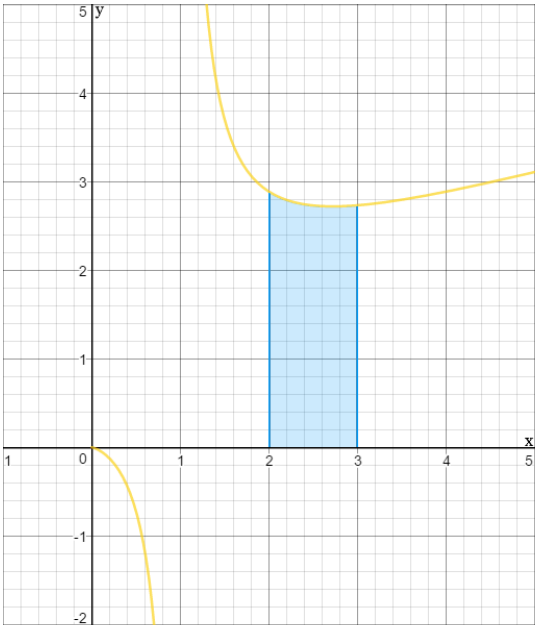
OCR MEI Numerical Methods coursework

Introduction:

I have chosen the problem of Numerical Integration.
I will attempt to solve:

$$\int_2^3 {}^{x}/_{\ln(x)}\,\mathrm{dx}$$

Fig. 1- The graph of $y = {}^{x}/_{\ln(x)}$:



Area in blue is what I am attempting to find (area under the graph between bounds $[2, 3]$).

I have chose this function as I cannot integrate functions involving logarithmic terms or terms with exponent '-1' analytically. I will therefore attempt to solve it numerically. This is unlikely to produce the exact answer, but it will give an accurate approximation of the solution.

Methods:

In order to carry out Numerical Integration, I will be adopting a variety of methods in order to make effective estimates for the integral to a high degree of accuracy. These methods include the Midpoint Rule and Trapezium Rule. The results of these methods will allow me to gain an estimate to a greater degree of accuracy via the Simpson's Rule.

$$M_n = h(f(m_1) + f(m_2) + \cdots + f(m_n))$$

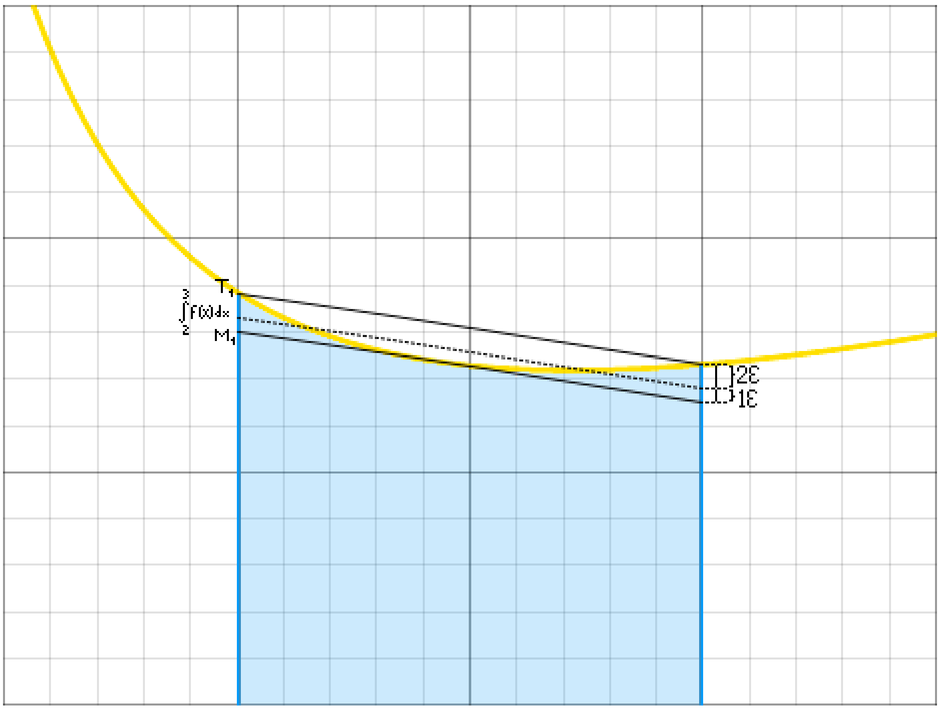
$$T_n = {}^{h}/_{2}\,(f(x_0) + 2\big(f(x_1) + f(x_2) + \cdots + f(x_{n-1})\big) + f(x_n))$$

n = number of strips
h = height of strips (horizontal)
$f(x) = {}^{x}/_{\ln(x)}$

The Midpoint and Trapezium rule will respectively provide under and over estimates in the case of my integral. This is because the region of graph that I am attempting to integrate is of Concave nature. I can therefore conclude that the solution of the integral will lay within the interval of values created by the Midpoint and Trapezium rules, such that

$$M_n < \int_2^3 {}^{x}/_{\ln(x)}\,\mathrm{dx} < \mathrm{T}_n$$

Fig. 2- Enlarged graph of $y = {}^{x}/_{\ln(x)}$ with Midpoint and Trapezium estimates illustrated.



$\varepsilon = |Error|$

$f(x) = {}^{x}/_{\ln(x)}$

The ratio between the absolute errors of the respective Midpoint and Trapezium are in the ratio of $1:2$. The Simpson's rule considers this, giving an estimate to a greater degree of accuracy through a weighted average, such that

$$S_n = {}^{2(M_n) + (T_n)}/_{3}$$

I will use between 1 and 64 strips for each method (where 'n' = number of strips, 'n' increments as a power of 2). The greater value of strips will provide the most accurate results. By carrying out the procedure for progressive numbers of strips, I will obtain the ratio of differences of the Simpson's method (most accurate). I will use this with extrapolation to find a solution that I deem suitable to a very high degree of accuracy.

My final solution will therefore result from the application of multiple methods, each with the intention of increasing the degree of accuracy by some magnitude and thus gaining confidence in the result.

Calculations and initial results:

I have used the excel spreadsheet package to carry out my investigation. This has imposed few limitations due to the ability to alter the number of decimal places by up to 30 digits, more than enough for my requirements (I have used 15 Decimal Places in my working, this seems sufficient as it allows the smaller differences in values from extrapolation to be seen). I will show particular parts of the spreadsheet throughout the document.

In order to apply formulas within my spreadsheet, I have used tables. Cells in the table reference other relevant cells via the cell formula, applying the appropriate arithmetical operations to give the intended solution, demonstrated in **Fig. 3**. Looking at the '$f(x)$' column of the formula view, '$B6$' refers to the cell seen storing value '2', this is used twice by the adjacent cell formula, which divides the value in '$B6$' by the 'LN' (Natural Log) of the value in '$B6$' to produce '$f(B6)$'. Similarly, '$B7$' refers to the adjacent value of 'x' which is manipulated in the same fashion to produce '$f(B7)$'.

Fig. 3- Table of 'x' and '$f(x)$' of the graph (standard and formula view):

x	f(x)=x/ln(x)
2	2.885390081777930
2.0078125	2.880459848665060
2.015625	2.875646573207410
…	…
2.984375	2.729468953610010
2.9921875	2.730086313530600
3	2.730717679880510

x	f(x)=x/ln(x)
2	=B6/LN(B6)
=B6+($F$12/2)	=B7/LN(B7)
=B7+($F$12/2)	=B8/LN(B8)
…	…
=B131+($F$12/2)	=B132/LN(B132)
=B132+($F$12/2)	=B133/LN(B133)
=B133+($F$12/2)	=B134/LN(B134)

Here, the interval between consecutive x values is ‘$^h/_2$’ for 64 strips, the maximum number of strips that I will be finding values for using this method.
Note: ‘$h = {}^w/_n$’, where ‘w’ is the range of the limits of integration (in this case $w = 1$).

Using the appropriate values of f(x), I have applied the midpoint and trapezium rules through to 64 strips (up to ‘$n = 64$’). I chose to calculate the trapezium rule in a similar fashion to the midpoint rule (through direct application of the respective formulas, with values acquired from **Fig. 3**). This was my chosen approach as it meant the ‘$x, f(x)$’ table would have a consistent ‘x’ interval throughout, from which all the values of ‘$f(x)$’ are used. However, I could have taken the approach of using the formula

All necessary values given for the application of the Midpoint and Trapezium rule with up to 64 strips, between the limits ‘2’ and ‘3’.

$$T_{2n} = {}^{(T_n + M_n)}/_2$$

This would have inevitably saved me some time once T_1 was calculated due to its simplicity over my chosen method, which involved over 64 terms at its worst.

Once confident that all calculations up to this point were correct, I progressed to find the Simpsons rule. I went about this through direct application of the Simpson’s rule formula with the relevant Midpoint and Trapezium values for each value of ‘n’.

Fig. 4- Table of calculated values for Midpoint, Trapezium and Simpson’s rules:

n	h	T_n	M_n	S_n
1	1	2.808053880829220	2.728391669843230	2.754945740171890
2	0.5	2.768222775336220	2.746527370992960	2.753759172440720
4	0.25	2.757375073164590	2.751802449337540	2.753659990613230
8	0.125	2.754588761251070	2.753185312822430	2.753653128965310
16	0.0625	2.753887037036750	2.753535512657630	2.753652687450670
32	0.03125	2.753711274847190	2.753623352045080	2.753652659645790
64	0.015625	2.753667313446140	2.753645330133900	2.753652657904640

The sequence of values produced by S_n comfortably converges to ‘2.753 …’, with the final two estimates both agreeing to a solution of ‘2.75365266’ ($9. S. F.$). This gives me confidence that this value is correct to 9 $significant\ figures$.

Extrapolating using approximations:

As is clear from the data, continuing this method for greater values of ‘n’ would allow me to give a solution to an even greater degree of accuracy. Rather than doing this, I have practiced the method of extrapolation. This is because it does not require any ‘new’ values for the Trapezium and Midpoint rules (to gain estimates for higher numbers of strips). Doing this would prove time consuming due to limitations in the formulas of the spreadsheet package (I would need to extend **Fig. 3**, adjusting all previous cell formulas for Trapezium and Midpoint rule estimates).

I have extrapolated all values up to S_{512}.

In order to extrapolate these values, I have acquired the ratio of differences, R of terms in the sequence S_n. The value of R will allow me to calculate what the sequence of S_n values is tending towards. I found that $R \approx 0.0626\ (3.S.F.)$ (see **Fig. 5**, later in document) from my best estimates by using the formula

$$R \approx {(S_{2n} - S_n)}\Big/{(S_n - S_{n/2})}$$

R = ratio of differences

This is very close to the expected value of 0.0625 and thus I believe that $R = 0.0625$. I expected this value as it is common of all fourth order methods, of which the Simpson's rule is an example. Values for the ratio of differences using lesser strips did not clearly show that '$R = 0.0625$'. Extrapolating using these estimates would not improve the accuracy of my result very much as consecutive values of S_n do not have consistent differences. For this reason, I have only extrapolated using estimates with the greatest number of strips.

By assuming that this value of R persists for future estimates of S_n, I can rearrange the formula to find the value of S_{2n}, such that

$$S_{2n} \approx S_n + R(S_n - S_{n/2})$$

I can continue using this formula to find the next value in the sequence of S_n

$$S_{4n} \approx S_{2n} + R(S_{2n} - S_n)$$

By substituting in the expression for S_{2n}, you find that

$$S_{4n} \approx (S_n + R(S_n - S_{n/2})) + R((S_n + R(S_n - S_{n/2})) - S_n)$$

This simplifies to

$$S_{4n} \approx S_n + R\left(S_n - S_{n/2}\right) + R^2\left(S_n - S_{n/2}\right)$$

If I were to continue the process and find the next value, S_{8n} then I would find that it is similar to the expression above but with a new term, '$R^3\left(S_n - S_{n/2}\right)$' added to the right-hand-side of the equation. This pattern proceeds for all following values of S_n, with a new term added which has an incremented R exponent from the previous term.

Using this knowledge of extrapolations, I can treat the terms with a factor of $(S_n - S_{n/2})$ as a series. By considering the value of S_∞, such that there are an infinite number of strips, I can calculate what the sequence of extrapolated estimates tends towards, and thus find a solution of the integral to a greater degree of accuracy. S_∞ is estimated using the infinite series of $(S_n - S_{n/2})$ terms, such that

$$S_\infty \approx S_n + {}^{R}/_{(1-R)}\left(S_n - S_{n/2}\right)$$

${}^{a}/_{(1-r)}$ = sum to infinity, where a = first term r = common ratio (In this case; a = R and r = R).

Fig. 5- Table of S_n and Table of extrapolated values respectively:

Exact		Extrapolated	
n	**S_n**	**R**	**S_n (using S_{32} & S_{64})**
1	2.754945740171890	-	-
2	2.753759172440720	-	-
4	2.753659990613230	0.083587160584827	-
8	2.753653128965310	0.069182511446405	-
16	2.753652687450670	0.064345278187509	-
32	2.753652659645790	0.062976125229237	-
64	2.753652657904640	0.062620087672416	-
128	-	0.0625	2.753652657795820
256	-		2.753652657789020
512	-		2.753652657788590
∞	-		2.753652657788570

The values for extrapolation up to S_∞ (far-right column) show the estimates of S tending towards '2.753652657788570'. This is an estimate for the value of S and so we cannot give this as a definite answer. The latter extrapolations agree with the value of '2.753652657789' ($13. S. F.$); I am confident in this value since the sequence is clearly converging. I can then go even further and decide whether to take the next digit of the integral to be '5' or '6'. '6' seems a reasonable choice since the final two extrapolations agree with this value. However, the sequence of extrapolations clearly shows the value of the integral tending towards '5' with greater values of 'n'. With both points above considered and with my previous experiences of extrapolating using estimates, it seems appropriate that '5' should be the value of this digit.

Interpretation and conclusion:

I have found the solution of

$$\int_{2}^{3} {}^{x}/_{\ln(x)}\,\mathrm{dx}$$

to be '2.7536526577885' to 14 $Significant\ Figures$.

Fig. 6- Interval of Midpoint and Trapezium rules (64 strip estimates):

$$2.753645330133900 < \int_{2}^{3} {}^{x}/_{\ln(x)}\,\mathrm{dx} < 2.753667313446140$$

I am confident in this being valid since it fits within my initial interval of values, set out by the Midpoint and Trapezium rules (see **Fig. 6** above). Furthermore, the method of extrapolation predicted convergence to this value.

The most significant limitation to my approach was the limit to the feasible number of strips that I could apply to the Simpson's rule. This is due to the method which I used (a single table for 'x, f(x)') to calculate consecutive values of the Midpoint and Trapezium rules.

I do not believe that the calculating power of the spreadsheet has limited my solution.