rapidity variable* y. The rapidity of a particle is defined in terms of its energy-momentum components p_0 and p_z by

$$y = \frac{1}{2} \ln \left(\frac{p_0 + p_z}{p_0 - p_z} \right) . \tag{2.14}$$

It is a dimensionless quantity related to the ratio of the forward light-cone momentum to the backward light-cone momentum. It can be either positive or negative. In the nonrelativistic limit, the rapidity of a particle travelling in the longitudinal direction is equal to the velocity of the particle in units of the speed of light (see Exercise 2.4). The rapidity variable depends on the frame of reference, but the dependence is very simple. The rapidity of the particle in one frame of reference is related to the rapidity in another Lorentz frame of reference by an additive constant (see Exercise 2.5).

If the particle c is a free particle, it is then on the mass shell. Its four-momentum has only three degrees of freedom and can be represented by $(y, \boldsymbol{p}_T)$. We obtain below a simple transformation which gives $(p_0, \boldsymbol{p})$ in terms of $(y, \boldsymbol{p}_T)$.

From the definition (2.14), we have

$$e^y = \sqrt{\frac{p_0 + p_z}{p_0 - p_z}} , \tag{2.15a}$$

and

$$e^{-y} = \sqrt{\frac{p_0 - p_z}{p_0 + p_z}} . \tag{2.15b}$$

Adding Eqs. (2.15a) and (2.15b), we get the relation between the energy p_0 and the rapidity y of the particle c:

$$p_0 = m_T \cosh y , \tag{2.16}$$

where m_T is the transverse mass of the particle:

$$m_T^2 = m^2 + \boldsymbol{p}_T^2 \,.$$

Subtracting Eq. (2.15b) from Eq. (2.15a), we obtain the relation between the longitudinal momentum p_z and the rapidity y of the particle

$$p_z = m_T \sinh y \,. \tag{2.17}$$

Equations (2.16) and (2.17) are useful relations linking the components of the momentum with the rapidity variable.

•⟦ Exercise 2.4.
What is the rapidity of a particle traveling in the positive z direction with a velocity β ?

◇ Solution:
The energy of the particle is

$$p_0 = \gamma m \,,$$

where m is the rest mass of the particle. The momentum of the particle in the longitudinal direction is

$$p_z = \gamma \beta m \,.$$

From the definition of the rapidity variable, Eq. (2.14), the rapidity of the particle traveling with a velocity β in the positive z direction is then

$$y_\beta = \frac{1}{2} \ln \left(\frac{1+\beta}{1-\beta} \right) . \tag{1}$$

Note that this is independent of the mass of the particle. When β is small, an expansion of y_β in terms of β leads to

$$y_\beta = \beta + O(\beta^3) \,.$$

Thus, in the nonrelativistic case, the rapidity of a particle is equal approximately to the longitudinal velocity of the particle. ⟧•

•⟦ Exercise 2.5.
Find the relationship between the rapidity y of a particle in a laboratory frame F and the rapidity y' of the particle in a boosted Lorentz frame F' which moves with a velocity β in the z-direction.

◇ Solution:
The rapidity y' of the particle c in the new frame F' is defined by

$$y' = \frac{1}{2} \ln \left(\frac{p_0' + p_z'}{p_0' - p_z'} \right) .$$

Under the Lorentz transformation, the energy p_0' and the longitudinal momentum p_z' in the frame F' are related to the energy p_0 and the longitudinal momentum p_z in the frame F' by

$$p_0' = \gamma(p_0 - \beta p_z)\,,$$
$$p_z' = \gamma(p_z - \beta p_0)\,,$$

where β is the velocity of F' relative to F. Therefore, the rapidity y' in the frame F' is

$$\begin{aligned} y' &= \frac{1}{2}\ln\left[\frac{\gamma(1-\beta)(p_0+p_z)}{\gamma(1+\beta)(p_0-p_z)}\right] \\ &= y + \frac{1}{2}\ln\left(\frac{1-\beta}{1+\beta}\right) \\ &= y - \frac{1}{2}\ln\left(\frac{1+\beta}{1-\beta}\right)\,, \end{aligned}$$

where y is the rapidity of the particle c in the laboratory frame F. (Note that here the symbol β is the velocity of the frame F' relative to the frame F, whereas in Exercise 2.4, β was the velocity of the particle in the frame F.)]•

From the above results in Exercise 2.5, we observe that under a *Lorentz transformation* from the laboratory frame F to a new coordinate frame F' moving with a velocity β in the z-direction, the rapidity y' of the particle in the new frame F' is related to the rapidity y in the old frame F by

$$y' = y - y_\beta\,, \tag{2.18}$$

where y_β is

$$y_\beta = \frac{1}{2}\ln\left(\frac{1+\beta}{1-\beta}\right)\,. \tag{2.19}$$

According to Eq. (1) of Exercise 2.4, the quantity y_β is the rapidity a particle would have in the frame F, if it were traveling with the velocity β of the moving frame. The quantity y_β can conveniently be called the 'rapidity of the moving frame'. Thus, the rapidity of a particle in a moving frame is equal to the rapidity in the rest frame minus the rapidity of the moving frame, much like the subtraction of the velocity of the moving frame in the nonrelativistic case. This similarity is not surprising because nonrelativistically, y is equal to the longitudinal velocity β, as shown in Exercise 2.4. It is often useful to treat the rapidity variable as a relativistic measure of the 'velocity' of a particle. This simple property of the rapidity variable under a Lorentz transformation makes it a suitable choice to describe the dynamics of relativistic particles. To go from a frame of reference at rest to a moving frame of reference, it is only necessary to find the rapidity of the moving frame y_β and change the rapidity variables by subtracting this constant y_β.

•⟦ Exercise 2.6.
In the collision of a *b*eam particle b with momentum b_z on a target particle a with momentum a_z, show that the initial rapidities of the particles are

$$y_a = \sinh^{-1}(a_z/m_a)$$

and

$$y_b = \sinh^{-1}(b_z/m_b)\,,$$

where m_a and m_b are the rest masses of particles a and b respectively. For the case when the rest mass of the projectile and the rest mass of the target particle are the same, show that the rapidity of the center-of-mass frame is given by

$$y_{cm} = (y_a + y_b)/2\,,$$

and, in the center-of-mass frame, the rapidities of a and b are

$$y_a^* = -(y_b - y_a)/2 \quad \text{and} \quad y_b^* = (y_b - y_a)/2\,.$$

◇ Solution:
The transverse momentum of the beam particle is zero. From Eq. (2.17), we have

$$b_z = m_b \sinh y_b,$$

or

$$y_b = \sinh^{-1}(b_z/m_b)\,.$$

From Eq. (2.16), we infer that the energy of the beam particle b in the laboratory frame is

$$b_0 = m_b \cosh y_b\,.$$

Similarly, because the target particle has no transverse momentum, the rapidity of the target particle a in the laboratory frame F is given by

$$y_a = \sinh^{-1}(a_z/m_a)\,,$$

where we have allowed the target particle to have a longitudinal momentum. The energy of the target particle a in the laboratory frame is

$$a_0 = m_a \cosh y_a\,.$$

The center-of-mass frame F_{CM} is obtained by boosting the laboratory frame F by a velocity of the center-of-mass frame β_{CM} such that the longitudinal momentum of the beam particle b_z^* and the longitudinal momentum of the target particle a_z^* are equal and opposite. Therefore, the velocity of the center-of-mass frame, β_{CM}, satisfies the following condition

$$a_z^* = \gamma_{CM}(a_z - \beta_{CM} a_0) = -b_z^* = -\gamma_{CM}(b_z - \beta_{CM} b_0)\,,$$

where

$$\gamma_{CM} = \frac{1}{\sqrt{1-\beta_{CM}^2}}\,.$$

Hence, we have the velocity of the center-of-mass frame given by

$$\beta_{CM} = \frac{a_z + b_z}{a_0 + b_0}\,. \tag{1}$$

From Eq. (2.19), the rapidity of the center-of-mass frame is

$$y_{CM} = \frac{1}{2}\ln\left(\frac{1+\beta_{CM}}{1-\beta_{CM}}\right)\,. \tag{2}$$

Substituting the velocity of the center-of-mass frame from Eq. (1) into Eq. (2), we obtain

$$y_{CM} = \frac{1}{2}\ln\left[\frac{a_0 + a_z + b_0 + b_z}{a_0 - a_z + b_0 - b_z}\right]\,.$$

Writing the energies and the momenta in terms of the rapidity variables in the frame F, we can express the rapidity of the center-of-mass frame y_{CM} in terms of the rapidities of the colliding particles as

$$\begin{aligned} y_{CM} &= \frac{1}{2}\ln\left[\frac{m_a e^{y_a} + m_b e^{y_b}}{m_a e^{-y_a} + m_b e^{-y_b}}\right], \\ &= \frac{1}{2}(y_a + y_b) + \frac{1}{2}\ln\left[\frac{m_a e^{y_a} + m_b e^{y_b}}{m_a e^{y_b} + m_b e^{y_a}}\right]\,. \end{aligned}$$

The above result is for the general case, when the rest masses of the beam particle b and the rest mass of the target particle a may be different. In the special case when the rest masses of a and b are equal, we have

$$y_{CM} = \frac{1}{2}(y_a + y_b)\,,$$

and the rapidities of a and b in the center-of-mass frame are

$$y_a^* = y_a - y_{CM} = -\frac{1}{2}(y_b - y_a)\,,$$

and

$$y_b^* = y_b - y_{CM} = \frac{1}{2}(y_b - y_a)\,.$$

]•

For a given incident energy, the rapidity of the projectile particles and the rapidity of the target particles can be determined easily (see Exercise 2.6). The greater the incident energy, the greater is the separation between the projectile rapidity and the target rapidity.

The region of rapidity about midway between the projectile rapidity and the target rapidity is called the *central rapidity* region. The rapidities of the produced particles lie mostly in this region. For example, in a pp collision at a laboratory momentum of 100 GeV/c, the beam rapidity y_b is 5.36 and the target rapidity y_a is 0. The central rapidity region is around $y \approx 2.7$.

While the light-cone variables x_+ or x_- are the ratios of two quantities which bear a (daughter)-(parent) relationship, the rapidity variable y of a particle is a kinematic variable of an individual particle in a given coordinate system. From the definitions of $x_\pm$ and y, we can relate $x_\pm$ to y. We consider a detected particle c which is found to have a rapidity y in a given frame of reference. Relative to the beam particle b, which has a beam rapidity y_b, the forward light-cone variable of c is x_+. From the definitions of x and y, we relate the forward light-cone variable x_+ to y by

$$x_+ = \frac{m_{c_T}}{m_b} e^{y-y_b} , \tag{2.20}$$

where m_{c_T} is the transverse mass of c. Conversely, we can express y in terms of the forward light-cone fraction x_+ as

$$y = y_b + \ln\ x_+ + \ln(m_b/m_{c_T}) . \tag{2.21}$$

Similarly, relative to the target particle a with a target rapidity y_a, the backward light-cone variable of the detected particle c is x_-. The variable x_- is related to y by

$$x_- = \frac{m_{c_T}}{m_a} e^{y_a-y} , \tag{2.22}$$

and conversely,

$$y = y_a - \ln\ x_- - \ln(m_a/m_{c_T}) . \tag{2.23}$$

We can illustrate the relationship between x_+ and y by using these quantities to describe a proton or a pion detected after a pp collision at an incident momentum of 100 GeV/c. These relations depend on the transverse momentum of the proton and the pion. For numerical purposes, we suppose that the detected proton acquires a transverse momentum with a magnitude 0.46 GeV/c, which is the average value for a proton detected after a collision at this incident momentum. Similarly, we suppose that the pion has a transverse momentum of magnitude 0.35 GeV/c, which is the average magnitude of p_T for produced pions. Figure 2.1 gives the light-cone variables x_+ for the

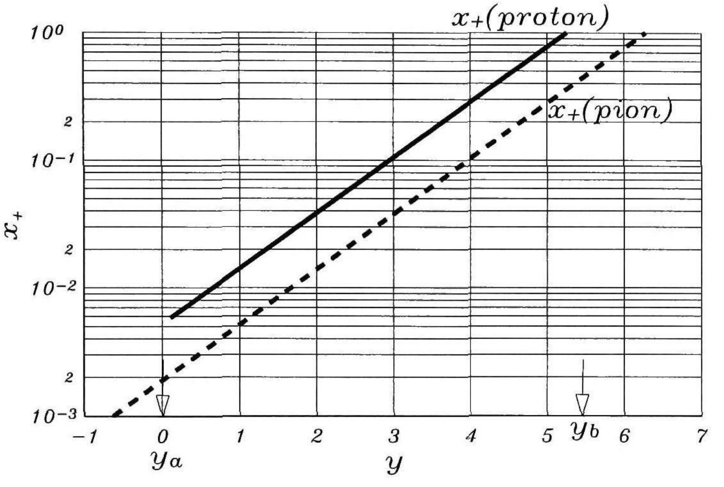

Fig. 2.1 The light-cone variables x_+ as a function of y, for a proton or a pion produced after a *pp* collision at an incident beam momentum of 100 GeV/c.

proton and the pion as a function of y. The arrows indicate the positions of the beam rapidity y_b and the target rapidity y_a. As one observes from Eq. (2.21) and Fig. 2.1, $\ln x_+$ is a linear function of y. A projectile fragmentation reaction is characterized by particles produced with light-cone variables close to unity. In this region of x_+ close to 1, a large fraction of the entire range of x_+ is covered by about one and a half units of rapidity, which comprise a small fraction of the whole range of the rapidity variable. Thus, for a reaction which leads to particles with a momentum close to the beam momentum, it is most appropriate to use the forward light-cone variable x_+ to describe those particles. Similarly, for reactions leading to particles with momentum close to the target momentum, the backward light-cone variable x_- is a better variable to describe those particles. On the other hand, for those particles detected in the central rapidity region, away from the beam rapidity and the target rapidity, a small region of the light-cone variable x_+ is mapped into a large region in the rapidity variable y, as is evident in Fig. 2.1. To examine particles in these regions, the rapidity variable y is a more appropriate kinematic variable. These relations indicate that a complete description of the full dynamical range of the produced particles requires $x_\pm$ and y variables.

§2.4 Pseudorapidity Variable

To characterize the rapidity of a particle, it is necessary to measure two quantities of the particle, such as its energy and its longitudinal momentum. In many experiments, it is only possible to measure the angle of the detected particle relative to the beam axis. In that case, it is convenient to utilize this information by using the *pseudorapidity variable* η to characterize the detected particle. The pseudorapidity variable of a particle c is defined as

$$\eta = -\ln[\tan(\theta/2)]\,, \tag{2.24}$$

where θ is the angle between the particle momentum $\boldsymbol{p}$ and the beam axis. In terms of the momentum, the pseudorapidity variable can be written as

$$\eta = \frac{1}{2}\ln\left(\frac{|\boldsymbol{p}| + p_z}{|\boldsymbol{p}| - p_z}\right). \tag{2.25}$$

By comparing Eqs. (2.14) and (2.25), it is easy to see that the pseudorapidity variable coincides with the rapidity variable when the momentum is large, that is, when $|\boldsymbol{p}| \approx p_0$.

We consider the change of variables from $(y, \boldsymbol{p}_T)$ to $(\eta, \boldsymbol{p}_T)$. It is easy to express y as a function of η, and vice versa. From the definition of η, we have

$$e^{\eta} = \sqrt{\frac{|\boldsymbol{p}| + p_z}{|\boldsymbol{p}| - p_z}} \tag{2.26}$$

and

$$e^{-\eta} = \sqrt{\frac{|\boldsymbol{p}| - p_z}{|\boldsymbol{p}| + p_z}}\,. \tag{2.27}$$

Adding Eqs. (2.26) and (2.27), we obtain the relation

$$|\boldsymbol{p}| = p_T \cosh\eta\,,$$

where p_T is the magnitude of the transverse momentum:

$$p_T = \sqrt{\boldsymbol{p}^2 - p_z^2}\,.$$

Subtracting Eq. (2.27) from (2.26), we obtain

$$p_z = p_T \sinh\eta\,. \tag{2.28}$$

Using these results, we can express the rapidity variable y in terms of the pseudorapidity variable η as

$$y = \frac{1}{2}\ln\left[\frac{\sqrt{p_T^2\cosh^2\eta + m^2} + p_T\sinh\eta}{\sqrt{p_T^2\cosh^2\eta + m^2} - p_T\sinh\eta}\right], \tag{2.29}$$

where m is the rest mass of the particle. Conversely, the pseudorapidity variable η can be expressed in terms of the rapidity variable y by

$$\eta = \frac{1}{2}\ln\left[\frac{\sqrt{m_T^2\cosh^2 y - m^2} + m_T\sinh y}{\sqrt{m_T^2\cosh^2 y - m^2} - m_T\sinh y}\right]. \tag{2.30}$$

If the particles have a distribution $dN/dy d\boldsymbol{p}_T$ in terms of the rapidity variable y, then the distribution in the pseudorapidity variable η is

$$\frac{dN}{d\eta d\boldsymbol{p}_T} = \sqrt{1 - \frac{m^2}{m_T^2\cosh^2 y}}\,\frac{dN}{dy d\boldsymbol{p}_T}. \tag{2.31}$$

In many experiments, only the pseudorapidity variable of the detected particles is measured to give $dN/d\eta$, which is the integral of $dN/d\eta d\boldsymbol{p}_T$ with respect to the transverse momentum. One can compare this quantity with dN/dy, which is the integral of $dN/dy d\boldsymbol{p}_T$ with respect to the transverse momentum. From Eq. (2.31), we can infer that in the region of y much greater than zero, $dN/d\eta$ and dN/dy are approximately the same, but in the region of y close to zero, there is a small depression of the $dN/d\eta$ distribution relative to dN/dy due to the above transformation (2.31). In experiments at high energies where dN/dy has a plateau shape, this transformation gives a small dip in $dN/d\eta$ around $\eta \approx 0$.

The transformation (2.31) reveals the difference in the maximum magnitude of the pseudorapidity distribution for $dN/d\eta$, whether η is measured in the laboratory frame or in the center-of-mass frame. In the center-of-mass frame, the peak of the distribution is located around $y \approx \eta \approx 0$, and the peak value of $dN/d\eta$ is smaller than the peak value of dN/dy by approximately the factor $(1 - m^2/< m_T^2 >)^{1/2}$. In the laboratory frame, the peak of the distribution is located around half of the beam rapidity $\eta \approx y_b/2$ for which the factor $[1 - m^2/< m_T^2 > \cosh^2(y_b/2)]^{1/2}$ is about unity. The peak value of

$dN/d\eta$ is approximately equal to the peak value of dN/dy. Because the shape of the rapidity distribution dN/dy does not change when one goes from the center-of-mass to the laboratory frame, the peak value of the pseudorapidity distribution in the center-of-mass frame is lower than the peak value of the pseudorapidity distribution in the laboratory frame.

§Reference for Chapter 2

1. W. Pauli, *Theory of Relativity*, Pergamon Press, N.Y., 1958, p. 25.