Mathematics for Physicists 2

Vector Calculus

Why?

Vector calculus is used in many equations in physics,

- Maxwell's equations (Electromagnetism)
- Continuity equation (Conservation)
- Schrodinger equation (Quantum mechanics)
- Wave equation (Optics etc.)
- Navier-Stokes equation (Fluid mechanics)

Calculating gradient, divergent, curl etc.

- A scalar field $\phi(x, y, z) = \phi(\boldsymbol{r})$ is a scalar quantity defined at each point in space $\boldsymbol{r} = (x, y, z)$
- A vector field $\boldsymbol{A}(x, y, z) = \boldsymbol{A}(\boldsymbol{r})$ is a vector quantity defined at each point in space. It can be represented by it's cartesian components,

$$\boldsymbol{A}(x, y, z) = A_x(x, y, z)\,\boldsymbol{i} + A_y(x, y, z)\,\boldsymbol{j} + A_z(x, y, z)\,\boldsymbol{k}$$

- The vector differential operator ∇ (del or nabla) is defined as

$$\boldsymbol{\nabla} = \boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial z}$$

- If we apply ∇ to a scalar function ϕ we get its gradient $\nabla\phi$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\boldsymbol{i} + \frac{\partial\phi}{\partial y}\boldsymbol{j} + \frac{\partial\phi}{\partial z}\boldsymbol{k}$$

- If we take the scalar product of ∇ with a vector function **A** we get its divergence $\nabla \cdot \mathbf{A}$

$$\boldsymbol{\nabla}\cdot\boldsymbol{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

- If we take the vector product of ∇ with a vector function **A** we get its curl $\nabla \times \mathbf{A}$

$$\nabla \times \mathbf{A} = \mathrm{Det}\left[\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{pmatrix}\right]$$

- The scalar product of ∇ with itself gives the Laplacian ∇^2

$$\boldsymbol{\nabla}^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

- The Laplacian can act on scalar and vector fields

Differentiation of Vectors

Consider a vector function of a single real variable, $\boldsymbol{r}(t)$. A good example of this is position of a particle as a function of time. We can describe the derivate as usual, using the definition of the derivative,

$$\frac{d\boldsymbol{r}}{\mathrm{dt}} = \lim_{\delta\mathrm{t}\to 0} \frac{\boldsymbol{r}(t + \delta\mathrm{t}) - \boldsymbol{r}(t)}{\delta\mathrm{t}}$$

The product rules still apply, for dot product, cross product, and scalar multiple of a vector,

$$\frac{d}{d\,t}\boldsymbol{a}(t)\cdot\boldsymbol{b}(t) = \frac{d\,\boldsymbol{a}}{d\,t}\cdot\boldsymbol{b} + \boldsymbol{a}\cdot\frac{d\,\boldsymbol{b}}{d\,t}$$

$$\frac{d}{d\,t}\boldsymbol{a}(t)\times\boldsymbol{b}(t) = \frac{d\,\boldsymbol{a}}{d\,t}\times\boldsymbol{b} + \boldsymbol{a}\times\frac{d\,\boldsymbol{b}}{d\,t}$$

$$\frac{d}{d\,t}f(t)\;\boldsymbol{a}(t) = \frac{d\,f}{d\,t}\boldsymbol{a} + f\,\frac{d\,\boldsymbol{a}}{d\,t}$$

If we have a vector $\boldsymbol{r}(t) = x(t)\,\boldsymbol{i} + y(t)\,\boldsymbol{j} + z(t)\,\boldsymbol{k}$, then

$$\frac{d\boldsymbol{r}}{\mathrm{dt}} = \frac{\mathrm{dx}}{\mathrm{dt}}\boldsymbol{i} + \frac{\mathrm{dy}}{\mathrm{dt}}\boldsymbol{j} + \frac{\mathrm{dz}}{\mathrm{dt}}\boldsymbol{k}$$