# SOLVING KEPLER PROBLEM WITH VECTOR CALCULUS

JOE BENTLEY

*Date*: October 9, 2014.

## 1. Conservation of Energy

Here we will arrive at a result which shows conservation of energy from Newton's second law of motion, $\mathbf{F} = m\mathbf{a}$

First, we need to set up a differential equation describing the situation,

$$m\frac{d\mathbf{v}}{dt} = -\frac{GMm}{r^3}\mathbf{r}. \tag{1}$$

We then take the scalar product of each side with $\mathbf{v}$,

$$m\mathbf{v} \cdot \frac{d\mathbf{v}}{dt} = -\frac{GMm}{r^3}\mathbf{r} \cdot \mathbf{v}. \tag{2}$$

In the next part we will consider the time derivative of $v^2$, we know that $v^2 = \mathbf{v} \cdot \mathbf{v}$, hence

$$\frac{d}{dt}(v^2) = \frac{d}{dt}(\mathbf{v} \cdot \mathbf{v}) = \mathbf{v} \cdot \frac{d\mathbf{v}}{dt} + \frac{d\mathbf{v}}{dt} \cdot \mathbf{v} = 2\mathbf{v} \cdot \frac{d\mathbf{v}}{dt}$$

In the second step we applied the product rule.

We can see that this is almost like the left hand of our equation, just with a factor of 2, and missing an $m$. The left hand side of our equation is then, in fact, the time derivative of the kinetic energy.

$$\frac{d}{dt}\left(\frac{1}{2}mv^2\right) = m\mathbf{v} \cdot \frac{d\mathbf{v}}{dt}$$

This is the left hand side of our equation. Now we will concentrate on the right hand side.

Clearly, $r^2 = \mathbf{r} \cdot \mathbf{r}$, and thus,

$$\begin{aligned} \frac{d}{dt}\left(r^2\right) &= \frac{d}{dt}\left(\mathbf{r} \cdot \mathbf{r}\right) \\ 2r\frac{dr}{dt} &= 2\mathbf{r} \cdot \frac{d\mathbf{r}}{dt} = 2\mathbf{r} \cdot \mathbf{v} \\ \mathbf{r} \cdot \mathbf{v} &= r\frac{dr}{dt} \end{aligned}$$

We can substitute this into eq. 2 so the right hand side becomes,

$$-\frac{GMm}{r^3}\left(r\frac{dr}{dt}\right) = -\frac{GMm}{r^2}\frac{dr}{dt} = \frac{d}{dt}\left[\frac{GMm}{r}\right]$$

Now we have both the left hand side and right hand side of our equation,

$$\frac{d}{dt}\left(\frac{1}{2}mv^2\right) = \frac{d}{dt}\left(\frac{GMm}{r}\right)$$

By minusing the RHS,

$$\frac{d}{dt}\left(\frac{1}{2}mv^2 - \frac{GMm}{r}\right) = 0$$

Thus we see conservation of energy, the total energy in the system (the kinetic energy minus the gravitational potential energy) is invariant in time.

## 2. Conservation of Angular Momentum

We start by taking the vector product of eq. 1 with $\mathbf{r}$,

$$\mathbf{r} \times \left(m\frac{d\mathbf{v}}{dt}\right) = \mathbf{r} \times \left(-\frac{GMm}{r^3}\right)\mathbf{r} = 0. \tag{3}$$

This is zero because $\mathbf{r} \times \mathbf{r}$ is zero. Also used here is $\mathbf{F} = m\mathbf{a}$ in the second argument to the vector product.

What is the left side in differential form? Well it kind of looks like the time derivative of the angular momentum, $\mathbf{L} = \mathbf{r} \times m\mathbf{v}$, so we will try that,

$$\frac{d}{dt}(\mathbf{r} \times m\mathbf{v}) = \frac{d\mathbf{r}}{dt} \times m\mathbf{v} + \mathbf{r} \times m\frac{d\mathbf{v}}{dt} = \mathbf{v} \times m\mathbf{v} + \mathbf{0} = \mathbf{0}$$

The $\mathbf{r} \times m\frac{d\mathbf{v}}{dt}$ is zero as we have shown in eq. 3.

Therefore we see that since the time derivative of the angular momentum is zero, it is conserved, as it is unchanging in time,

$$\frac{d}{dt}(\mathbf{L}) = \frac{d}{dt}(\mathbf{r} \times m\mathbf{v}) = 0.$$

Since $\mathbf{r}$ is perpendicular to $\mathbf{L}$, and $\mathbf{L}$ is constant, the motion takes place in the plane perpendicular to $\mathbf{L}$, which also contains the origin of rotation.

## 3. Laplace-Runge-Lenz Vector

To find a new unknown conserved quantity we take the vector product of $\mathbf{F} = m\mathbf{a}$ with $\mathbf{L}$,

$$m\frac{d\mathbf{v}}{dt} \times \mathbf{L} = -\frac{GMm}{r^3}\mathbf{r} \times \mathbf{L}$$

and then dividing by $m$,

$$\frac{d\mathbf{v}}{dt} \times \mathbf{L} = -\frac{GM}{r^3}\mathbf{r} \times \mathbf{L} \tag{4}$$

The left hand side looks kind of like the time derivative of $\mathbf{v} \times \mathbf{L}$,

$$\frac{d}{dt}(\mathbf{v} \times \mathbf{L}) = \frac{d\mathbf{v}}{dt} \times \mathbf{L} + \mathbf{v} \times \frac{d\mathbf{L}}{dt} = \frac{d\mathbf{v}}{dt} \times \mathbf{L}$$

Since angular momentum $\mathbf{L}$ is conserved, we see know that $\frac{d\mathbf{L}}{dt}$ is zero. Therefore we now know that the left hand side is equal to $\frac{d}{dt}(\mathbf{v}\times\mathbf{L})$.

Now for the right hand side, we plug back in the definition of $\mathbf{L}$,

$$-\frac{GM}{r^3}\mathbf{r}\times\mathbf{L} = -\frac{GMm}{r^3}\mathbf{r}\times(\mathbf{r}\times\mathbf{v})$$

This is just the vector triple product,

$$\begin{aligned}
&-\frac{GMm}{r^3}\left[(\mathbf{r}\cdot\mathbf{v})\,\mathbf{r}-(\mathbf{r}\cdot\mathbf{r})\,\mathbf{v}\right]\\
&= \frac{GMm}{r^3}\left[r^2\mathbf{v}-\left(r\frac{dr}{dt}\right)\mathbf{r}\right]\\
&= \frac{GMm}{r}\mathbf{v}-\frac{GMm}{r^2}\frac{dr}{dt}\mathbf{r}\\
&= \frac{GMm}{r}\frac{d\mathbf{r}}{dt}+\frac{d}{dt}\left(\frac{GMm}{r}\right)\mathbf{r}\\
&= \frac{d}{dt}\left(\frac{GMm}{r}\mathbf{r}\right)
\end{aligned}$$

The last line was by seeing that the second to last line is just the product rule, so we just inversed it into the $\frac{d}{dt}$

We therefore have finally that,

$$\frac{d}{dt}(\mathbf{v}\times\mathbf{L}) = \frac{d}{dt}\left(\frac{GMm}{r}\mathbf{r}\right)$$

And then by minusing the right side,

$$\frac{d}{dt}\left(\mathbf{v}\times\mathbf{L}-\frac{GMm}{r}\mathbf{r}\right) = \mathbf{0}$$

Which is conserved as we've shown here, as it is invariant in time. This will be written in it's final form as a vector,

$$\mathbf{A} = \mathbf{v}\times\mathbf{L}-\frac{GMm}{r}\mathbf{r}$$

This is called the Laplace-Runge-Lenz vector.

## 4. Shape of the Orbit

The shape of the orbit can be found by taking the scalar product of $\mathbf{A}$ with $\mathbf{r}$,

$$\mathbf{r}\cdot\mathbf{A} = \mathbf{r}\cdot(\mathbf{v}\times\mathbf{L})-\frac{GMm}{r}\mathbf{r}\cdot\mathbf{r}$$

In the first time on the right hand side of the equation we have the scalar triple product, so $\mathbf{r} \cdot (\mathbf{v} \times \mathbf{L}) = (\mathbf{r} \times \mathbf{v}) \cdot \mathbf{L}$. $\mathbf{r} \times \mathbf{v}$ is just equal to the angular momentum divided by the mass, therefore $\mathbf{r} \times \mathbf{v} = \frac{\mathbf{L}}{m}$. Therefore we have,

$$\begin{aligned} rA\cos\theta &= (\mathbf{r} \times \mathbf{v}) \cdot \mathbf{L} - GMmr \\ &= \frac{L^2}{m} - GMmr \\ \implies \frac{L^2}{GMm^2} &= r\left[1 + \frac{A}{GMm}\cos\theta\right] \end{aligned}$$

Now we shall make a few definitions,

$$\begin{aligned} p &= \frac{L^2}{GMm^2}, \\ e &= \frac{A}{GMm} \end{aligned}$$

where $p$ is known as the semi-latus rectum, and $e$ is known as the eccentricity.

Now, if we arrange for $r$, we have the equation for a conic section,

$$r = \frac{p}{1 + e\cos\theta}$$

## 5. Energy and the LRL Vector

The magnitude of $A$, and hence the eccentricity $e$, can be related to the energy $E$ of the orbit.

$$\begin{aligned} A^2 = \mathbf{A} \cdot \mathbf{A} &= \|\mathbf{v} \times \mathbf{L}\|^2 + \left(\frac{GMm}{r}\right)^2 \mathbf{r} \cdot \mathbf{r} - 2\frac{GMm}{r}\mathbf{r} \cdot (\mathbf{v} \times \mathbf{L}) \\ &= GMm^2 - 2\frac{GM}{r}L^2 + \|\mathbf{v} \times \mathbf{L}\|^2 \end{aligned}$$

The angular momentum $L^2$ appears in the final line because of the scalar triple product at the end of the first line being equal to $\frac{L^2}{m^2}$.

The term $\|\mathbf{v} \times \mathbf{L}\|^2$ can be rewritten, which will be shown generally here,

$$\begin{aligned} \mathbf{a} \cdot \mathbf{b} &= ab\cos\theta \\ \mathbf{a} \times \mathbf{b} &= ab\sin\theta\hat{\mathbf{n}} \\ \|\mathbf{a} \times \mathbf{b}\| &= ab\sin\theta \\ \|\mathbf{a} \times \mathbf{b}\|^2 + (\mathbf{a} \cdot \mathbf{b})^2 &= a^2b^2\left[\cos^2\theta + \sin^2\theta\right] = a^2b^2 \\ \implies (\mathbf{v} \cdot \mathbf{L})^2 + \|\mathbf{v} \times \mathbf{L}\|^2 &= v^2L^2 \end{aligned}$$

But, since $\mathbf{v}$ and $\mathbf{L}$ are perpendicular by definition of $\mathbf{L}$, we see that,

$$\|\mathbf{v} \times \mathbf{L}\|^2 = v^2 L^2$$

Now we can substitute this in to our equation from earlier to get a relation between $A^2$ and $E = \frac{1}{2}mv^2 - \frac{GMm}{r}$,

$$\begin{aligned} A^2 &= (GMm)^2 + v^2L^2 - 2\frac{GM}{r}L^2 \\ &= (GMm)^2 + \frac{2L^2}{m}\left[\frac{1}{2}mv^2 - \frac{GMm}{r}\right] \\ &= (GMm)^2 + \frac{2L^2E}{m} \end{aligned}$$

We now have an expression relating the energy of the orbit to the square of the Laplace-Runge-Lenz vector.