DISTRIBUTIONS (GENERALISED FUNCTIONS)

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1. Dirac Delta Function

Consider a vector function defined as $\mathbf{A} = \frac{\mathbf{r}}{r^3}$. In spherical polar coordinates this can be written as $\mathbf{A} = \frac{1}{r^2}\mathbf{e_r}$. Next we calculate its divergence using the formula proved in previous notes,

$$\nabla \cdot \mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{1}{r^2}\right) = 0$$

Consider next the divergence theorem for the same vector field $\mathbf{A}$, also known as Gauss' theorem, which was shown to be,

$$\int_V \nabla \cdot \mathbf{A} dV = \oint_S \mathbf{A} \cdot d\mathbf{S}$$

By integrating over a sphere of radius a centered at the origin we see,

$$\oint_S \mathbf{A} \cdot d\mathbf{S} = \frac{1}{a^2} 4\pi a^2 = 4\pi$$

There appears to be a contradiction as we have just proved that $\nabla \cdot \mathbf{A} = 0$, but this integral tells us that since the surface integral is non-zero, then so is the volume integral. It turns out however that we have only really proved that $\nabla \cdot \mathbf{A} = 0$ when $\mathbf{r} \neq \mathbf{0}$. Therefore we must have that $\nabla \cdot \mathbf{A}$ suddenly blows up at $\mathbf{r} = \mathbf{0}$ in such a way that the integral over it has the value 4π. This result is given by a function called the Dirac delta function, in this case in 3 dimensions,

$$\nabla \cdot \mathbf{A} = 4\pi\delta^3(\mathbf{r})$$

where

$$\delta^3(\mathbf{r}) = \begin{cases} 0 & \mathbf{r} \neq \mathbf{0} \\ \infty & \mathbf{r} = \mathbf{0} \end{cases}$$

which also has the constraint,

$$\int_V \delta^3(\mathbf{r}) d^3\mathbf{r} = 1$$

For example, the electric field due to a point charge at the origin is given by,

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0}\frac{\mathbf{r}}{r^3}$$

The divergence of this can thus be written as,

$$\nabla \cdot \mathbf{E} = \frac{q}{\epsilon_0}\delta^3(\mathbf{r}) = \frac{\rho(\mathbf{r})}{\epsilon_0}$$

Here, $\rho(\mathbf{r}) = q\delta^3(\mathbf{r})$ is the charge density of a point charge q at the origin. It is zero at $\mathbf{r} \neq \mathbf{0}$ and infinite at $\mathbf{r} = \mathbf{0}$ such that the integral over its volume gives q.

The Dirac delta function shows up every time we mix discrete and continuous objects.

2. The 1D Dirac Delta Function

The one-dimensional Dirac delta function $\delta(x)$ is defined,

$$\delta(x) = \begin{cases} 0 & x \neq 0 \\ \infty & x = 0 \end{cases}$$

Under the constraint,

$$\int_{-\infty}^{\infty} \delta(x) dx = 1$$

As in the three-dimensional case, it may be defined as the limit of a sequence of functions that become increasingly peaked, getting more and more narrow. A good function to choose for this is a Gaussian distribution,

$$\delta_n(x) = \sqrt{\frac{n}{\pi}} e^{-nx^2}$$

We divide by $\sqrt{\frac{n}{\pi}}$ to normalise the distribution so that if we integrate over it we get one regardless of the value of n. We can see that the limits give us back the Dirac delta function,

$$\begin{aligned} &\lim_{n\to\infty} \delta_n(x) = 0 && \text{for } x \neq 0 \\ &\lim_{n\to\infty} \delta_n(x) = \infty && \text{for } x = 0 \\ &\lim_{n\to\infty} \int_{-\infty}^{\infty} \delta_n(x) = 1 \end{aligned}$$

We can therefore think of the Dirac delta function $\delta(x)$ as the limit of $\delta_n(x)$ as $n \to \infty$.

The key property of the delta function is that,

$$\int_a^b f(x)\delta(x - x_0) dx = \begin{cases} f(x_0) & a < x_0 < b \\ 0 & \text{otherwise} \end{cases}$$

This is because the delta function effectively picks out the value of $f(x)$ where $x = x_0$. This is because the only place where the delta function isn't zero, is when $x = x_0$. Since the integral over the entire Dirac delta function is one, that means we just get the function $f(x_0)$ back.

We may also write this as,

$$f(x)\delta(x - x_0) = f(x_0)\delta(x - x_0)$$

Using some intuition it is clear to see that they are equal, as the product will be zero for all $x \neq x_0$, so we can just replace $f(x)$ with $f(x_0)$

We can relate the one-dimensional and three-dimensional Dirac delta functions by,

$$\delta^3(\mathbf{r}) = \delta(x)\delta(y)\delta(z)$$

Or for a given offset $\mathbf{r_0}$,

$$\delta^3(\mathbf{r} - \mathbf{r_0}) = \delta(x - x_0)\delta(y - y_0)\delta(z - z_0)$$