Mechanics

Equations of Motion

This text is an exercise in learning mathematica for taking notes, and testing my algebra skills. There may be formatting errors, but I am exploring this to hopefully allow me to take mathematics notes in the future.

Acceleration is the derivative of velocity, and velocity is the derivative of displacement. From these two differential equations (and some careful substitution) we can set up all of our equations of motion that we need for any circumstance.

$$a = \frac{dv}{dt}$$

We can solve this basic differential equation by seperating the variables and then integrating.

$$a\,dt = dv$$
$$\int a\,dt = \int dv$$

We obtain the well known equation, which we will call eq. 1,

$$v = \mathrm{at} + v_0$$

By re-writing this as a differential equation we can obtain an expression for displacement at a given time *t*,

$$\frac{dx}{dt} = \mathrm{at} + v_0$$
$$dx = dt\,(\mathrm{at} + v_0)$$
$$\int dx = \int dt\,(\mathrm{at} + v_0)$$

We obtain a relation between distance and time, eq. 2,

$$x = \frac{1}{2}\mathrm{at}^2 + v_0\,t + x_0$$

Then by rearranging and substitution we can arrive at all other equations.
If we rearrange eq. 1 for *a*, and substitute into eq. 2,

$$a = \frac{v - v_0}{t}$$
$$x = \frac{1}{2}\left(\frac{v - v_0}{t}\right)t^2 + v_0\,t + x_0$$
$$x = \frac{1}{2}(v - v_0)\,t + v_0\,t + x_0$$
$$x = x_0 + \frac{1}{2}(v + v_0)\,t$$

If we rearrange eq. 1 for *t*, and sub. into eq. 2,

$$t = \frac{v - v_0}{a}$$

$$x = \frac{1}{2} a \left(\frac{v - v_0}{a}\right)^2 + v_0 \left(\frac{v - v_0}{a}\right)$$

$$x = \frac{1}{2a} \left(v^2 - 2 v_0 v + v_0^2\right) + v_0 \left(\frac{v - v_0}{a}\right)$$

$$x = \frac{1}{2a} \left(v^2 - v_0^2\right)$$

$$v^2 = v_0^2 + 2ax$$

Which is a well known, and very useful equation.

We have one final equation left to find, which will be found by rearranging eq. 1 for v_0 and then substituting in eq. 2

$$v_0 = v - at$$

$$x = \frac{1}{2} at^2 + (v - at)\, t + x_0$$

$$x = \frac{1}{2} at^2 + vt - at^2 + x_0$$

$$x = -\frac{1}{2} at^2 + vt + x_0$$

This completes our set of possible equations of motion for each case, and shows how to derive each one cleanly.