FOURIER SERIES

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1. Periodicity

A function $f(x)$ is said to be periodic with period L if it has the same value at x and $x+L$,

$$f(x+L) = f(x)$$

If a function has a period L then it clearly also has period nL for any positive integer n, as shown by,

$$f(x+2L) = f\left((x+L)+L\right) = f(x+L) = f(x)$$

The minimum period of a periodic function, which cannot be divided into any more sections, is called the fundamental period of the function. For example the fundamental period of $f(x) = \sin x$ is 2π.

If we know the value of a periodic function $f(x)$ over an interval of the length of its fundamental period L, we know the value of the function everywhere, as it is just that interval repeated.

2. Fourier Series

Sine and cosine are periodic functions with a fundamental period of 2π. These functions are useful as we know much about them, for example we can integrate and differentiate them, and we can evaluate them for any value of x. Therefore it would be useful to be able to use them to represent a periodic function, much as we do with the maclaurin series except that instead of using a polynomial, we are using sines and cosines.

$\sin nx$ and $\cos nx$ are also periodic over 2π, as we have shown earlier, as long as n is a positive integer. It follows that $\sin \frac{2\pi nx}{L}$ and $\cos \frac{2\pi nx}{L}$ are periodic over length L. We can see this because if $x = L$, then it cancels to give $\sin 2\pi n$, which we know represents a full cycle as the period of $\sin nx$ is 2π. Think of this like the wavenumber, $k = \frac{2\pi}{\lambda}$.

As we have mentioned, the idea of the Fourier series is to be able to write function $f(x)$ of period L in terms of sines and cosines of the same period L but different amplitudes.

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left[a_n \cos \frac{2\pi nx}{L} + b_n \sin \frac{2\pi nx}{L} \right]$$

We will show that we can find these coefficients for a function $f(x)$ by evaluating the overlap integral over a full period L,

$$a_0 = \frac{2}{L}\int_0^L f(x)dx$$
$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{2\pi nx}{L}dx$$
$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{2\pi nx}{L}dx$$

Note that the limits do not have to be from 0 to L, but just need to be over any length of the fundamental period, so for example can be from $-L/2$ to $L/2$.

3. Orthogonality Relations

In this section we will prove that the functions $\sin\frac{2\pi nx}{L}$ (where $n = 1, 2, 3\dots$) and $\cos\frac{2\pi nx}{L}$ (where $n = 0, 1, 2\dots$) can be considered mutually orthogonal, such that if $\phi(x)$ and $\psi(x)$ are two different functions from the set then,

$$\int_0^L \phi(x)\psi(x)dx = 0$$

To show this we will consider the three cases where, $\phi = \cos$ and $\psi = \cos$, $\phi = \sin$ and $\psi = \sin$, and $\phi = \sin$ and $\psi = \cos$.

First consider the integral,

$$I_{mn} = \int_0^L \cos\frac{2\pi mx}{L}\cos\frac{2\pi nx}{L}dx$$

For this we will use a trigonometric product identity (which can be obtained from the addition formula),

$$\cos A\cos B = \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]$$

Plugging this into our integral,

$$\begin{aligned} I_{mn} &= \frac{1}{2}\int_0^L \left[\cos\frac{2\pi(m+n)x}{L} + \cos\frac{2\pi(m-n)x}{L}\right]dx \\ &= \frac{L}{4\pi}\left[\frac{1}{m+n}\sin\frac{2\pi(m+n)x}{L} + \frac{1}{m-n}\sin\frac{2\pi(m-n)x}{L}\right]_0^L \end{aligned}$$

This has different values for different m and n. For $m \neq n$, $I_{mn} = 0$, since both of the sine functions are periodic over the limits of L. For $m = n \neq 0$, $I_{mn} = \frac{1}{2}L$, and finally for $m = n = 0$, $I_{mn} = L$.

Next we need to evaulate the integral,

$$J_{mn} = \int_0^L \sin\frac{2\pi mx}{L}\sin\frac{2\pi nx}{L}dx$$

This time we need the product identity for sine,

$$\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$$

The result is simply the negative of what we had before,

$$J_{mn} = \frac{1}{2}\int_0^L \left[\cos\frac{2\pi(m-n)x}{L} - \cos\frac{2\pi(m+n)x}{L}\right]dx$$

This time the results are the same unless $m = n = 0$. We have for $m \neq n$, that $J_{mn} = 0$. For $m = n$ we have $J_{mn} = \frac{1}{2}L$ (even if $m = n = 0$).

Finally we need to consider the integral for sine and cosine. This time we need the product identity $\sin A \cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$,

$$\begin{aligned}
K_{mn} &= \int_0^L \sin\frac{2\pi mx}{L}\cos\frac{2\pi nx}{L}dx \\
&= \frac{1}{2}\int_0^L \left[\sin\frac{2\pi(m+n)x}{L} + \sin\frac{2\pi(m-n)x}{L}\right]dx \\
&= \frac{L}{4\pi}\left[-\frac{1}{m+n}\cos\frac{2\pi(m+n)x}{L} - \frac{1}{m-n}\cos\frac{2\pi(m-n)x}{L}\right]_0^L \\
&= 0
\end{aligned}$$

Again, by periodicity, $K_{mn} = 0$, for all m and n.

We have shown therefore that the sine and cosine functions are orthogonal. This is similar to the scalar product, for example let's say we have a vector $\mathbf{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$, if we want the x component of the vector we can use the scalar product $\hat{\imath}\cdot\mathbf{A} = A_x$. We can use this similarly to find the coefficients in the fourier series, but instead using an integral.

If we take the formula for the Fourier series and multiply both sides by $\cos\frac{2\pi nx}{L}$ and then integrate,

$$\begin{aligned}
\int_0^L f(x)\cos\frac{2\pi nx}{L}dx = \frac{1}{2}a_0\int_0^L \cos\frac{2\pi nx}{L}dx \\
+ \sum_{m=1}^{\infty}\left[a_m\int_0^L \cos\frac{2\pi mx}{L}\cos\frac{2\pi nx}{L}dx + b_m\int_0^L \sin\frac{2\pi mx}{L}\cos\frac{2\pi nx}{L}dx\right]
\end{aligned}$$

We know that the first term goes to zero due to periodicity. We also know that the last term inside the sum will be zero for all m and n. Finally we know that the first term in the sum will be zero if $m \neq n$, but if $m = n \neq 0$

it will be $\frac{1}{2}L$, so we can represent this using the Kronecker delta $\delta_m n$. Now the integral goes to,

$$\int_0^L f(x)\cos\frac{2\pi nx}{L}dx = \sum_{m=1}^{\infty} a_m \frac{1}{2}\delta mn$$
$$= \frac{1}{2}La_n$$

Therefore we have shown that the coefficient a_n can be written in the form,

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{2\pi nx}{L}dx$$

for $n = 1, 2, 3 \ldots$.

If we instead multiply by $\sin\frac{2\pi nx}{L}$ and integrate, we get the same result,

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{2\pi nx}{L}dx$$

for $n = 0, 1, 2 \ldots$.

To find the coefficient a_0, we take the formula for the Fourier series and integrate it alone,

$$\int_0^L f(x)dx = \frac{1}{2}a_0\int_0^L dx = \frac{1}{2}La_0$$

Therefore,

$$a_0 = \frac{2}{L}\int_0^L f(x)dx$$

4. Finding the Fourier Series

In the following sections we will apply what we know to a few functions and evaluate their Fourier series.

First let's consider what happens if our function $f(x)$, which we wish to evaluate the Fourier series for, is even such that $f(x) = f(-x)$. In this case,

$$b_n = \frac{2}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} f(x)\sin\frac{2\pi nx}{L}dx = 0$$

so only a_n terms remain.

Similarly, if $f(x)$ is odd such that $f(x) = -f(-x)$,

$$a_n = \frac{2}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} f(x)\cos\frac{2\pi nx}{L}dx = 0$$

so only b_n terms remain.

5. Finding the Fourier Series: Example 1

Consider the periodic function which is defined over the interval $x \in \left(-\frac{L}{2}, \frac{L}{2}\right)$ as $f(x) = |x|$. This can be written equivalently as,

$$f(x) = \begin{cases} -x - \frac{L}{2} \leq & x \leq 0 \\ x0 \leq & x \leq \frac{L}{2} \end{cases}$$

This function is clearly even, as $f(x) = f(-x)$, so we only get cosine (a_n) coefficients,

$$a_n = \frac{2}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} |x|\cos\frac{2\pi nx}{L}dx$$

Since the function is even, this means that it is symmetric over the limits, so we can just integrate over half the limits (for example from 0 to $L/2$ instead of $-L/2$ to $L/2$) and then multiply by two,

$$a_n = \frac{4}{L}\int_{0}^{\frac{L}{2}} x\cos\frac{2\pi nx}{L}dx$$

To simplify solving this we make the solution $y = \frac{2\pi nx}{L}$ and thus $x = \frac{L}{2\pi n}y$ and $dx = \frac{L}{2\pi n}dy$. The integral therefore becomes,

$$\begin{aligned} a_n &= \frac{4}{L}\left(\frac{L}{2\pi n}\right)^2 \int_0^{\pi n} y\cos y dy \\ &= \frac{L}{\pi^2 n^2}[y\sin y + \cos y]_0^{\pi n} \\ &= \frac{L}{\pi^2 n^2}[\cos\pi n - 1] \\ &= \frac{L}{\pi^2 n^2}[(-1)^n - 1] \end{aligned}$$

In the third line we noted that $\cos\pi n$ will be negative for odd integers of n, and positive for even integers of n. If it is positive then it cancels out with the other 1 to give zero in brackets. If it is negative it will add up to give two inside the brackets. This behaviour is summarized here,

$$a_n = \begin{cases} -\frac{2L}{\pi^2 n^2} & \text{n is odd} \\ 0 & \text{n is even} \end{cases}$$

Finally we need to calculate our zeroth coefficient, a_0,

$$a_0 = \frac{4}{L}\int_0^{\frac{L}{2}} x dx = \frac{4}{L}\left[\frac{1}{2}x^2\right]_0^{\frac{L}{2}} = \frac{L}{2}$$

We now have all the information we need to subsitute into the Fourier series,

$$\begin{aligned} f(x) &= \frac{L}{4} - \frac{2L}{\pi^2}\sum_{\text{n odd}} \frac{1}{n^2}\cos\frac{2\pi n x}{L} \\ &= \frac{L}{4} - \frac{2L}{\pi^2}\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}\cos\frac{2\pi(2k+1)x}{L} \end{aligned}$$

By letting $x = 0$ we can find the value of the sum of $\frac{1}{(2k+1)^2}$,

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8}$$

6. The Riemann Zeta Function

In this section we will explore the Riemann Zeta function, but this is not examinable material. The Riemann Zeta function is defined as,

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} \qquad \Re z > 1$$

For other values of z the value of ζz can be found by using analytic continuoations. From the definition we see that,

$$\begin{aligned} \zeta(2) &= 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} + \dots \\ \frac{1}{4}\zeta(2) &= \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{5^2} + \dots \\ \zeta(2) - \frac{1}{4}\zeta(2) = \frac{3}{4}\zeta(2) &= 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots \end{aligned}$$

We can see that the last sequence is just equal to the sum that we found at the end of the last section, as it is just hte sum of all odd n squared. Therefore $\frac{3}{4}\zeta(2) = \frac{\pi^2}{8}$, and it follows that,

$$\zeta(2) = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots = \frac{\pi^2}{6}$$

$\zeta(2n)$ where n is a positive integer, is always a rational multiple of π^{2n}. The general formula is given by,

$$\zeta(2n) = \frac{2^{2n-1}|B_{2n}|}{2n!}\pi^{2n}$$

where B_k are the Bernoulli numbers defined as,

$$\frac{z}{e_z - 1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} z^k$$

The only non-zero values of B_k for odd k is B_1, all others are odd. The first few Bernoulli numbers are,

$$B_0 = 1 \qquad B_1 = -\frac{1}{2} \qquad B_2 = \frac{1}{6} \qquad B_4 = -\frac{1}{30} \qquad B_6 = \frac{1}{42}$$

7. Parseval's Theorem

In this section we return to examinable stuff. Consider the Fourier series of a periodic function $f(x)$ given by,

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left[a_n \cos\frac{2\pi nx}{L} + b_n \sin\frac{2\pi nx}{L} \right]$$

If we square $f(x)$ we get a complicated double series. It we then take the integral over a full period length L then orthogonality means that we only get the diagonal (squared) terms, as we have shown that the integral over a full period of a product of two different cosines, two different sines, or a sine and a cosine is zero,

$$\int_0^L f^2(x)dx = \frac{1}{4}a_0^2 \int_0^L dx + \sum_{n=1}^{\infty} \left[\int_0^L a_n^2 \cos^2\frac{2\pi nx}{L} dx + \int_0^L b_n^2 \sin^2\frac{2\pi nx}{L} dx \right]$$

Or since $\sin^2 x + \cos^2 x = 1$ this can be written as,

$$\int_0^L f^2(x)dx = \frac{1}{2}L \left[\frac{1}{2}a_0^2 + \sum_{n=1}^{\infty} \left(a_n^2 + b_n^2 \right) \right]$$

Note that the limits on the integral can be over any full period length, and don't have to be between 0 and L.

8. Finding the Fourier Series: Example 3

The periodic function $h(x)$ is defined over the interval $x \in (-\frac{L}{2}, \frac{L}{2})$ as,

$$h(x) = \frac{1}{2}x^2 sgnx - \frac{1}{4}Lx$$

The function is clearly odd, therefore we only need to calculate the b_n coefficients, as the a_n coefficients will all be zero. Using our formula from before to find b_n, as well as using the substitution $y = \frac{2\pi nx}{L}$ again from example one,

$$\begin{aligned}
b_n &= \frac{2}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} h(x)\sin\frac{2\pi nx}{L}dx \\
&= \frac{4}{L}\int_{0}^{\frac{L}{2}} h(x)\sin\frac{2\pi nx}{L}dx \\
&= \frac{4}{L}\int_{0}^{\frac{L}{2}} \left[\frac{1}{2}x^2 - \frac{1}{4}Lx\right]\sin\frac{2\pi nx}{L}dx \\
&= \frac{4}{L}\int_{0}^{\pi n} \left[\frac{1}{2}\left(\frac{L}{2\pi n}\right)^3 y^2\sin y - \frac{1}{4}L\left(\frac{L}{2\pi n}\right)^2 y\sin y\right] dy \\
&= \frac{L^2}{4\pi^3 n^3}\int_{0}^{\pi n} y^2 \sin y dy - \frac{L^2}{4\pi^2 n^2}\int_{0}^{\pi n} y\sin y dy \\
&= \frac{L^2}{4\pi^3 n^3}\left[-y^2\cos y + 2y\sin y + 2\cos y\right]_0^{\pi n} - \frac{L^2}{4\pi^2 n^2}\left[-y\cos y + \sin y\right]_0^{\pi n} \\
&= \frac{L^2}{4\pi^3 n^3}\left[-(\pi n)^2\cos\pi n + 2\cos\pi n - 2\right] - \frac{L^2}{4\pi^2 n^2}\left[-\pi n\cos\pi n\right] \\
&= \frac{L^2}{2\pi^3 n^3}\left[\cos\pi n - 1\right] \\
&= \frac{L^2}{2\pi^3 n^3}\left[(-1)^2 - 1\right]
\end{aligned}$$

We therefore have that the coefficients of the sine term are given by,

$$a_n = \begin{cases} -\frac{L^2}{\pi^3 n^3} & \text{n odd} \\ 0 & \text{n even} \end{cases}$$

The Fourier series can then be written as,

$$\begin{aligned} h(x) &= -\frac{L^2}{\pi^3}\sum_{\text{n odd}}\frac{1}{n^3}\sin\frac{2\pi nx}{L} \\ &= -\frac{L^2}{\pi^3}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^3}\sin\frac{2\pi(2k+1)x}{L} \end{aligned}$$

Parseval's theorem for this function is given by,

$$\begin{aligned} \int_{-\frac{L}{2}}^{\frac{L}{2}} h^2(x)dx &= \frac{1}{2}L\sum_{n=1}^{\infty}b_n^2 \\ &= \frac{1}{2}L\left(\frac{L^4}{\pi^6}\right)\sum_{\text{n odd}}\frac{1}{n^6} \\ &= \frac{L^5}{2\pi^6}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^6} \end{aligned}$$

By evaluating the left side of the equation we get,

$$\begin{aligned} \int_{-\frac{L}{2}}^{\frac{L}{2}} h^2(x)dx &= 2\int_0^{\frac{L}{2}}\left[\frac{1}{2}x^2-\frac{1}{4}Lx\right]^2 dx \\ &= 2\int_0^{\frac{L}{2}}\left[\frac{1}{4}x^6-\frac{1}{4}Lx^3+\frac{1}{16}L^2x^2\right] dx \\ &= 2\left[\frac{1}{20}x^5-\frac{1}{16}Lx^4+\frac{1}{48}L^2x^3\right]_0^{\frac{L}{2}} \\ &= 2\left[\frac{1}{20}\frac{L^5}{32}-\frac{1}{16}\frac{L^3}{16}+\frac{1}{48}\frac{L^5}{8}\right] \\ &= \frac{L^5}{1920} \end{aligned}$$

Therefore we have,

$$\frac{L^5}{1920} = \frac{L^5}{2\pi^6}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^6}$$

Which we can rearrange to find an expression for the sum alone,

$$\sum_{k=0}^{\infty}\frac{1}{(2k+1)^6} = \frac{\pi^6}{960}$$

9. Finding the Fourier series: Example 4

Consider the periodic function defined on the interval $x \in (0, 2\pi)$ as $f(x) = e^{ax}$. This function is neither even nor odd so we have to calculate both a_n and b_n coefficients. Our a_n (cosine) coefficients are given by,

$$\begin{aligned}
a_n &= \frac{2}{2\pi}\int_0^{2\pi} e^{ax}\cos nx dx \\
&= \frac{1}{\pi}\Re\int_0^{2\pi} e^{ax}\cos nx dx \\
&= \frac{1}{\pi}\Re\int_0^{2\pi} e^{(a+in)x} dx \\
&= \frac{1}{\pi}\Re\left[\frac{e^{(a+in)x}}{a+in}\right]_0^{2\pi} \\
&= \frac{1}{\pi}\Re\left[\frac{e^{2\pi(a+in)}-1}{a+in}\right] \\
&= \frac{1}{\pi}\Re\left[\frac{e^{2\pi a}e^{2\pi in}-1}{a+in}\right] \\
&= \frac{e^{2\pi a}-1}{\pi}\Re\left[\frac{1}{a+in}\right] \\
&= \frac{e^{2\pi a}-1}{\pi}\Re\left[\frac{a-in}{a^2+n^2}\right]
\end{aligned}$$

The cosine coefficients are therefore given by,

$$a_n = \frac{1}{\pi}\left[e^{2\pi a}-1\right]\frac{a}{n^2+a^2}$$

To find the sine (b_n) coefficients we just need to take the imaginary part of the integral instead,

$$\begin{aligned}
b_n = \frac{1}{\pi}\int_0^{2\pi} e^{ax}\sin nx dx &= \frac{1}{\pi}\Im\int_0^{2\pi} e^{(a+in)x} dx \\
&= \frac{1}{\pi}\left[e^{2\pi a}-1\right]\Im\left[\frac{a-in}{a^2+n^2}\right] \\
&= -\frac{1}{\pi}\left[e^{2\pi a}-1\right]\frac{n}{n^2+a^2}
\end{aligned}$$

Finally we need to find the coefficient when $n = 0$ which can be found simply by setting $n = 0$ in our expression for a_n,

$$a_0 = \frac{1}{\pi a}\left[e^{2\pi a} - 1\right]$$

The Fourier series for this function is therefore,

$$f(x) = \frac{1}{\pi}\left[e^{2\pi a} - 1\right]\left\{\frac{1}{2a} + \sum_{n=1}^{\infty}\frac{a\cos nx - n\sin nx}{n^2 + a^2}\right\}$$

The function is discontinuous at $x = 0$, as it has a sharp change. By setting $x = 0$ the Fourier series converges to the average value of the function,

$$\frac{1}{2}\left[f(0+) + f(0-)\right] = \frac{1}{2}\left[1 + e^{2\pi a}\right]$$

Therefore by plugging $x = 0$ into our Fourier series we get,

$$\frac{1}{2}\left[e^{2\pi a} + 1\right] = \frac{1}{\pi}\left[e^{2\pi a} + 1\right]\left\{\frac{1}{2a} + \sum_{n=1}^{\infty}\frac{a}{n^2 + a^2}\right\}$$

By rearranging we can find that,

$$\frac{\pi\left[e^{2\pi a} + 1\right]}{\left[e^{2\pi a} - 1\right]} = \frac{1}{a} + \sum_{n=1}^{\infty}\frac{2a}{n^2 + a^2}$$

Multiplying the top and bottom of the left hand side by $e^{-\pi a}$,

$$\begin{aligned}
\frac{\pi\left[e^{\pi a} + e^{-\pi a}\right]}{\left[e^{\pi a} - e^{-\pi a}\right]} &= \frac{1}{a} + \sum_{n=1}^{\infty}\frac{2a}{n^2 + a^2} \\
\pi\coth\pi a &= \frac{1}{a} + \sum_{n=1}^{\infty}\frac{2a}{n^2 + a^2} \\
\pi a\coth\pi a &= 1 + \sum_{n=1}^{\infty}\frac{2a^2}{n^2 + a^2}
\end{aligned}$$

Using this we can find an expression for the Riemann Zeta $\zeta(2k)$. First we let a be small, so that we can expand using the Maclaurin series,

$$\frac{a^2}{n^2 + a^2} = \frac{a^2/n^2}{1 + a^2/n^2} = \frac{a^2}{n^2}\left[1 - \frac{a^2}{n^2} + \frac{a^4}{n^4} - \frac{a^6}{n^6}\ldots\right]$$

Our previous equation therefore becomes,

$$\begin{aligned}\pi a \coth \pi a &= 1 + 2\sum_{n=1}^{\infty}\left[\frac{a^2}{n^2} - \frac{a^4}{n^4} + \frac{a^6}{n^6}\dots\right] \\ &= 1 + 2\left[a^2\zeta(2) - a^4\zeta(4) + a^6\zeta(6)\dots\right] \\ &= 1 + 2\sum_{n=1}^{\infty}(-1)^{k+1}a^{2k}\zeta(2k)\end{aligned}$$

Next look at the definition of the Bernoulli numbers, which gives us,

$$\begin{aligned}\frac{x}{e^x - 1} &= \sum_{n=0}^{\infty}\frac{B_n}{n!}x^n \\ &= x\frac{e^{-\frac{x}{2}}}{e^{\frac{x}{2}} - e^{-\frac{x}{2}}} \\ &= \frac{1}{2}x\left[\frac{e^{\frac{x}{2}} + e^{-\frac{x}{2}}}{e^{\frac{x}{2}} - e^{-\frac{x}{2}}} - 1\right] \\ &= \frac{1}{2}x\left[\coth\frac{x}{2} - 1\right]\end{aligned}$$

Therefore we have,

$$\frac{1}{2}x\left[\coth\frac{x}{2} - 1\right] = \sum_{n=0}^{\infty}\frac{B_n}{n!}x^n$$

Setting $x = 2\pi a$ gives us the left hand side the same as our equation earlier,

$$\pi a \coth \pi a - \pi a = \sum_{n=0}^{\infty}\frac{B_n}{n!}(2\pi a)^n$$

By comparing the coefficients of this equation with the other equation equal to $\pi a \coth \pi a - \pi a$ we find that the only term of the Bernoulli number expression that survives is when $n = 2k$, such that,

$$2(-1)^{k+1}\zeta(2k) = \frac{B_{2k}}{(2k)!}(2\pi)^{2k}$$

Therefore we can now find an expression for the zeta,

$$\zeta(2k) = \frac{|B_{2k}|2^{2k-1}\pi^{2k}}{(2k)!}$$

10. Complex Fourier Series

So far we have written the Fourier series using sines and cosines to represent our function. Because of the deep relation between sines and cosines this means that we can write the Fourier series in terms of complex exponentials,

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{\frac{2\pi inx}{L}}$$

The complex exponential functions $\phi_n(x) = e^{2\pi inx}L$ are orthogonal such that,

$$\begin{aligned}\int_{x_0}^{x_0+L} \phi_m^*(x)\phi_n(x)dx &= \int_{x_0}^{x_0+L} e^{-\frac{2\pi imx}{L}} e^{\frac{2\pi inx}{L}} dx \\ &= \int_{x_0}^{x_0+L} e^{\frac{2\pi i(n-m)x}{L}} dx \\ &= \begin{cases} 0 & n \neq m \\ L & n = m \end{cases}\end{aligned}$$

The integral is zero over the whole period because of the periodicity of the function $e^{\frac{2\pi ix}{L}}$. Therefore we have shown that the functions are orthogonal, which can be expressed by,

$$\int_{x_0}^{x_0+L} \phi_m^*(x)\phi_n(x)dx = L\delta_{mn}$$

The coefficients C_n can be extracted using these relations by performing an overlap integral of the form,

$$\begin{aligned}\int_{x_0}^{x_0+L} f(x)\phi^*(x)dx &= \sum_{m=-\infty}^{\infty} C_m \int_{x_0}^{x_0+L} \phi_n^*(x)\phi_m(x)dx \\ &= \sum_{m=-\infty}^{\infty} C_m L\delta_{mn} = LC_n\end{aligned}$$

The coefficients are therefore given by,

$$C_n = \frac{1}{L}\int_{x_0}^{x_0+L} f(x)e^{-\frac{2\pi inx}{L}} dx$$

Parseval's theorem for this series can be found in the same way as usual, but we multiply the function by it's complex conjugate instead of just squaring it,

$$f(x)f^*(x) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} C_m^* C_n \phi_m^*(x) \phi_n(x)$$

Parseval's theorem is found by integrating over this over a whole period,

$$\begin{aligned}
\int_{x_0}^{x_0+L} |f(x)|^2 dx &= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} C_m^* C_n \int_{x_0}^{x_0+L} \phi_m^* \phi_n dx \\
&= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} C_m^* C_N L \delta_{mn} \\
&= L \sum_{n=-\infty}^{\infty} |C_n|^2
\end{aligned}$$

We can manipulate the complex Taylor series to find the real Taylor series quite easily. First we note that the complex conjugate of the coefficients C_n are given by,

$$C_n^* = \frac{1}{L} \int_{x_0}^{\infty} f(x) e^{\frac{2\pi i n x}{L}} dx$$

Therefore, if $f(x)$ is real the coefficients are related by $C_n^* = C_{-n}$. It is possible to write the complex coefficients in terms of the real coefficients such that $C_n = \frac{1}{2}(a_n - ib_n)$ as shown by,

$$\begin{aligned}
C_n &= \frac{1}{L} \int_{x_0}^{x_0+L} f(x) e^{-\frac{2\pi i n x}{L}} dx \\
&= \frac{1}{L} \int_{x_0}^{x_0+L} f(x) \left[\cos\frac{2\pi n x}{L} - i \sin\frac{2\pi n x}{L}\right] dx \\
&= \frac{1}{2}(a_n - ib_n)
\end{aligned}$$

The complex conjugate of the coefficients is thus $C^* = C_{-n} = \frac{1}{2}(a_n + ib_n)$. Adding together the C_n and C_{-n} terms of the Taylor series gives us,

$$\begin{aligned}
C_n e^{\frac{2\pi i n x}{L}} + C_{-n} e^{-\frac{2\pi i n x}{L}} &= C_n e^{\frac{2\pi i n x}{L}} + C_{-n}^* e^{-\frac{2\pi i n x}{L}} \\
&= 2\Re C_n e^{\frac{2\pi i n x}{L}} \\
&= \Re(a_n - ib_n)\left(\cos\frac{2\pi n x}{L} + i \sin\frac{2\pi n x}{L}\right) \\
&= a_n \cos\frac{2\pi n x}{L} + b_n \sin\frac{2\pi n x}{L}
\end{aligned}$$

Which is just our real Taylor series.