Finding the optimal detector

Optimal detector design using QCRB

Joe Bentley 14th September 2020

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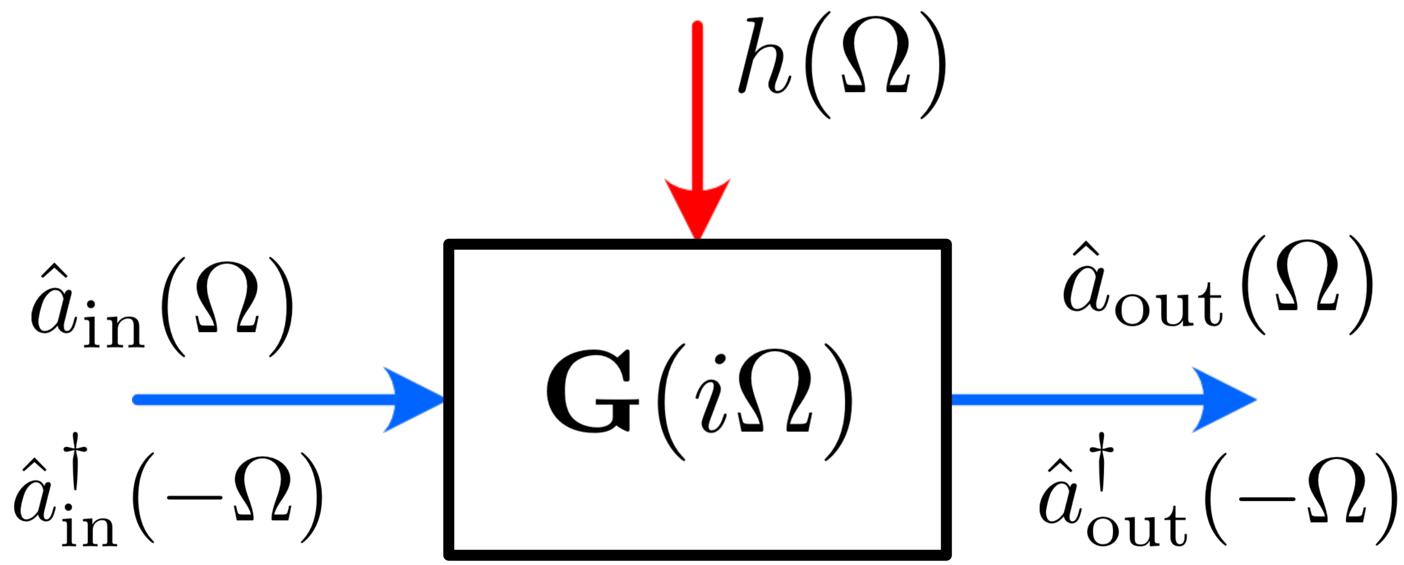
Overview of this talk

- The overall objective is to create a holistic framework for the design of optimal linear quantum measurement devices
- In this talk I will give an introduction to how this can be achieved using quantum network synthesis and the quantum Cramér-Rao bound (QCRB)
- I will start by giving an overview of our model of a detector
- Then discuss how quantum systems can be realised from their input-output transfer matrices
- Then discuss how the QCRB can be used to evaluate the detector performance
- I will then give three examples with one, two, and three degrees of freedom

Our picture of a detector



We consider the detector as a “black-box” quantum filter

$$\hat{H}_{\text{int}} = -\hat{F}h(t)$$

The signal is linearly coupled to an internal degree of freedom

Quantum Cramér-Rao Bound

Fundamental limit on variance of dispersive measurement

$$\sigma_{hh}^{\mathrm{QCRB}}(\Omega) = \frac{\hbar^2}{4S_{FF}(\Omega)}$$

More fluctuation in signal-coupled degree-of-freedom is better!

Fluctuation determined by transfer func. from input

$$S_{FF}(\Omega) = |G_{\mathrm{input}\to F}(\Omega)|^2$$

Want to maximise over broadband

$$\mathcal{S} = \int_{-\infty}^{\infty} \mathrm{d}\Omega \; |G_{\mathrm{input}\to F}(\Omega)|^2$$

Transfer matrix to physical system

Quantum network synthesis framework:

infer quantum system directly from transfer matrix

https://arxiv.org/abs/2002.07644

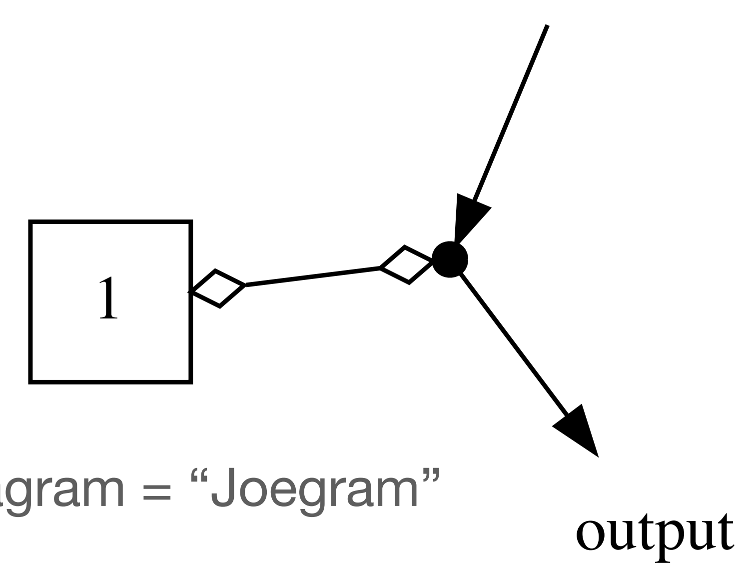
Example for 1 degree of freedom

$$\begin{bmatrix} \hat{a}^1_{\mathrm{out}}(\Omega) \\ \hat{a}^2_{\mathrm{out}}(\Omega) \end{bmatrix} = \mathbf{G}_q(i\Omega) \begin{bmatrix} \hat{a}^1_{\mathrm{in}}(\Omega) \\ \hat{a}^2_{\mathrm{in}}(\Omega) \end{bmatrix}$$

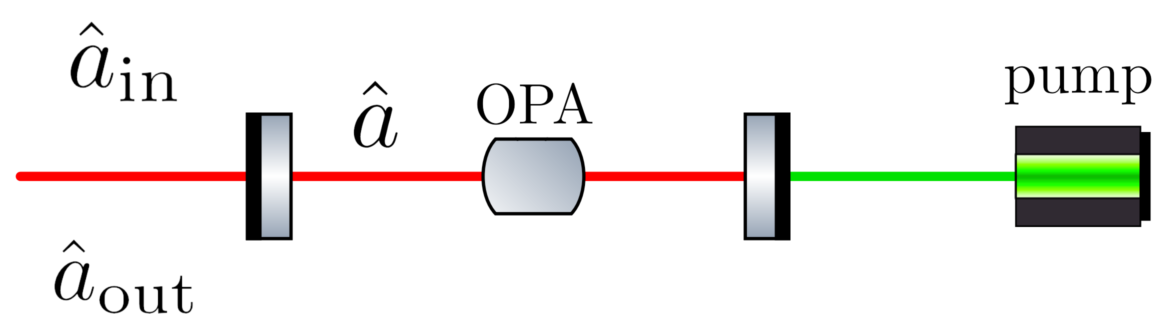
(Quadrature picture)

$$\mathbf{G}_q(i\Omega) = \begin{bmatrix} \frac{\gamma + \sqrt{\gamma s_0} + i\Omega}{\gamma - \sqrt{\gamma s_0} - i\Omega} & 0 \\ 0 & \frac{\gamma - \sqrt{\gamma s_0} + i\Omega}{\gamma + \sqrt{\gamma s_0} - i\Omega} \end{bmatrix}$$

apply framework



Joe diagram = “Joegram”



See my software (Simba): https://github.com/joebentley/simba

Physical realisability restriction

Naturally, *there is a restriction on the transfer matrix*

we musn't break quantum mechanics by not preserving the commutation relations :)

NB back in sideband picture

$$\mathbf{G}^{\dagger}(-i\Omega)\, J\, \mathbf{G}(i\Omega) = J$$

$$J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Implication?

System must be lossless with unity gain (or lossy noise channels must be added, see Caves' phase-insensitive amplifier model)

“Passive” systems

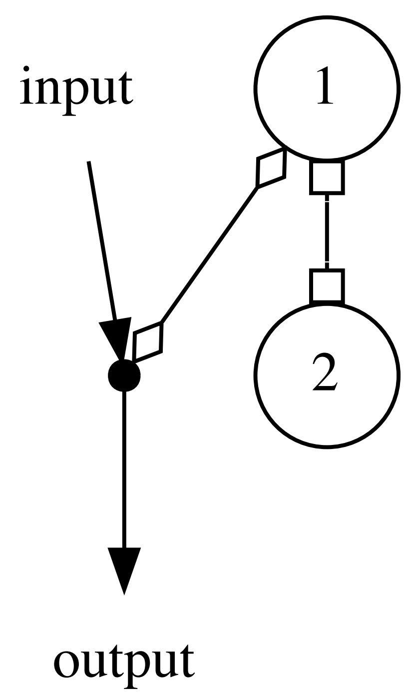
First consider systems that are diagonal in the sideband picture

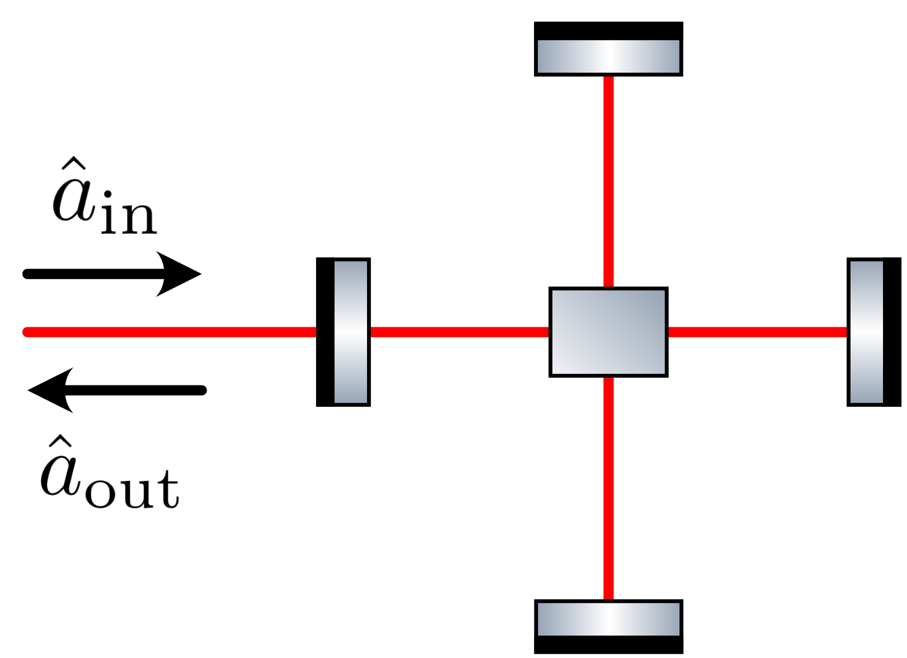
$$\mathbf{G}(i\Omega) = \begin{bmatrix} G(i\Omega) & 0 \\ 0 & G^\dagger(-i\Omega) \end{bmatrix}$$

Example: active coupled-cavity

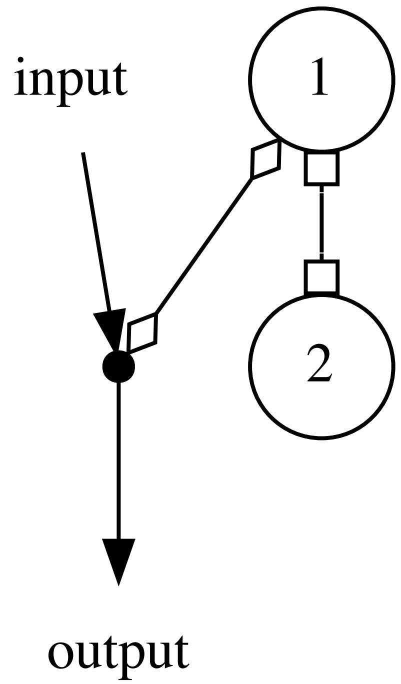
$$\mathbf{G}(i\Omega) = \frac{\Omega^2 - i\Omega\gamma_f + g^2 - \omega_s^2}{\Omega^2 + i\Omega\gamma_f + g^2 - \omega_s^2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

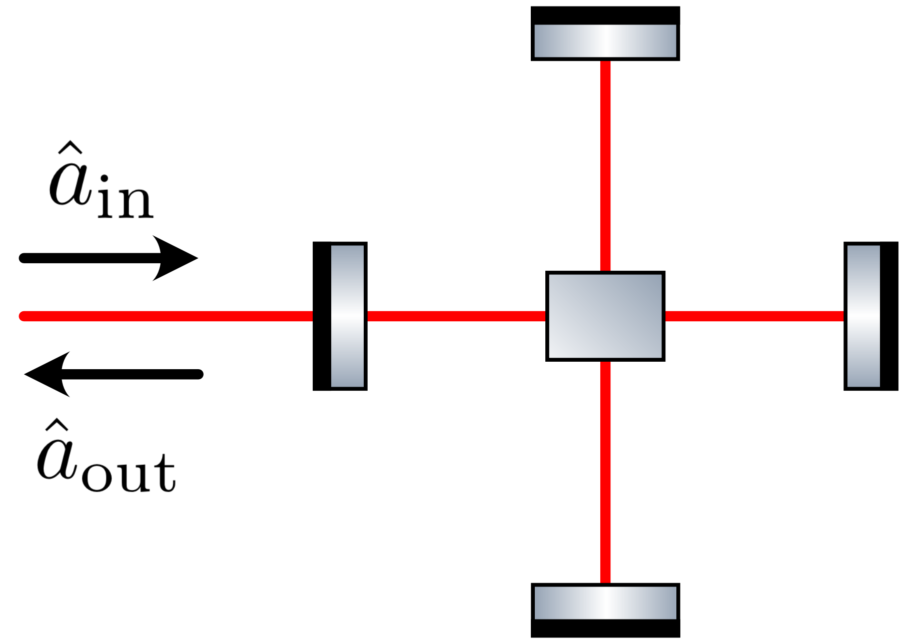
apply framework





“Passive” systems





Get $\mathcal{S} = \int_{-\infty}^{\infty} \mathrm{d}\Omega \, |G_{\text{input}\to F}(\Omega)|^2 = \pi$

For both F = 1, 2

This is the Mizuno limit: no broadband improvement

My unproven conjecture (TODO)

All physical realisations of transfer matrices that are diagonal in the sideband picture *do not surpass the Mizuno limit*

Diagonal in quadrature picture

What about systems that are diagonal in the sideband picture?

$$\mathbf{G}_q(i\Omega) = \begin{bmatrix} G_{11}(i\Omega) & 0 \\ 0 & G_{22}(i\Omega) \end{bmatrix}$$

Similar condition to before:

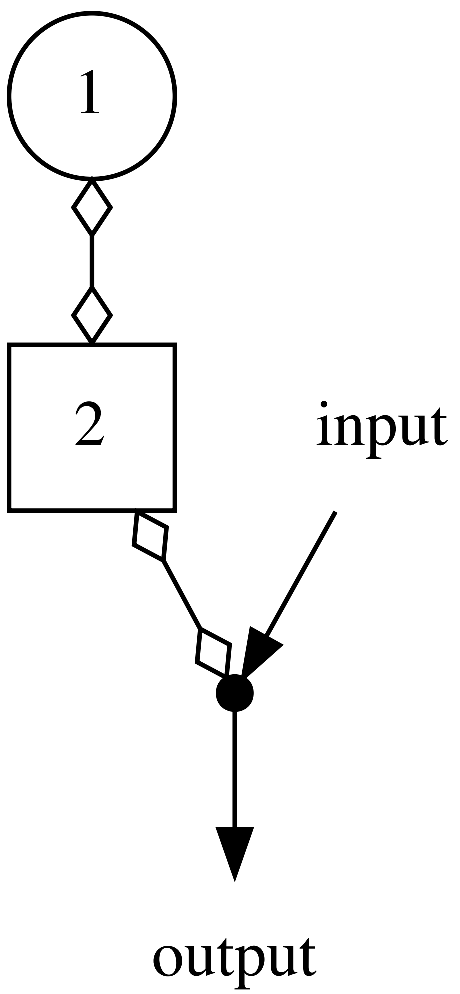
$$\mathbf{G}_q^\dagger(-i\Omega)\Theta\mathbf{G}_q(i\Omega) = \Theta$$

$$\Theta = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$$

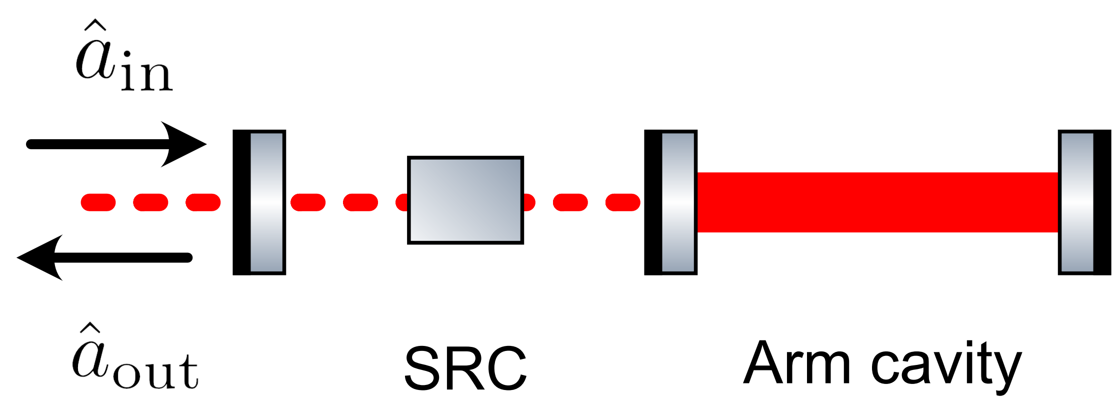
Example: Quantum Expander

[Mikhail et.al. Nature 2019]

$$\mathbf{G}_q(i\Omega) = \begin{bmatrix} \frac{\Omega(\gamma-\chi)+i\left(\Omega^2-\omega_s^2\right)}{\Omega(\gamma+\chi)-i(\Omega^2-\omega_s^2)} & 0 \\ 0 & \frac{\Omega(\gamma+\chi)+i\left(\Omega^2-\omega_s^2\right)}{\Omega(\gamma-\chi)-i(\Omega^2-\omega_s^2)} \end{bmatrix}$$

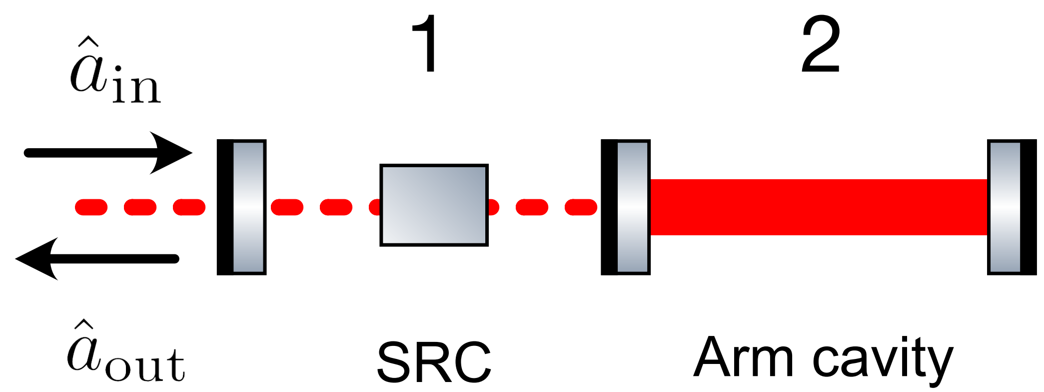


apply framework



Quantum expander

Calculate $\mathcal{S}$ for both modes 1 and 2



For mode 1 phase quadrature

$$\mathcal{S} = \frac{2\pi\gamma}{\gamma + \chi}$$

Mizuno limited

For mode 1 amplitude quadrature

$$\mathcal{S} = \frac{2\pi\gamma}{|\gamma - \chi|}$$

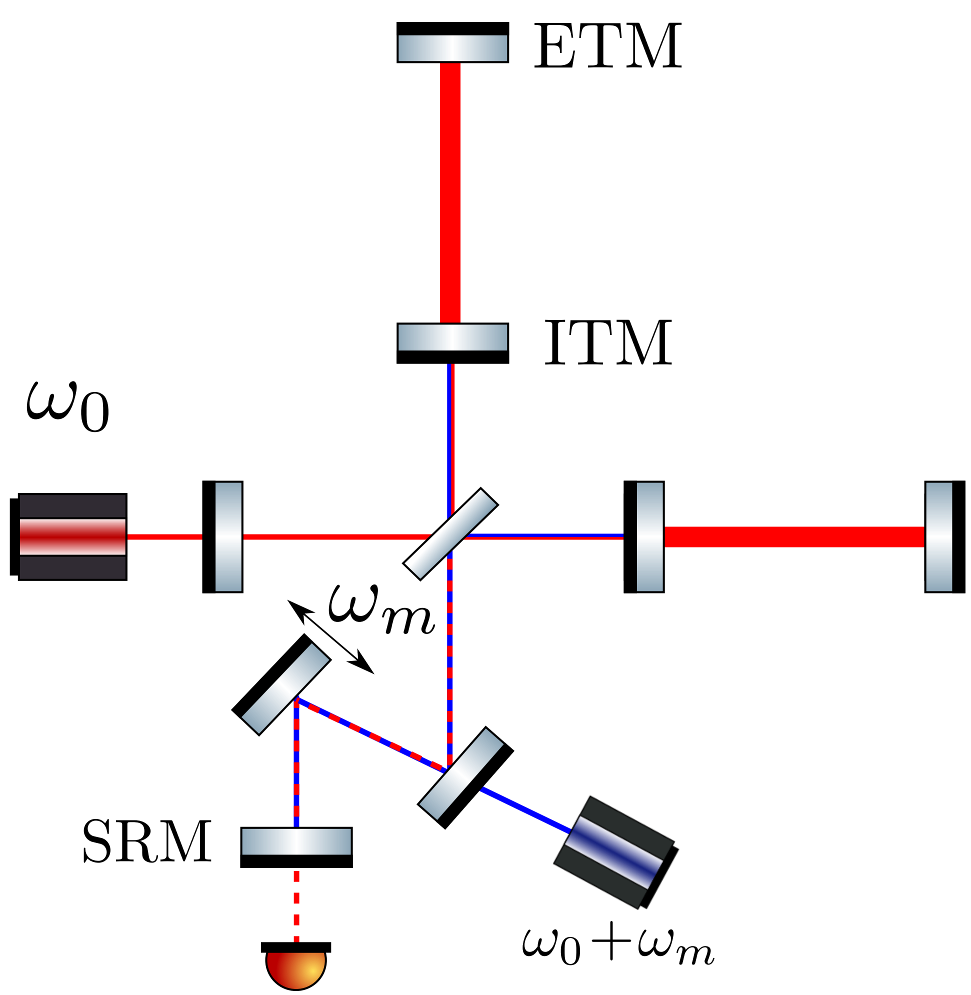
Diverges as $\chi \rightarrow \gamma$ ✓

So should couple signal to amplitude quadrature

(For mode 2 amplitude and phase swapped)

Transmission-readout setup

- More complicated third-order detector design
- Uses mechanically suspended mirror to implement bandwidth broadening

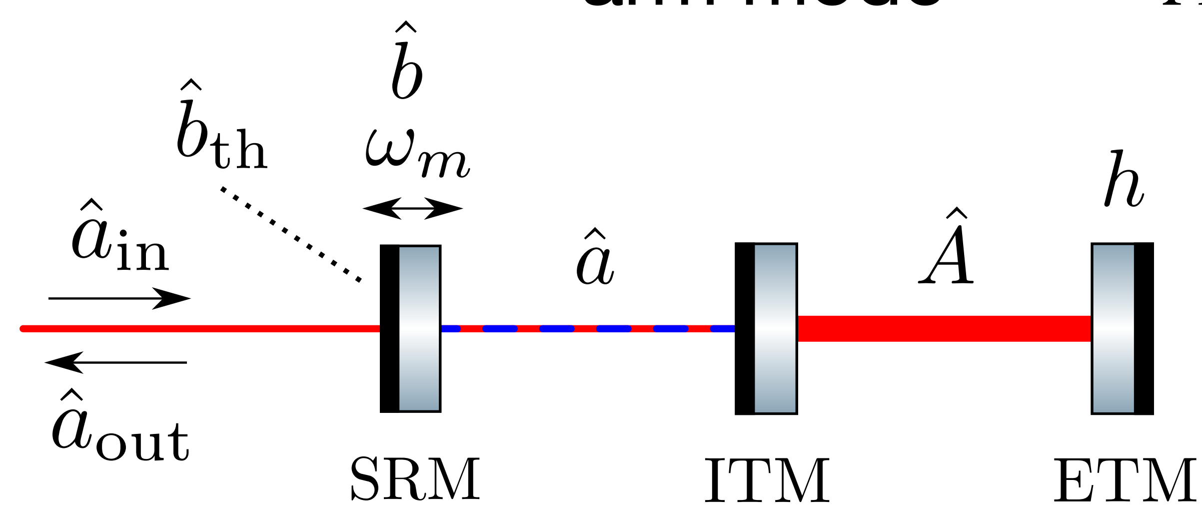


3 modes:

mech. oscillator $\hat{b}$

SRC mode $\hat{a}$

arm mode $\hat{A}$



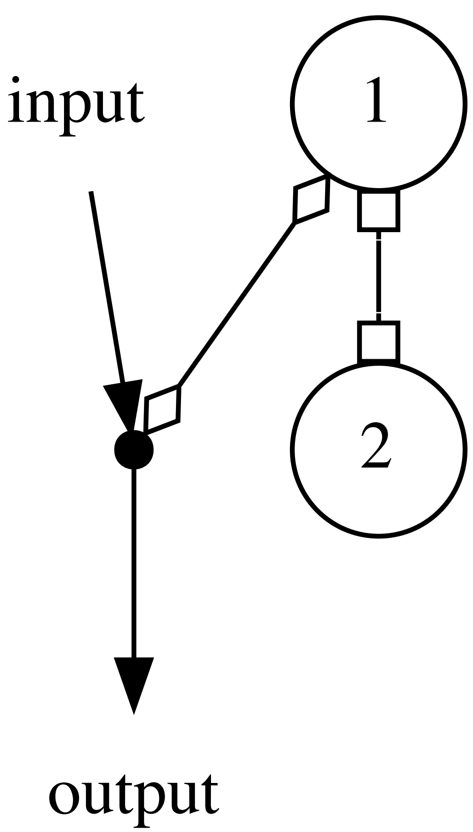
Transmission-readout setup

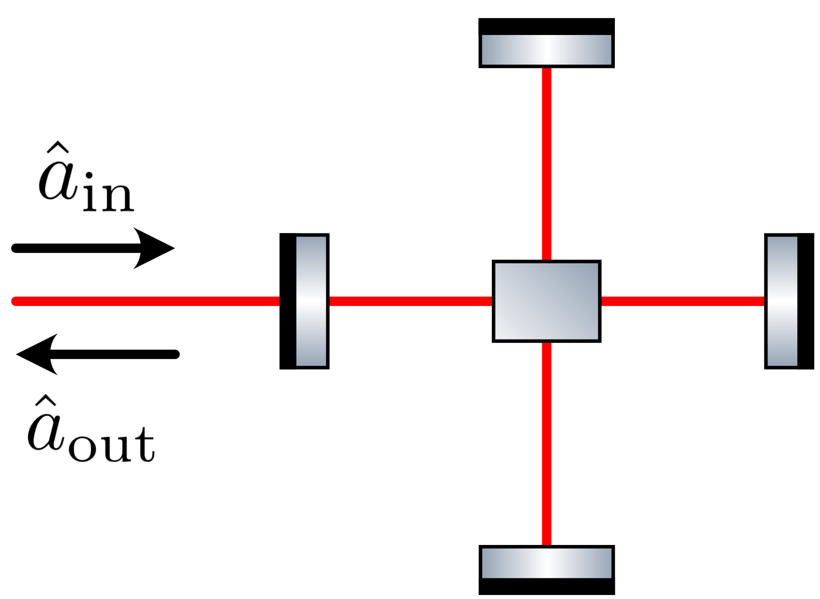
If we ignore mech. oscillator damping γ_m then we get

$$\mathbf{G}(i\Omega) = \frac{\Omega^2 - i\Omega\gamma_f + g^2 - \omega_s^2}{\Omega^2 + i\Omega\gamma_f + g^2 - \omega_s^2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

apply framework

Only 2 degrees of freedom





Same as earlier slide

No broadband improvement

Transmission-readout setup

- So the 2x2 transfer matrix *does not result in the bandwidth broadening*

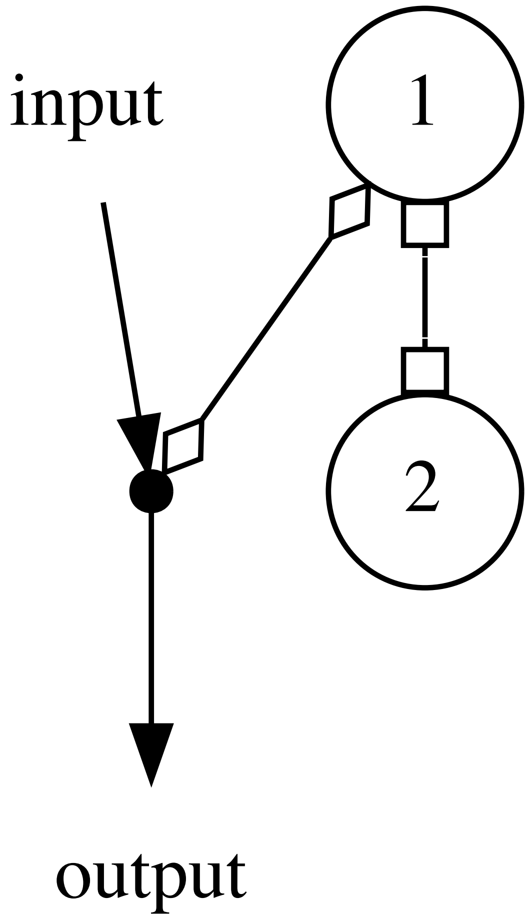
- The transfer function is second order so one of the 3 modes is missing, i.e. the mechanical damping γ_m cannot be ignored

- Instead have to consider full transfer matrix:

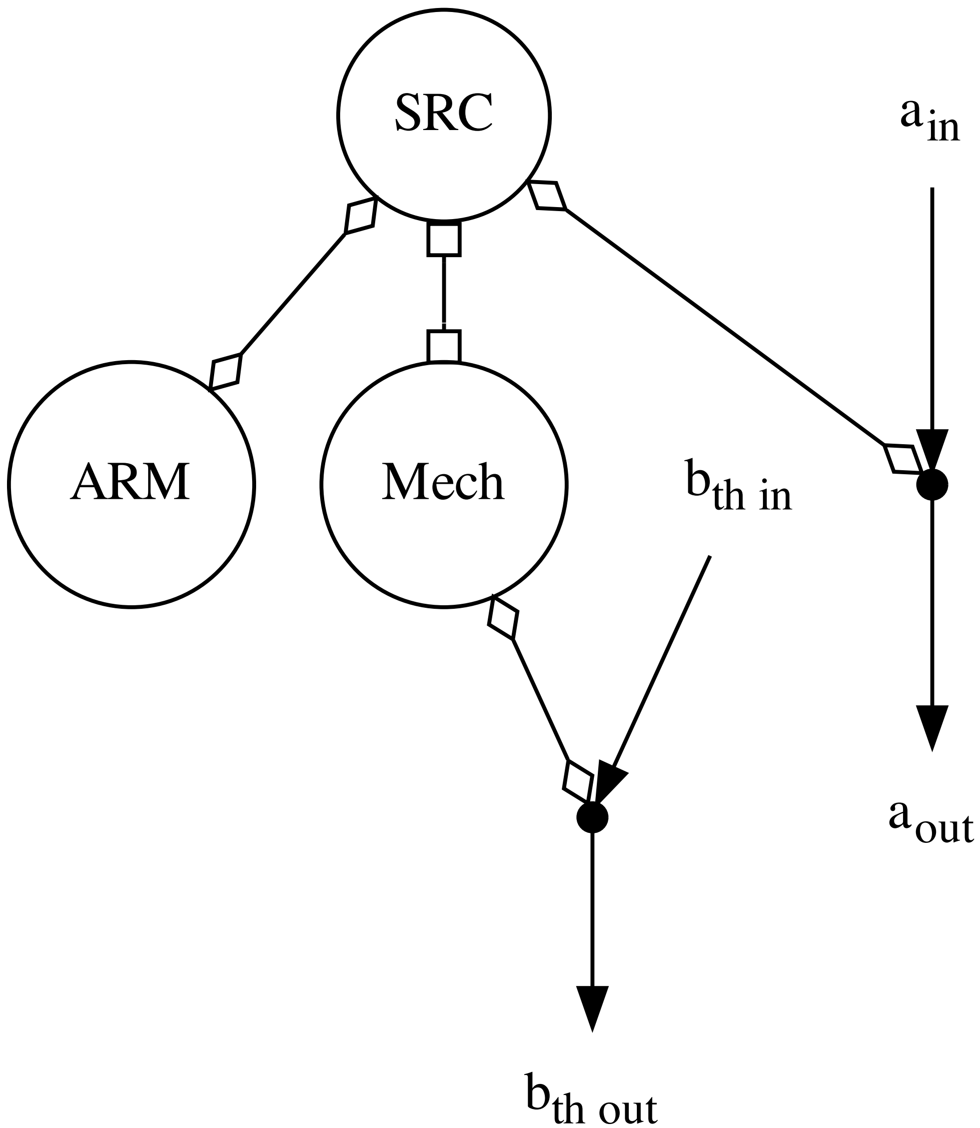
$$\begin{bmatrix} \hat{a}_{\mathrm{out}}(\Omega) \\ \hat{a}^\dagger_{\mathrm{out}}(-\Omega) \\ \hat{b}_{\mathrm{th,out}}(\Omega) \\ \hat{b}^\dagger_{\mathrm{th,out}}(-\Omega) \end{bmatrix} = \begin{bmatrix} A & 0 & 0 & -B \\ 0 & A & B & 0 \\ 0 & -B & C & 0 \\ B & 0 & 0 & C \end{bmatrix} \begin{bmatrix} \hat{a}_{\mathrm{in}}(\Omega) \\ \hat{a}^\dagger_{\mathrm{in}}(-\Omega) \\ \hat{b}_{\mathrm{th,in}}(\Omega) \\ \hat{b}^\dagger_{\mathrm{th,in}}(-\Omega) \end{bmatrix}$$

Transmission-readout setup

from 2dof tf matrix



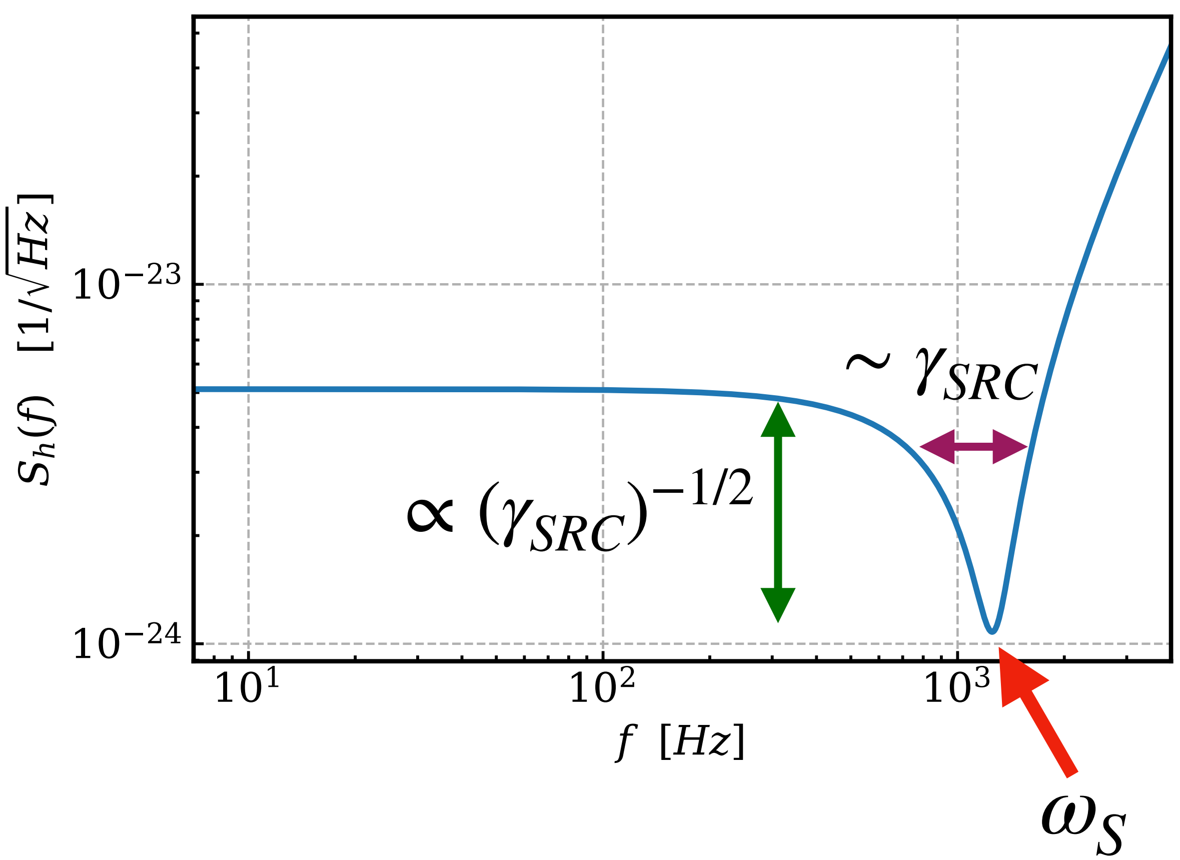
expected from 3dof tf matrix (TODO)

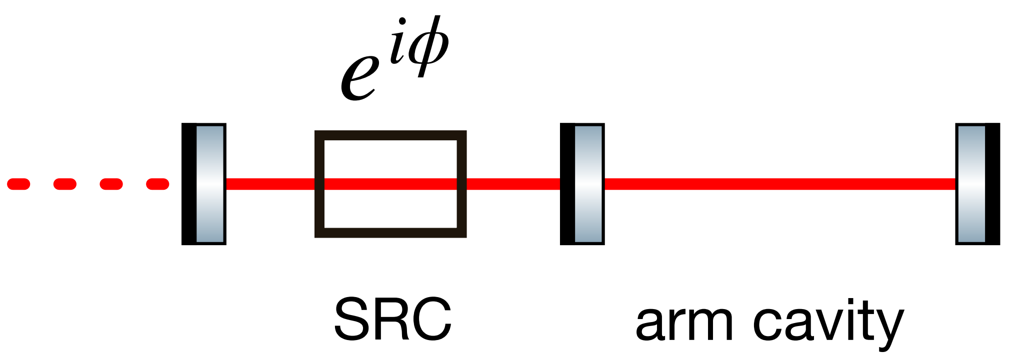


The computation proves difficult with third-order MIMO transfer matrix

Coupled-cavity broadener design

Can use framework to broaden coupled cavity resonance





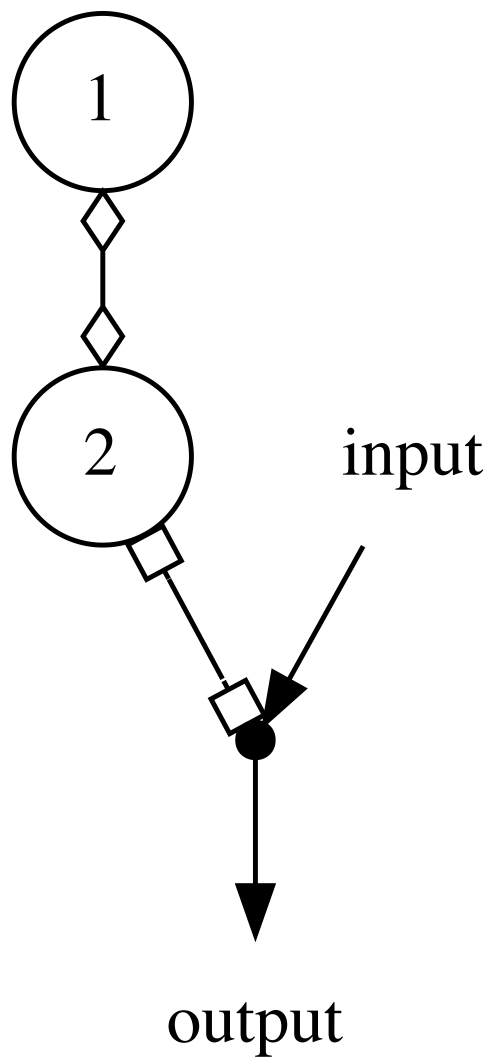
$$G(i\Omega) = e^{i\phi} \approx \frac{2\gamma_1\Omega - i\tau_1(\Omega^2 - \omega_s^2)(\gamma_1 - \gamma_1')}{2\gamma_1\Omega + i\tau_1(\Omega^2 - \omega_s^2)(\gamma_1 - \gamma_1')}$$

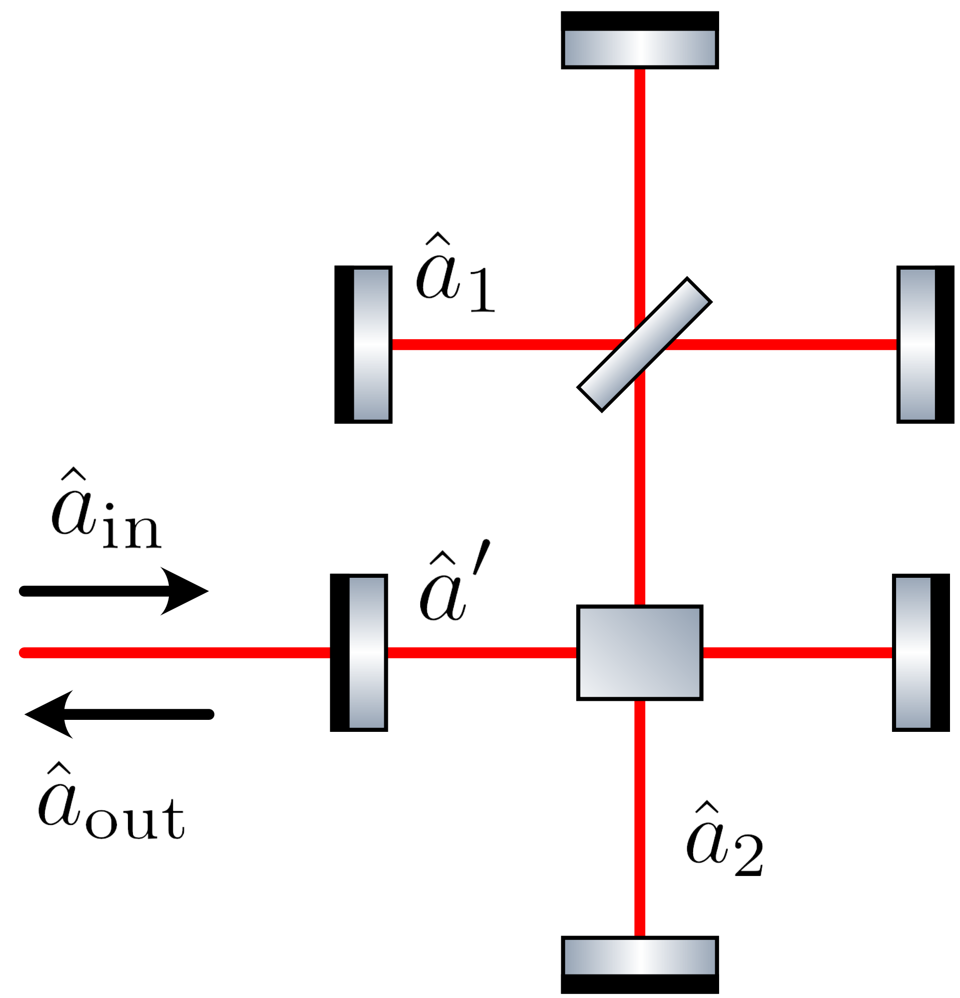
Would like to boost width of dip *without sacrificing depth*

How to realise the filter $G(i\Omega)$*?*

Coupled-cavity broadener realisation

The realisation is another coupled cavity but with an active element





Summary

- I described a novel method for realising quantum optical systems from their frequency-domain transfer matrices
- This method has wide-ranging implications for how future quantum measurement devices could be designed
- I have shown how the QCRB can be minimised for the resulting realisation to produce an optimal detector design
- I have applied this to investigate first & second order transfer matrices, recovering well-known designs
- I will apply this to a third order MIMO transfer matrix to recover the transmission-readout setup design

Supplementary slides

General second order TF (TODO)

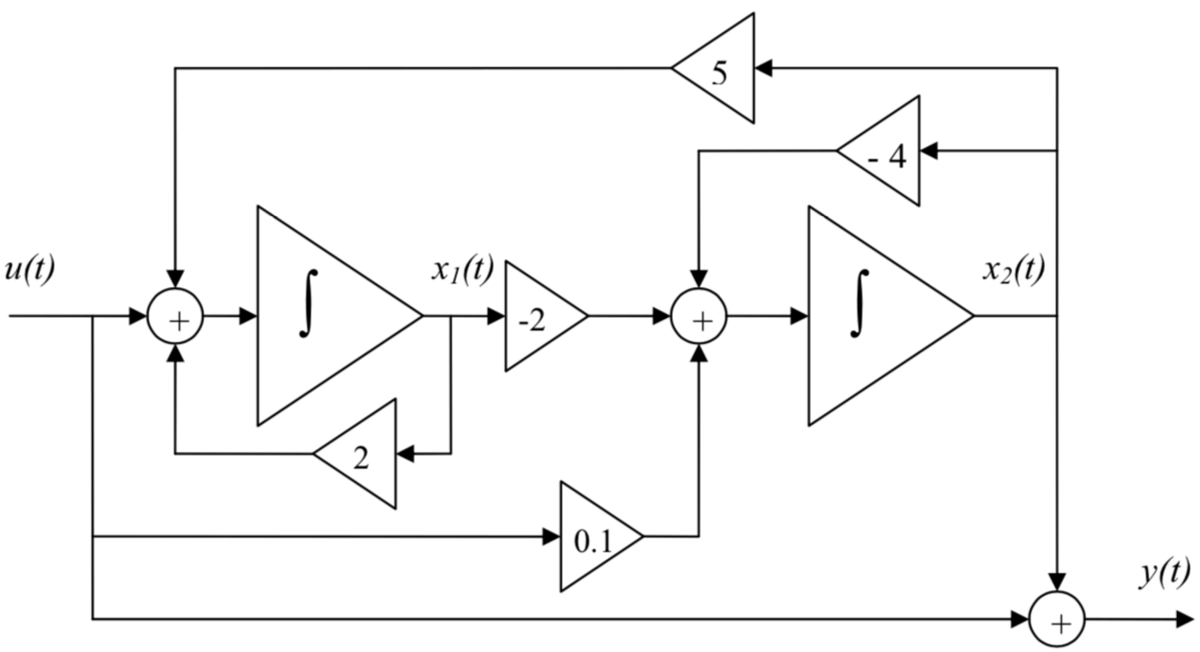
Want to recover quantum expander from general second order transfer matrix with params α, β, γ

$$\mathbf{G}_q(s) = \frac{s^2 + \alpha s + \beta}{s^2 + \alpha s + \gamma}$$

Realizing classical systems

- For *classical systems* it is easy to realize *arbitrary* state-space representations using integrators and feedback

$$
\begin{aligned}
\frac{dx(t)}{dt} &= \begin{bmatrix} 2 & 5 \\ -2 & -4 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0.1 \end{bmatrix} u(t), \\
y(t) &= \begin{bmatrix} 0 & 1 \end{bmatrix} x(t) + u(t),
\end{aligned}
$$



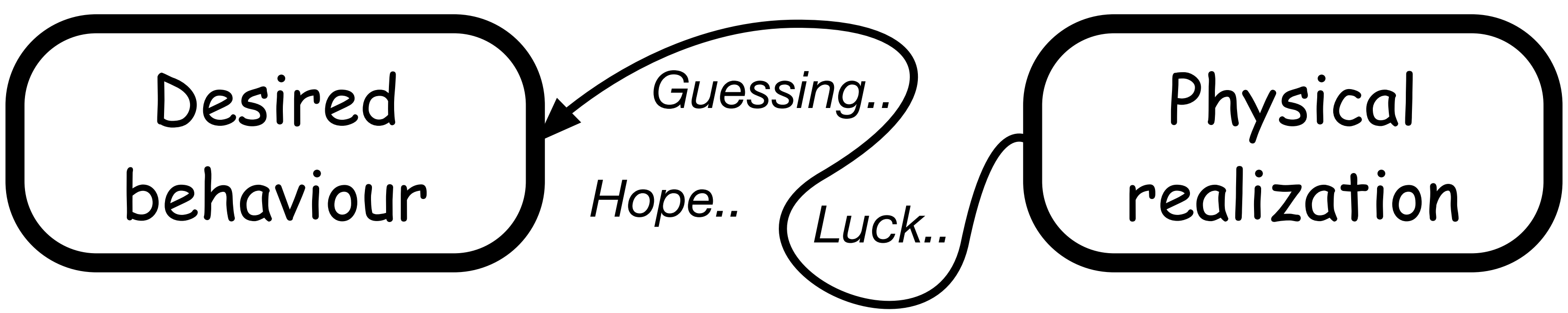
- For *quantum systems* most state-space representations <u>not physically possible</u>

Need to conserve $[x_i, x_j]$

Current limits of our quantum realization techniques

Say we want to build quantum system with a *desired transfer function* or general behaviour…

Current methods



New method

Quick intro to state-space representation

Used to dealing with frequency-domain transfer functions

$$y(s) = G(s)u(s)$$

y outputs u inputs

Control theorists prefer time-domain *state-space representation*

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

y outputs u inputs

x internal system state

A system dynamics matrix

B input coupling matrix

C output coupling matrix

D "Direct feed" matrix

(everything linear here)

State-space degeneracy

- **(A, B, C, D) -> G(s)** is *many-to-one* mapping
  - *Many* state-space reps, *(even non-physical ones)*, correspond to *one* transfer function **G(s)**

- Therefore, actual (A, B, C, D) gives physical insight
  - *bijection* exist between *(A, B, C, D)* and *full Hamiltonian for system*

Finding the state-space rep for transfer func.

Identify desired transfer function **(freq. domain)**

Find a *state-space* representation **(time domain)**

Find the physically realizable one

Example: tuned cavity

$$G(i\omega) = \frac{i\omega - \gamma}{i\omega + \gamma}$$

$$\dot{\vec{x}} = A\vec{x} + B\vec{u}$$

$$\vec{y} = C\vec{x} + D\vec{u}$$

No unique mapping from G to (A, B, C, D)!

Need to ensure that

$$d[x_i, x_j] = 0$$

Constrains (A, B, C, D)

Constraints on (A, B, C, D) for physical realizability

It can be shown that $d[x_i, x_j] = 0$ *(quantum îto product)*

implies that $AJ + JA^\dagger + BJB^\dagger = 0$

$$JC^\dagger + BJD^\dagger = 0$$

if we use cavity mode operators (annihilation & creation)

$x_i = a, \quad x_j = a^\dagger$ and $[a, a^\dagger] = 1$

then $J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ J is *"commutation matrix"*

Example: unstable filter

- Unstable filter = optomechanical device w/ *negative dispersion* $\phi \propto -\Omega\tau$

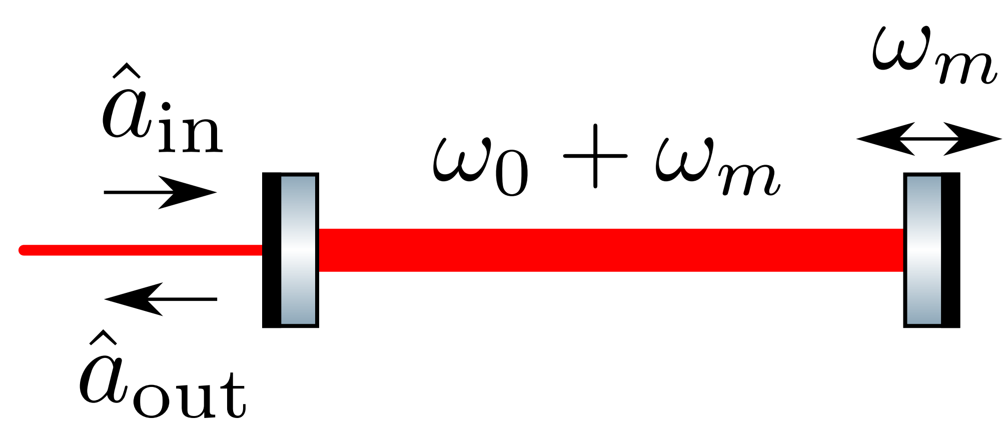
- Can be used to broaden bandwidth of GW detector *without sacrificing sensitivity*

$$G(s) = \frac{s + 2}{s - 2}$$

opp. sign to tuned cavity

Known physical realization:

cavity coupled to mirror via off-resonant pump $\omega_0 + \omega_m$



Finding realizable state-space representation

Guess a *state-space representation*, not necessarily physical

$$\begin{bmatrix} \dot{a} \\ \dot{a}^\dagger \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} a \\ a^\dagger \end{bmatrix} + \begin{bmatrix} u \\ u^\dagger \end{bmatrix}$$

$$\begin{bmatrix} Y \\ Y^\dagger \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} a \\ a^\dagger \end{bmatrix} + \begin{bmatrix} u \\ u^\dagger \end{bmatrix}$$

Found using *"Controllable Canonical Form"*

Transform matrices to obey

$$AJ + JA^\dagger + BJB^\dagger = 0$$
$$JC^\dagger + BJD^\dagger = 0$$

(I omitted details but it is easy to do)

Find *physically realizable form*

$$\begin{bmatrix} \dot{a} \\ \dot{a}^\dagger \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} a \\ a^\dagger \end{bmatrix} + 2 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} u \\ u^\dagger \end{bmatrix}$$

$$\begin{bmatrix} Y \\ Y^\dagger \end{bmatrix} = 2 \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ a^\dagger \end{bmatrix} + \begin{bmatrix} u \\ u^\dagger \end{bmatrix}$$

Final steps

From this (A, B, C, D) can find (H, L)

H internal system Hamiltonian

L linear coupling matrix

(Usually) have a clear physical realization

For our state-space find $(\hbar = 1)$

$H = 0$ (chose *rotating frame* w.r.t resonant freq.)

$$L = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix} \implies H_{int} = \frac{\sqrt{\gamma_2}}{2}(ab + a^\dagger b^\dagger)$$

Same physics as unstable filter

adiabatically eliminated high frequency auxiliary mode
(e.g. mechanically suspended mirror)