Using framework to find the optimal detector

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As shown previously, the limit on the variance of an estimator of a classical signal $x(t)$ coupled to the detector linearly via $\hat{H}_{\text{int}} = -\hat{F}x(t)$ is given by,

$$\sigma_{xx}^{\text{QCRB}}(\omega) = \frac{\hbar^2}{4\bar{S}_{FF}(\omega)}, \tag{1}$$

where $\bar{S}_{FF}(\omega)$ is the symmetrised power spectral density describing the quantum fluctuations of $\hat{F}$. Ignoring losses, for measurement shot noise this is given by,

$$\bar{S}_{FF}(\omega) = \bar{S}_{uu}(\omega)|T_{uF}(\omega)|^2 = |T_{uF}(\omega)|^2, \tag{2}$$

where $T_{uF}(\omega)$ is the transfer function from the vacuum quantum noise input $\hat{u}$ to the internal degree of freedom $\hat{F}$, and we have used the fact that the noise spectrum of the input vacuum noise is unity.

For the optimal GW strain detector we should maximise the SNR,

$$\mathcal{S} = \int_{f_1}^{f_2} \mathrm{d}f \frac{|h(f)|^2}{\bar{S}_{hh}(f)} \leq \int_{f_1}^{f_2} \mathrm{d}f \frac{|h(f)|^2}{\bar{S}_{hh}^{\text{QCRB}}(f)} \leq \int_{f_1}^{f_2} \mathrm{d}f \bar{S}_{FF}(f)|h(f)|^2 \tag{3}$$

where f_1 and f_2 are the lower and upper bounds of the frequency range of interest, $h(f)$ is the Fourier transform of the strain signal $h(t)$, $\bar{S}_{hh}(f)$ is the strain signal spectral density, and $\bar{S}_{hh}^{\text{QCRB}}(f) = \sigma_{xx}^{\text{QCRB}}(f)/L^2$, where L is the arm length. So by maximising $\mathcal{S}$ we also maximise the QCRB.

We now know that we can synthesise any n degree-of-freedom system directly from its transfer function. A general lossless ($|T_{uy}(\omega)| = 1$, where $T_{uy}(\omega)$ is the transfer function from the input $\hat{u}$ to the output $\hat{y}$) n degree-of-freedom system's transfer function is given by,

$$T_{uy}(\omega) = \prod_{j=0}^{n} \frac{\omega + \Delta_j + i\gamma_j}{\omega + \Delta_j - i\gamma_j}, \tag{4}$$

where Δ_j describes the detuning and γ_j describes the bandwidth of the j-th degree-of-freedom.

From this we construct the system using the synthesis theorem. **(isn't it just a pure cascade realisation?)** Labelling each internal degree of freedom $\hat{F}_i$ we can then calculate the transfer functions from the input to those degrees of freedom, $T_{uF_i}(\omega)$. We then choose the $\hat{F}_i$ that minimises $\mathcal{S}$, and this is the degree of freedom to which we should couple our classical signal $x(t)$.