Equations of motion from Hamiltonian matrix

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We have a state vector of the form $\vec{a} = (\hat{a}_1, \hat{a}_1^\dagger; \ldots; \hat{a}_n, \hat{a}_n^\dagger)$ with commutation relations $[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij}$, $[\hat{a}_i, \hat{a}_j] = 0$. The Hamiltonian is given by $\hat{H} = \sum_{ij} \vec{a}_i^\dagger H_{ij} \vec{a}_j$ where H_{ij} is the Hamiltonian matrix.

Further note that $[\vec{a}_k, \vec{a}_i] = J_{ki}$ where,

$$J = \mathrm{diag}(J_1, \ldots, J_1) \in \mathbb{R}^{2n \times 2n}, \quad J_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{1}$$

and $[\vec{a}_j^\dagger, \vec{a}_i] = K_{ji}$, where $K = -\mathrm{diag}(1, -1, \ldots, 1, -1) \in \mathbb{R}^{2n \times 2n}$.

The Heisenberg equation of motion is given by,

$$\begin{aligned} \dot{\vec{a}}_i = i[\hat{H}, \vec{a}_i] &= i[\sum_{jk} \vec{a}_j^\dagger H_{jk} \vec{a}_k, \vec{a}_i] = i \sum_{jk} H_{jk} [\vec{a}_j^\dagger \vec{a}_k, \vec{a}_i] \\ &= i \sum_{jk} H_{jk} (\vec{a}_j^\dagger [\vec{a}_k, \vec{a}_i] + [\vec{a}_j^\dagger, \vec{a}_i] \vec{a}_k) \\ &= i \sum_{jk} (\vec{a}_j^\dagger H_{jk} J_{ki} + K_{ij} H_{jk} \vec{a}_k), \end{aligned}$$

So that,

$$\dot{\vec{a}} = i((\vec{a}^\dagger H J)^T + K H \vec{a}) = i(J^T H^T \Theta + K H)\vec{a}, \tag{2}$$

where H here is the Hamiltonian matrix and,

$$\Theta = \mathrm{diag}(\Theta_1, \ldots, \Theta_1) \in \mathbb{R}^{2n \times 2n}, \quad \Theta_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{3}$$