LINE AND SURFACE INTEGRALS

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Date: October 16, 2014.

1. 3D Curves and Polar Coordinate Review

We can represent a 3D curve as a vector function of a single parameter $\mathbf{r} = \mathbf{r}(u)$ for some interval $u_1 \leq u \leq u_2$. A curve such as this can be represented generally as,

$$\mathbf{r}(u) = x(u)\hat{\imath} + y(u)\hat{\jmath} + z(u)\hat{k}$$

A quick recap of polar coordinates. For example, with a flat circle in the xy plane of radius a, we can represent the cartesian coordinates as,

$$\begin{aligned} x &= a\cos\theta \\ y &= a\sin\theta \\ z &= 0 \end{aligned}$$

for $0 \leq \theta \leq 2\pi$.

2. Integral Over a Line

Diagram of line.

For a function $\mathbf{r}(u)$ giving a value at a point u on a line, we can say that there is a small change along the line which we call $d\mathbf{r}$ corresponding to a change in u. When u increases to $u + du$, then $\mathbf{r}(u)$ changes to,

$$\mathbf{r}(u + du) = \mathbf{r}(u) + d\mathbf{r} = \mathbf{r}(u) + \frac{d\mathbf{r}}{du}du$$

The integral of a vector field $\mathbf{F}(\mathbf{r})$ along a curve C defined by the vector function $\mathbf{r}(u)$ is given by,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{u_1}^{u_2} \mathbf{F}(\mathbf{r}(u)) \cdot \frac{d\mathbf{r}}{du}du$$

A physical example of this is the work done W performed by a force $\mathbf{F}$ in moving an object along a path C,

$$W = \int_C \mathbf{F} \cdot d\mathbf{r}$$

3. Example: Integral Over a Line

In this section we will take a simple example of an integral over a closed line of a function $\mathbf{F}(\mathbf{r}) = -y\hat{\imath} + x\hat{\jmath}$. The integral we need to evaluate is $\oint_C \mathbf{F} \cdot d\mathbf{r}$. C is a circle with radius a in the xy plane which will be travered by θ anti-clockwise. We define a point $\mathbf{r}$ on the circumference on the circle as,

$$\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a\cos\theta \\ a\sin\theta \\ 0 \end{pmatrix}$$

where $0 \leq \theta \leq 2\pi$.

We can see that $d\mathbf{r} = \frac{d\mathbf{r}}{d\theta}d\theta$, so we can use this to evaluate our integral. First we need to take the dot product, $\mathbf{F}\cdot d\mathbf{r}$,

$$\begin{aligned}
\mathbf{F}\cdot d\mathbf{r} &= \begin{pmatrix} -a\sin\theta \\ a\cos\theta \end{pmatrix} \cdot \begin{pmatrix} -a\sin\theta \\ a\cos\theta \end{pmatrix} d\theta \\
&= a^2\left(\sin^2\theta + \cos^2\theta\right)d\theta \\
&= a^2 d\theta
\end{aligned}$$

Now the integral can be evaluated,

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} a^2 d\theta = 2\pi a^2$$

So we have now evaluated the integral of our function over the circle C.

4. Kinetic Energy

Suppose a body of mass m moves along a trajectory $\mathbf{r}(t)$ under the influence of some force $\mathbf{F}(\mathbf{r})$. We know that the work done moving from $\mathbf{r}(t_1) = \mathbf{r_1}$ at time t_1, to point $\mathbf{r}(t_2) = \mathbf{r_2}$ at time t_2 is,

$$\begin{aligned}
W &= \int_C \mathbf{F}\cdot d\mathbf{r} \\
&= \int_C m\frac{d\mathbf{v}}{dt}\cdot d\mathbf{r} \\
&= \int_{t_1}^{t_2} m\frac{d\mathbf{v}}{dt}\cdot\frac{d\mathbf{r}}{dt}dt \\
&= \int_{t_1}^{t_2} m\frac{d\mathbf{v}}{dt}\cdot\mathbf{v}dt \\
&= \int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{1}{2}m\mathbf{v}\cdot\mathbf{v}\right)dt \\
W &= \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2
\end{aligned}$$

arriving at an expression for the work done moving through a force between two points. This can be called the kinetic energy of a body. If the force is derived from a potential energy V, then $\mathbf{F} = -\nabla V$,

$$\begin{aligned} W &= \int_C \mathbf{F} \cdot d\mathbf{r} \\ &= \int_{t_1}^{t_2} -\nabla V \cdot \frac{d\mathbf{r}}{dt} dt \end{aligned}$$

5. Surface Integrals

A surface can be represented by a vector function of two parameters,

$$\mathbf{r} = \mathbf{r}(u, v) = x(u, v)\hat{\imath} + y(u, v)\hat{\jmath} + z(u, v)\hat{k}$$

For example, we can describe a sphere of radius a centered at the origin by the following parameterisation,

$$\begin{aligned} x &= a \sin\theta \cos\theta \\ y &= a \sin\theta \sin\phi \\ z &= a \cos\theta \\ 0 \leq \theta &\leq \pi \\ 0 \leq \phi &\leq 2\pi \end{aligned}$$

so in this case $\mathbf{r}$ can be represented as,

$$\mathbf{r} = a \begin{pmatrix} \sin\theta\cos\theta \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}$$

To evaluate a surface integral, we need an expression for a small element of surface area dS. We form this by considering points $\mathbf{r}(u, v)$ when u ranges from u to $u + du$ and v ranges from v to $v + dv$. This element is a parallelogram with sides $d\mathbf{r_1}$ and $d\mathbf{r_2}$ given by,

$$\begin{aligned} d\mathbf{r_1} &= \frac{\partial \mathbf{r}}{\partial u} du \\ d\mathbf{r_2} &= \frac{\partial \mathbf{r}}{\partial v} dv \end{aligned}$$

The area of this small element dS is given by the cross product, $\mathbf{A} = ab \sin\theta \hat{n} = \mathbf{a} \times \mathbf{b}$.

6. Changing Variable in the Double Integral

We can view the double integral,

$$\iint_S f(x, y) dx dy$$

as a surface integral over a surface S in the xy plane. We can parameterise the surface S by a function of two variables, $\mathbf{r} = \mathbf{r}(u, v)$. This is equivalent to changing the variables $x, y \to u, v$. For the surface S we can do this as follows,

$$\begin{aligned} d\mathbf{S} &= \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} dudv \\ &= \begin{pmatrix} \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial u} \\ 0 \end{pmatrix} \times \begin{pmatrix} \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial v} \\ 0 \end{pmatrix} \\ &= \left[\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \right] \hat{k} dudv \end{aligned}$$

Since $d\mathbf{S} = \hat{k} dxdy$ it follows that,

$$dxdy = \left[\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \right] dudv$$

Just like for a line integral paramterisation $dx = \frac{dx}{du} du$, here $dxdy = Jdudv$ where J is the differential term in square brackets above. This term J is called the Jacobian determinant, as we see that it is the determinant of a 2×2 matrix,

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

The formula for changing variables in a double integrals is thus given by,

$$\iint_S f(x, y) dxdy = \iint_S f\left(x(u, v), y(u, v)\right) \frac{\partial(x, y)}{\partial(u, v)} dudv$$

7. Example: 2D Polars

We know that the x and y components can be parameterized in terms of r and θ,

$$x = r\cos\theta y \qquad = r\sin\theta$$

Then we can calculate the Jacobian as follows,

$$\begin{aligned} \frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} &= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} \\ &= r\cos^2\theta + r\sin^2\theta = r \end{aligned}$$

Therefore we have an expression for $dxdy$,

$$dxdy = rdrd\theta$$

This is what we expect when we parameterize to evaluate the double integral. For example, we will use this to evaluate, by way of parameterisation,

$$\begin{aligned}
&\iint e^{-(x^2+y^2)}dxdy \\
&= \int_{-\infty}^{\infty} e^{-x^2}dx \int_{-\infty}^{\infty} e^{-y^2}dy \\
&= \int_0^{\infty} rdr \int_0^{2\pi} e^{-r^2}d\theta \\
&= \int_0^{\infty} 2\pi r e^{-r^2}dr = \left[-\pi e^{-r^2}\right]_0^{\infty} = \pi \\
&\implies \int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}
\end{aligned}$$

8. Changing Variable in the Triple Integral

We already know how to perform volume integrals as it is equivalent to simply taking the triple integral over a region V,

$$\iiint_V f(x,y,z)dxdydz$$

We can parameteries the volume V by some vector function $\mathbf{r}(u,v,w)$ for uvw in some region $\overline{V}$ in uvw-space. Therefore we are just changing the variables again from $x,y,z \to u,v,w$. The element of volume is a parallelepiped given by the vectors $d\mathbf{r_1}$, $d\mathbf{r_2}$, $d\mathbf{r_3}$,

$$d\mathbf{r_1} = \frac{\partial \mathbf{r}}{\partial u}du \qquad d\mathbf{r_2} = \frac{\partial \mathbf{r}}{\partial v}dv \qquad d\mathbf{r_3} = \frac{\partial \mathbf{r}}{\partial w}dw$$

Now we can represent an element of volume (as given by the formula for the volume of the parallelepiped) as,

$$\begin{aligned}
dV &= d\mathbf{r_1} \cdot (d\mathbf{r_2} \times d\mathbf{r_3}) \\
&= \frac{\partial \mathbf{r}}{\partial u} \cdot \left(\frac{\partial \mathbf{r}}{\partial v} \times \frac{\partial \mathbf{r}}{\partial w}\right) dudvdw
\end{aligned}$$

Here we have the scalar triple product, and since the scalar triple product can be written as a determinant, we now have the Jacobian determinant in three dimensions,

$$\frac{\partial \mathbf{r}}{\partial u} \cdot \left(\frac{\partial \mathbf{r}}{\partial v} \times \frac{\partial \mathbf{r}}{\partial w} \right) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$