THE ART OF COMPUTER PROGRAMMING

PRE-FASCICLE 3B

A DRAFT OF SECTIONS 7.2.1.4–5: GENERATING ALL PARTITIONS

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Internet page `http://www-cs-faculty.stanford.edu/~knuth/taocp.html` contains current information about this book and related books.

See also `http://www-cs-faculty.stanford.edu/~knuth/sgb.html` for information about *The Stanford GraphBase*, including downloadable software for dealing with the graphs used in many of the examples in Chapter 7.

See also `http://www-cs-faculty.stanford.edu/~knuth/mmixware.html` for downloadable software to simulate the `MMIX` computer.

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PREFACE

Quhen a word fales to be divyded at the end of a lyne,
the partition must be made at the end of a syllab.
— ALEXANDER HUME, *Orthographie . . . of the Britan Tongue* (c. 1620)

THIS BOOKLET contains draft material that I'm circulating to experts in the field, in hopes that they can help remove its most egregious errors before too many other people see it. I am also, however, posting it on the Internet for courageous and/or random readers who don't mind the risk of reading a few pages that have not yet reached a very mature state. *Beware:* This material has not yet been proofread as thoroughly as the manuscripts of Volumes 1, 2, and 3 were at the time of their first printings. And those carefully-checked volumes, alas, were subsequently found to contain thousands of mistakes.

Given this caveat, I hope that my errors this time will not be so numerous and/or obtrusive that you will be discouraged from reading the material carefully. I did try to make it both interesting and authoritative, as far as it goes. But the field is so vast, I cannot hope to have surrounded it enough to corral it completely. Therefore I beg you to let me know about any deficiencies you discover.

To put the material in context, this pre-fascicle contains Sections 7.2.1.4 and 7.2.1.5 of a long, long chapter on combinatorial algorithms. Chapter 7 will eventually fill three volumes (namely Volumes 4A, 4B, and 4C), assuming that I'm able to remain healthy. It will begin with a short review of graph theory, with emphasis on some highlights of significant graphs in The Stanford GraphBase, from which I will be drawing many examples. Then comes Section 7.1, which deals with the topic of bitwise manipulations. (I drafted about 60 pages about that subject in 1977, but those pages need extensive revision; meanwhile I've decided to work for awhile on the material that follows it, so that I can get a better feel for how much to cut.) Section 7.2 is about generating all possibilities, and it begins with Section 7.2.1: Generating Basic Combinatorial Patterns— which, in turn, begins with Section 7.2.1.1, "Generating all n-tuples," Section 7.2.1.2, "Generating all permutations," and Section 7.2.1.3, "Generating all combinations." (Readers of the present booklet should have already looked at those sections, drafts of which are available as Pre-Fascicles 2A, 2B, and 3A.) The stage is now set for the main contents of this booklet, Section 7.2.1.4: "Generating all partitions," and Section 7.2.1.5: "Generating all set partitions." Then will come Section 7.2.1.6 (about trees), etc. Section 7.2.2 will deal with backtracking in general. And so it will go on, if all goes well; an outline of the entire Chapter 7 as currently envisaged appears on the `taocp` webpage that is cited on page ii.

Even the apparently lowly topic of partition generation turns out to be surprisingly rich, with ties to Sections 1.2.5, 1.2.6, 1.2.9, 1.2.10, 1.2.11.2, 1.3.3, 2.3.3, 2.3.4.2, 2.3.4.4, 2.3.4.5, 3.3.2, 3.3.3, 3.4.1, 4.5.4, 4.6.2, 4.7, 5, 5.1.1, 5.1.2, 5.1.3, 5.1.4, 5.2.2, 5.2.3, and 5.2.5 of the first three volumes. I strongly believe in building up a firm foundation, so I have discussed this topic much more thoroughly than I will be able to do with material that is newer or less basic. Indeed, the theory of partitions is one of the nicest chapters in all of mathematics. To my surprise, I came up with 154 exercises, even though — believe it or not — I had to eliminate quite a bit of the interesting material that appears in my files.

Some of the things presented are new, to the best of my knowledge, although I will not be at all surprised to learn that my own little "discoveries" have been discovered before. Please look, for example, at the exercises that I've classed as research problems (rated with difficulty level 46 or higher), namely exercises 7.2.1.4–51, 62, 63, 71, and 7.2.1.5–18, 66, 74, 77; I've also implicitly posed additional unsolved questions in the answers to exercises 7.2.1.4–48 and 69. Are those problems still open? Please let me know if you know of a solution to any of these intriguing questions. And of course if no solution is known today but you do make progress on any of them in the future, I hope you'll let me know.

I urgently need your help also with respect to some exercises that I made up as I was preparing this material. I certainly don't like to get credit for things that have already been published by others, and most of these results are quite natural "fruits" that were just waiting to be "plucked." Therefore please tell me if you know who I should have credited, with respect to the ideas found in exercises 7.2.1.4–20, 27, 48, 49, 50, 56; 7.2.1.5–2, 6, 8, 9, 25, 26, 35, 38(e), 47, 50, 52, 56, and/or 76.

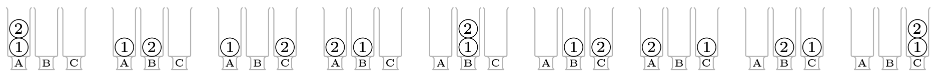
I shall happily pay a finder's fee of $2.56 for each error in this draft when it is first reported to me, whether that error be typographical, technical, or historical. The same reward holds for items that I forgot to put in the index. And valuable suggestions for improvements to the text are worth 32¢ each. (Furthermore, if you find a better solution to an exercise, I'll actually reward you with immortal glory instead of mere money, by publishing your name in the eventual book:–)

Cross references to yet-unwritten material sometimes appear as '00'; this impossible value is a placeholder for the actual numbers to be supplied later.

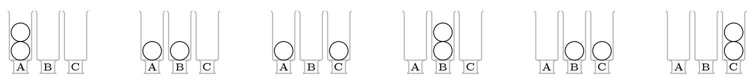
Happy reading!

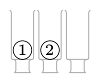
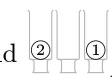
Stanford, California D. E. K.
14 February 2004

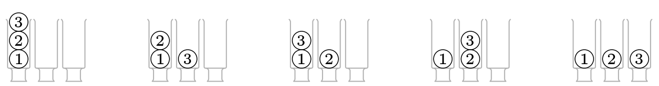
7.2.1.4. Generating all partitions. Richard Stanley's magnificent book *Enumerative Combinatorics* (1986) begins by discussing The Twelvefold Way, a $2 \times 2 \times 3$ array of basic combinatorial problems that arise frequently in practice (see Table 1). All twelve of Stanley's basic problems can be described in terms of the ways that a given number of balls can be placed into a given number of urns. For example, there are nine ways to put 2 balls into 3 urns if the balls and urns are labeled:



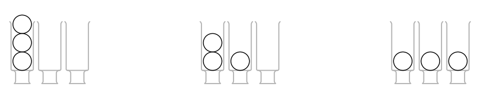
(The order of balls *within* an urn is ignored.) But if the balls are unlabeled, some of these arrangements are indistinguishable, so only six different ways are possible:

 (1)

If the urns are unlabeled, arrangements like  and  are essentially the same, hence only two of the original nine arrangements are distinguishable. And if we have three labeled balls, the only distinct ways to place them into three unlabeled urns are

 (2)

Finally, if neither balls nor urns are labeled, these five possibilities reduce to only three:

 (3)

The Twelvefold Way considers all arrangements that are possible when balls and urns are labeled or unlabeled, and when the urns may optionally be required to contain at least one ball or at most one ball.

Table 1

THE TWELVEFOLD WAY

balls per urn	unrestricted	≤ 1	≥ 1
n labeled balls, m labeled urns	n-tuples of m things	n-permutations of m things	partitions of $\{1,\ldots,n\}$ into m ordered parts
n unlabeled balls, m labeled urns	n-multicombinations of m things	n-combinations of m things	compositions of n into m parts
n labeled balls, m unlabeled urns	partitions of $\{1,\ldots,n\}$ into $\le m$ parts	n pigeons into m holes	partitions of $\{1,\ldots,n\}$ into m parts
n unlabeled balls, m unlabeled urns	partitions of n into $\le m$ parts	n pigeons into m holes	partitions of n into m parts

We've learned about n-tuples, permutations, combinations, and compositions in previous sections of this chapter; and two of the twelve entries in Table 1 are trivial (namely the ones related to "pigeons"). So we can complete our study of classical combinatorial mathematics by learning about the remaining five entries in the table, which all involve *partitions*.

> *Let us begin by acknowledging that the word "partition" has numerous meanings in mathematics. Any time a division of some object into subobjects is undertaken, the word partition is likely to pop up.*
>
> — GEORGE ANDREWS, *The Theory of Partitions* (1976)

Two quite different concepts share the same name: The *partitions of a set* are the ways to subdivide it into disjoint subsets; thus (2) illustrates the five partitions of $\{1,2,3\}$, namely

$$\{1,2,3\}, \qquad \{1,2\}\{3\}, \qquad \{1,3\}\{2\}, \qquad \{1\}\{2,3\}, \qquad \{1\}\{2\}\{3\}. \tag{4}$$

And the *partitions of an integer* are the ways to write it as a sum of positive integers, disregarding order; thus (3) illustrates the three partitions of 3, namely

$$3, \qquad\qquad 2+1, \qquad\qquad 1+1+1. \tag{5}$$

We shall follow the common practice of referring to integer partitions as simply "partitions," without any qualifying adjective; the other kind will be called "set partitions" in what follows, to make the distinction clear. Both kinds of partitions are important, so we'll study each of them in turn.

Generating all partitions of an integer. A partition of n can be defined formally as a sequence of nonnegative integers $a_1 \geq a_2 \geq \cdots$ such that $n = a_1 + a_2 + \cdots$; for example, one partition of 7 has $a_1 = a_2 = 3$, $a_3 = 1$, and $a_4 = a_5 = \cdots = 0$. The number of nonzero terms is called the number of *parts*, and the zero terms are usually suppressed. Thus we write $7 = 3 + 3 + 1$, or simply 331 to save space when the context is clear.

The simplest way to generate all partitions, and one of the fastest, is to visit them in reverse lexicographic order, starting with 'n' and ending with '$11\ldots1$'. For example, the partitions of 8 are

$$\begin{gathered} 8,\ 71,\ 62,\ 611,\ 53,\ 521,\ 5111,\ 44,\ 431,\ 422,\ 4211,\ 41111,\ 332,\ 3311, \\ 3221,\ 32111,\ 311111,\ 2222,\ 22211,\ 221111,\ 2111111,\ 11111111, \end{gathered} \tag{6}$$

when listed in this order.

If a partition isn't all 1s, it ends with $(x+1)$ followed by zero or more 1s, for some $x \geq 1$; therefore the next smallest partition in lexicographic order is obtained by replacing the suffix $(x+1)1\ldots1$ by $x\ldots xr$ for some appropriate remainder $r \leq x$. The process is quite efficient if we keep track of the largest subscript q such that $a_q \neq 1$, as suggested by J. K. S. McKay [*CACM* **13** (1970), 52]:

Algorithm P (*Partitions in reverse lexicographic order*). This algorithm generates all partitions $a_1 \ge a_2 \ge \cdots \ge a_m \ge 1$ with $a_1 + a_2 + \cdots + a_m = n$ and $1 \le m \le n$, assuming that $n \ge 1$.

P1. [Initialize.] Set $a_0 \leftarrow 0$ and $m \leftarrow 1$.

P2. [Store the final part.] Set $a_m \leftarrow n$ and $q \leftarrow m - [n=1]$.

P3. [Visit.] Visit the partition $a_1 a_2 \ldots a_m$. Then go to P5 if $a_q \ne 2$.

P4. [Change 2 to 1+1.] Set $a_q \leftarrow 1$, $q \leftarrow q-1$, $m \leftarrow m+1$, $a_m \leftarrow 1$, and return to P3.

P5. [Decrease a_q.] Terminate the algorithm if $q = 0$. Otherwise set $x \leftarrow a_q - 1$, $a_q \leftarrow x$, $n \leftarrow m - q + 1$, and $m \leftarrow q + 1$.

P6. [Copy x if necessary.] If $n \le x$, return to step P2. Otherwise set $a_m \leftarrow x$, $m \leftarrow m + 1$, $n \leftarrow n - x$, and repeat this step. ▮

Notice that the operation of going from one partition to the next is particularly easy if a 2 is present; then step P4 simply changes the rightmost 2 to a 1 and appends another 1 at the right. This happy situation is, fortunately, the most common case. For example, nearly 79% of all partitions contain a 2 when $n = 100$.

Another simple algorithm is available when we want to generate all partitions of n into a fixed number of parts. The following method, which was featured in C. F. Hindenburg's 18th-century dissertation [*Infinitinomii Dignitatum Exponentis Indeterminati* (Göttingen, 1779), 73–91], visits the partitions in *colex* order, namely in lexicographic order of the reflected sequence $a_m \ldots a_2 a_1$:

Algorithm H (*Partitions into m parts*). This algorithm generates all integer m-tuples $a_1 \ldots a_m$ such that $a_1 \ge \cdots \ge a_m \ge 1$ and $a_1 + \cdots + a_m = n$, assuming that $n \ge m \ge 2$.

H1. [Initialize.] Set $a_1 \leftarrow n - m + 1$ and $a_j \leftarrow 1$ for $1 < j \le m$. Also set $a_{m+1} \leftarrow -1$.

H2. [Visit.] Visit the partition $a_1 \ldots a_m$. Then go to H4 if $a_2 \ge a_1 - 1$.

H3. [Tweak a_1 and a_2.] Set $a_1 \leftarrow a_1 - 1$, $a_2 \leftarrow a_2 + 1$, and return to H2.

H4. [Find j.] Set $j \leftarrow 3$ and $s \leftarrow a_1 + a_2 - 1$. Then, if $a_j \ge a_1 - 1$, set $s \leftarrow s + a_j$, $j \leftarrow j + 1$, and repeat until $a_j < a_1 - 1$. (Now $s = a_1 + \cdots + a_{j-1} - 1$.)

H5. [Increase a_j.] Terminate if $j > m$. Otherwise set $x \leftarrow a_j + 1$, $a_j \leftarrow x$, $j \leftarrow j - 1$.

H6. [Tweak $a_1 \ldots a_j$.] While $j > 1$, set $a_j \leftarrow x$, $s \leftarrow s - x$, and $j \leftarrow j - 1$. Finally set $a_1 \leftarrow s$ and return to H2. ▮

For example, when $n = 11$ and $m = 4$ the successive partitions visited are

$$8111,\ 7211,\ 6311,\ 5411,\ 6221,\ 5321,\ 4421,\ 4331,\ 5222,\ 4322,\ 3332. \qquad (7)$$

The basic idea is that colex order goes from one partition $a_1 \ldots a_m$ to the next by finding the smallest j such that a_j can be increased without changing $a_{j+1} \ldots a_m$. The new partition $a'_1 \ldots a'_m$ will have $a'_1 \ge \cdots \ge a'_j = a_j + 1$ and $a'_1 + \cdots + a'_j =$

$a_1 + \cdots + a_j$, and these conditions are achievable if and only if $a_j < a_1 - 1$. Furthermore, the smallest such partition $a'_1 \ldots a'_m$ in colex order has $a'_2 = \cdots = a'_j = a_j + 1$.

Step H3 handles the simple case $j = 2$, which is by far the most common. And indeed, the value of j almost always turns out to be quite small; we will prove later that the total running time of Algorithm H is at most a small constant times the number of partitions visited, plus $O(m)$.

Other representations of partitions. We've defined a partition as a sequence of nonnegative integers $a_1 a_2 \ldots$ with $a_1 \geq a_2 \geq \cdots$ and $a_1 + a_2 + \cdots = n$, but we can also regard it as an n-tuple of nonnegative integers $c_1 c_2 \ldots c_n$ such that

$$c_1 + 2c_2 + \cdots + nc_n \;=\; n. \tag{8}$$

Here c_j is the number of times the integer j appears in the sequence $a_1 a_2 \ldots$; for example, the partition 331 corresponds to the counts $c_1 = 1$, $c_2 = 0$, $c_3 = 2$, $c_4 = c_5 = c_6 = c_7 = 0$. The number of parts is then $c_1 + c_2 + \cdots + c_n$. A procedure analogous to Algorithm P can readily be devised to generate partitions in part-count form; see exercise 5.

We have already seen the part-count representation implicitly in formulas like Eq. 1.2.9–(38), which expresses the symmetric function

$$h_n = \sum_{N \geq d_n \geq \cdots \geq d_2 \geq d_1 \geq 1} x_{d_1} x_{d_2} \ldots x_{d_n} \tag{9}$$

as

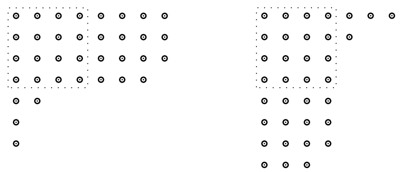
$$\sum_{\substack{c_1, c_2, \ldots, c_n \geq 0 \\ c_1 + 2c_2 + \cdots + nc_n = n}} \frac{S_1^{c_1}}{1^{c_1} c_1!} \, \frac{S_2^{c_2}}{2^{c_2} c_2!} \cdots \frac{S_n^{c_n}}{n^{c_n} c_n!}, \tag{10}$$

where S_j is the symmetric function $x_1^j + x_2^j + \cdots + x_N^j$. The sum in (9) is essentially taken over all n-multicombinations of N, while the sum in (10) is taken over all partitions of n. Thus, for example, $h_3 = \frac{1}{6} S_1^3 + \frac{1}{2} S_1 S_2 + \frac{1}{3} S_3$, and when $N = 2$ we have

$$x^3 + x^2 y + x y^2 + y^3 = \tfrac{1}{6}(x+y)^3 + \tfrac{1}{2}(x+y)(x^2+y^2) + \tfrac{1}{3}(x^3+y^3).$$

Other sums over partitions appear in exercises 1.2.5–21, 1.2.9–10, 1.2.9–11, 1.2.10–12, etc.; for this reason partitions are of central importance in the study of symmetric functions, a class of functions that pervades mathematics in general. [Chapter 7 of Richard Stanley's *Enumerative Combinatorics* **2** (1999) is an excellent introduction to advanced aspects of symmetric function theory.]

Partitions can be visualized in an appealing way by considering an array of n dots, having a_1 dots in the top row and a_2 in the next row, etc. Such an arrangement of dots is called the *Ferrers diagram* of the partition, in honor of N. M. Ferrers [see *Philosophical Mag.* **5** (1853), 199–202]; and the largest square subarray of dots that it contains is called the *Durfee square*, after W. P. Durfee [see *Johns Hopkins Univ. Circular* **2** (December 1882), 23]. For example, the Ferrers diagram of 8887211 is shown with its 4×4 Durfee square in Fig. 28(a).



(a) 8887211 (b) 75444443

Fig. 28. The Ferrers diagrams and Durfee squares of two conjugate partitions.

The Durfee square contains k^2 dots when k is the largest subscript such that $a_k \geq k$; we may call k the *trace* of the partition.

If α is any partition $a_1 a_2 \ldots$, its *conjugate* $\alpha^T = b_1 b_2 \ldots$ is obtained by transposing the rows and columns of the corresponding Ferrers diagram. For example, Fig. 28(b) shows that $(8887211)^T = 75444443$. When $\beta = \alpha^T$ we obviously have $\alpha = \beta^T$; the partition β has a_1 parts and α has b_1 parts. Indeed, there's a simple relation between the part-count representation $c_1 \ldots c_n$ of α and the conjugate partition $b_1 b_2 \ldots$, namely

$$b_j - b_{j+1} \;=\; c_j \qquad \text{for all } j \geq 1. \tag{11}$$

This relation makes it easy to compute the conjugate of a given partition, or to write it down by inspection (see exercise 6).

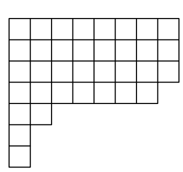
The notion of conjugation often explains properties of partitions that would otherwise be quite mysterious. For example, now that we know the definition of α^T, we can easily see that the value of $j-1$ in step H5 of Algorithm H is just the second-smallest part of the conjugate partition $(a_1 \ldots a_m)^T$. Therefore the average amount of work that needs to be done in steps H4 and H6 is essentially proportional to the average size of the second-smallest part of a random partition whose largest part is m. And we will see below that the second-smallest part is almost always quite small.

Moreover, *Algorithm H produces partitions in lexicographic order of their conjugates.* For example, the respective conjugates of (7) are

$$\begin{gathered} 41111111,\ 4211111,\ 422111,\ 42221,\ 431111, \\ 43211,\ 4322,\ 4331,\ 44111,\ 4421,\ 443; \end{gathered} \tag{12}$$

these are the partitions of $n = 11$ with largest part 4. One way to generate all partitions of n is to start with the trivial partition 'n', then run Algorithm H for $m = 2, 3, \ldots, n$ in turn; this process yields all α in lexicographic order of α^T (see exercise 7). Thus Algorithm H can be regarded as a dual of Algorithm P.

There is at least one more useful way to represent partitions, called the *rim representation.* Suppose we replace the dots of a Ferrers diagram by boxes, thereby obtaining a tableau shape as we did in Section 5.1.4; for example, the partition 8887211 of Fig. 28(a) becomes



. (13)

The right-hand boundary of this shape can be regarded as a path from the lower left corner to the upper right corner of an $n \times n$ square, and we know from Table 7.2.1.3–1 that such a path corresponds to an (n, n)-combination.

For example, (13) corresponds to the 70-bit string

$$0\ldots01001011111010001\ldots1 \;=\; 0^{28}1^{1}0^{2}1^{1}0^{1}1^{5}0^{1}1^{1}0^{3}1^{27}, \qquad (14)$$

where we place enough 0s at the beginning and 1s at the end to make exactly n of each. The 0s represent upward steps of the path, and the 1s represent rightward steps. It is easy to see that the bit string defined in this way has exactly n inversions; conversely, every permutation of the multiset $\{n\cdot 0,\ n\cdot 1\}$ that has exactly n inversions corresponds to a partition of n. When the partition has t different parts, its bit string can be written in the form

$$0^{n-q_1-q_2-\cdots-q_t}\,1^{p_1}\,0^{q_1}\,1^{p_2}\,0^{q_2}\ldots 1^{p_t}\,0^{q_t}\,1^{n-p_1-p_2-\cdots-p_t}, \qquad (15)$$

where the exponents p_j and q_j are positive integers. Then the partition's standard representation is

$$a_1a_2\ldots \;=\; (p_1+\cdots+p_t)^{q_t}\,(p_1+\cdots+p_{t-1})^{q_{t-1}}\ldots(p_1)^{q_1}, \qquad (16)$$

namely $(1{+}1{+}5{+}1)^3(1{+}1{+}5)^1(1{+}1)^1(1)^2 = 8887211$ in our example.

The number of partitions. Inspired by a question that was posed to him by Philipp Naudé in 1740, Leonhard Euler wrote two fundamental papers in which he counted partitions of various kinds by studying their generating functions [*Commentarii Academiæ Scientiarum Petropolitanæ* **13** (1741), 64–93; *Novi Comment. Acad. Sci. Pet.* **3** (1750), 125–169]. He observed that the coefficient of z^n in the infinite product

$$(1{+}z{+}z^2{+}\cdots{+}z^j{+}\cdots)(1{+}z^2{+}z^4{+}\cdots{+}z^{2k}{+}\cdots)(1{+}z^3{+}z^6{+}\cdots{+}z^{3l}{+}\cdots)\ldots$$

is the number of nonnegative integer solutions to the equation $j{+}2k{+}3l{+}\cdots = n$; and $1 + z^m + z^{2m} + \cdots$ is $1/(1-z^m)$. Therefore if we write

$$P(z) \;=\; \prod_{m=1}^{\infty}\frac{1}{1-z^m} \;=\; \sum_{n=0}^{\infty} p(n)\,z^n, \qquad (17)$$

the number of partitions of n is $p(n)$. This function $P(z)$ turns out to have an amazing number of subtle mathematical properties.

For example, Euler discovered that massive cancellation occurs when the denominator of $P(z)$ is multiplied out:

$$\begin{aligned}(1{-}z)(1{-}z^2)(1{-}z^3)\ldots &= 1 - z - z^2 + z^5 + z^7 - z^{12} - z^{15} + z^{22} + z^{26} - \cdots\\ &= \sum_{-\infty<n<\infty}(-1)^n z^{(3n^2+n)/2}. \qquad (18)\end{aligned}$$

A combinatorial proof of this remarkable identity, based on Ferrers diagrams, appears in exercise 5.1.1–14; we can also prove it by setting $u = z$ and $v = z^2$ in the even more remarkable identity of Jacobi,

$$\prod_{k=1}^{\infty}(1-u^k v^{k-1})(1-u^{k-1}v^k)(1-u^k v^k) \;=\; \sum_{n=-\infty}^{\infty}(-1)^n u^{\binom{n}{2}} v^{\binom{-n}{2}}, \qquad (19)$$

because the left-hand side becomes $\prod_{k=1}^{\infty}(1-z^{3k-2})(1-z^{3k-1})(1-z^{3k})$; see exercise 5.1.1–20. Euler's identity (18) implies that the partition numbers satisfy the recurrence

$$p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7) + p(n-12) + p(n-15) - \cdots, \qquad (20)$$

from which we can compute their values more rapidly than by performing the power series calculations in (17):

$n =$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$p(n) =$	1	1	2	3	5	7	11	15	22	30	42	56	77	101	135	176

We know from Section 1.2.8 that solutions to the Fibonacci recurrence $f(n) = f(n-1) + f(n-2)$ grow exponentially, with $f(n) = \Theta(\phi^n)$ when $f(0)$ and $f(1)$ are positive. The additional terms '$-\,p(n-5) - p(n-7)$' in (20) have a dampening effect on partition numbers, however; in fact, if we were to stop the recurrence there, the resulting sequence would oscillate between positive and negative values. Further terms '$+p(n-12)+p(n-15)$' reinstate exponential growth.

The actual growth rate of $p(n)$ turns out to be of order $A^{\sqrt{n}}/n$ for a certain constant A. For example, exercise 33 proves directly that $p(n)$ grows at least as fast as $e^{2\sqrt{n}}/n$. And one fairly easy way to obtain a decent *upper* bound is to take logarithms in (17),

$$\ln P(z) \;=\; \sum_{m=1}^{\infty} \ln\frac{1}{1-z^m} \;=\; \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{z^{mn}}{n}, \qquad (21)$$

and then to look at the behavior near $z = 1$ by setting $z = e^{-t}$:

$$\ln P(e^{-t}) = \sum_{m,n\ge 1}\frac{e^{-mnt}}{n} = \sum_{n\ge 1}\frac{1}{n}\,\frac{1}{e^{tn}-1} < \sum_{n\ge 1}\frac{1}{n^2 t} = \frac{\zeta(2)}{t}. \qquad (22)$$

Consequently, since $p(n) \le p(n+1) < p(n+2) < \cdots$ and $e^t > 1$, we have

$$\frac{p(n)}{1-e^{-t}} \;<\; \sum_{k=0}^{\infty} p(k)e^{(n-k)t} \;=\; e^{nt}P(e^{-t}) \;<\; e^{nt+\zeta(2)/t} \qquad (23)$$

for all $t > 0$. Setting $t = \sqrt{\zeta(2)/n}$ gives

$$p(n) \;<\; Ce^{2C\sqrt{n}}/\sqrt{n}, \qquad \text{where } C = \sqrt{\zeta(2)} = \pi/\sqrt{6}. \qquad (24)$$

We can obtain more accurate information about the size of $\ln P(e^{-t})$ by using Euler's summation formula (Section 1.2.11.2) or Mellin transforms (Section 5.2.2); see exercise 25. But the methods we have seen so far aren't powerful enough to deduce the precise behavior of $P(e^{-t})$, so it is time for us to add a new weapon to our arsenal of techniques.

Euler's generating function $P(z)$ is ideally suited to the *Poisson summation formula* [*J. École Royale Polytechnique* **12** (1823), 404–509, §63], according to which

$$\sum_{n=-\infty}^{\infty} f(n+\theta) \;=\; \lim_{M\to\infty} \sum_{m=-M}^{M} e^{2\pi m i\theta} \int_{-\infty}^{\infty} e^{-2\pi m i y} f(y)\,dy, \qquad (25)$$

whenever f is a "well-behaved" function. This formula is based on the fact that the left-hand side is a periodic function of θ, and the right-hand side is the expansion of that function as a Fourier series. The function f is sufficiently nice if, for example, $\int_{-\infty}^{\infty} \bigl|f(y)\bigr|\,dy < \infty$ and either

i) $f(n+\theta)$ is an analytic function of the complex variable θ in the region $|\Im\theta| \le \epsilon$ for some $\epsilon > 0$ and $0 \le \Re\theta \le 1$, and the left-hand side converges uniformly in that rectangle; or

ii) $f(\theta) = \frac{1}{2}\lim_{\epsilon\to 0}\bigl(f(\theta-\epsilon)+f(\theta+\epsilon)\bigr) = g(\theta) - h(\theta)$ for all real numbers θ, where g and h are monotone increasing and $g(\pm\infty)$, $h(\pm\infty)$ are finite.

[See Peter Henrici, *Applied and Computational Complex Analysis* **2** (New York: Wiley, 1977), Theorem 10.6.2.] Poisson's formula is not a panacea for summation problems of every kind; but when it does apply the results can be spectacular, as we will see.

Let us multiply Euler's formula (18) by $z^{1/24}$ in order to "complete the square":

$$\frac{z^{1/24}}{P(z)} \;=\; \sum_{n=-\infty}^{\infty} (-1)^n\, z^{\frac{3}{2}(n+\frac{1}{6})^2}. \qquad (26)$$

Then for all $t > 0$ we have $e^{-t/24}/P(e^{-t}) = \sum_{n=-\infty}^{\infty} f(n)$, where

$$f(y) \;=\; e^{-\frac{3}{2}t(y+\frac{1}{6})^2} \cos \pi y; \qquad (27)$$

and this function f qualifies for Poisson's summation formula under both of the criteria (i) and (ii) stated above. Therefore we can try to integrate $e^{-2\pi m i y} f(y)$, and for $m = 0$ the result is

$$\int_{-\infty}^{\infty} f(y)\,dy \;=\; \sqrt{\frac{\pi}{2t}}\, e^{-\pi^2/6t}. \qquad (28)$$

To this we must add

$$\sum_{m=1}^{\infty} \int_{-\infty}^{\infty} (e^{2\pi m i y} + e^{-2\pi m i y})\, f(y)\,dy \;=\; 2\sum_{m=1}^{\infty} \int_{-\infty}^{\infty} f(y)\cos 2\pi m y\,dy; \qquad (29)$$

again the integral turns out to be doable. And the results (see exercise 27) fit together quite beautifully, giving

$$\frac{e^{-t/24}}{P(e^{-t})} \;=\; \sqrt{\frac{2\pi}{t}} \sum_{n=-\infty}^{\infty} (-1)^n e^{-6\pi^2(n+\frac{1}{6})^2/t} \;=\; \sqrt{\frac{2\pi}{t}}\, \frac{e^{-\pi^2/6t}}{P(e^{-4\pi^2/t})}. \qquad (30)$$

Surprise! We have proved another remarkable fact about $P(z)$:

Theorem D. *The generating function* (17) *for partitions satisfies the functional relation*

$$\ln P(e^{-t}) \;=\; \frac{\zeta(2)}{t} + \frac{1}{2}\ln\frac{t}{2\pi} - \frac{t}{24} + \ln P(e^{-4\pi^2/t}) \tag{31}$$

when $\Re t > 0$. ▮

This theorem was discovered by Richard Dedekind [*Crelle* **83** (1877), 265–292, §6], who wrote $\eta(\tau)$ for the function $z^{1/24}/P(z)$ when $z = e^{2\pi i\tau}$; his proof was based on a much more complicated theory of elliptic functions. Notice that when t is a small positive number, $\ln P(e^{-4\pi^2/t})$ is *extremely* tiny; for example, when $t = 0.1$ we have $\exp(-4\pi^2/t) \approx 3.5 \times 10^{-172}$. Therefore Theorem D tells us essentially everything we need to know about the value of $P(z)$ when z is near 1.

G. H. Hardy and S. Ramanujan used this knowledge to deduce the asymptotic behavior of $p(n)$ for large n, and their work was extended many years later by Hans Rademacher, who discovered a series that is not only asymptotic but convergent [*Proc. London Math. Soc.* (2) **17** (1918), 75–115; **43** (1937), 241–254]. The Hardy–Ramanujan–Rademacher formula for $p(n)$ is surely one of the most astonishing identities ever discovered; it states that

$$p(n) \;=\; \frac{\pi}{2^{5/4}3^{3/4}(n-1/24)^{3/4}} \sum_{k=1}^{\infty} \frac{A_k(n)}{k} I_{3/2}\biggl(\sqrt{\frac{2}{3}}\,\frac{\pi}{k}\sqrt{n-1/24}\biggr). \tag{32}$$

Here $I_{3/2}$ denotes the modified spherical Bessel function

$$I_{3/2}(z) \;=\; \Bigl(\frac{z}{2}\Bigr)^{3/2} \sum_{k=0}^{\infty} \frac{1}{\Gamma(k+5/2)} \frac{(z^2/4)^k}{k!} \;=\; \sqrt{\frac{2z}{\pi}}\Bigl(\frac{\cosh z}{z} - \frac{\sinh z}{z^2}\Bigr); \tag{33}$$

and the coefficient $A_k(n)$ is defined by the formula

$$A_k(n) \;=\; \sum_{h=0}^{k-1} [h \perp k] \exp\Bigl(2\pi i\Bigl(\frac{\sigma(h,k,0)}{24} - \frac{nh}{k}\Bigr)\Bigr) \tag{34}$$

where $\sigma(h,k,0)$ is the Dedekind sum defined in Eq. 3.3.3–(16). We have

$$A_1(n) = 1, \qquad A_2(n) = (-1)^n, \qquad A_3(n) = 2\cos\frac{(24n+1)\pi}{18}, \tag{35}$$

and in general $A_k(n)$ lies between $-k$ and k.

A proof of (32) would take us far afield, but the basic idea is to use the "saddle point method" discussed in Section 7.2.1.5. The term for $k = 1$ is derived from the behavior of $P(z)$ when z is near 1; and the next term is derived from the behavior when z is near -1, where a transformation similar to (31) can be applied. In general, the kth term of (32) takes account of the way $P(z)$ behaves when z approaches $e^{2\pi ih/k}$ for irreducible fractions h/k with denominator k; every kth root of unity is a pole of each of the factors $1/(1-z^k)$, $1/(1-z^{2k})$, $1/(1-z^{3k})$, … in the infinite product for $P(z)$.

The leading term of (32) can be simplified greatly, if we merely want a rough approximation:

$$p(n) \;=\; \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}\bigl(1+O(n^{-1/2})\bigr). \tag{36}$$

Or, if we choose to retain a few more details,

$$p(n) \;=\; \frac{e^{\pi\sqrt{2n'/3}}}{4n'\sqrt{3}}\left(1-\frac{1}{\pi}\sqrt{\frac{3}{2n'}}\right)\left(1+O\bigl(e^{-\pi\sqrt{n/6}}\bigr)\right), \quad n' = n-\frac{1}{24}. \tag{37}$$

For example, $p(100)$ has the exact value 190,569,292; formula (36) tells us that $p(100) \approx 1.993 \times 10^8$, while (37) gives the far better estimate 190,568,944.783.

Andrew Odlyzko has observed that, when n is large, the Hardy–Ramanujan–Rademacher formula actually gives a near-optimum way to compute the precise value of $p(n)$, because the arithmetic operations can be carried out in nearly $O\bigl(\log p(n)\bigr) = O(n^{1/2})$ steps. The first few terms of (32) give the main contribution; then the series settles down to terms that are of order $k^{-3/2}$ and usually of order k^{-2}. Furthermore, about half of the coefficients $A_k(n)$ turn out to be zero (see exercise 28). For example, when $n = 10^6$, the terms for $k = 1$, 2, and 3 are $\approx 1.47 \times 10^{1107}$, 1.23×10^{550}, and -1.23×10^{364}, respectively. The sum of the first 250 terms is $\approx 1471684986\ldots73818.01$, while the true value is $1471684986\ldots73818$; and 123 of those 250 terms are zero.

The number of parts. It is convenient to introduce the notation

$$\left|{n \atop m}\right| \tag{38}$$

for the number of partitions of n that have exactly m parts. Then the recurrence

$$\left|{n \atop m}\right| \;=\; \left|{n-1 \atop m-1}\right| + \left|{n-m \atop m}\right| \tag{39}$$

holds for all integers m and n, because $\left|{n-1 \atop m-1}\right|$ counts the partitions whose smallest part is 1 and $\left|{n-m \atop m}\right|$ counts the others. (If the smallest part is 2 or more, we can subtract 1 from each part and get a partition of $n-m$ into m parts.) By similar reasoning we can conclude that $\left|{m+n \atop m}\right|$ is the number of partitions of n into *at most* m parts, namely into m nonnegative summands. We also know, by considering Ferrers diagrams, that $\left|{n \atop m}\right|$ is the number of partitions of n whose *largest* part is m. Thus $\left|{n \atop m}\right|$ is a good number to know. The boundary conditions

$$\left|{n \atop 0}\right| = \delta_{n0} \qquad \text{and} \qquad \left|{n \atop m}\right| = 0 \quad \text{for } m<0 \text{ or } n<0 \tag{40}$$

make it easy to tabulate $\left|{n \atop m}\right|$ for small values of the parameters, and we obtain an array of numbers analogous to the familiar triangles for $\binom{n}{m}$, $\left[{n \atop m}\right]$, $\left\{{n \atop m}\right\}$, and $\left\langle{n \atop m}\right\rangle$ that we've seen before; see Table 2. The generating function is

$$\sum_n \left|{n \atop m}\right| z^n \;=\; \frac{z^m}{(1-z)(1-z^2)\ldots(1-z^m)}. \tag{41}$$

Table 2

PARTITION NUMBERS

n	$\left\lvert{n \atop 0}\right\rvert$	$\left\lvert{n \atop 1}\right\rvert$	$\left\lvert{n \atop 2}\right\rvert$	$\left\lvert{n \atop 3}\right\rvert$	$\left\lvert{n \atop 4}\right\rvert$	$\left\lvert{n \atop 5}\right\rvert$	$\left\lvert{n \atop 6}\right\rvert$	$\left\lvert{n \atop 7}\right\rvert$	$\left\lvert{n \atop 8}\right\rvert$	$\left\lvert{n \atop 9}\right\rvert$	$\left\lvert{n \atop 10}\right\rvert$	$\left\lvert{n \atop 11}\right\rvert$
0	1	0	0	0	0	0	0	0	0	0	0	0
1	0	1	0	0	0	0	0	0	0	0	0	0
2	0	1	1	0	0	0	0	0	0	0	0	0
3	0	1	1	1	0	0	0	0	0	0	0	0
4	0	1	2	1	1	0	0	0	0	0	0	0
5	0	1	2	2	1	1	0	0	0	0	0	0
6	0	1	3	3	2	1	1	0	0	0	0	0
7	0	1	3	4	3	2	1	1	0	0	0	0
8	0	1	4	5	5	3	2	1	1	0	0	0
9	0	1	4	7	6	5	3	2	1	1	0	0
10	0	1	5	8	9	7	5	3	2	1	1	0
11	0	1	5	10	11	10	7	5	3	2	1	1

Almost all partitions of n have $\Theta(\sqrt{n}\log n)$ parts. This fact, discovered by P. Erdős and J. Lehner [*Duke Math. J.* **8** (1941), 335–345], has a very instructive proof:

Theorem E. *Let $C = \pi/\sqrt{6}$ and $m = \frac{1}{2C}\sqrt{n}\ln n + x\sqrt{n} + O(1)$. Then*

$$\frac{1}{p(n)}\left\lvert{m+n \atop m}\right\rvert \;=\; F(x)\bigl(1+O(n^{-1/2+\epsilon})\bigr) \qquad (42)$$

for all $\epsilon > 0$ and all fixed x as $n \to \infty$, where

$$F(x) \;=\; e^{-e^{-Cx}/C}. \qquad (43)$$

This function $F(x)$ approaches 0 quite rapidly when $x \to -\infty$, and it rapidly increases to 1 when $x \to +\infty$; so it is a probability distribution function. Figure 29(b) shows that the corresponding density function $f(x) = F'(x)$ is largely concentrated in the region $-2 \le x \le 4$. The values of $\left\lvert{n \atop m}\right\rvert = \left\lvert{m+n \atop m}\right\rvert - \left\lvert{m-1+n \atop m-1}\right\rvert$ are shown in Fig. 29(a) for comparison when $n = 100$; in this case $\frac{1}{2C}\sqrt{n}\ln n \approx 18$.

Proof. We will use the fact that $\left\lvert{m+n \atop m}\right\rvert$ is the number of partitions of n whose largest part is $\le m$. Then, by the principle of inclusion and exclusion, Eq. 1.3.3–(29), we have

$$\left\lvert{m+n \atop m}\right\rvert = p(n) - \sum_{j>m} p(n-j) + \sum_{j_2>j_1>m} p(n-j_1-j_2) - \sum_{j_3>j_2>j_1>m} p(n-j_1-j_2-j_3) + \cdots,$$

because $p(n-j_1-\cdots-j_r)$ is the number of partitions of n that use each of the parts $\{j_1,\ldots,j_r\}$ at least once. Let us write this as

$$\frac{1}{p(n)}\left\lvert{m+n \atop m}\right\rvert = 1 - \Sigma_1 + \Sigma_2 - \Sigma_3 + \cdots, \quad \Sigma_r = \sum_{j_r>\cdots>j_1>m} \frac{p(n-j_1-\cdots-j_r)}{p(n)}. \qquad (44)$$

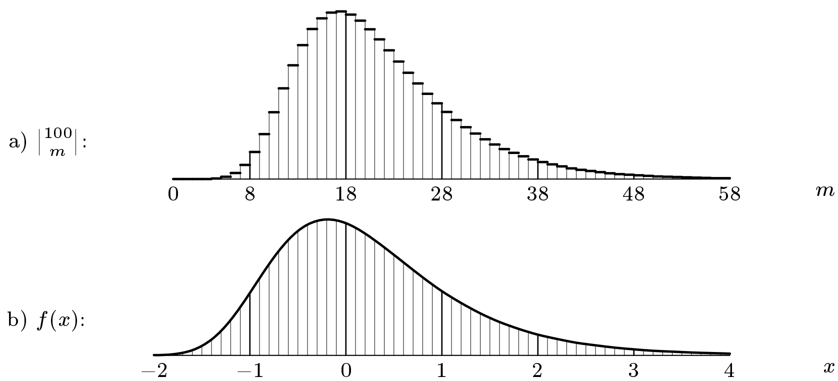


Fig. 29. Partitions of n with m parts, when (a) $n = 100$; (b) $n \to \infty$. (See Theorem E.)

In order to evaluate Σ_r we need to have a good estimate of the ratio $p(n-t)/p(n)$. And we're in luck, because Eq. (36) implies that

$$\begin{aligned}\frac{p(n-t)}{p(n)} &= \exp\bigl(2C\sqrt{n-t} - \ln(n-t) + O\bigl((n-t)^{-1/2}\bigr) - 2C\sqrt{n} + \ln n\bigr)\\ &= \exp\bigl(-Ctn^{-1/2} + O(n^{-1/2+2\epsilon})\bigr) \qquad \text{if } 0 \le t \le n^{1/2+\epsilon}. \end{aligned} \tag{45}$$

Furthermore, if $t \ge n^{1/2+\epsilon}$ we have $p(n-t)/p(n) \le p(n-n^{1/2+\epsilon})/p(n) \approx \exp(-Cn^{\epsilon})$, a value that is asymptotically smaller than any power of n. Therefore we may safely use the approximation

$$\frac{p(n-t)}{p(n)} \approx \alpha^t, \qquad \alpha = \exp(-Cn^{-1/2}), \tag{46}$$

for all values of $t \ge 0$. For example, we have

$$\begin{aligned}\Sigma_1 = \sum_{j>m} \frac{p(n-j)}{p(n)} &= \frac{\alpha^{m+1}}{1-\alpha}\bigl(1 + O(n^{-1/2+2\epsilon})\bigr) + \sum_{n \ge j > n^{1/2+\epsilon}} \frac{p(n-j)}{p(n)}\\ &= \frac{e^{-Cx}}{C}\bigl(1 + O(n^{-1/2+2\epsilon})\bigr) + O(ne^{-Cn^{\epsilon}}),\end{aligned}$$

because $\alpha/(1-\alpha) = n^{1/2}/C + O(1)$ and $\alpha^m = n^{-1/2}e^{-Cx}$. A similar argument (see exercise 36) proves that, if $r = O(\log n)$,

$$\Sigma_r \;=\; \frac{e^{-Crx}}{C^r r!}\bigl(1 + O(n^{-1/2+2\epsilon})\bigr) + O(e^{-n^{\epsilon/2}}). \tag{47}$$

Finally — and this is a wonderful property of the inclusion-exclusion principle in general — the partial sums of (44) always "bracket" the true value, in the sense that

$$1 - \Sigma_1 + \Sigma_2 - \cdots - \Sigma_{2r-1} \le \frac{1}{p(n)}\left|{m+n \atop m}\right| \le 1 - \Sigma_1 + \Sigma_2 - \cdots - \Sigma_{2r-1} + \Sigma_{2r} \tag{48}$$

for all r. (See exercise 37.) When $2r$ is near $\ln n$ and n is large, the term Σ_{2r} is extremely tiny; therefore we obtain (42), except with 2ϵ in place of ϵ. ▮

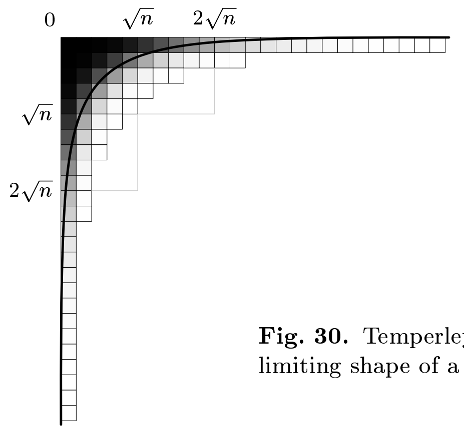


Fig. 30. Temperley's curve (49) for the limiting shape of a random partition.

Theorem E tells us that the largest part of a random partition almost always is $\frac{1}{2C}\sqrt{n}\ln n + O(\sqrt{n}\,)$, and when n is reasonably large the other parts tend to be predictable as well. Suppose, for example, that we take all the partitions of 25 and superimpose their Ferrers diagrams, changing dots to boxes as in the rim representation. Which cells are occupied most often? Figure 30 shows the result: A random partition tends to have a typical shape that approaches a limiting curve as $n \to \infty$.

H. N. V. Temperley [*Proc. Cambridge Philos. Soc.* **48** (1952), 683–697] gave heuristic reasons to believe that most parts a_k of a large random partition $a_1 \ldots a_m$ will satisfy the approximate law

$$e^{-Ck/\sqrt{n}} + e^{-Ca_k/\sqrt{n}} \;\approx\; 1, \tag{49}$$

and his formula has subsequently been verified in a strong form. For example, a theorem of Boris Pittel [*Advances in Applied Math.* **18** (1997), 432–488] allows us to conclude that the trace of a random partition is almost always $\frac{\ln 2}{C}\sqrt{n} \approx 0.54\sqrt{n}$, in accordance with (49), with an error of at most $O(\sqrt{n}\ln n)^{1/2}$; thus about 29% of all the Ferrers dots tend to lie in the Durfee square.

If, on the other hand, we look only at partitions of n with m parts, where m is fixed, the limiting shape is rather different: Almost all such partitions have

$$a_k \;\approx\; \frac{n}{m}\ln\frac{m}{k}, \tag{50}$$

if m is reasonably large. Figure 31 illustrates the case $n = 50$, $m = 5$. In fact, the same limit holds when m grows with n, but at a slower rate than $\sqrt{n}$ [see Vershik and Yakubovich, *Moscow Math. J.* **1** (2001), 457–468].

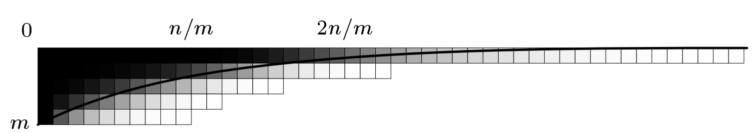


Fig. 31. The limiting shape (50) when there are m parts.

The rim representation of partitions gives us further information about partitions that are *doubly* bounded, in the sense that we not only restrict the number of parts but also the size of each part. A partition that has at most m parts, each of size at most l, fits inside an $m \times l$ box. All such partitions correspond to permutations of the multiset $\{m \cdot 0, l \cdot 1\}$ that have exactly n inversions, and we have studied the inversions of multiset permutations in exercise 5.1.2–16. In particular, that exercise derives a nonobvious formula for the number of ways n inversions can happen:

Theorem C. *The number of partitions of n that have no more than m parts and no part larger than l is*

$$[z^n]\binom{l+m}{m}_z \;=\; [z^n]\,\frac{(1-z^{l+1})}{(1-z)}\frac{(1-z^{l+2})}{(1-z^2)}\cdots\frac{(1-z^{l+m})}{(1-z^m)}. \qquad (51)$$

This result is due to A. Cauchy, *Comptes Rendus Acad. Sci.* **17** (Paris, 1843), 523–531. Notice that when $l \to \infty$ the numerator becomes simply 1. An interesting combinatorial proof of a more general result appears in exercise 39 below. ▮

Analysis of the algorithms. Now we know more than enough about the quantitative aspects of partitions to deduce the behavior of Algorithm P quite precisely. Suppose steps P1, ..., P6 of that algorithm are executed respectively $T_1(n), \ldots, T_6(n)$ times. We obviously have $T_1(n) = 1$ and $T_3(n) = p(n)$; furthermore Kirchhoff's law tells us that $T_2(n) = T_5(n)$ and $T_4(n) + T_5(n) = T_3(n)$. We get to step P4 once for each partition that contains a 2; and this is clearly $p(n-2)$.

Thus the only possible mystery about the running time of Algorithm P is the number of times we must perform step P6, which loops back to itself. A moment's thought, however, reveals that the algorithm stores a value ≥ 2 into the array $a_1a_2\ldots$ only in steps P2 and P6; and every such value is eventually decreased by 1, either in step P4 or step P5. Therefore

$$T_2''(n) \;+\; T_6(n) \;=\; p(n) - 1, \qquad (52)$$

where $T_2''(n)$ is the number of times step P2 sets a_m to a value ≥ 2. Let $T_2(n) = T_2'(n) + T_2''(n)$, so that $T_2'(n)$ is the number of times step P2 sets $a_m \leftarrow 1$. Then $T_2'(n) + T_4(n)$ is the number of partitions that end in 1, hence

$$T_2'(n) \;+\; T_4(n) = p(n-1). \qquad (53)$$

Aha! We've found enough equations to determine all of the required quantities:

$$\bigl(T_1(n), \ldots, T_6(n)\bigr) \;=\; \bigl(1,\; p(n)-p(n-2),\; p(n),\; p(n-2),\; p(n)-p(n-2),\; p(n-1)-1\bigr). \qquad (54)$$

And from the asymptotics of $p(n)$ we also know the average amount of computation per partition:

$$\left(\frac{T_1(n)}{p(n)}, \ldots, \frac{T_6(n)}{p(n)}\right) \;=\; \left(0,\, \frac{2C}{\sqrt{n}},\, 1,\, 1-\frac{2C}{\sqrt{n}},\, \frac{2C}{\sqrt{n}},\, 1-\frac{C}{\sqrt{n}}\right) \;+\; O\!\left(\frac{1}{n}\right), \qquad (55)$$

where $C = \pi/\sqrt{6} \approx 1.283$. (See exercise 45.) The total number of memory accesses per partition therefore comes to only $4 - 3C/\sqrt{n} + O(1/n)$.

> *Whoever wants to go about generating all partitions*
> *not only immerses himself in immense labor,*
> *but also must take pains to keep fully attentive,*
> *so as not to be grossly deceived.*
>
> — LEONHARD EULER, *De Partitione Numerorum* (1750)

Algorithm H is more difficult to analyze, but we can at least prove a decent upper bound on its running time. The key quantity is the value of j, the smallest subscript for which $a_j < a_1 - 1$. The successive values of j when $m = 4$ and $n = 11$ are $(2, 2, 2, 3, 2, 2, 3, 4, 2, 3, 5)$, and we have observed that $j = b_{l-1} + 1$ when $b_1 \ldots b_l$ is the conjugate partition $(a_1 \ldots a_m)^T$. (See (7) and (12).) Step H3 singles out the case $j = 2$, because this case is not only the most common, it is also especially easy to handle.

Let $c_m(n)$ be the accumulated total value of $j - 1$, summed over all of the $\genfrac{|}{|}{0pt}{}{n}{m}$ partitions generated by Algorithm H. For example, $c_4(11) = 1 + 1 + 1 + 2 + 1 + 1 + 2 + 3 + 1 + 2 + 4 = 19$. We can regard $c_m(n)/\genfrac{|}{|}{0pt}{}{n}{m}$ as a good indication of the running time per partition, because the time to perform the most costly steps, H4 and H6, is roughly proportional to $j - 2$. This ratio $c_m(n)/\genfrac{|}{|}{0pt}{}{n}{m}$ is *not* bounded, because $c_m(m) = m$ while $\genfrac{|}{|}{0pt}{}{m}{m} = 1$. But the following theorem shows that Algorithm H is efficient nonetheless:

Theorem H. *The cost measure $c_m(n)$ for Algorithm H is at most $3\genfrac{|}{|}{0pt}{}{n}{m} + m$.*

Proof. We can readily verify that $c_m(n)$ satisfies the same recurrence as $\genfrac{|}{|}{0pt}{}{n}{m}$, namely

$$c_m(n) \;=\; c_{m-1}(n-1) \;+\; c_m(n-m), \qquad \text{for } m, n \ge 1, \tag{56}$$

if we artificially define $c_m(n) = 1$ when $1 \le n < m$; see (39). But the boundary conditions are now different:

$$c_m(0) = [m > 0]; \qquad\qquad c_0(n) = 0. \tag{57}$$

Table 3 shows how $c_m(n)$ behaves when m and n are small.

To prove the theorem, we will actually prove a stronger result,

$$c_m(n) \;\le\; 3\left|{n \atop m}\right| + 2m - n - 1 \qquad \text{for } n \ge m \ge 2. \tag{58}$$

Exercise 50 shows that this inequality holds when $m \le n \le 2m$, so the proof will be complete if we can prove it when $n > 2m$. In the latter case we have

$$\begin{aligned} c_m(n) &= c_1(n-m) + c_2(n-m) + c_3(n-m) + \cdots + c_m(n-m) \\ &\le 1 + \Bigl(3\genfrac{|}{|}{0pt}{}{n-m}{2} + 3-n+m\Bigr) + \Bigl(3\genfrac{|}{|}{0pt}{}{n-m}{3} + 5-n+m\Bigr) + \cdots \\ &\qquad\qquad\qquad\qquad + \Bigl(3\genfrac{|}{|}{0pt}{}{n-m}{m} + 2m-1-n+m\Bigr) \\ &= 3\genfrac{|}{|}{0pt}{}{n-m}{1} + 3\genfrac{|}{|}{0pt}{}{n-m}{2} + \cdots + 3\genfrac{|}{|}{0pt}{}{n-m}{m} - 3 + m^2 - (m-1)(n-m) \\ &= 3\genfrac{|}{|}{0pt}{}{n}{m} + 2m^2 - m - (m-1)n - 3 \end{aligned}$$

by induction; and $2m^2 - m - (m-1)n - 3 \le 2m - n - 1$ because $n \ge 2m + 1$. ▮

Table 3

COSTS IN ALGORITHM H

n	$c_0(n)$	$c_1(n)$	$c_2(n)$	$c_3(n)$	$c_4(n)$	$c_5(n)$	$c_6(n)$	$c_7(n)$	$c_8(n)$	$c_9(n)$	$c_{10}(n)$	$c_{11}(n)$
0	0	1	1	1	1	1	1	1	1	1	1	1
1	0	1	1	1	1	1	1	1	1	1	1	1
2	0	1	2	1	1	1	1	1	1	1	1	1
3	0	1	2	3	1	1	1	1	1	1	1	1
4	0	1	3	3	4	1	1	1	1	1	1	1
5	0	1	3	4	4	5	1	1	1	1	1	1
6	0	1	4	6	5	5	6	1	1	1	1	1
7	0	1	4	7	7	6	6	7	1	1	1	1
8	0	1	5	8	11	8	7	7	8	1	1	1
9	0	1	5	11	12	12	9	8	8	9	1	1
10	0	1	6	12	16	17	13	10	9	9	10	1
11	0	1	6	14	19	21	18	14	11	10	10	11

***A Gray code for partitions.** When partitions are generated in part-count form $c_1 \ldots c_n$ as in exercise 5, at most four of the c_j values change at each step. But we might prefer to minimize the changes to the individual parts, generating partitions in such a way that the successor of $a_1 a_2 \ldots a_n$ is always obtained by simply setting $a_j \leftarrow a_j + 1$ and $a_k \leftarrow a_k - 1$ for some j and k, as in the "revolving door" algorithms of Section 7.2.1.3. It turns out that this is always possible; in fact, there is a unique way to do it when $n = 6$:

$$111111,\ 21111,\ 3111,\ 2211,\ 222,\ 321,\ 33,\ 42,\ 411,\ 51,\ 6. \tag{59}$$

And in general, the $\left|{m+n \atop m}\right|$ partitions of n into at most m parts can always be generated by a suitable Gray path.

Notice that $\alpha \to \beta$ is an allowable transition from one partition to another if and only if we get the Ferrers diagram for β by moving just one dot in the Ferrers diagram for α. Therefore $\alpha^T \to \beta^T$ is also an allowable transition. It follows that every Gray code for partitions into at most m parts corresponds to a Gray code for partitions into parts that do not exceed m. We shall work with the latter constraint.

The total number of Gray codes for partitions is vast: There are 52 when $n = 7$, and 652 when $n = 8$; there are 298,896 when $n = 9$, and 2,291,100,484 when $n = 10$. But no really simple construction is known. The reason is probably that a few partitions have only two neighbors, namely the partitions $d^{n/d}$ when $1 < d < n$ and d is a divisor of n. Such partitions must be preceded and followed by $\{(d{+}1)d^{n/d-2}(d{-}1),\ d^{n/d-1}(d{-}1)1\}$, and this requirement seems to rule out any simple recursive approach.

Carla D. Savage [*J. Algorithms* **10** (1989), 577–595] found a way to surmount the difficulties with only a modest amount of complexity. Let

$$\mu(m,n) \;=\; \overbrace{m\ \ m\ \ \ldots\ \ m}^{\lfloor n/m \rfloor}\ \ (n \bmod m) \tag{60}$$

be the lexicographically largest partition of n with parts $\le m$; our goal will be to construct recursively defined Gray paths $L(m,n)$ and $M(m,n)$ from the partition 1^n to $\mu(m,n)$, where $L(m,n)$ runs through all partitions whose parts are bounded by m while $M(m,n)$ runs through those partitions and a few more: $M(m,n)$ also includes partitions whose largest part is $m+1$, provided that the other parts are all strictly less than m. For example, $L(3,8)$ is 11111111, 2111111, 311111, 221111, 22211, 2222, 3221, 32111, 3311, 332, while $M(3,8)$ is

$$\begin{gathered}11111111,\ 2111111,\ 221111,\ 22211,\ 2222,\ 3221,\\ 3311,\ 32111,\ 311111,\ 41111,\ 4211,\ 422,\ 332;\end{gathered}\tag{61}$$

the additional partitions starting with 4 will give us "wiggle room" in other parts of the recursion. We will define $L(m,n)$ for all $n \ge 0$, but $M(m,n)$ only for $n > 2m$.

The following construction, illustrated for $m = 5$ to simplify the notation, *almost* works:

$$L(5) = \begin{Bmatrix} L(3) \\ 4L(\infty)^R \\ 5L(\infty) \end{Bmatrix} \text{ if } n \le 7;\quad \begin{Bmatrix} L(3) \\ 4L(2)^R \\ 5L(2) \\ 431 \\ 44 \\ 53 \end{Bmatrix} \text{ if } n = 8;\quad \begin{Bmatrix} M(4) \\ 54L(4)^R \\ 55L(5) \end{Bmatrix} \text{ if } n \ge 9;\tag{62}$$

$$M(5) = \begin{Bmatrix} L(4) \\ 5L(4)^R \\ 6L(3) \\ 64L(\infty)^R \\ 55L(\infty) \end{Bmatrix} \text{ if } 11 \le n \le 13;\qquad \begin{Bmatrix} L(4) \\ 5M(4)^R \\ 6L(4) \\ 554L(4)^R \\ 555L(5) \end{Bmatrix} \text{ if } n \ge 14.\tag{63}$$

Here the parameter n in $L(m,n)$ and $M(m,n)$ has been omitted because it can be deduced from the context; each L or M is supposed to generate partitions of whatever amount remains after previous parts have been subtracted. Thus, for example, (63) specifies that

$$M(5,14) \;=\; L(4,14),\ 5M(4,9)^R,\ 6L(4,8),\ 554L(4,0)^R,\ 555L(5,-1);$$

the sequence $L(5,-1)$ is actually empty, and $L(4,0)$ is the empty string, so the final partition of $M(5,14)$ is $554 = \mu(5,14)$ as it should be. The notation $L(\infty)$ stands for $L(\infty,n) = L(n,n)$, the Gray path of all partitions of n, starting with 1^n and ending with n^1.

In general, $L(m)$ and $M(m)$ are defined for all $m \ge 3$ by essentially the same rules, if we replace the digits 2, 3, 4, 5, and 6 in (62) and (63) by $m-3$, $m-2$, $m-1$, m, and $m+1$, respectively. The ranges $n \le 7$, $n = 8$, $n \ge 9$ become $n \le 2m-3$, $n = 2m-2$, $n \ge 2m-1$; the ranges $11 \le n \le 13$ and $n \ge 14$ become $2m+1 \le n \le 3m-2$ and $n \ge 3m-1$. The sequences $L(0)$, $L(1)$, $L(2)$ have obvious definitions because the paths are unique when $m \le 2$. The sequence $M(2)$ is 1^n, 21^{n-2}, 31^{n-3}, 221^{n-4}, 2221^{n-6}, …, $\mu(2,n)$ for $n \ge 5$.

Theorem S. *Gray paths $L'(m,n)$ for $m,n \ge 0$ and $M'(m,n)$ for $n \ge 2m+1 \ge 5$ exist for all partitions with the properties described above, except in the case $L'(4,6)$. Furthermore, L' and M' obey the mutual recursions (62) and (63) except in a few cases.*

Proof. We noted above that (62) and (63) *almost* work; the reader may verify that the only glitch occurs in the case $L(4,6)$, when (62) gives

$$\begin{aligned} L(4,6) &= L(2,6),\ 3L(1,3)^R,\ 4L(1,2),\ 321,\ 33,\ 42 \\ &= 111111,\ 21111,\ 2211,\ 222,\ 3111,\ 411,\ 321,\ 33,\ 42. \end{aligned} \tag{64}$$

If $m > 4$, we're OK because the transition from the end of $L(m-2, 2m-2)$ to the beginning of $(m-1)L(m-3, m-1)^R$ is from $(m-2)(m-2)2$ to $(m-1)(m-3)2$. There is no satisfactory path $L(4,6)$, because all Gray codes through those nine partitions must end with either 411, 33, 3111, 222, or 2211.

In order to neutralize this anomaly we need to patch the definitions of $L(m,n)$ and $M(m,n)$ at eight places where the "buggy subroutine" $L(4,6)$ is invoked. One simple way is to make the following definitions:

$$\begin{aligned} L'(4,6) &= 111111, 21111, 3111, 411, 321, 33, 42; \\ L'(3,5) &= 11111, 2111, 221, 311, 32. \end{aligned} \tag{65}$$

Thus, we omit 222 and 2211 from $L(4,6)$; we also reprogram $L(3,5)$ so that 2111 is adjacent to 221. Then exercise 60 shows that it is always easy to "splice in" the two partitions that are missing from $L(4,6)$. ▌

EXERCISES

▸ **1.** [*M21*] Give formulas for the total number of possibilities in each problem of The Twelvefold Way. For example, the number of n-tuples of m things is m^n. (Use the notation (38) when appropriate, and be careful to make your formulas correct even when $m = 0$ or $n = 0$.)

▸ **2.** [*20*] Show that a small change to step H1 yields an algorithm that will generate all partitions of n into *at most* m parts.

3. [*M17*] A partition $a_1 + \cdots + a_m$ of n into m parts $a_1 \ge \cdots \ge a_m$ is *optimally balanced* if $|a_i - a_j| \le 1$ for $1 \le i, j \le m$. Prove that there is exactly one such partition, whenever $n \ge m \ge 1$, and give a simple formula that expresses the jth part a_j as a function of j, m, and n.

4. [*M22*] (Gideon Ehrlich, 1974.) What is the lexicographically smallest partition of n in which all parts are $\ge r$? For example, when $n = 19$ and $r = 5$ the answer is 766.

▸ **5.** [*23*] Design an algorithm that generates all partitions of n in the part-count form $c_1 \ldots c_n$ of (8). Generate them in colex order, namely in the lexicographic order of $c_n \ldots c_1$, which is equivalent to lexicographic order of the corresponding partitions $a_1 a_2 \ldots$. For efficiency, maintain also a table of links $l_0 l_1 \ldots l_n$ so that, if the distinct values of k for which $c_k > 0$ are $k_1 < \cdots < k_t$, we have

$$l_0 = k_1, \quad l_{k_1} = k_2, \quad \ldots, \quad l_{k_{t-1}} = k_t, \quad l_{k_t} = 0.$$

(Thus the partition 331 would be represented by $c_1 \ldots c_7 = 1020000$, $l_0 = 1$, $l_1 = 3$, and $l_3 = 0$; the other links l_2, l_4, l_5, l_7 can be set to any convenient values.)

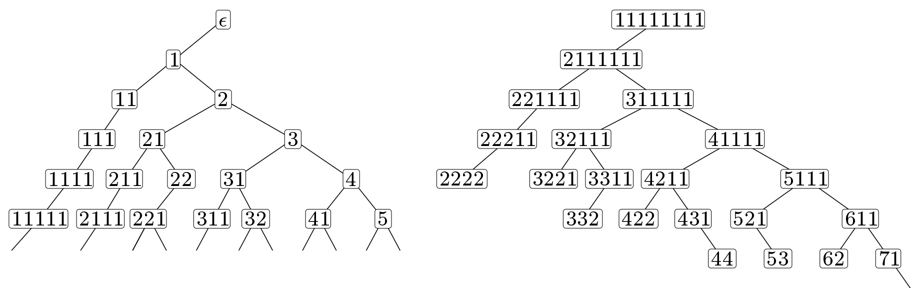
6. [*20*] Design an algorithm to compute $b_1 b_2 \ldots = (a_1 a_2 \ldots)^T$, given $a_1 a_2 \ldots$.

7. [*M20*] Suppose $a_1 \ldots a_n$ and $a'_1 \ldots a'_n$ are partitions of n with $a_1 \geq \cdots \geq a_n \geq 0$ and $a'_1 \geq \cdots \geq a'_n \geq 0$, and let their respective conjugates be $b_1 \ldots b_n = (a_1 \ldots a_n)^T$, $b'_1 \ldots b'_n = (a'_1 \ldots a'_n)^T$. Show that $b_1 \ldots b_n < b'_1 \ldots b'_n$ if and only if $a_n \ldots a_1 < a'_n \ldots a'_1$.

8. [*15*] When $(p_1 \ldots p_t, q_1 \ldots q_t)$ is the rim representation of a partition $a_1 a_2 \ldots$ as in (15) and (16), what is the conjugate partition $(a_1 a_2 \ldots)^T = b_1 b_2 \ldots$?

9. [*22*] If $a_1 a_2 \ldots a_m$ and $b_1 b_2 \ldots b_m = (a_1 a_2 \ldots a_m)^T$ are conjugate partitions, show that the multisets $\{a_1 + 1, a_2 + 2, \ldots, a_m + m\}$ and $\{b_1 + 1, b_2 + 2, \ldots, b_m + m\}$ are equal.

10. [*21*] Two simple kinds of binary trees are sometimes helpful for reasoning about partitions: (a) a tree that includes all partitions of all integers, and (b) a tree that includes all partitions of a given integer n, illustrated here for $n = 8$:



(a) (b)

Deduce the general rules underlying these constructions. What order of tree traversal corresponds to lexicographic order of the partitions?

11. [*M22*] How many ways are there to pay one euro, using coins worth 1, 2, 5, 10, 20, 50, and/or 100 cents? What if you are allowed to use at most two of each coin?

▸ **12.** [*M21*] (L. Euler, 1750.) Use generating functions to prove that the number of ways to partition n into *distinct* parts is the number of ways to partition n into *odd* parts. For example, $5 = 4 + 1 = 3 + 2$; $5 = 3 + 1 + 1 = 1 + 1 + 1 + 1 + 1$.

[*Note:* The next two exercises use combinatorial techniques to prove extensions of this famous theorem.]

▸ **13.** [*M22*] (F. Franklin, 1882.) Find a one-to-one correspondence between partitions of n that have exactly k parts repeated more than once and partitions of n that have exactly k even parts. (The case $k = 0$ corresponds to Euler's result.)

▸ **14.** [*M28*] (J. J. Sylvester, 1882.) Find a one-to-one correspondence between partitions of n into distinct parts $a_1 > a_2 > \cdots > a_m$ that have exactly k "gaps" where $a_j > a_{j+1} + 1$, and partitions of n into odd parts that have exactly $k + 1$ different values. (For example, when $k = 0$ this construction proves that the number of ways to write n as a sum of consecutive integers is the number of odd divisors of n.)

15. [*M20*] (J. J. Sylvester.) Find a generating function for the number of partitions that are *self-conjugate* (namely, partitions such that $\alpha = \alpha^T$).

16. [*M21*] Find the generating function for partitions of trace k, and sum it on k to obtain a nontrivial identity.

17. [*M26*] A *joint partition* of n is a pair of sequences $(a_1,\dots,a_r;\ b_1,\dots,b_s)$ of positive integers for which we have

$$a_1 \ge \cdots \ge a_r, \quad b_1 > \cdots > b_s, \quad \text{and} \quad a_1+\cdots+a_r+b_1+\cdots+b_s = n.$$

Thus it is an ordinary partition if $s=0$, and a partition into distinct parts if $r=0$.

a) Find a simple formula for the generating function $\sum u^{r+s}v^s z^n$, summed over all joint partitions of n with r ordinary parts a_i and s distinct parts b_j.

b) Similarly, find a simple formula for $\sum v^s z^n$ when the sum is over all joint partitions that have exactly $r+s=t$ total parts, given the value of t.

c) What identity do you deduce?

▸ **18.** [*M23*] (Doron Zeilberger.) Show that there is a one-to-one correspondence between pairs of integer sequences $(a_1,a_2,\dots,a_r;\ b_1,b_2,\dots,b_s)$ such that

$$a_1 \ge a_2 \ge \cdots \ge a_r, \qquad b_1 > b_2 > \cdots > b_s,$$

and pairs of integer sequences $(c_1,c_2,\dots,c_{r+s};\ d_1,d_2,\dots,d_{r+s})$ such that

$$c_1 \ge c_2 \ge \cdots \ge c_{r+s}, \qquad d_j \in \{0,1\} \quad \text{for } 1 \le j \le r+s,$$

related by the multiset equations

$$\{a_1,a_2,\dots,a_r\} = \{c_j \mid d_j = 0\} \quad \text{and} \quad \{b_1,b_2,\dots,b_s\} = \{c_j+r+s-j \mid d_j = 1\}.$$

Consequently we obtain the interesting identity

$$\sum_{\substack{a_1\ge\cdots\ge a_r>0\\ b_1>\cdots>b_s>0}} u^{r+s}v^s z^{a_1+\cdots+a_r+b_1+\cdots+b_s} = \sum_{\substack{c_1\ge\cdots\ge c_t>0\\ d_1,\dots,d_t\in\{0,1\}}} u^t v^{d_1+\cdots+d_t} z^{c_1+\cdots+c_t+(t-1)d_1+\cdots+d_{t-1}}.$$

19. [*M21*] (E. Heine, 1847.) Prove the four-parameter identity

$$\prod_{m=1}^{\infty}\frac{(1-wxz^m)(1-wyz^m)}{(1-wz^m)(1-wxyz^m)} = \sum_{k=0}^{\infty}\frac{w^k(x-1)(x-z)\dots(x-z^{k-1})(y-1)(y-z)\dots(y-z^{k-1})z^k}{(1-z)(1-z^2)\dots(1-z^k)(1-wz)(1-wz^2)\dots(1-wz^k)}.$$

Hint: Carry out the sum over either k or l in the formula

$$\sum_{k,l\ge 0} u^k v^l z^{kl}\,\frac{(z-az)(z-az^2)\dots(z-az^k)}{(1-z)(1-z^2)\dots(1-z^k)}\,\frac{(z-bz)(z-bz^2)\dots(z-bz^l)}{(1-z)(1-z^2)\dots(1-z^l)}$$

and consider the simplifications that occur when $b=auz$.

▸ **20.** [*M21*] Approximately how long does it take to compute a table of the partition numbers $p(n)$ for $1 \le n \le N$, using Euler's recurrence (20)?

21. [*M21*] (L. Euler.) Let $q(n)$ be the number of partitions into distinct parts. What is a good way to compute $q(n)$ if you already know the values of $p(1)$, …, $p(n)$?

22. [*HM21*] (L. Euler.) Let $\sigma(n)$ be the sum of all positive divisors of the positive integer n. Thus, $\sigma(n)=n+1$ when n is prime, and $\sigma(n)$ can be significantly larger than n when n is highly composite. Prove that, in spite of this rather chaotic behavior, $\sigma(n)$ satisfies almost the same recurrence (20) as the partition numbers:

$$\sigma(n) = \sigma(n-1)+\sigma(n-2)-\sigma(n-5)-\sigma(n-7)+\sigma(n-12)+\sigma(n-15)-\cdots$$

for $n \ge 1$, except that when a term on the right is '$\sigma(0)$' the value 'n' is used instead. For example, $\sigma(11) = 1+11 = \sigma(10)+\sigma(9)-\sigma(6)-\sigma(4) = 18+13-12-7$; $\sigma(12) = 1+2+3+4+6+12 = \sigma(11)+\sigma(10)-\sigma(7)-\sigma(5)+12 = 12+18-8-6+12$.

23. [*HM25*] Use Jacobi's triple product identity (19) to prove another formula that he discovered:

$$\prod_{k=1}^{\infty}(1-z^k)^3 \;=\; 1-3z+5z^3-7z^6+9z^{10}-\cdots \;=\; \sum_{n=0}^{\infty}(-1)^n(2n+1)z^{\binom{n+1}{2}}.$$

24. [*M26*] (S. Ramanujan, 1919.) Let $A(z)=\prod_{k=1}^{\infty}(1-z^k)^4$.

a) Prove that $[z^n]\,A(z)$ is a multiple of 5 when $n \bmod 5 = 4$.

b) Prove that $[z^n]\,A(z)B(z)^5$ has the same property, if B is any power series with integer coefficients.

c) Therefore $p(n)$ is a multiple of 5 when $n \bmod 5 = 4$.

25. [*HM27*] Improve on (22) by using (a) Euler's summation formula and (b) Mellin transforms to estimate $\ln P(e^{-t})$. *Hint:* The dilogarithm function $\mathrm{Li}_2(x)=x/1^2+x^2/2^2+x^3/3^2+\cdots$ satisfies $\mathrm{Li}_2(x)+\mathrm{Li}_2(1-x)=\zeta(2)-(\ln x)\ln(1-x)$.

26. [*HM22*] In exercises 5.2.2–44 and 5.2.2–51 we studied two ways to prove that

$$\sum_{k=1}^{\infty} e^{-k^2/n} \;=\; \frac{1}{2}(\sqrt{\pi n}-1)+O(n^{-M}) \qquad \text{for all } M>0.$$

Show that Poisson's summation formula gives a much stronger result.

27. [*HM23*] Evaluate (29) and complete the calculations leading to Theorem D.

28. [*HM42*] (D. H. Lehmer.) Show that the Hardy–Ramanujan–Rademacher coefficients $A_k(n)$ defined in (34) have the following remarkable properties:

a) If k is odd, then $A_{2k}\bigl(km+4n+(k^2-1)/8\bigr)=A_2(m)A_k(n)$.

b) If p is prime, $p^e>2$, and $k\perp 2p$, then

$$A_{p^ek}\bigl(k^2m+p^{2e}n-(k^2+p^{2e}-1)/24\bigr) \;=\; (-1)^{[p^e=4]}A_{p^e}(m)A_k(n).$$

In this formula $k^2+p^{2e}-1$ is a multiple of 24 if p or k is divisible by 2 or 3; otherwise division by 24 should be done modulo p^ek.

c) If p is prime, $|A_{p^e}(n)|<2^{[p>2]}p^{e/2}$.

d) If p is prime, $A_{p^e}(n)\neq 0$ if and only if $1-24n$ is a quadratic residue modulo p and either $e=1$ or $24n \bmod p \neq 1$.

e) The probability that $A_k(n)=0$, when k is divisible by exactly t primes ≥ 5 and n is a random integer, is approximately $1-2^{-t}$.

▸ **29.** [*M16*] Generalizing (41), evaluate the sum $\sum_{a_1\geq a_2\geq\cdots\geq a_m\geq 1} z_1^{a_1}z_2^{a_2}\ldots z_m^{a_m}$.

30. [*M17*] Find closed forms for the sums

$$\text{(a)}\quad \sum_{k\geq 0}\left|{n-km \atop m-1}\right| \qquad \text{and} \qquad \text{(b)}\quad \sum_{k\geq 0}\left|{n \atop m-k}\right|$$

(which are finite, because the terms being summed are zero when k is large).

31. [*M24*] (A. De Morgan, 1843.) Show that $\left|{n \atop 2}\right|=\lfloor n/2\rfloor$ and $\left|{n \atop 3}\right|=\lfloor (n^2+6)/12\rfloor$; find a similar formula for $\left|{n \atop 4}\right|$.

32. [*M15*] Prove that $\left|{n \atop m}\right|\leq p(n-m)$ for all $m,n\geq 0$. When does equality hold?

33. [*HM20*] Use the fact that there are exactly $\binom{n-1}{m-1}$ *compositions* of n into m parts, Eq. 7.2.1.3–(9), to prove a lower bound on $\left|{n \atop m}\right|$. Then set $m=\lfloor\sqrt{n}\,\rfloor$ to obtain an elementary lower bound on $p(n)$.

▶ **34.** [*HM21*] Show that $\left|{n-m(m-1)/2 \atop m}\right|$ is the number of partitions of n into m distinct parts. Consequently

$$\left|{n \atop m}\right| \;=\; \frac{n^{m-1}}{m!\,(m-1)!}\Bigl(1+O\Bigl(\frac{m^3}{n}\Bigr)\Bigr) \qquad \text{when } m \le n^{1/3}.$$

35. [*HM21*] In the Erdős–Lehner probability distribution (43), what value of x is (a) most probable? (b) the median? (c) the mean? (d) What is the standard deviation?

36. [*HM24*] Prove the key estimate (47) that is needed in Theorem E.

37. [*M22*] Prove the inclusion-exclusion bracketing lemma (48), by analyzing how many times a partition that has exactly q different parts exceeding m is counted in the rth partial sum.

38. [*M20*] What is the generating function for the partitions of n that have exactly m parts, and largest part l?

▶ **39.** [*M25*] (F. Franklin.) Generalizing Theorem C, show that, for $0 \le k \le m$,

$$[z^n]\,\frac{(1-z^{l+1})\ldots(1-z^{l+k})}{(1-z)(1-z^2)\ldots(1-z^m)}$$

is the number of partitions $a_1a_2\ldots$ of n into m or fewer parts with the property that $a_1 \le a_{k+1}+l$.

40. [*M22*] (A. Cauchy.) What is the generating function for partitions into m parts, all *distinct* and less than l?

41. [*HM42*] Extend the Hardy–Ramanujan–Rademacher formula (32) to obtain a convergent series for partitions of n into at most m parts, with no part exceeding l.

42. [*HM42*] Find the limiting shape, analogous to (49), for random partitions of n into at most $\theta\sqrt{n}$ parts, with no part exceeding $\varphi\sqrt{n}$, assuming that $\theta\varphi > 1$.

43. [*M21*] Given n and k, how many partitions of n have $a_1 > a_2 > \cdots > a_k$?

▶ **44.** [*M22*] How many partitions of n have their two smallest parts equal?

45. [*HM21*] Compute the asymptotic value of $p(n-1)/p(n)$, with relative error $O(n^{-2})$.

46. [*M20*] In the text's analysis of Algorithm P, which is larger, $T_2'(n)$ or $T_2''(n)$?

▶ **47.** [*HM22*] (A. Nijenhuis and H. S. Wilf, 1975.) The following simple algorithm, based on a table of the partition numbers $p(0)$, $p(1)$, …, $p(n)$, generates a random partition of n using the part-count representation $c_1\ldots c_n$ of (8). Prove that it produces each partition with equal probability.

N1. [Initialize.] Set $m \leftarrow n$ and $c_1\ldots c_n \leftarrow 0\ldots0$.

N2. [Done?] Terminate if $m = 0$.

N3. [Generate.] Generate a random integer M in the range $0 \le M < mp(m)$.

N4. [Choose parts.] Set $s \leftarrow 0$. Then for $j = 1, 2, \ldots, n$ and for $k = 1, 2, \ldots, \lfloor m/j \rfloor$, repeatedly set $s \leftarrow s + kp(m-jk)$ until $s > M$.

N5. [Update.] Set $c_k \leftarrow c_k + j$, $m \leftarrow m - jk$, and return to N2. ▮

Hint: Step N4, which is based on the identity

$$\sum_{j=1}^{m}\sum_{k=1}^{\lfloor m/j\rfloor} kp(m-jk) \;=\; mp(m),$$

chooses each particular pair of values (j,k) with probability $kp(m-jk)/(mp(m))$.

48. [*HM40*] Analyze the running time of the algorithm in the previous exercise.

▸ **49.** [*HM26*] (a) What is the generating function $F(z)$ for the sum of the smallest parts of all partitions of n? (The series begins $z + 3z^2 + 5z^3 + 9z^4 + 12z^5 + \cdots$.)

(b) Find the asymptotic value of $[z^n]\,F(z)$, with relative error $O(n^{-1})$.

50. [*HM33*] Let $c(m) = c_m(2m)$ in the recurrence (56), (57).

a) Prove that $c_m(m+k) = m - k + c(k)$ for $0 \le k \le m$.
b) Consequently (58) holds for $m \le n \le 2m$ if $c(m) < 3p(m)$ for all m.
c) Show that $c(m) - m$ is the sum of the second-smallest parts of all partitions of m.
d) Find a one-to-one correspondence between all partitions of n with second-smallest part k and all partitions of numbers $\le n$ with smallest part $k+1$.
e) Describe the generating function $\sum_{m\ge 0} c(m)z^m$.
f) Conclude that $c(m) < 3p(m)$ for all $m \ge 0$.

51. [*M46*] Make a detailed analysis of Algorithm H.

▸ **52.** [*M21*] What is the millionth partition generated by Algorithm P when $n = 64$? *Hint:* $p(64) = 1741630 = 1000000 + \left|{77 \atop 13}\right| + \left|{60 \atop 10}\right| + \left|{47 \atop 8}\right| + \left|{35 \atop 5}\right| + \left|{27 \atop 3}\right| + \left|{22 \atop 2}\right| + \left|{18 \atop 1}\right| + \left|{15 \atop 0}\right|$.

▸ **53.** [*M21*] What is the millionth partition generated by Algorithm H when $m = 32$ and $n = 100$? *Hint:* $999999 = \left|{80 \atop 12}\right| + \left|{66 \atop 11}\right| + \left|{50 \atop 7}\right| + \left|{41 \atop 6}\right| + \left|{33 \atop 5}\right| + \left|{26 \atop 4}\right| + \left|{21 \atop 4}\right|$.

▸ **54.** [*M30*] The partition $\alpha = a_1a_2\ldots$ is said to *majorize* the partition $\beta = b_1b_2\ldots$, written $\alpha \succeq \beta$ or $\beta \preceq \alpha$, if $a_1 + \cdots + a_k \ge b_1 + \cdots + b_k$ for all $k \ge 0$.

a) True or false: $\alpha \succeq \beta$ implies $\alpha \ge \beta$ (lexicographically).
b) True or false: $\alpha \succeq \beta$ implies $\beta^T \succeq \alpha^T$.
c) Show that any two partitions of n have a greatest lower bound $\alpha \wedge \beta$ such that $\alpha \succeq \gamma$ and $\beta \succeq \gamma$ if and only if $\alpha \wedge \beta \succeq \gamma$. Explain how to compute $\alpha \wedge \beta$.
d) Similarly, explain how to compute a least upper bound $\alpha \vee \beta$ such that $\gamma \succeq \alpha$ and $\gamma \succeq \beta$ if and only if $\gamma \succeq \alpha \vee \beta$.
e) If α has l parts and β has m parts, how many parts do $\alpha \wedge \beta$ and $\alpha \vee \beta$ have?
f) True or false: If α has distinct parts and β has distinct parts, then so do $\alpha \wedge \beta$ and $\alpha \vee \beta$.

▸ **55.** [*M37*] Continuing the previous exercise, say that α *covers* β if $\alpha \succeq \beta$, $\alpha \ne \beta$, and $\alpha \succeq \gamma \succeq \beta$ implies $\gamma = \alpha$ or $\gamma = \beta$. For example, Fig. 32 illustrates the covering relations between partitions of 12.

a) Let us write $\alpha \vartriangleright \beta$ if $\alpha = a_1a_2\ldots$ and $\beta = b_1b_2\ldots$ are partitions for which $b_k = a_k - [k=l] + [k=l+1]$ for all $k \ge 1$ and some $l \ge 1$. Prove that α covers β if and only if $\alpha \vartriangleright \beta$ or $\beta^T \vartriangleright \alpha^T$.
b) Show that there is an easy way to tell if α covers β by looking at the rim representations of α and β.
c) Let $n = \binom{n_2}{2} + \binom{n_1}{1}$ where $n_2 > n_1 \ge 0$. Show that no partition of n covers more than $n_2 - 2$ partitions.
d) Say that the partition μ is *minimal* if there is no partition λ with $\mu \vartriangleright \lambda$. Prove that μ is minimal if and only if μ^T has distinct parts.
e) Suppose $\alpha = \alpha_0 \vartriangleright \alpha_1 \vartriangleright \cdots \vartriangleright \alpha_k$ and $\alpha = \alpha'_0 \vartriangleright \alpha'_1 \vartriangleright \cdots \vartriangleright \alpha'_{k'}$, where α_k and $\alpha'_{k'}$ are minimal partitions. Prove that $k = k'$ and $\alpha_k = \alpha'_{k'}$.
f) Explain how to compute the lexicographically smallest partition into distinct parts that majorizes a given partition α.
g) Describe λ_n, the lexicographically smallest partition of n into distinct parts. What is the length of all paths $n^1 = \alpha_0 \vartriangleright \alpha_1 \vartriangleright \cdots \vartriangleright \lambda_n^T$?

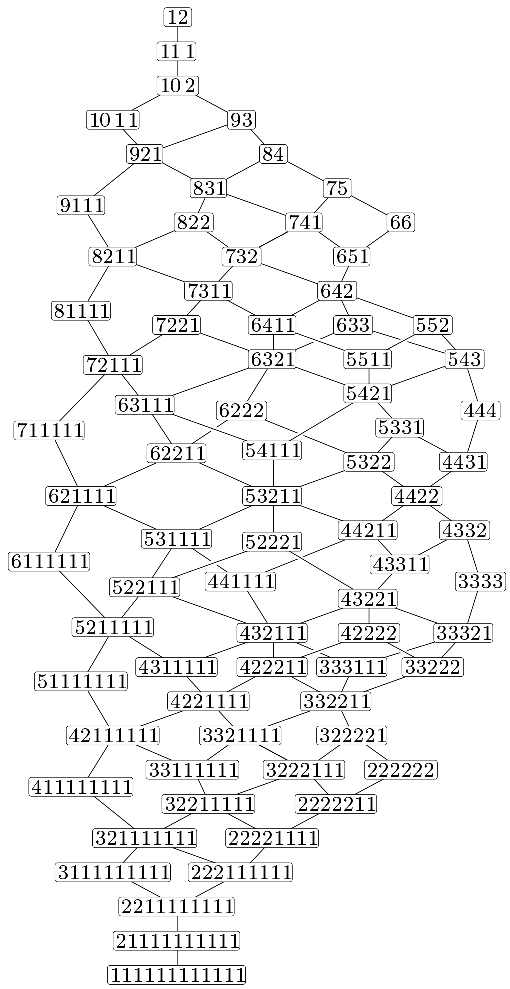


Fig. 32. The majorization lattice for partitions of 12. (See exercises 54–58.)

h) What are the lengths of the longest and shortest paths of the form $n^1 = \alpha_0, \alpha_1, \ldots, \alpha_l = 1^n$, where α_j covers α_{j+1} for $0 \le j < l$?

▶ **56.** [*M27*] Design an algorithm to generate all partitions α such that $\lambda \preceq \alpha \preceq \mu$, given partitions λ and μ with $\lambda \preceq \mu$.

Note: Such an algorithm has numerous applications. For example, to generate all partitions that have m parts and no part exceeding l, we can let λ be the smallest such partition, namely $\lceil n/m \rceil \ldots \lfloor n/m \rfloor$ as in exercise 3, and let μ be the largest, namely $\bigl((n-m+1)1^{m-1}\bigr) \wedge \bigl(l^{\lfloor n/l \rfloor}(n \bmod l)\bigr)$. Similarly, according to a well-known theorem of H. G. Landau [*Bull. Math. Biophysics* **15** (1953), 143–148], the partitions of $\binom{m}{2}$ such that

$$\left\lfloor \frac{m}{2} \right\rfloor^{\lfloor m/2 \rfloor} \left\lfloor \frac{m-1}{2} \right\rfloor^{\lceil m/2 \rceil} \preceq \alpha \preceq (m-1)(m-2)\ldots 21$$

are the possible "score vectors" of a round-robin tournament, namely the partitions $a_1 \ldots a_m$ such that the jth strongest player wins a_j games.

57. [*M22*] Suppose a matrix (a_{ij}) of 0s and 1s has row sums $r_i = \sum_j a_{ij}$ and column sums $c_j = \sum_i a_{ij}$. Then $\lambda = r_1 r_2 \ldots$ and $\mu = c_1 c_2 \ldots$ are partitions of $n = \sum_{i,j} a_{ij}$. Prove that such a matrix exists if and only if $\lambda \preceq \mu^T$.

58. [*M23*] (*Symmetrical means.*) Let $\alpha = a_1 \ldots a_m$ and $\beta = b_1 \ldots b_m$ be partitions of n. Prove that the inequality

$$\frac{1}{m!}\sum x_{p_1}^{a_1} \ldots x_{p_m}^{a_m} \;\geq\; \frac{1}{m!}\sum x_{p_1}^{b_1} \ldots x_{p_m}^{b_m}$$

holds for all nonnegative values of the variables $(x_1, \ldots, x_m)$, where the sums range over all $m!$ permutations of $\{1, \ldots, m\}$, if and only if $\alpha \succeq \beta$. (For example, this inequality reduces to $(y_1 + \cdots + y_n)/n \geq (y_1 \ldots y_n)^{1/n}$ in the special case $m = n$, $\alpha = n0\ldots0$, $\beta = 11\ldots1$, $x_j = y_j^{1/n}$.)

59. [*M22*] The Gray path (59) is symmetrical in the sense that the reversed sequence 6, 51, ..., 111111 is the same as conjugate sequence $(111111)^T$, $(21111)^T$, ..., $(6)^T$. Find all Gray paths $\alpha_1, \ldots, \alpha_{p(n)}$ that are symmetrical in this way.

60. [*23*] Complete the proof of Theorem S by modifying the definitions of $L(m,n)$ and $M(m,n)$ in all places where $L(4,6)$ is called in (62) and (63).

61. [*26*] Implement a partition-generation scheme based on Theorem S, always specifying the two parts that have changed between visits.

62. [*46*] Prove or disprove: For all sufficiently large integers n and $3 \leq m < n$ such that $n \bmod m \neq 0$, and for all partitions α of n with $a_1 \leq m$, there is a Gray path for all partitions with parts $\leq m$, beginning at 1^n and ending at α, unless $\alpha = 1^n$ or $\alpha = 21^{n-2}$.

63. [*47*] For which partitions λ and μ is there a Gray code through all partitions α such that $\lambda \preceq \alpha \preceq \mu$?

▸ **64.** [*32*] (*Binary partitions.*) Design a loopless algorithm that visits all partitions of n into powers of 2, where each step replaces $2^k + 2^k$ by 2^{k+1} or vice versa.

65. [*23*] It is well known that every commutative group of m elements can be represented as a discrete torus $T(m_1, \ldots, m_n)$ with the addition operation of 7.2.1.3–(66), where $m = m_1 \ldots m_n$ and m_j is a multiple of m_{j+1} for $1 \leq j < n$. For example, when $m = 360 = 2^3 \cdot 3^2 \cdot 5^1$ there are six such groups, corresponding to the factorizations $(m_1, m_2, m_3) = (30, 6, 2)$, $(60, 6, 1)$, $(90, 2, 2)$, $(120, 3, 1)$, $(180, 2, 1)$, and $(360, 1, 1)$.

Explain how to generate all such factorizations systematically with an algorithm that changes exactly two of the factors m_j at each step.

▸ **66.** [*M25*] (*P-partitions.*) Instead of insisting that $a_1 \geq a_2 \geq \cdots$, suppose we want to consider all nonnegative compositions of n that satisfy a given *partial* order. For example, P. A. MacMahon observed that all solutions to the "up-down" inequalities $a_4 \leq a_2 \geq a_3 \leq a_1$ can be divided into five nonoverlapping types:

$$a_1 \geq a_2 \geq a_3 \geq a_4; \quad a_1 \geq a_2 \geq a_4 > a_3;$$
$$a_2 > a_1 \geq a_3 \geq a_4; \quad a_2 > a_1 \geq a_4 > a_3; \quad a_2 \geq a_4 > a_1 \geq a_3.$$

Each of these types is easily enumerated since, for example, $a_2 > a_1 \geq a_4 > a_3$ is equivalent to $a_2 - 2 \geq a_1 - 1 \geq a_4 - 1 \geq a_3$; the number of solutions with $a_3 \geq 0$ and $a_1 + a_2 + a_3 + a_4 = n$ is the number of partitions of $n-1-2-0-1$ into at most four parts.

Explain how to solve a general problem of this kind: Given any partial order relation $\prec$ on m elements, consider all m-tuples $a_1 \ldots a_m$ with the property that $a_j \geq a_k$

when $j \prec k$. Assuming that the subscripts have been chosen so that $j \prec k$ implies $j \le k$, show that all of the desired m-tuples fall into exactly N classes, one for each of the outputs of the topological sorting algorithm 7.2.1.2V. What is the generating function for all such $a_1 \ldots a_m$ that are nonnegative and sum to n? How could you generate them all?

67. [*M25*] (P. A. MacMahon, 1886.) A *perfect partition* of n is a multiset that has exactly $n+1$ submultisets, and these multisets are partitions of the integers 0, 1, ..., n. For example, the multisets $\{1,1,1,1,1\}$, $\{2,2,1\}$, and $\{3,1,1\}$ are perfect partitions of 5.

Explain how to construct the perfect partitions of n that have fewest elements.

68. [*M23*] What partition of n into m parts has the largest product $a_1 \ldots a_m$, when (a) m is given; (b) m is arbitrary?

69. [*M30*] Find all $n < 10^9$ such that the equation $x_1 + x_2 + \cdots + x_n = x_1 x_2 \ldots x_n$ has only one solution in positive integers $x_1 \ge x_2 \ge \cdots \ge x_n$. (There is, for example, only one solution when $n = 2$, 3, or 4; but $5+2+1+1+1 = 5 \cdot 2 \cdot 1 \cdot 1 \cdot 1$ and $3+3+1+1+1 = 3 \cdot 3 \cdot 1 \cdot 1 \cdot 1$ and $2+2+2+1+1 = 2 \cdot 2 \cdot 2 \cdot 1 \cdot 1$.)

70. [*M30*] ("Bulgarian solitaire.") Take n cards and divide them arbitrarily into one or more piles. Then repeatedly remove one card from each pile and form a new pile.

Show that if $n = 1 + 2 + \cdots + m$, this process always reaches a self-repeating state with piles of sizes $\{m, m-1, \ldots, 1\}$. For example, if $n = 10$ and if we start with piles whose sizes are $\{3,3,2,2\}$, we get the sequence of partitions

$$3322 \to 42211 \to 5311 \to 442 \to 3331 \to 4222 \to 43111 \to 532 \to 4321 \to 4321 \to \cdots.$$

What cycles of states are possible for other values of n?

71. [*M46*] Continuing the previous problem, what is the maximum number of steps that can occur before n-card Bulgarian solitaire reaches a cyclic state?

72. [*M25*] Suppose we write down all partitions of n, for example

$$6,\ 51,\ 42,\ 411,\ 33,\ 321,\ 3111,\ 222,\ 2211,\ 21111,\ 111111$$

when $n = 6$, and change each jth occurrence of k to j:

$$1,\ 11,\ 11,\ 112,\ 12,\ 111,\ 1123,\ 123,\ 1212,\ 11234,\ 123456.$$

a) Prove that this operation yields a permutation of the individual elements.
b) How many times does the element k appear altogether?

7.2.1.5. Generating all set partitions. Now let's shift gears and concentrate on a rather different kind of partition. The *partitions of a set* are the ways to regard that set as a union of nonempty, disjoint subsets called *blocks*. For example, we listed the five essentially different partitions of $\{1,2,3\}$ at the beginning of the previous section, in 7.2.1.4–(2) and 7.2.1.4–(4). Those five partitions can also be written more compactly in the form

$$123,\quad 12|3,\quad 13|2,\quad 1|23,\quad 1|2|3, \tag{1}$$

using a vertical line to separate one block from another. In this list the elements of each block could have been written in any order, and so could the blocks themselves, because '13|2' and '31|2' and '2|13' and '2|31' all represent the same partition. But we can standardize the representation by agreeing, for example, to list the elements of each block in increasing order, and to arrange the blocks in increasing order of their smallest elements. With this convention the partitions of $\{1,2,3,4\}$ are

$$\begin{gathered}1234,\ 123|4,\ 124|3,\ 12|34,\ 12|3|4,\ 134|2,\ 13|24,\ 13|2|4,\\ 14|23,\ 1|234,\ 1|23|4,\ 14|2|3,\ 1|24|3,\ 1|2|34,\ 1|2|3|4,\end{gathered} \tag{2}$$

obtained by placing 4 among the blocks of (1) in all possible ways.

Set partitions arise in many different contexts. Political scientists and economists, for example, often see them as "coalitions"; computer system designers may consider them to be "cache-hit patterns" for memory accesses; poets know them as "rhyme schemes" (see exercises 34–37). We saw in Section 2.3.3 that any *equivalence relation* between objects — namely any binary relation that is reflexive, symmetric, and transitive — defines a partition of those objects into so-called "equivalence classes." Conversely, every set partition defines an equivalence relation: If Π is a partition of $\{1,2,\ldots,n\}$ we can write

$$j \equiv k \quad (\text{modulo } \Pi) \tag{3}$$

whenever j and k belong to the same block of Π.

One of the most convenient ways to represent a set partition inside a computer is to encode it as a *restricted growth string*, namely as a string $a_1a_2\ldots a_n$ in which we have

$$a_1 = 0 \quad \text{and} \quad a_{j+1} \le 1 + \max(a_1,\ldots,a_j) \text{ for } 1 \le j < n. \tag{4}$$

The idea is to set $a_j = a_k$ if and only if $j \equiv k$, and to choose the smallest available number for a_j whenever j is smallest in its block. For example, the restricted growth strings for the fifteen partitions in (2) are respectively

$$\begin{gathered}0000,\ 0001,\ 0010,\ 0011,\ 0012,\ 0100,\ 0101,\ 0102,\\ 0110,\ 0111,\ 0112,\ 0120,\ 0121,\ 0122,\ 0123.\end{gathered} \tag{5}$$

This convention suggests the following simple generation scheme, due to George Hutchinson [*CACM* **6** (1963), 613–614]:

Algorithm H (*Restricted growth strings in lexicographic order*). Given $n \ge 2$, this algorithm generates all partitions of $\{1,2,\ldots,n\}$ by visiting all strings $a_1a_2\ldots a_n$ that satisfy the restricted growth condition (4). We maintain an auxiliary array $b_1b_2\ldots b_n$, where $b_{j+1} = 1 + \max(a_1,\ldots,a_j)$; the value of b_n is actually kept in a separate variable, m, for efficiency.

H1. [Initialize.] Set $a_1\ldots a_n \leftarrow 0\ldots 0$, $b_1\ldots b_{n-1} \leftarrow 1\ldots 1$, and $m \leftarrow 1$.

H2. [Visit.] Visit the restricted growth string $a_1\ldots a_n$, which represents a partition into $m + [a_n = m]$ blocks. Then go to H4 if $a_n = m$.

H3. [Increase a_n.] Set $a_n \leftarrow a_n + 1$ and return to H2.

H4. [Find j.] Set $j \leftarrow n-1$; then, while $a_j = b_j$, set $j \leftarrow j-1$.

H5. [Increase a_j.] Terminate if $j = 1$. Otherwise set $a_j \leftarrow a_j + 1$.

H6. [Zero out $a_{j+1} \ldots a_n$.] Set $m \leftarrow b_j + [a_j = b_j]$ and $j \leftarrow j + 1$. Then, while $j < n$, set $a_j \leftarrow 0$, $b_j \leftarrow m$, and $j \leftarrow j + 1$. Finally set $a_n \leftarrow 0$ and go back to H2. ▌

Exercise 47 proves that steps H4–H6 are rarely necessary, and that the loops in H4 and H6 are almost always short. A linked-list variant of this algorithm appears in exercise 2.

Gray codes for set partitions. One way to pass quickly through all set partitions is to change just one digit of the restricted growth string $a_1 \ldots a_n$ at each step, because a change to a_j simply means that element j moves from one block to another. An elegant way to arrange such a list was proposed by Gideon Ehrlich [*JACM* **20** (1973), 507–508]: We can successively append the digits

$$0,\ m,\ m-1,\ \ldots,\ 1 \qquad \text{or} \qquad 1,\ \ldots,\ m-1,\ m,\ 0 \tag{6}$$

to each string $a_1 \ldots a_{n-1}$ in the list for partitions of $n-1$ elements, where $m = 1 + \max(a_1, \ldots, a_{n-1})$, alternating between the two cases. Thus the list '00, 01' for $n = 2$ becomes '000, 001, 011, 012, 010' for $n = 3$; and that list becomes

$$\begin{gathered} 0000,\ 0001,\ 0011,\ 0012,\ 0010,\ 0110,\ 0112,\ 0111, \\ 0121,\ 0122,\ 0123,\ 0120,\ 0100,\ 0102,\ 0101 \end{gathered} \tag{7}$$

when we extend it to the case $n = 4$. Exercise 14 shows that Ehrlich's scheme leads to a simple algorithm that achieves this Gray-code order without doing much more work than Algorithm H.

Suppose, however, that we aren't interested in *all* of the partitions; we might want only the ones that have exactly m blocks. Can we run through this smaller collection of restricted growth strings, still changing only one digit at a time? Yes; a very pretty way to generate such a list has been discovered by Frank Ruskey [*Lecture Notes in Comp. Sci.* **762** (1993), 205–206]. He defined two such sequences, A_{mn} and A'_{mn}, both of which start with the lexicographically smallest m-block string $0^{n-m}01 \ldots (m-1)$. The difference between them, if $n > m + 1$, is that A_{mn} ends with $01 \ldots (m-1)0^{n-m}$ while A'_{mn} ends with $0^{n-m-1}01 \ldots (m-1)0$. Here are Ruskey's recursive rules, when $1 < m < n$:

$$A_{m(n+1)} = \begin{cases} A_{(m-1)n}(m-1), A^R_{mn}(m-1), \ldots, A^R_{mn}1, A_{mn}0, & \text{if } m \text{ is even;} \\ A'_{(m-1)n}(m-1), A_{mn}(m-1), \ldots, A^R_{mn}1, A_{mn}0, & \text{if } m \text{ is odd;} \end{cases} \tag{8}$$

$$A'_{m(n+1)} = \begin{cases} A'_{(m-1)n}(m-1), A_{mn}(m-1), \ldots, A_{mn}1, A^R_{mn}0, & \text{if } m \text{ is even;} \\ A_{(m-1)n}(m-1), A^R_{mn}(m-1), \ldots, A_{mn}1, A^R_{mn}0, & \text{if } m \text{ is odd.} \end{cases} \tag{9}$$

Of course the base cases are simply one-element lists,

$$A_{1n} = A'_{1n} = \{0^n\} \qquad \text{and} \qquad A_{nn} = \{01 \ldots (n-1)\}. \tag{10}$$

With these definitions the $\left\{{5 \atop 3}\right\} = 25$ partitions of $\{1,2,3,4,5\}$ into three blocks are

$$\begin{gathered}00012,\ 00112,\ 01112,\ 01012,\ 01002,\ 01102,\ 00102,\\ 00122,\ 01122,\ 01022,\ 01222,\ 01212,\ 01202,\\ 01201,\ 01211,\ 01221,\ 01021,\ 01121,\ 00121,\\ 00120,\ 01120,\ 01020,\ 01220,\ 01210,\ 01200.\end{gathered} \tag{11}$$

(See exercise 17 for an efficient implementation.)

In Ehrlich's scheme (7) the rightmost digits of $a_1 \ldots a_n$ vary most rapidly, but in Ruskey's scheme most of the changes occur near the left. In both cases, however, each step affects just one digit a_j, and the changes are quite simple: Either a_j changes by ± 1, or it jumps between the two extreme values 0 and $1 + \max(a_1, \ldots, a_{j-1})$. Under the same constraints, the sequence A'_{1n}, A'_{2n}, …, A'_{nn} runs through *all* partitions, in increasing order of the number of blocks.

The number of set partitions. We've seen that there are 5 partitions of $\{1,2,3\}$ and 15 of $\{1,2,3,4\}$. A quick way to compute these counts was discovered by C. S. Peirce, who presented the following triangle of numbers in the *American Journal of Mathematics* **3** (1880), page 48:

$$\begin{array}{cccccc} 1 \\ 2 & 1 \\ 5 & 3 & 2 \\ 15 & 10 & 7 & 5 \\ 52 & 37 & 27 & 20 & 15 \\ 203 & 151 & 114 & 87 & 67 & 52 \end{array} \tag{12}$$

Here the entries ϖ_{n1}, ϖ_{n2}, …, ϖ_{nn} of the nth row obey the simple recurrence

$$\varpi_{nk} = \varpi_{(n-1)k} + \varpi_{n(k+1)} \text{ if } 1 \le k < n; \qquad \varpi_{nn} = \varpi_{(n-1)1} \text{ if } n > 1; \tag{13}$$

and $\varpi_{11} = 1$. Peirce's triangle has many remarkable properties, some of which are surveyed in exercises 26–31. For example, ϖ_{nk} is the number of partitions of $\{1, 2, \ldots, n\}$ in which k is the smallest of its block.

The entries on the diagonal and in the first column of Peirce's triangle, which tell us the total number of set partitions, are commonly known as *Bell numbers*, because E. T. Bell wrote several influential papers about them [*AMM* **41** (1934), 411–419; *Annals of Math.* **35** (1934), 258–277; **39** (1938), 539–557]. We shall denote Bell numbers by ϖ_n, following the lead of Louis Comtet, in order to avoid confusion with the Bernoulli numbers B_n. The first few cases are

$$\begin{array}{rccccccccccccc} n & = 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ \varpi_n & = 1 & 1 & 2 & 5 & 15 & 52 & 203 & 877 & 4140 & 21147 & 115975 & 678570 & 4213597 \end{array}$$

Notice that this sequence grows rapidly, but not as fast as $n!$; we will prove below that $\varpi_n = \Theta(n/\log n)^n$.

The Bell numbers $\varpi_n = \varpi_{n1}$ for $n \ge 0$ must satisfy the recurrence formula

$$\varpi_{n+1} \;=\; \varpi_n + \binom{n}{1}\varpi_{n-1} + \binom{n}{2}\varpi_{n-2} + \cdots \;=\; \sum_k \binom{n}{k}\varpi_{n-k}, \tag{14}$$

because every partition of $\{1,\ldots,n+1\}$ is obtained by choosing k elements of $\{1,\ldots,n\}$ to put in the block containing $n+1$ and by partitioning the remaining elements in ϖ_{n-k} ways, for some k. This recurrence, found by Yoshisuke Matsunaga in the 18th century (see Section 7.2.1.7), leads to a nice generating function,

$$\Pi(z) \;=\; \sum_{n=0}^{\infty} \varpi_n \frac{z^n}{n!} \;=\; e^{e^z-1}, \tag{15}$$

discovered by W. A. Whitworth [*Choice and Chance*, 3rd edition (1878), 3.XXIV]. For if we multiply both sides of (14) by $z^n/n!$ and sum on n we get

$$\Pi'(z) = \sum_{n=0}^{\infty} \varpi_{n+1}\frac{z^n}{n!} = \Bigl(\sum_{k=0}^{\infty}\frac{z^k}{k!}\Bigr)\Bigl(\sum_{m=0}^{\infty}\varpi_m\frac{z^m}{m!}\Bigr) = e^z\,\Pi(z),$$

and (15) is the solution to this differential equation with $\Pi(0)=1$.

The numbers ϖ_n had been studied for many years because of their curious properties related to this formula, long before Whitworth pointed out their combinatorial connection with set partitions. For example, we have

$$\varpi_n = \frac{n!}{e}\,[z^n]\,e^{e^z} = \frac{n!}{e}\,[z^n]\sum_{k=0}^{\infty}\frac{e^{kz}}{k!} = \frac{1}{e}\sum_{k=0}^{\infty}\frac{k^n}{k!} \tag{16}$$

[*Mat. Sbornik* **3** (1868), 62; **4** (1869), 39; G. Dobiński, *Archiv der Math. und Physik* **61** (1877), 333–336; **63** (1879), 108–110]. Christian Kramp discussed the expansion of e^{e^z} in *Der polynomische Lehrsatz*, ed. by C. F. Hindenburg (Leipzig: 1796), 112–113; he mentioned two ways to compute the coefficients, namely either to use (14) or to use a summation of $p(n)$ terms, one for each ordinary partition of n. (See Arbogast's formula, exercise 1.2.5–21. Kramp, who came close to discovering that formula, seemed to prefer his partition-based method, not realizing that it would require more than polynomial time as n got larger and larger; and he computed 116015, not 115975, for the coefficient of z^{10}.)

***Asymptotic estimates.** We can learn how fast ϖ_n grows by using one of the most basic principles of complex residue theory: If the power series $\sum_{k=0}^{\infty} a_k z^k$ converges whenever $|z|<r$, then

$$a_{n-1} \;=\; \frac{1}{2\pi i}\oint \frac{a_0+a_1z+a_2z^2+\cdots}{z^n}\,dz, \tag{17}$$

if the integral is taken along a simple closed path that goes counterclockwise around the origin and stays inside the circle $|z|=r$. Let $f(z)=\sum_{k=0}^{\infty} a_k z^{k-n}$ be the integrand. We're free to choose any such path, but special techniques often apply when the path goes through a point z_0 at which the derivative $f'(z_0)$ is zero, because we have

$$f(z_0+\epsilon e^{i\theta}) \;=\; f(z_0)+\frac{f''(z_0)}{2}\epsilon^2 e^{2i\theta}+O(\epsilon^3) \tag{18}$$

in the vicinity of such a point. If, for example, $f(z_0)$ and $f''(z_0)$ are real and positive, say $f(z_0)=u$ and $f''(z_0)=2v$, this formula says that the value of

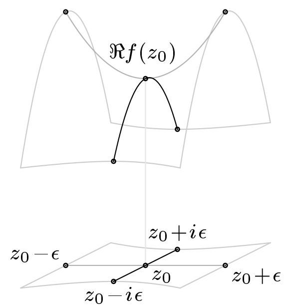


Fig. 33. The behavior of an analytic function near a saddle point.

$f(z_0 \pm \epsilon)$ is approximately $u + v\epsilon^2$ while $f(z_0 \pm i\epsilon)$ is approximately $u - v\epsilon^2$. If z moves from $z_0 - i\epsilon$ to $z_0 + i\epsilon$, the value of $f(z)$ rises to a maximum value u, then falls again; but the larger value $u + v\epsilon^2$ occurs both to the left and to the right of this path. In other words, a mountaineer who goes hiking on the complex plane, when the altitude at point z is $\Re f(z)$, encounters a "pass" at z_0; the terrain looks like a saddle at that point. The overall integral of $f(z)$ will be the same if taken around any path, but a path that doesn't go through the pass won't be as nice because it will have to cancel out some higher values of $f(z)$ that could have been avoided. Therefore we tend to get best results by choosing a path that goes through z_0, in the direction of increasing imaginary part. This important technique, due to P. Debye [*Math. Annalen* **67** (1909), 535–558], is called the "saddle point method."

Let's get familiar with the saddle point method by starting with an example for which we already know the answer:

$$\frac{1}{(n-1)!} \;=\; \frac{1}{2\pi i}\oint \frac{e^z}{z^n}\,dz. \tag{19}$$

Our goal is to find a good approximation for the value of the integral on the right when n is large. It will be convenient to deal with $f(z) = e^z/z^n$ by writing it as $e^{g(z)}$ where $g(z) = z - n\ln z$; then the saddle point occurs where $g'(z_0) = 1 - n/z_0$ is zero, namely at $z_0 = n$. If $z = n + it$ we have

$$\begin{aligned} g(z) &= g(n) + \sum_{k=2}^{\infty} \frac{g^{(k)}(n)}{k!}(it)^k \\ &= n - n\ln n - \frac{t^2}{2n} + \frac{it^3}{3n^2} + \frac{t^4}{4n^3} - \frac{it^5}{5n^4} + \cdots \end{aligned}$$

because $g^{(k)}(z) = (-1)^k(k-1)!\,n/z^k$ when $k \ge 2$. Let's integrate $f(z)$ on a rectangular path from $n - im$ to $n + im$ to $-n + im$ to $-n - im$ to $n - im$:

$$\begin{aligned} \frac{1}{2\pi i}\oint \frac{e^z}{z^n}\,dz = \frac{1}{2\pi}\int_{-m}^{m} f(n+it)\,dt + \frac{1}{2\pi i}\int_{n}^{-n} f(t+im)\,dt \\ + \frac{1}{2\pi}\int_{m}^{-m} f(-n+it)\,dt + \frac{1}{2\pi i}\int_{-n}^{n} f(t-im)\,dt. \end{aligned}$$

Clearly $|f(z)| \le 2^{-n} f(n)$ on the last three sides of this path if we choose $m = 2n$, because $|e^z| = e^{\Re z}$ and $|z| \ge \max(\Re z, \Im z)$; so we're left with

$$\frac{1}{2\pi i}\oint \frac{e^z}{z^n}\,dz \;=\; \frac{1}{2\pi}\int_{-m}^{m} e^{g(n+it)}\,dt + O\Bigl(\frac{ne^n}{2^n n^n}\Bigr).$$

Now we fall back on a technique that we've used several times before— for example to derive Eq. 5.1.4–(53): If $\hat f(t)$ is a good approximation to $f(t)$ when $t \in A$, and if the sums $\sum_{t\in B\setminus A} f(t)$ and $\sum_{t\in C\setminus A}\hat f(t)$ are both small, then $\sum_{t\in C}\hat f(t)$ is a good approximation to $\sum_{t\in B} f(t)$. The same idea applies to integrals as well as sums. [This general method, introduced by Laplace in 1782, is often called "trading tails"; see *CMath* §9.4.] If $|t| \le n^{1/2+\epsilon}$ we have

$$\begin{aligned} e^{g(n+it)} &= \exp\Bigl(g(n) - \frac{t^2}{2n} + \frac{it^3}{3n^2} + \cdots\Bigr)\\ &= \frac{e^n}{n^n}\exp\Bigl(-\frac{t^2}{2n} + \frac{it^3}{3n^2} + \frac{t^4}{4n^3} + O(n^{5\epsilon-3/2})\Bigr)\\ &= \frac{e^n}{n^n}e^{-t^2/(2n)}\Bigl(1 + \frac{it^3}{3n^2} + \frac{t^4}{4n^3} - \frac{t^6}{18n^4} + O(n^{9\epsilon-3/2})\Bigr). \end{aligned}$$

And when $|t| > n^{1/2+\epsilon}$ we have

$$|e^{g(n+it)}| < |f(n+in^{1/2+\epsilon})| = \frac{e^n}{n^n}\exp\Bigl(-\frac{n}{2}\ln(1+n^{2\epsilon-1})\Bigr) = O\Bigl(\frac{e^{n-n^{-2\epsilon}/2}}{n^n}\Bigr).$$

Furthermore the incomplete gamma function

$$\int_{n^{1/2+\epsilon}}^{\infty} e^{-t^2/(2n)} t^k\,dt = 2^{(k-1)/2} n^{(k+1)/2}\,\Gamma\Bigl(\frac{k+1}{2}, \frac{n^{2\epsilon}}{2}\Bigr) = O(n^{O(1)} e^{-n^{2\epsilon}/2})$$

is negligible. Thus we can trade tails and obtain the approximation

$$\begin{aligned} \frac{1}{2\pi i}\oint \frac{e^z}{z^n}\,dz &= \frac{e^n}{2\pi n^n}\int_{-\infty}^{\infty} e^{-t^2/(2n)}\Bigl(1 + \frac{it^3}{3n^2} + \frac{t^4}{4n^3} - \frac{t^6}{18n^4} + O(n^{9\epsilon-3/2})\Bigr)dt\\ &= \frac{e^n}{2\pi n^n}\Bigl(I_0 + \frac{i}{3n^2}I_3 + \frac{1}{4n^3}I_4 - \frac{1}{18n^4}I_6 + O(n^{9\epsilon-3/2})\Bigr), \end{aligned}$$

where $I_k = \int_{-\infty}^{\infty} e^{-t^2/(2n)} t^k\,dt$. Of course $I_k = 0$ when k is odd. Otherwise we can evaluate I_k by using the well-known fact that

$$\int_{-\infty}^{\infty} e^{-at^2} t^{2l}\,dt = \frac{\Gamma\bigl((2l+1)/2\bigr)}{a^{(2l+1)/2}} = \frac{\sqrt{2\pi}}{(2a)^{(2l+1)/2}}\prod_{j=1}^{l}(2j-1) \tag{20}$$

when $a > 0$; see exercise 39. Putting everything together gives us, for all $\epsilon > 0$, the asymptotic estimate

$$\frac{1}{(n-1)!} = \frac{e^n}{\sqrt{2\pi}\,n^{n-1/2}}\Bigl(1 + 0 + \frac{3}{4n} - \frac{15}{18n} + O(n^{9\epsilon-3/2})\Bigr); \tag{21}$$

this result agrees perfectly with Stirling's approximation, which we derived by quite different methods in 1.2.11.2–(19). Further terms in the expansion of

$g(n+it)$ would allow us to prove that the true error in (21) is only $O(n^{-2})$, because the same procedure yields an asymptotic series of the general form $e^n/(\sqrt{2\pi}n^{n-1/2})\bigl(1+c_1/n+c_2/n^2+\cdots+c_m/n^m+O(n^{-m-1})\bigr)$ for all m.

Our derivation of this result has glossed over an important technicality: The function $\ln z$ is not single-valued along the path of integration, because it grows by $2\pi i$ when we loop around the origin. Indeed, this fact underlies the basic mechanism that makes the residue theorem work. But our reasoning was valid because the ambiguity of the logarithm does not affect the integrand $f(z)=e^z/z^n$ when n is an integer. Furthermore, if n were not an integer, we could have adapted the argument and kept it rigorous by choosing to carry out the integral (19) along a path that starts at $-\infty$, circles the origin counterclockwise and returns to $-\infty$. That would have given us Hankel's integral for the gamma function, Eq. 1.2.5–(17); we could thereby have derived the asymptotic formula

$$\frac{1}{\Gamma(x)} \;=\; \frac{1}{2\pi i}\oint \frac{e^z}{z^x}\,dz \;=\; \frac{e^x}{\sqrt{2\pi}\,x^{x-1/2}}\Bigl(1-\frac{1}{12x}+O(x^{-2})\Bigr), \tag{22}$$

valid for all real x as $x\to\infty$.

So the saddle point method seems to work — although it isn't the simplest way to get this particular result. Let's apply it now to deduce the approximate size of the Bell numbers:

$$\frac{\varpi_{n-1}}{(n-1)!}=\frac{1}{2\pi ie}\oint e^{g(z)}\,dz, \qquad g(z)=e^z-n\ln z. \tag{23}$$

A saddle point now occurs at the point $z_0=\xi>0$, where

$$\xi e^{\xi} \;=\; n. \tag{24}$$

(We should actually write $\xi(n)$ to indicate that ξ depends on n; but that would clutter up the formulas below.) Let's assume for the moment that a little bird has told us the value of ξ. Then we want to integrate on a path where $z=\xi+it$, and we have

$$g(\xi+it)=e^{\xi}-n\Bigl(\ln\xi-\frac{(it)^2}{2!}\frac{\xi+1}{\xi^2}-\frac{(it)^3}{3!}\frac{\xi^2-2!}{\xi^3}-\frac{(it)^4}{4!}\frac{\xi^3+3!}{\xi^4}+\cdots\Bigr).$$

By integrating on a suitable rectangular path, we can prove as above that the integral in (23) is well approximated by

$$\int_{-n^{\epsilon-1/2}}^{n^{\epsilon-1/2}} e^{g(\xi)-na_2t^2-nia_3t^3+na_4t^4+\cdots}\,dt, \qquad a_k=\frac{\xi^{k-1}+(-1)^k(k-1)!}{k!\,\xi^k}; \tag{25}$$

see exercise 43. Noting that a_kt^k is $O(n^{k\epsilon-k/2})$ inside this integral, we obtain an asymptotic expansion of the form

$$\varpi_{n-1}=\frac{e^{e^{\xi}-1}(n-1)!}{\xi^{n-1}\sqrt{2\pi n(\xi+1)}}\Bigl(1+\frac{b_1}{n}+\frac{b_2}{n^2}+\cdots+\frac{b_m}{n^m}+O\Bigl(\frac{\log n}{n}\Bigr)^{m+1}\Bigr), \tag{26}$$

where $(\xi+1)^{3k}b_k$ is a polynomial of degree $4k$ in ξ. (See exercise 44.) For example,

$$b_1 = -\frac{2\xi^4-3\xi^3-20\xi^2-18\xi+2}{24(\xi+1)^3}; \qquad (27)$$

$$b_2 = \frac{4\xi^8-156\xi^7-695\xi^6-696\xi^5+1092\xi^4+2916\xi^3+1972\xi^2-72\xi+4}{1152(\xi+1)^6}. \qquad (28)$$

Stirling's approximation (21) can be used in (26) to prove that

$$\varpi_{n-1} = \exp\Bigl(n\Bigl(\xi-1+\frac{1}{\xi}\Bigr)-\xi-\frac{1}{2}\ln(\xi+1)-1-\frac{\xi}{12n}+O\Bigl(\frac{\log n}{n}\Bigr)^2\Bigr); \qquad (29)$$

and exercise 45 proves the similar formula

$$\varpi_n = \exp\Bigl(n\Bigl(\xi-1+\frac{1}{\xi}\Bigr)-\frac{1}{2}\ln(\xi+1)-1-\frac{\xi}{12n}+O\Bigl(\frac{\log n}{n}\Bigr)^2\Bigr). \qquad (30)$$

Consequently we have $\varpi_n/\varpi_{n-1} \approx e^\xi = n/\xi$. More precisely,

$$\frac{\varpi_{n-1}}{\varpi_n} = \frac{\xi}{n}\Bigl(1+O\Bigl(\frac{1}{n}\Bigr)\Bigr). \qquad (31)$$

But what is the asymptotic value of ξ? The definition (24) implies that

$$\begin{aligned}\xi &= \ln n - \ln\xi = \ln n - \ln(\ln n - \ln\xi)\\ &= \ln n - \ln\ln n + O\Bigl(\frac{\log\log n}{\log n}\Bigr);\end{aligned} \qquad (32)$$

and we can go on in this vein, as shown in exercise 49. But the asymptotic series for ξ developed in this way never gives better accuracy than $O(1/(\log n)^m)$ for larger and larger m; so it is hugely inaccurate when multiplied by n in formula (29) for ϖ_{n-1} or formula (30) for ϖ_n.

Thus if we want to use (29) or (30) to calculate good numerical approximations to Bell numbers, our best strategy is to start by computing a good numerical value for ξ, without using a slowly convergent series. Newton's rootfinding method, discussed in the remarks preceding Algorithm 4.7N, yields the efficient iterative scheme

$$\xi_0 = \ln n, \qquad \xi_{k+1} = \frac{\xi_k}{\xi_k+1}(1+\xi_0-\ln\xi_k), \qquad (33)$$

which converges rapidly to the correct value. For example, when $n=100$ the fifth iterate

$$\xi_5 = 3.38563\,01402\,90050\,18488\,82443\,64529\,72686\,74917- \qquad (34)$$

is already correct to 40 decimal places. Using this value in (29) gives us successive approximations

$$(1.6176088053\ldots, 1.6187421339\ldots, 1.6187065391\ldots, 1.6187060254\ldots)\times 10^{114}$$

when we take terms up to b_0, b_1, b_2, b_3 into account; the true value of ϖ_{99} is the 115-digit integer $16187060274460\ldots20741$.

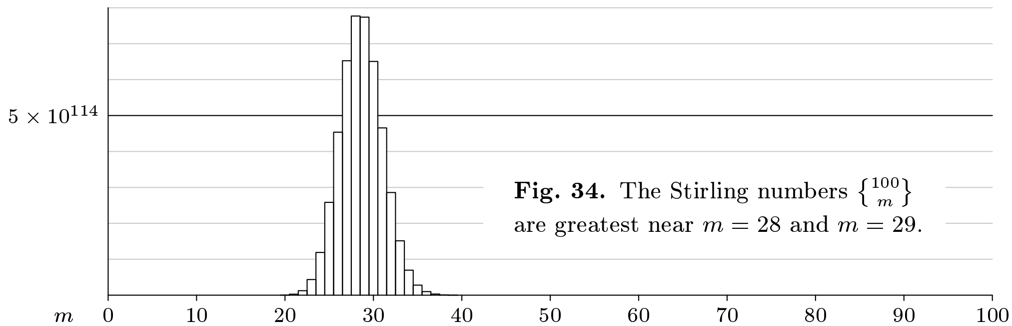


Fig. 34. The Stirling numbers ${100 \brace m}$ are greatest near $m = 28$ and $m = 29$.

Now that we know the number of set partitions ϖ_n, let's try to figure out how many of them have exactly m blocks. It turns out that nearly all partitions of $\{1, \ldots, n\}$ have roughly $n/\xi = e^\xi$ blocks, with about ξ elements per block. For example, Fig. 34 shows a histogram of the numbers ${n \brace m}$ when $n = 100$ and $e^\xi \approx 29.54$.

We can investigate the size of ${n \brace m}$ by applying the saddle point method to formula 1.2.9–(23), which states that

$$\left\{ {n \atop m} \right\} \;=\; \frac{n!}{m!}\,[z^n]\,(e^z-1)^m \;=\; \frac{n!}{m!}\,\frac{1}{2\pi i}\oint e^{m\ln(e^z-1)-(n+1)\ln z}\,dz. \tag{35}$$

Let $\alpha = (n+1)/m$. The function $g(z) = \alpha^{-1}\ln(e^z-1) - \ln z$ has a saddle point at $\sigma > 0$ when

$$\frac{\sigma}{1-e^{-\sigma}} \;=\; \alpha. \tag{36}$$

Notice that $\alpha > 1$ for $1 \le m \le n$. This special value σ is given by

$$\sigma = \alpha - \beta, \qquad \beta = T(\alpha e^{-\alpha}), \tag{37}$$

where T is the tree function of Eq. 2.3.4.4–(30). Indeed, β is the value between 0 and 1 for which we have

$$\beta e^{-\beta} \;=\; \alpha e^{-\alpha}; \tag{38}$$

the function xe^{-x} increases from 0 to e^{-1} when x increases from 0 to 1, then it decreases to 0 again. Therefore β is uniquely defined, and we have

$$e^\sigma \;=\; \frac{\alpha}{\beta}. \tag{39}$$

All such pairs α and β are obtainable by using the inverse formulas

$$\alpha \;=\; \frac{\sigma e^\sigma}{e^\sigma - 1}, \qquad \beta \;=\; \frac{\sigma}{e^\sigma-1}; \tag{40}$$

for example, the values $\alpha = \ln 4$ and $\beta = \ln 2$ correspond to $\sigma = \ln 2$.

We can show as above that the integral in (35) is asymptotically equivalent to an integral of $e^{(n+1)g(z)}\,dz$ over the path $z = \sigma + it$. (See exercise 58.) Exercise 56 proves that the Taylor series about $z=\sigma$,

$$g(\sigma+it) \;=\; g(\sigma)-\frac{t^2(1-\beta)}{2\sigma^2}-\sum_{k=3}^{\infty}\frac{(it)^k}{k!}\,g^{(k)}(\sigma), \qquad (41)$$

has the property that

$$|g^{(k)}(\sigma)| < 2(k-1)!\,(1-\beta)/\sigma^k \qquad \text{for all } k>0. \qquad (42)$$

Therefore we can conveniently remove a factor of $N=(n+1)(1-\beta)$ from the power series $(n+1)g(z)$, and the saddle point method leads to the formula

$$\left\{ {n \atop m} \right\} = \frac{n!}{m!}\frac{1}{(\alpha-\beta)^{n-m}\beta^m\sqrt{2\pi N}}\left(1+\frac{b_1}{N}+\frac{b_2}{N^2}+\cdots+\frac{b_l}{N^l}+O\Big(\frac{1}{N^{l+1}}\Big)\right) \quad (43)$$

as $N\to\infty$, where $(1-\beta)^{2k}b_k$ is a polynomial in α and β. (The quantity $(\alpha-\beta)^{n-m}\beta^m$ in the denominator comes from the fact that $(e^\sigma-1)^m/\sigma^n = (\alpha/\beta-1)^m/(\alpha-\beta)^n$, by (37) and (39).) For example,

$$b_1 \;=\; \frac{6-\beta^3-4\alpha\beta^2-\alpha^2\beta}{8(1-\beta)}-\frac{5(2-\beta^2-\alpha\beta)^2}{24(1-\beta)^2}. \qquad (44)$$

Exercise 57 proves that $N\to\infty$ if and only if $n-m\to\infty$. An asymptotic expansion for $\left\{ {n \atop m} \right\}$ similar to (43), but somewhat more complicated, was first obtained by Leo Moser and Max Wyman, *Duke Math. J.* **25** (1957), 29–43.

Formula (43) looks a bit scary because it is designed to apply over the entire range of block counts m. Significant simplifications are possible when m is relatively small or relatively large (see exercises 60 and 61); but the simplified formulas don't give accurate results in the important cases when $\left\{ {n \atop m} \right\}$ is largest. Let's look at those crucial cases more closely now, so that we can account for the sharp peak illustrated in Fig. 34.

Let $\xi e^\xi = n$ as in (24), and suppose $m=\exp(\xi+r/\sqrt{n}) = ne^{r/\sqrt{n}}/\xi$; we will assume that $|r|\le n^\epsilon$, so that m is near e^ξ. The leading term of (43) can be rewritten

$$\frac{n!}{m!}\frac{1}{(\alpha-\beta)^{n-m}\beta^m\sqrt{2\pi(n+1)(1-\beta)}} \;=\; \frac{m^n}{m!}\frac{(n+1)!}{(n+1)^{n+1}}\frac{e^{n+1}}{\sqrt{2\pi(n+1)}}\Big(1-\frac{\beta}{\alpha}\Big)^{m-n}\frac{e^{-\beta m}}{\sqrt{1-\beta}}, \quad (45)$$

and Stirling's approximation for $(n+1)!$ is evidently ripe for cancellation in the midst of this expression. With the help of computer algebra we find

$$\frac{m^n}{m!} \;=\; \frac{1}{\sqrt{2\pi}}\exp\bigg(n\Big(\xi-1+\frac{1}{\xi}\Big)-\frac{1}{2}\Big(\xi+r^2+\frac{r^2}{\xi}\Big) - \Big(\frac{r}{2}+\frac{r^3}{6}+\frac{r^3}{3\xi}\Big)\frac{1}{\sqrt{n}}+O(n^{4\epsilon-1})\bigg);$$

and the relevant quantities related to α and β are

$$\frac{\beta}{\alpha} = \frac{\xi}{n} + \frac{r\xi^2}{n\sqrt{n}} + O(\xi^3 n^{2\epsilon-2});$$
$$e^{-\beta m} = \exp\Bigl(-\xi - \frac{r\xi^2}{\sqrt{n}} + O(\xi^3 n^{2\epsilon-1})\Bigr);$$
$$\Bigl(1-\frac{\beta}{\alpha}\Bigr)^{m-n} = \exp\Bigl(\xi - 1 + \frac{r(\xi^2-\xi-1)}{\sqrt{n}} + O(\xi^3 n^{2\epsilon-1})\Bigr).$$

Therefore the overall result is

$$\left\{ {n \atop e^{\xi+r/\sqrt{n}}} \right\} = \frac{1}{\sqrt{2\pi}} \exp\Bigl(n\Bigl(\xi - 1 + \frac{1}{\xi}\Bigr) - \frac{\xi}{2} - 1 - \frac{\xi+1}{2\xi}\Bigl(r + \frac{3\xi(2\xi+3)+(\xi+2)r^2}{6(\xi+1)\sqrt{n}}\Bigr)^2 + O(\xi^3 n^{4\epsilon-1})\Bigr). \quad (46)$$

The squared expression on the last line is zero when

$$r = -\frac{\xi(2\xi+3)}{2(\xi+1)\sqrt{n}} + O(\xi^2 n^{-3/2});$$

thus the maximum occurs when the number of blocks is

$$m \;=\; \frac{n}{\xi} - \frac{3+2\xi}{2+2\xi} + O\Bigl(\frac{\xi}{n}\Bigr). \quad (47)$$

By comparing (47) to (30) we see that the largest Stirling number $\left\{ {n \atop m} \right\}$ for a given value of n is approximately equal to $\xi\varpi_n/\sqrt{2\pi n}$.

The saddle point method applies to problems that are considerably more difficult than the ones we have considered here. Excellent expositions of advanced techniques can be found in several books: N. G. de Bruijn, *Asymptotic Methods in Analysis* (1958), Chapters 5 and 6; F. W. J. Olver, *Asymptotics and Special Functions* (1974), Chapter 4; R. Wong, *Asymptotic Approximations of Integrals* (2001), Chapters 2 and 7.

***Random set partitions.** The sizes of blocks in a partition of $\{1,\ldots,n\}$ constitute by themselves an ordinary partition of the number n. Therefore we might wonder what sort of partition they are likely to be. Figure 30 in Section 7.2.1.4 showed the result of superimposing the Ferrers diagrams of all $p(25) = 1958$ partitions of 25; those partitions tended to follow the symmetrical curve of Eq. 7.2.1.4–(49). By contrast, Fig. 35 shows what happens when we superimpose the corresponding diagrams of all $\varpi_{25} \approx 4.6386 \times 10^{18}$ partitions of the set $\{1,\ldots,25\}$. Evidently the "shape" of a random set partition is quite different from the shape of a random integer partition.

This change is due to the fact that some integer partitions occur only a few times as block sizes of set partitions, while others are extremely common. For example, the partition $n = 1+1+\cdots+1$ arises in only one way, but if n is

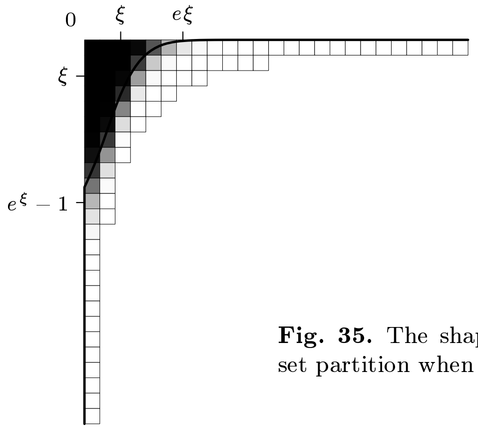


Fig. 35. The shape of a random set partition when $n = 25$.

even the partition $n = 2 + 2 + \cdots + 2$ arises in $(n-1)(n-3)\ldots(1)$ ways. When $n = 25$, the integer partition

$$25 \;=\; 4+4+3+3+3+2+2+2+1+1$$

actually occurs in more than 2% of all possible set partitions. (This particular partition turns out to be most common in the case $n = 25$. The answer to exercise 1.2.5–21 explains that exactly

$$\frac{n!}{c_1!\,1!^{c_1}\,c_2!\,2!^{c_2}\ldots c_n!\,n!^{c_n}} \tag{48}$$

set partitions correspond to the integer partition $n = c_1\cdot 1 + c_2\cdot 2 + \cdots + c_n\cdot n$.)

We can easily determine the average number of k-blocks in a random partition of $\{1,\ldots,n\}$: If we write out all ϖ_n of the possibilities, every particular k-element block occurs exactly ϖ_{n-k} times. Therefore the average number is

$$\binom{n}{k}\frac{\varpi_{n-k}}{\varpi_n}\,. \tag{49}$$

An extension of Eq. (31) above, proved in exercise 64, shows moreover that

$$\frac{\varpi_{n-k}}{\varpi_n} \;=\; \Bigl(\frac{\xi}{n}\Bigr)^k\biggl(1+\frac{k\xi(k\xi+k+1)}{2(\xi+1)^2n}+O\Bigl(\frac{k^3}{n^2}\Bigr)\biggr) \qquad \text{if } k\le n^{2/3}, \tag{50}$$

where ξ is defined in (24). Therefore if, say, $k \le n^\epsilon$, formula (49) simplifies to

$$\frac{n^{\underline{k}}}{k!}\Bigl(\frac{\xi}{n}\Bigr)^k\Bigl(1+O\Bigl(\frac{1}{n}\Bigr)\Bigr) \;=\; \frac{\xi^k}{k!}\bigl(1+O(n^{2\epsilon-1})\bigr). \tag{51}$$

There are, on average, about ξ blocks of size 1, and $\xi^2/2!$ blocks of size 2, etc.

The variance of these quantities is small (see exercise 65), and it turns out that a random partition behaves essentially as if the number of k-blocks were a Poisson deviate with mean $\xi^k/k!$. The smooth curve shown in Fig. 35 runs through the points $\bigl(f(k),k\bigr)$ in Ferrers-like coordinates, where

$$f(k) \;=\; \xi^{k+1}/(k+1)! + \xi^{k+2}/(k+2)! + \xi^{k+3}/(k+3)! + \cdots \tag{52}$$

is the approximate distance from the top line corresponding to block size $k \ge 0$. (This curve becomes more nearly vertical when n is larger.)

The largest block tends to contain approximately $e\xi$ elements. Furthermore, the probability that the block containing element 1 has size less than $\xi + a\sqrt{\xi}$ approaches the probability that a normal deviate is less than a. [See John Haigh, *J. Combinatorial Theory* **A13** (1972), 287–295; V. N. Sachkov, *Probabilistic Methods in Combinatorial Analysis* (1997), Chapter 4, translated from a Russian book published in 1978; Yu. Yakubovich, *J. Mathematical Sciences* **87** (1997), 4124–4137, translated from a Russian paper published in 1995; B. Pittel, *J. Combinatorial Theory* **A79** (1997), 326–359.]

A nice way to generate random partitions of $\{1, 2, \ldots, n\}$ was introduced by A. J. Stam in the *Journal of Combinatorial Theory* **A35** (1983), 231–240: Let M be a random integer that takes the value m with probability

$$p_m \;=\; \frac{m^n}{e\,m!\,\varpi_n}; \tag{53}$$

these probabilities sum to 1 because of (16). Once M has been chosen, generate a random n-tuple $X_1X_2\ldots X_n$, where each X_j is uniformly and independently distributed between 0 and $M-1$. Then let $i \equiv j$ in the partition if and only if $X_i = X_j$. This procedure works because each k-block partition is obtained with probability $\sum_{m\ge 0}(m^{\underline{k}}/m^n)p_m = 1/\varpi_n$.

For example, if $n = 25$ we have

$p_4 \approx .00000372$	$p_9 \approx .15689865$	$p_{14} \approx .04093663$	$p_{19} \approx .00006068$
$p_5 \approx .00019696$	$p_{10} \approx .21855285$	$p_{15} \approx .01531445$	$p_{20} \approx .00001094$
$p_6 \approx .00313161$	$p_{11} \approx .21526871$	$p_{16} \approx .00480507$	$p_{21} \approx .00000176$
$p_7 \approx .02110279$	$p_{12} \approx .15794784$	$p_{17} \approx .00128669$	$p_{22} \approx .00000026$
$p_8 \approx .07431024$	$p_{13} \approx .08987171$	$p_{18} \approx .00029839$	$p_{23} \approx .00000003$

and the other probabilities are negligible. So we can usually get a random partition of 25 elements by looking at a random 25-digit integer in radix 9, 10, 11, or 12. The number M can be generated using 3.4.1–(3); it tends to be approximately $n/\xi = e^{\xi}$ (see exercise 67).

***Partitions of a multiset.** The partitions of an integer and the partitions of a set are just the extreme cases of a far more general problem, the partitions of a multiset. Indeed, the partitions of n are essentially the same as the partitions of $\{1, 1, \ldots, 1\}$, where there are n 1s.

From this standpoint there are essentially $p(n)$ different multisets with n elements. For example, five different cases of multiset partitions arise when $n = 4$:

$$\begin{gathered}
1234,\ 123|4,\ 124|3,\ 12|34,\ 12|3|4,\ 134|2,\ 13|24,\ 13|2|4,\\
14|23,\ 14|2|3,\ 1|234,\ 1|23|4,\ 1|24|3,\ 1|2|34,\ 1|2|3|4;\\
1123,\ 112|3,\ 113|2,\ 11|23,\ 11|2|3,\ 123|1,\ 12|13,\ 12|1|3,\ 13|1|2,\ 1|1|23,\ 1|1|2|3;\\
1122,\ 112|2,\ 11|22,\ 11|2|2,\ 122|1,\ 12|12,\ 12|1|2,\ 1|1|22,\ 1|1|2|2;\\
1112,\ 111|2,\ 112|1,\ 11|12,\ 11|1|2,\ 12|1|1,\ 1|1|1|2;\\
1111,\ 111|1,\ 11|11,\ 11|1|1,\ 1|1|1|1.
\end{gathered} \tag{54}$$

When the multiset contains m distinct elements, with n_1 of one kind, n_2 of another, ..., and n_m of the last, we write $p(n_1, n_2, \ldots, n_m)$ for the total number of partitions. Thus the examples in (54) show that

$$p(1,1,1,1) = 15, \quad p(2,1,1) = 11, \quad p(2,2) = 9, \quad p(3,1) = 7, \quad p(4) = 5. \tag{55}$$

Partitions with $m = 2$ are often called "bipartitions"; those with $m = 3$ are "tripartitions"; and in general these combinatorial objects are known as *multipartitions*. The study of multipartitions was inaugurated long ago by P. A. MacMahon [*Philosophical Transactions* **181** (1890), 481–536; **217** (1917), 81–113; *Proc. Cambridge Philos. Soc.* **22** (1925), 951–963]; but the subject is so vast that many unsolved problems remain. In the remainder of this section and in the exercises below we shall take a glimpse at some of the most interesting and instructive aspects of the theory that have been discovered so far.

In the first place it is important to notice that multipartitions are essentially the partitions of *vectors* with nonnegative integer components, namely the ways to decompose such a vector as a sum of such vectors. For example, the nine partitions of $\{1,1,2,2\}$ listed in (54) are the same as the nine partitions of the bipartite column vector $\begin{smallmatrix}2\\2\end{smallmatrix}$, namely

$$\begin{smallmatrix}2\\2\end{smallmatrix},\ \begin{smallmatrix}2&0\\1&1\end{smallmatrix},\ \begin{smallmatrix}2&0\\0&2\end{smallmatrix},\ \begin{smallmatrix}2&0&0\\0&1&1\end{smallmatrix},\ \begin{smallmatrix}1&1\\2&0\end{smallmatrix},\ \begin{smallmatrix}1&1\\1&1\end{smallmatrix},\ \begin{smallmatrix}1&1&0\\1&0&1\end{smallmatrix},\ \begin{smallmatrix}1&1&0\\0&0&2\end{smallmatrix},\ \begin{smallmatrix}1&1&0&0\\0&0&1&1\end{smallmatrix}. \tag{56}$$

(We drop the + signs for brevity, as in the case of one-dimensional integer partitions.) Each partition can be written in canonical form if we list its parts in nonincreasing lexicographic order.

A simple algorithm suffices to generate the partitions of any given multiset. In the following procedure we represent partitions on a stack that contains triples of elements (c, u, v), where c denotes a component number, $u > 0$ denotes the yet-unpartitioned amount remaining in component c, and $v \le u$ denotes the c component of the current part. Triples are actually kept in three arrays $(c_0, c_1, \ldots)$, $(u_0, u_1, \ldots)$, and $(v_0, v_1, \ldots)$ for convenience, and a "stack frame" array $(f_0, f_1, \ldots)$ is also maintained so that the $(l+1)$st vector of the partition consists of elements f_l through $f_{l+1} - 1$ in the c, u, and v arrays. For example, the following arrays would represent the bipartition $\begin{smallmatrix}3&2&2&1&1&0&0\\1&2&0&1&1&3&1\end{smallmatrix}$:

j	0	1	2	3	4	5	6	7	8	9	10	
c_j	1	2	1	2	1	1	2	1	2	2	2	
u_j	9	9	6	8	4	2	6	1	5	4	1	
v_j	3	1	2	2	2	1	1	1	1	3	1	
	$f_0 = 0$		$f_1 = 2$		$f_2 = 4$	$f_3 = 5$		$f_4 = 7$		$f_5 = 9$	$f_6 = 10$	$f_7 = 11$

(57)

Algorithm M (*Multipartitions in decreasing lexicographic order*). Given a multiset $\{n_1 \cdot 1, \ldots, n_m \cdot m\}$, this algorithm visits all of its partitions using arrays $f_0 f_1 \ldots f_n$, $c_0 c_1 \ldots c_n$, $u_0 u_1 \ldots u_n$, and $v_0 v_1 \ldots v_n$ as described above, where $n = n_1 + \cdots + n_m$. We assume that $m > 0$ and $n_1, \ldots, n_m > 0$.

M1. [Initialize.] Set $c_j \leftarrow j+1$ and $u_j \leftarrow v_j \leftarrow n_{j+1}$ for $0 \le j < m$; also set $f_0 \leftarrow a \leftarrow l \leftarrow 0$ and $f_1 \leftarrow b \leftarrow m$. (In the following steps, the current stack frame runs from a to $b-1$, inclusive.)

M2. [Subtract v from u.] (At this point we want to find all partitions of the vector u in the current frame, into parts that are lexicographically $\le v$. First we will use v itself.) Set $j \leftarrow a$ and $k \leftarrow b$. Then while $j < b$ do the following: Set $u_k \leftarrow u_j - v_j$, and if $u_k \ge v_j$ set $c_k \leftarrow c_j$, $v_k \leftarrow v_j$, $k \leftarrow k+1$, $j \leftarrow j+1$. But if u_k is less than v_j after it has been decreased, the action changes: First set $c_k \leftarrow c_j$, $v_k \leftarrow u_k$, and $k \leftarrow k+1$ if u_k was nonzero; then set $j \leftarrow j+1$. While $j < b$, set $u_k \leftarrow u_j - v_j$, $c_k \leftarrow c_j$, $v_k \leftarrow u_k$, and $k \leftarrow k+1$ if $u_j \ne v_j$; then again $j \leftarrow j+1$, until $j = b$.

M3. [Push if nonzero.] If $k > b$, set $a \leftarrow b$, $b \leftarrow k$, $l \leftarrow l+1$, $f_{l+1} \leftarrow b$, and return to M2.

M4. [Visit a partition.] Visit the partition represented by the $l+1$ vectors currently in the stack. (For $0 \le k \le l$, the vector has v_j in component c_j, for $f_k \le j < f_{k+1}$.)

M5. [Decrease v.] Set $j \leftarrow b-1$, and if $v_j = 0$ set $j \leftarrow j-1$ until $v_j > 0$. Then if $j = a$ and $v_j = 1$, go to M6. Otherwise set $v_j \leftarrow v_j - 1$, and $v_k \leftarrow u_k$ for $j < k < b$. Return to M2.

M6. [Backtrack.] Terminate if $l = 0$. Otherwise set $l \leftarrow l-1$, $b \leftarrow a$, $a \leftarrow f_l$, and return to M5. ▮

The key to this algorithm is step M2, which decreases the current residual vector, u, by the largest permissible part, v; that step also decreases v, if necessary, to the lexicographically largest vector $\le v$ that is less than or equal to the new residual amount in every component.

Let us conclude this section by discussing an amusing connection between multipartitions and the least-significant-digit-first procedure for radix sorting (Algorithm 5.2.5R). The idea is best understood by considering an example. See Table 1, where Step (0) shows nine 4-partite column vectors in lexicographic order. Serial numbers ①–⑨ have been attached at the bottom for identification. Step (1) performs a stable sort of the vectors, bringing their fourth (least significant) entries into decreasing order; similarly, Steps (2), (3), and (4) do a stable sort on the third, second, and top rows. The theory of radix sorting tells us that the original lexicographic order is thereby restored.

Suppose the serial number sequences after these stable sorting operations are respectively α_4, $\alpha_3\alpha_4$, $\alpha_2\alpha_3\alpha_4$, and $\alpha_1\alpha_2\alpha_3\alpha_4$, where the α's are permutations; Table 1 shows the values of α_4, α_3, α_2, and α_1 in parentheses. And now comes the point: Wherever the permutation α_j has a descent, the numbers in row j after sorting must also have a descent, because the sorting is stable. (These descents are indicated by caret marks in the table.) For example, where α_3 has 8 followed by 7, we have 5 followed by 3 in row 3. Therefore the entries $a_1 \ldots a_9$ in row 3 after Step (2) are not an arbitrary partition of their sum; they must satisfy

$$a_1 \ge a_2 \ge a_3 \ge a_4 > a_5 \ge a_6 > a_7 \ge a_8 \ge a_9. \tag{58}$$

Table 1

RADIX SORTING AND MULTIPARTITIONS

Step (0): Original partition

$$\begin{array}{ccccccccc} 6&5&5&4&3&2&1&0&0\\ 3&2&1&0&4&5&6&4&2\\ 6&6&3&1&1&5&2&0&7\\ 4&2&1&3&3&1&1&2&5\\ \textcircled{1}&\textcircled{2}&\textcircled{3}&\textcircled{4}&\textcircled{5}&\textcircled{6}&\textcircled{7}&\textcircled{8}&\textcircled{9} \end{array}$$

Step (1): Sort row 4

$$\begin{array}{ccccccccc} 0&6&4&3&5&0&5&2&1\\ 2&3&0&4&2&4&1&5&6\\ 7&6&1&1&6&0&3&5&2\\ 5_\wedge&4&3&3_\wedge&2&2_\wedge&1&1&1\\ \textcircled{9}&\textcircled{1}&\textcircled{4}&\textcircled{5}&\textcircled{2}&\textcircled{8}&\textcircled{3}&\textcircled{6}&\textcircled{7} \end{array}$$

$$\alpha_4 = (\,9_\wedge 1\;\;4\;\;5_\wedge 2\;\;8_\wedge 3\;\;6\;\;7\,)$$

Step (2): Sort row 3

$$\begin{array}{ccccccccc} 0&6&5&2&5&1&4&3&0\\ 2&3&2&5&1&6&0&4&4\\ 7&6&6&5_\wedge&3&2_\wedge&1&1&0\\ 5&4&2&1&1&1&3&3&2\\ \textcircled{9}&\textcircled{1}&\textcircled{2}&\textcircled{6}&\textcircled{3}&\textcircled{7}&\textcircled{4}&\textcircled{5}&\textcircled{8} \end{array}$$

$$\alpha_3 = (\,1\;\;2\;\;5\;\;8_\wedge 7\;\;9_\wedge 3\;\;4\;\;6\,)$$

Step (3): Sort row 2

$$\begin{array}{ccccccccc} 1&2&3&0&6&0&5&5&4\\ 6_\wedge&5&4&4_\wedge&3_\wedge&2&2&1&0\\ 2&5&1&0&6&7&6&3&1\\ 1&1&3&2&4&5&2&1&3\\ \textcircled{7}&\textcircled{6}&\textcircled{5}&\textcircled{8}&\textcircled{1}&\textcircled{9}&\textcircled{2}&\textcircled{3}&\textcircled{4} \end{array}$$

$$\alpha_2 = (\,6_\wedge 4\;\;8\;\;9_\wedge 2_\wedge 1\;\;3\;\;5\;\;7\,)$$

Step (4): Sort row 1

$$\begin{array}{ccccccccc} 6&5&5&4_\wedge&3_\wedge&2_\wedge&1&0&0\\ 3&2&1&0&4&5&6&4&2\\ 6&6&3&1&1&5&2&0&7\\ 4&2&1&3&3&1&1&2&5\\ \textcircled{1}&\textcircled{2}&\textcircled{3}&\textcircled{4}&\textcircled{5}&\textcircled{6}&\textcircled{7}&\textcircled{8}&\textcircled{9} \end{array}$$

$$\alpha_1 = (\,5\;\;7\;\;8\;\;9_\wedge 3_\wedge 2_\wedge 1\;\;4\;\;6\,)$$

But the numbers $(a_1-2, a_2-2, a_3-2, a_4-2, a_5-1, a_6-1, a_7, a_8, a_9)$ do form an essentially arbitrary partition of the original sum, minus $(4+6)$. The amount of decrease, $4+6$, is the sum of the indices where descents occur; this number is what we called $\operatorname{ind}\alpha_3$, the "index" of α_3, in Section 5.1.1.

Thus we see that any given partition of an m-partite number into at most r parts, with extra zeros added so that the number of columns is exactly r, can be encoded as a sequence of permutations $\alpha_1, \ldots, \alpha_m$ of $\{1, \ldots, r\}$ such that the product $\alpha_1 \ldots \alpha_m$ is the identity, together with a sequence of ordinary one-dimensional partitions of the numbers $(n_1 - \operatorname{ind}\alpha_1, \ldots, n_m - \operatorname{ind}\alpha_m)$ into at most r parts. For example, the vectors in Table 1 represent a partition of $(26, 27, 31, 22)$ into 9 parts; the permutations $\alpha_1, \ldots, \alpha_4$ appear in the table, and we have $(\operatorname{ind}\alpha_1, \ldots, \operatorname{ind}\alpha_4) = (15, 10, 10, 11)$; the partitions are respectively

$$26-15 = (322111100), \qquad 27-10 = (332222210),$$
$$31-10 = (544321110), \qquad 22-11 = (221111111).$$

Conversely, any such permutations and partitions will yield a multipartition of $(n_1, \ldots, n_m)$. If r and m are small, it can be helpful to consider these $r!^{m-1}$ sequences of one-dimensional partitions when listing or reasoning about multipartitions, especially in the bipartite case. [This construction is due to Basil Gordon, *J. London Math. Soc.* **38** (1963), 459–464.]

A good summary of early work on multipartitions, including studies of partitions into distinct parts and/or strictly positive parts, appears in a paper by M. S. Cheema and T. S. Motzkin, *Proc. Symp. Pure Math.* **19** (Amer. Math. Soc., 1971), 39–70.

EXERCISES

1. [*20*] (G. Hutchinson.) Show that a simple modification to Algorithm H will generate all partitions of $\{1, \ldots, n\}$ into *at most* r blocks, given n and $r \geq 2$.

▶ **2.** [*22*] When set partitions are used in practice, we often want to link the elements of each block together. Thus it is convenient to have an array of links $l_1 \ldots l_n$ and an array of headers $h_1 \ldots h_t$ so that the elements of the jth block of a t-block partition are $i_1 > \cdots > i_k$, where

$$i_1 = h_j, \quad i_2 = l_{i_1}, \quad \ldots, \quad i_k = l_{i_{k-1}}, \quad \text{and} \quad l_{i_k} = 0.$$

For example, the representation of 137|25|489|6 would have $t = 4$, $l_1 \ldots l_9 = 001020348$, and $h_1 \ldots h_4 = 7596$.

Design a variant of Algorithm H that generates partitions using this representation.

3. [*M23*] What is the millionth partition of $\{1, \ldots, 12\}$ generated by Algorithm H?

▶ **4.** [*21*] If $x_1 \ldots x_n$ is any string, let $\rho(x_1 \ldots x_n)$ be the restricted growth string that corresponds to the equivalence relation $j \equiv k \iff x_j = x_k$. Classify each of the five-letter English words in the Stanford GraphBase by applying this ρ function; for example, $\rho(\texttt{tooth}) = 01102$. How many of the 52 set partitions of five elements are representable by English words in this way? What's the most common word of each type?

5. [*22*] Guess the next elements of the following two sequences: (a) 0, 1, 1, 1, 12, 12, 12, 12, 12, 12, 100, 121, 122, 123, 123, . . . ; (b) 0, 1, 12, 100, 112, 121, 122, 123, ...

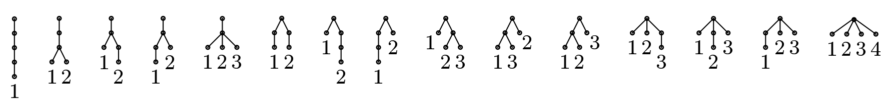
▶ **6.** [*25*] Suggest an algorithm to generate all partitions of $\{1, \ldots, n\}$ in which there are exactly c_1 blocks of size 1, c_2 blocks of size 2, etc.

7. [*M20*] How many permutations $a_1 \ldots a_n$ of $\{1, \ldots, n\}$ have the property that $a_{k-1} > a_k > a_j$ implies $j > k$?

8. [*20*] Suggest a way to generate all permutations of $\{1, \ldots, n\}$ that have exactly m left-to-right minima.

9. [*M20*] How many restricted growth strings $a_1 \ldots a_n$ contain exactly k_j occurrences of j, given the integers k_0, k_1, . . . , k_{n-1}?

10. [*25*] A *semilabeled tree* is an oriented tree in which the leaves are labeled with the integers $\{1, \ldots, k\}$, but the other nodes are unlabeled. Thus there are 15 semilabeled trees with 5 vertices:

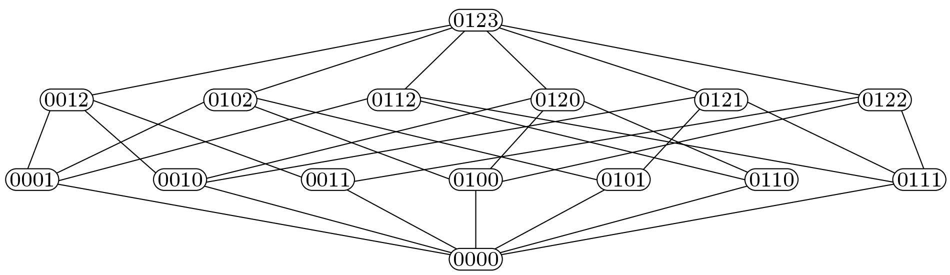


Find a one-to-one correspondence between partitions of $\{1, \ldots, n\}$ and semilabeled trees with $n + 1$ vertices.

▶ **11.** [*28*] We observed in Section 7.2.1.2 that Dudeney's famous problem $\texttt{send}+\texttt{more} = \texttt{money}$ is a "pure" alphametic, namely an alphametic with a unique solution. His puzzle corresponds to a set partition on 13 digit positions, for which the restricted growth string $\rho(\texttt{sendmoremoney})$ is 0123456145217; and we might wonder how lucky he had to be in order to come up with such a construction. How many restricted growth strings of length 13 define pure alphametics of the form $a_1a_2a_3a_4 + a_5a_6a_7a_8 = a_9a_{10}a_{11}a_{12}a_{13}$?

12. [*M31*] (*The partition lattice.*) If Π and Π' are partitions of the same set, we write $\Pi \preceq \Pi'$ if $x \equiv y$ (modulo Π) whenever $x \equiv y$ (modulo Π'). In other words, $\Pi \preceq \Pi'$ means that Π' is a "refinement" of Π, obtained by partitioning zero or more of the latter's blocks; and Π is a "crudification" or *coalescence* of Π', obtained by merging zero or more blocks together. This partial ordering is easily seen to be a lattice, with

$\Pi \vee \Pi'$ the greatest common refinement of Π and Π', and with $\Pi \wedge \Pi'$ their least common coalescence. For example, the lattice of partitions of $\{1,2,3,4\}$ is



if we represent partitions by restricted growth strings $a_1a_2a_3a_4$; upward paths in this diagram take each partition into its refinements. Partitions with t blocks appear on level t from the bottom, and their descendants form the partition lattice of $\{1,\ldots,t\}$.

a) Explain how to compute $\Pi \vee \Pi'$, given $a_1 \ldots a_n$ and $a'_1 \ldots a'_n$.
b) Explain how to compute $\Pi \wedge \Pi'$, given $a_1 \ldots a_n$ and $a'_1 \ldots a'_n$.
c) When does Π' cover Π in this lattice? (See exercise 7.2.1.4–55.)
d) If Π has t blocks of sizes $s_1, \ldots, s_t$, how many partitions does it cover?
e) If Π has t blocks of sizes $s_1, \ldots, s_t$, how many partitions cover it?
f) True or false: If $\Pi \vee \Pi'$ covers Π, then Π' covers $\Pi \wedge \Pi'$.
g) True or false: If Π' covers $\Pi \wedge \Pi'$, then $\Pi \vee \Pi'$ covers Π.
h) Let $b(\Pi)$ denote the number of blocks of Π. Prove that

$$b(\Pi) + b(\Pi') \;\le\; b(\Pi \vee \Pi') + b(\Pi \wedge \Pi').$$

13. [*M28*] (Stephen C. Milne, 1977.) If A is a set of partitions of $\{1,\ldots,n\}$, its *shadow* ∂A is the set of all partitions Π' such that Π covers Π' for some $\Pi \in A$. (We considered the analogous concept for the subset lattice in 7.2.1.3–(54).)

Let $\Pi_1, \Pi_2, \ldots$ be the partitions of $\{1,\ldots,n\}$ into t blocks, in lexicographic order of their restricted growth strings; and let $\Pi'_1, \Pi'_2, \ldots$ be the $(t-1)$-block partitions, also in lexicographic order. Prove that there is a function $f_{nt}(N)$ such that

$$\partial\{\Pi_1,\ldots,\Pi_N\} = \{\Pi'_1,\ldots,\Pi'_{f_{nt}(N)}\} \qquad \text{for } 0 \le N \le \left\{ {n \atop t} \right\}.$$

Hint: The diagram in exercise 12 shows that $(f_{43}(0),\ldots,f_{43}(6)) = (0,3,5,7,7,7,7)$.

14. [*23*] Design an algorithm to generate set partitions in Gray-code order like (7).

15. [*M21*] What is the final partition generated by the algorithm of exercise 14?

16. [*16*] The list (11) is Ruskey's A_{35}; what is A'_{35}?

17. [*26*] Implement Ruskey's Gray code (8) for all m-block partitions of $\{1,\ldots,n\}$.

18. [*M46*] For which n is it possible to generate all restricted growth strings $a_1 \ldots a_n$ in such a way that some a_j changes by ± 1 at each step?

19. [*28*] Prove that there's a Gray code for restricted growth strings in which, at each step, some a_j changes by either ± 1 or ± 2, when (a) we want to generate all ϖ_n strings $a_1 \ldots a_n$; or (b) we want to generate only the $\left\{ {n \atop m} \right\}$ cases with $\max(a_1,\ldots,a_n) = m-1$.

20. [*17*] If Π is a partition of $\{1,\ldots,n\}$, its conjugate Π^T is defined by the rule

$$j \equiv k \pmod{\Pi^T} \qquad\iff\qquad n+1-j \equiv n+1-k \pmod{\Pi}.$$

Suppose Π has the restricted growth string 001010202013; what is the restricted growth string of Π^T?

21. [*M27*] How many partitions of $\{1,\ldots,n\}$ are self-conjugate?

22. [*M23*] If X is a random variable with a given distribution, the expected value of X^n is called the nth *moment* of that distribution. What is the nth moment when X is (a) a Poisson deviate with mean 1 (Eq. 3.4.1–(40))? (b) the number of fixed points of a random permutation of $\{1,\ldots,m\}$, when $m \ge n$ (Eq. 1.3.3–(27))?

23. [*HM30*] If $f(x) = \sum a_k x^k$ is a polynomial, let $f(\varpi)$ stand for $\sum a_k \varpi_k$.

a) Prove the symbolic formula $f(\varpi+1) = \varpi f(\varpi)$. (For example, if $f(x)$ is the polynomial x^2, this formula states that $\varpi_2 + 2\varpi_1 + \varpi_0 = \varpi_3$.)

b) Similarly, prove that $f(\varpi+k) = \varpi^{\underline{k}} f(\varpi)$ for all positive integers k.

c) If p is prime, prove that $\varpi_{n+p} \equiv \varpi_n + \varpi_{n+1} \pmod{p}$. *Hint:* Show first that $x^{\underline{p}} \equiv x^p - x$.

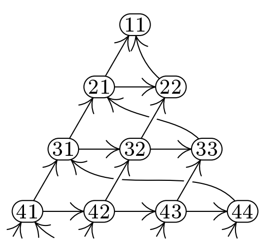
d) Consequently $\varpi_{n+N} \equiv \varpi_n \pmod{p}$ when $N = p^{p-1} + p^{p-2} + \cdots + p + 1$.

24. [*HM35*] Continuing the previous exercise, prove that the Bell numbers satisfy the periodic law $\varpi_{n+p^{e-1}N} \equiv \varpi_n \pmod{p^e}$, if p is an odd prime. *Hint:* Show that

$$x^{\underline{p^e}} \equiv g_e(x) + 1 \bigl(\text{modulo } p^e,\ p^{e-1}g_1(x),\ \ldots,\ \text{and } pg_{e-1}(x)\bigr), \text{ where } g_j(x) = (x^p - x - 1)^{p^j}.$$

25. [*M27*] Prove that $\varpi_n/\varpi_{n-1} \le \varpi_{n+1}/\varpi_n \le \varpi_n/\varpi_{n-1} + 1$.

▶ **26.** [*M22*] According to the recurrence equations (13), the numbers ϖ_{nk} in Peirce's triangle count the paths from ⓝⓚ to ⑪ in the infinite directed graph



Explain why each path from ⓝ① to ⑪ corresponds to a partition of $\{1,\ldots,n\}$.

▶ **27.** [*M35*] A "vacillating tableau loop" of order n is a sequence of integer partitions $\lambda_k = a_{k1}a_{k2}a_{k3}\ldots$ with $a_{k1} \ge a_{k2} \ge a_{k3} \ge \cdots$ for $0 \le k \le 2n$, such that $\lambda_0 = \lambda_{2n} = e_0$ and $\lambda_k = \lambda_{k-1} + (-1)^k e_{t_k}$ for $1 \le k \le 2n$ and for some $t_k \ge 0$; here e_t denotes the unit vector $0^{t-1}10^{n-t}$ when $t > 0$, and e_0 is all zeros.

a) List all the vacillating tableau loops of order 4. [*Hint:* There are 15 altogether.]

b) Prove that exactly ϖ_{nk} vacillating tableau loops of order n have $t_{2k-1} = 0$.

▶ **28.** [*M25*] (*Generalized rook polynomials.*) Consider an arrangement of $a_1 + \cdots + a_m$ square cells in rows and columns, where row k contains cells in columns $1, \ldots, a_k$. Place zero or more "rooks" into the cells, with at most one rook in each row and at most one in each column. An empty cell is called "free" if there is no rook to its right and no rook below. For example, Fig. 35 shows two such placements, one with four rooks in rows of lengths (3,1,4,1,5,9,2,6,5), and another with nine on a 9×9 square board. Rooks are indicated by solid circles; hollow circles have been placed above and to the left of each rook, thereby leaving the free cells blank.

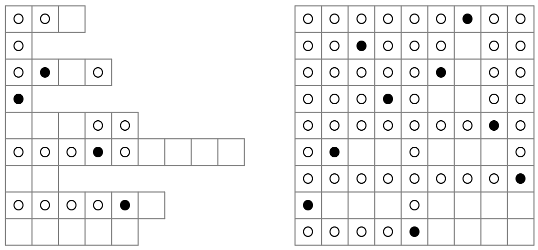


Fig. 35. Rook placements and free cells.

Let $R(a_1, \ldots, a_m)$ be the polynomial in x and y obtained by summing $x^r y^f$ over all legal rook placements, where r is the number of rooks and f is the number of free cells; for example, the left-hand placement in Fig. 35 contributes $x^4 y^{17}$ to the polynomial $R(3, 1, 4, 1, 5, 9, 2, 6, 5)$.

a) Prove that we have $R(a_1, \ldots, a_m) = R(a_1, \ldots, a_{j-1}, a_{j+1}, a_j, a_{j+2}, \ldots, a_m)$; in other words, the order of the row lengths is irrelevant, and we can assume that $a_1 \geq \cdots \geq a_m$ as in a Ferrers diagram like 7.2.1.4–(13).

b) If $a_1 \geq \cdots \geq a_m$ and if $b_1 \ldots b_n = (a_1 \ldots a_m)^T$ is the conjugate partition, prove that $R(a_1, \ldots, a_m) = R(b_1, \ldots, b_n)$.

c) Find a recurrence for evaluating $R(a_1, \ldots, a_m)$ and use it to compute $R(3, 2, 1)$.

d) Generalize Peirce's triangle (12) by changing the addition rule (13) to

$$\varpi_{nk}(x, y) \;=\; x\varpi_{(n-1)k}(x, y) + y\varpi_{n(k+1)}(x, y), \qquad 1 \leq k < n.$$

Thus $\varpi_{21}(x, y) = x+y$, $\varpi_{32}(x, y) = x+xy+y^2$, $\varpi_{31}(x, y) = x^2+2xy+xy^2+y^3$, etc. Prove that the resulting quantity $\varpi_{nk}(x, y)$ is the rook polynomial $R(a_1, \ldots, a_{n-1})$ where $a_j = n - j - [j < k]$.

e) The polynomial $\varpi_{n1}(x, y)$ in part (d) can be regarded as a generalized Bell number $\varpi_n(x, y)$, representing paths from (n1) to (11) in the digraph of exercise 26 that have a given number of "x steps" to the northeast and a given number of "y steps" to the east. Prove that

$$\varpi_n(x, y) \;=\; \sum_{a_1 \ldots a_n} x^{n-1-\max(a_1, \ldots, a_n)} y^{a_1 + \cdots + a_n}$$

summed over all restricted growth strings $a_1 \ldots a_n$ of length n.

29. [*M26*] Continuing the previous exercise, let $R_r(a_1, \ldots, a_m) = [x^r]\, R(a_1, \ldots, a_m)$ be the polynomial in y that enumerates free cells when r rooks are placed.

a) Show that the number of ways to place n rooks on an $n \times n$ board, leaving f cells free, is the number of permutations of $\{1, \ldots, n\}$ that have f inversions. Thus, by Eq. 5.1.1–(8) and exercise 5.1.2–16, we have

$$R_n(\overbrace{n, \ldots, n}^{n}) \;=\; n!_y \;=\; \prod_{k=1}^{n} (1 + y + \cdots + y^{k-1}).$$

b) What is $R_r(\overbrace{n, \ldots, n}^{m})$, the generating function for r rooks on an $m \times n$ board?

c) If $a_1 \geq \cdots \geq a_m$ and t is a nonnegative integer, prove the general formula

$$\prod_{j=1}^{m} \frac{1 - y^{a_j + m - j + t}}{1 - y} \;=\; \sum_{k=0}^{m} \frac{t!_y}{(t-k)!_y}\, R_{m-k}(a_1, \ldots, a_m).$$

[*Note:* The quantity $t!_y/(t{-}k)!_y = \prod_{j=0}^{k-1}((1-y^{t-j})/(1-y))$ is zero when $k > t \ge 0$. Thus, for example, when $t = 0$ the right-hand side reduces to $R_m(a_1, \ldots, a_m)$. We can compute R_m, R_{m-1}, ..., R_0 by successively setting $t = 0, 1, \ldots, m$.]

d) If $a_1 \ge a_2 \ge \cdots \ge a_m \ge 0$ and $a'_1 \ge a'_2 \ge \cdots \ge a'_m \ge 0$, show that we have $R(a_1, a_2 \ldots, a_m) = R(a'_1, a'_2, \ldots, a'_m)$ if and only if the associated multisets $\{a_1{+}m, a_2{+}m{-}1, \ldots, a_m{+}1\}$ and $\{a'_1{+}m, a'_2{+}m{-}1, \ldots, a'_m{+}1\}$ are the same.

30. [*HM30*] The generalized Stirling number $\left\{ {n \atop m} \right\}_q$ is defined by the recurrence

$$\left\{ {n+1 \atop m} \right\}_q = (1 + q + \cdots + q^{m-1})\left\{ {n \atop m} \right\}_q + \left\{ {n \atop m-1} \right\}_q; \qquad \left\{ {0 \atop m} \right\}_q = \delta_{m0}.$$

Thus $\left\{ {n \atop m} \right\}_q$ is a polynomial in q; and $\left\{ {n \atop m} \right\}_1$ is the ordinary Stirling number $\left\{ {n \atop m} \right\}$, because it satisfies the recurrence relation in Eq. 1.2.6–(46).

a) Prove that the generalized Bell number $\varpi_n(x, y) = R(n{-}1, \ldots, 1)$ of exercise 28(e) has the explicit form

$$\varpi_n(x, y) = \sum_{m=0}^{n} x^{n-m} y^{\binom{m}{2}} \left\{ {n \atop m} \right\}_y.$$

b) Show that generalized Stirling numbers also obey the recurrence

$$q^m \left\{ {n+1 \atop m+1} \right\}_q = q^n \left\{ {n \atop m} \right\}_q + \binom{n}{1} q^{n-1} \left\{ {n-1 \atop m} \right\}_q + \cdots = \sum_k \binom{n}{k} q^k \left\{ {k \atop m} \right\}_q.$$

c) Find generating functions for $\left\{ {n \atop m} \right\}_q$, generalizing 1.2.9–(23) and 1.2.9–(28).

31. [*HM23*] Generalizing (15), show that the elements of Peirce's triangle have a simple generating function, if we compute the sum

$$\sum_{n,k} \varpi_{nk} \frac{w^{n-k}}{(n-k)!} \frac{z^{k-1}}{(k-1)!}.$$

32. [*M22*] Let δ_n be the number of restricted growth strings $a_1 \ldots a_n$ for which the sum $a_1 + \cdots + a_n$ is even minus the number for which $a_1 + \cdots + a_n$ is odd. Prove that

$$\delta_n = (1, 0, -1, -1, 0, 1) \qquad \text{when} \qquad n \bmod 6 = (1, 2, 3, 4, 5, 0).$$

Hint: See exercise 28(e).

33. [*M21*] How many partitions of $\{1, 2, \ldots, n\}$ have $1 \not\equiv 2$, $2 \not\equiv 3$, ..., $k-1 \not\equiv k$?

34. [*14*] Many poetic forms involve *rhyme schemes*, which are partitions of the lines of a stanza with the property that $j \equiv k$ if and only if line j rhymes with line k. For example, a "limerick" is generally a 5-line poem with certain rhythmic constraints and with a rhyme scheme described by the restricted growth string 00110.

What rhyme schemes were used in the classical *sonnets* by (a) Guittone d'Arezzo (c. 1270)? (b) Petrarch (c. 1350)? (c) Spenser (1595)? (d) Shakespeare (1609)? (e) Elizabeth Barrett Browning (1850)?

35. [*M21*] Let ϖ'_n be the number of schemes for n-line poems that are "completely rhymed," in the sense that every line rhymes with at least one other. Thus we have $\langle \varpi'_0, \varpi'_1, \varpi'_2, \ldots \rangle = \langle 1, 0, 1, 1, 4, 11, 41, \ldots \rangle$. Give a combinatorial proof of the fact that $\varpi'_n + \varpi'_{n+1} = \varpi_n$.

36. [*M22*] Continuing exercise 35, what is the generating function $\sum_n \varpi'_n z^n/n!$?

37. [*M18*] Alexander Pushkin adopted an elaborate structure in his poetic novel *Eugene Onegin* (1833), based not only on "masculine" rhymes in which the sounds of accented final syllables agree with each other (pain–gain, form–warm, pun–fun, bucks–crux), but also on "feminine" rhymes in which one or two unstressed syllables also participate (humor–tumor, tetrameter–pentameter, lecture–conjecture, iguana–piranha). Every stanza of *Eugene Onegin* is a sonnet with the strict scheme 01012233455477, where the rhyme is feminine or masculine according as the digit is even or odd. Several modern translators of Pushkin's novel have also succeeded in retaining the same form in English and German.

How do I justify this stanza? / These feminine rhymes? My wrinkled muse?
This whole passé extravaganza? / How can I (careless of time) use
The dusty bread molds of Onegin / In the brave bakery of Reagan?
The loaves will surely fail to rise / Or else go stale before my eyes.
The truth is, I can't justify it. / But as no shroud of critical terms
Can save my corpse from boring worms, / I may as well have fun and try it.
If it works, good; and if not, well, / A theory won't postpone its knell.

— VIKRAM SETH, *The Golden Gate* (1986)

A 14-line poem might have any of $\varpi'_{14} = 24{,}011{,}157$ complete rhyme schemes, according to exercise 35. But how many schemes are possible if we are allowed to specify, for each block, whether its rhyme is to be feminine or masculine?

▶ **38.** [*M30*] Let σ_k be the cyclic permutation $(1, 2, \ldots, k)$. The object of this exercise is to study the sequences $k_1 k_2 \ldots k_n$, called *σ-cycles*, for which $\sigma_{k_1}\sigma_{k_2}\ldots\sigma_{k_n}$ is the identity permutation. For example, when $n = 4$ there are exactly 15 σ-cycles, namely

$$1111, 1122, 1212, 1221, 1333, 2112, 2121, 2211, 2222, 2323, 3133, 3232, 3313, 3331, 4444.$$

a) Find a one-to-one correspondence between partitions of $\{1, 2, \ldots, n\}$ and σ-cycles of length n.
b) How many σ-cycles of length n have $1 \le k_1, \ldots, k_n \le m$, given m and n?
c) How many σ-cycles of length n have $k_i = j$, given i, j, and n?
d) How many σ-cycles of length n have $k_1, \ldots, k_n \ge 2$?
e) How many partitions of $\{1, \ldots, n\}$ have $1 \not\equiv 2$, $2 \not\equiv 3$, $\ldots$, $n-1 \not\equiv n$, and $n \not\equiv 1$?

39. [*HM16*] Evaluate $\int_0^\infty e^{-t^{p+1}} t^q\,dt$ when p and q are nonnegative integers. *Hint:* See exercise 1.2.5–20.

40. [*HM20*] Suppose the saddle point method is used to estimate $[z^{n-1}]\,e^{cz}$. The text's derivation of (21) from (19) deals with the case $c = 1$; how should that derivation change if c is an arbitrary positive constant?

41. [*HM21*] Solve the previous exercise when $c = -1$.

42. [*HM23*] Use the saddle point method to estimate $[z^{n-1}]\,e^{z^2}$ with relative error $O(1/n^2)$.

43. [*HM22*] Justify replacing the integral in (23) by (25).

44. [*HM22*] Explain how to compute $b_1, b_2, \ldots$ in (26) from $a_2, a_3, \ldots$ in (25).

▶ **45.** [*HM23*] Show that, in addition to (26), we also have the expansion

$$\varpi_n = \frac{e^{e^\xi - 1}\, n!}{\xi^n \sqrt{2\pi n(\xi+1)}}\left(1 + \frac{b'_1}{n} + \frac{b'_2}{n^2} + \cdots + \frac{b'_m}{n^m} + O\Big(\frac{1}{n^{m+1}}\Big)\right),$$

where $b'_1 = -(2\xi^4 + 9\xi^3 + 16\xi^2 + 6\xi + 2)/(24(\xi+1)^3)$.

46. [*HM25*] Estimate the value of ϖ_{nk} in Peirce's triangle when $n \to \infty$.

47. [*M21*] Analyze the running time of Algorithm H.

48. [*HM25*] If n is not an integer, the integral in (23) can be taken over a Hankel contour to define a generalized Bell number ϖ_x for all real $x > 0$. Show that, as in (16),

$$\varpi_x \;=\; \frac{1}{e}\sum_{k=0}^{\infty}\frac{k^x}{k!}.$$

▶ **49.** [*HM35*] Prove that, for large n, the number ξ defined in Eq. (24) is equal to

$$\ln n - \ln\ln n + \sum_{j,k\ge 0}\left[{j+k \atop j+1}\right]\alpha^j\frac{\beta^k}{k!}, \qquad \alpha = -\frac{1}{\ln n}, \quad \beta = \frac{\ln\ln n}{\ln n}.$$

▶ **50.** [*HM21*] If $\xi(n)e^{\xi(n)} = n$ and $\xi(n) > 0$, how does $\xi(n+k)$ relate to $\xi(n)$?

51. [*HM27*] Use the saddle point method to estimate $t_n = n!\,[z^n]\,e^{z+z^2/2}$, the number of *involutions* on n elements (aka partitions of $\{1,\ldots,n\}$ into blocks of sizes ≤ 2).

52. [*HM22*] The *cumulants* of a probability distribution are defined in Eq. 1.2.10–(23). What are the cumulants, when the probability that a random integer equals k is (a) $e^{1-e^{\xi}}\varpi_k\xi^k/k!$? (b) $\sum_j \left\{{k \atop j}\right\} e^{\,e^{-1}-1-j}/k!$?

▶ **53.** [*HM30*] Let $G(z) = \sum_{k=0}^{\infty} p_k z^k$ be the generating function for a discrete probability distribution, converging for $|z| < 1+\delta$; thus the coefficients p_k are nonnegative, $G(1) = 1$, and the mean and variance are respectively $\mu = G'(1)$ and $\sigma^2 = G''(1) + G'(1) - G'(1)^2$. If $X_1, \ldots, X_n$ are independent random variables having this distribution, the probability that $X_1 + \cdots + X_n = m$ is $[z^m]\,G(z)^n$, and we often want to estimate this probability when m is near the mean value μn.

Assume that $p_0 \ne 0$ and that no integer $d > 1$ is a common divisor of all subscripts k with $p_k \ne 0$; this assumption means that m does not have to satisfy any special congruence conditions mod d when n is large. Prove that

$$[z^{\mu n+r}]\,G(z)^n \;=\; \frac{e^{-r^2/(2\sigma^2 n)}}{\sigma\sqrt{2\pi n}} + O\Big(\frac{1}{n}\Big) \qquad \text{as } n \to \infty,$$

when $\mu n + r$ is an integer. *Hint:* Integrate $G(z)^n/z^{\mu n+r}$ on the circle $|z| = 1$.

54. [*HM20*] If α and β are defined by (40), show that their arithmetic and geometric means are respectively $\frac{\alpha+\beta}{2} = s\coth s$ and $\sqrt{\alpha\beta} = s\,\mathrm{csch}\,s$, where $s = \sigma/2$.

55. [*HM20*] Suggest a good way to compute the number β needed in (43).

▶ **56.** [*HM26*] Let $g(z) = \alpha^{-1}\ln(e^z - 1) - \ln z$ and $\sigma = \alpha - \beta$ as in (37).

a) Prove that $(-\sigma)^{n+1}g^{(n+1)}(\sigma) = n! - \sum_{k=0}^{n}\left\langle{n \atop k}\right\rangle\alpha^k\beta^{n-k}$, where the Eulerian numbers $\left\langle{n \atop k}\right\rangle$ are defined in Section 5.1.3.

b) Prove that $\frac{\beta}{\alpha}n! < \sum_{k=0}^{n}\left\langle{n \atop k}\right\rangle\alpha^k\beta^{n-k} < n!$ for all $\sigma > 0$. *Hint:* See exercise 5.1.3–25.

c) Now verify the inequality (42).

57. [*HM22*] In the notation of (43), prove that (a) $n+1-m < 2N$; (b) $N < 2(n+1-m)$.

58. [*HM31*] Complete the proof of (43) as follows.

a) Show that for all $\sigma > 0$ there is a number $\tau \ge 2\sigma$ such that τ is a multiple of 2π and $|e^{\sigma+it} - 1|/|\sigma + it|$ is monotone decreasing for $0 \le t \le \tau$.

b) Prove that $\int_{-\tau}^{\tau}\exp\big((n+1)g(\sigma+it)\big)\,dt$ leads to (43).

c) Show that the corresponding integrals over the straight-line paths $z = t \pm i\tau$ for $-n \le t \le \sigma$ and $z = -n \pm it$ for $-\tau \le t \le \tau$ are negligible.

▶ **59.** [*HM23*] What does (43) predict for the approximate value of $\left\{{n \atop n}\right\}$?

60. [*HM25*] (a) Show that the partial sums in the identity

$$\left\{{n \atop m}\right\} = \frac{m^n}{m!} - \frac{(m-1)^n}{1!\,(m-1)!} + \frac{(m-2)^n}{2!\,(m-2)!} - \cdots + (-1)^m \frac{0^n}{m!\,0!}$$

alternately overestimate and underestimate the final value. (b) Conclude that

$$\left\{{n \atop m}\right\} = \frac{m^n}{m!}\bigl(1 - O(ne^{-n^\epsilon})\bigr) \qquad \text{when } m \le n^{1-\epsilon}.$$

(c) Derive a similar result from (43).

61. [*HM26*] Prove that if $m = n - r$ where $r \le n^\epsilon$ and $\epsilon \le n^{1/2}$, Eq. (43) yields

$$\left\{{n \atop n-r}\right\} = \frac{n^{2r}}{2^r\,r!}\Bigl(1 + O(n^{2\epsilon-1}) + O\Bigl(\frac{1}{r}\Bigr)\Bigr).$$

62. [*HM40*] Prove rigorously that if $\xi e^\xi = n$, the maximum $\left\{{n \atop m}\right\}$ occurs either when $m = \lfloor e^\xi - 1 \rfloor$ or when $m = \lceil e^\xi - 1 \rceil$.

▶ **63.** [*M35*] (J. Pitman.) Prove that there is an elementary way to locate the maximum Stirling numbers, and many similar quantities, as follows: Suppose $0 \le p_j \le 1$.

a) Let $f(z) = (1+p_1(z-1)) \ldots (1+p_n(1-z))$ and $a_k = [z^k]\,f(z)$; thus a_k is the probability that k heads turn up after n independent coin flips with the respective probabilities $p_1, \ldots, p_n$. Prove that $a_{k-1} < a_k$ whenever $k \le \mu = p_1 + \cdots + p_n$, $a_k \ne 0$.

b) Similarly, prove that $a_{k+1} < a_k$ whenever $k \ge \mu$ and $a_k \ne 0$.

c) If $f(x) = a_0 + a_1 x + \cdots + a_n x^n$ is any nonzero polynomial with nonnegative coefficients and with n real roots, prove that $a_{k-1} < a_k$ when $k \le \mu$ and $a_{k+1} < a_k$ when $k \ge \mu$, where $\mu = f'(1)/f(1)$. Therefore if $a_m = \max(a_0, \ldots, a_n)$ we must have either $m = \lfloor \mu \rfloor$ or $m = \lceil \mu \rceil$.

d) Under the hypotheses of (c), and with $a_j = 0$ when $j < 0$ or $j > n$, show that there are indices $s \le t$, such that $a_{k+1} - a_k < a_k - a_{k-1}$ if and only if $s \le k \le t$. (Thus, a histogram of the sequence $(a_0, a_1, \ldots, a_n)$ is always "bell-shaped.")

e) What do these results tell us about Stirling numbers?

64. [*HM21*] Prove the approximate ratio (50), using (30) and exercise 50.

▶ **65.** [*HM22*] What is the variance of the number of blocks of size k in a random partition of $\{1, \ldots, n\}$?

66. [*M46*] What partition of n leads to the most partitions of $\{1, \ldots, n\}$?

67. [*HM20*] What are the mean and variance of M in Stam's method (53)?

68. [*20*] How large can the stack get in Algorithm M, when it is generating all $p(n_1, \ldots, n_m)$ partitions of $\{n_1 \cdot 1, \ldots, n_m \cdot m\}$?

▶ **69.** [*21*] Modify Algorithm M so that it produces only partitions into at most r parts.

▶ **70.** [*M22*] Analyze the number of r-block partitions possible in the n-element multisets (a) $\{0, \ldots, 0, 1\}$; (b) $\{1, 2, \ldots, n-1, n-1\}$. What is the total, summed over r?

71. [*M20*] How many partitions of $\{n_1 \cdot 1, \ldots, n_m \cdot m\}$ have exactly 2 parts?

72. [*M26*] Can $p(n, n)$ be evaluated in polynomial time?

▶ **73.** [*M32*] Can $p(2, \ldots, 2)$ be evaluated in polynomial time when there are n 2s?

74. [*M46*] Can $p(n, \ldots, n)$ be evaluated in polynomial time when there are n ns?

75. [*HM41*] Find the asymptotic value of $p(n, n)$.

76. [*HM36*] Find the asymptotic value of $p(2, \ldots, 2)$ when there are n 2s.

77. [*HM46*] Find the asymptotic value of $p(n, \ldots, n)$ when there are n ns.

78. [*20*] What partition of $(15, 10, 10, 11)$ leads to the permutations α_1, α_2, α_3, and α_4 shown in Table 1?

79. [*22*] A sequence u_1, u_2, u_3, $\ldots$ is called *universal* for partitions of $\{1, \ldots, n\}$ if its subsequences $(u_{m+1}, u_{m+2}, \ldots, u_{m+n})$ for $0 \le m < \varpi_n$ represent all possible set partitions under the convention "$j \equiv k$ if and only if $u_{m+j} = u_{m+k}$." For example, $(0, 0, 0, 1, 0, 2, 2)$ is a universal sequence for partitions of $\{1, 2, 3\}$.

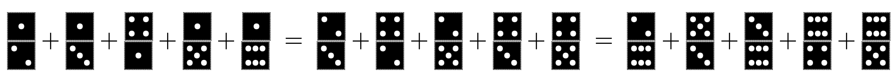
Write a program to find all universal sequences for partitions of $\{1, 2, 3, 4\}$ with the properties that (i) $u_1 = u_2 = u_3 = u_4 = 0$; (ii) the sequence has restricted growth; (iii) $0 \le u_j \le 3$; and (iv) $u_{16} = u_{17} = u_{18} = 0$ (hence the sequence is essentially *cyclic*).

80. [*M28*] Prove that universal cycles for partitions of $\{1, 2, \ldots, n\}$ exist in the sense of the previous exercise whenever $n \ge 4$.

81. [*29*] Find a way to arrange an ordinary deck of 52 playing cards so that the following trick is possible: Five players each cut the deck (applying a cyclic permutation) as often as they like. Then each player takes a card from the top. A magician tells them to look at their cards and to form affinity groups, joining with others who hold the same suit: Everybody with clubs gets together, everybody with diamonds forms another group, and so on. (The Jack of Spades is, however, considered to be a "joker"; its holder, if any, should remain aloof.)

Observing the affinity groups, but not being told any of the suits, the magician can name all five cards, if the cards were suitably arranged in the first place.

82. [*22*] In how many ways can the following 15 dominoes, optionally rotated, be partitioned into three sets of five having the same sum when regarded as fractions?



SECTION 7.2.1.4

1.

m^n	$m^{\underline{n}}$	$m!\left\{{n \atop m}\right\}$
$\binom{m+n-1}{n}$	$\binom{m}{n}$	$\binom{n-1}{n-m}$
$\left\{{n \atop 0}\right\}+\cdots+\left\{{n \atop m}\right\}$	$[m\geq n]$	$\left\{{n \atop m}\right\}$
$\left\lvert{m+n \atop m}\right\rvert$	$[m\geq n]$	$\left\lvert{n \atop m}\right\rvert$

2. In general, given any integers $x_1 \geq \cdots \geq x_m$, we obtain all integer m-tuples $a_1 \ldots a_m$ such that $a_1 \geq \cdots \geq a_m$, $a_1+\cdots+a_m = x_1+\cdots+x_m$, and $a_m \ldots a_1 \geq x_m \ldots x_1$ by initializing $a_1 \ldots a_m \leftarrow x_1 \ldots x_m$ and $a_{m+1} \leftarrow x_m - 2$. In particular, if c is any integer constant, we obtain all integer m-tuples such that $a_1 \geq \cdots \geq a_m \geq c$ and $a_1 + \cdots + a_m = n$ by initializing $a_1 \leftarrow n - mc + c$, $a_j \leftarrow c$ for $1 < j \leq m$, and $a_{m+1} \leftarrow c - 2$, assuming that $n \geq cm$.

3. $a_j = \lfloor (n+m-j)/m \rfloor = \lceil (n+1-j)/m \rceil$, for $1 \leq j \leq m$; see *CMath* §3.4.

4. We must have $a_m \geq a_1 - 1$; therefore $a_j = \lfloor (n+m-j)/m \rfloor$ for $1 \leq j \leq m$, where m is the largest integer with $\lfloor n/m \rfloor \geq r$, namely $m = \lfloor n/r \rfloor$.

5. [See Eugene M. Klimko, *BIT* **13** (1973), 38–49.]

C1. [Initialize.] Set $c_0 \leftarrow 1$, $c_1 \leftarrow n$, $c_2 \ldots c_n \leftarrow 0 \ldots 0$, $l_0 \leftarrow 1$, $l_1 \leftarrow 0$. (We assume that $n > 0$.)

C2. [Visit.] Visit the partition represented by part counts $c_1 \ldots c_n$ and links $l_0 l_1 \ldots l_n$.

C3. [Branch.] Set $j \leftarrow l_0$ and $k \leftarrow l_j$. If $c_j = 1$, go to C6; otherwise, if $j > 1$, go to C5.

C4. [Change 1+1 to 2.] Set $c_1 \leftarrow c_1 - 2$, $c_2 \leftarrow c_2 + 1$. Then if $c_1 = 0$, set $l_0 \leftarrow 2$, and set $l_2 \leftarrow l_1$ if $k \neq 2$. If $c_1 > 0$ and $k \neq 2$, set $l_2 \leftarrow l_1$ and $l_1 \leftarrow 2$. Return to C2.

C5. [Change $j \cdot c_j$ to $(j{+}1) + 1 + \cdots + 1$.] Set $c_1 \leftarrow j(c_j - 1) - 1$ and go to C7.

C6. [Change $k \cdot c_k + j$ to $(k{+}1) + 1 + \cdots + 1$.] Terminate if $k = 0$. Otherwise set $c_j \leftarrow 0$; then set $c_1 \leftarrow k(c_k - 1) + j - 1$, $j \leftarrow k$, and $k \leftarrow l_k$.

C7. [Adjust links.] If $c_1 > 0$, set $l_0 \leftarrow 1$, $l_1 \leftarrow j + 1$; otherwise set $l_0 \leftarrow j + 1$. Then set $c_j \leftarrow 0$ and $c_{j+1} \leftarrow c_{j+1} + 1$. If $k \neq j + 1$, set $l_{j+1} \leftarrow k$. Return to C2. ▌

Notice that this algorithm is *loopless*; but it isn't really faster than Algorithm P. Steps C4, C5, and C6 are performed respectively $p(n-2)$, $2p(n) - p(n+1) - p(n-2)$, and $p(n+1) - p(n)$ times; thus step C4 is most important when n is large. (See exercise 45 and the detailed analysis by Fenner and Loizou in *Acta Inf.* **16** (1981), 237–252.)

6. Set $k \leftarrow a_1$ and $j \leftarrow 1$. Then, while $k > a_{j+1}$, set $b_k \leftarrow j$ and $k \leftarrow k - 1$ until $k = a_{j+1}$. If $k > 0$, set $j \leftarrow j + 1$ and repeat until $k = 0$. (We have used (11) in the dual form $a_j - a_{j+1} = d_j$, where $d_1 \ldots d_n$ is the part-count representation of $b_1 b_2 \ldots$. Notice that the running time of this algorithm is essentially proportional to $a_1 + b_1$, the length of the output plus the length of the input.)

7. We have $b_1 \ldots b_n = n^{a_n}(n{-}1)^{a_{n-1}-a_n} \ldots 1^{a_1-a_2} 0^{n-a_1}$, by the dual of (11).

8. Transposing the Ferrers diagram corresponds to reflecting and complementing the bit string (15). So we simply interchange and reverse the p's and q's, getting the partition $b_1 b_2 \ldots = (q_t + \cdots + q_1)^{p_1}(q_t + \cdots + q_2)^{p_2} \ldots (q_t)^{p_t}$.

9. By induction: If $a_k = l-1$ and $b_l = k-1$, increasing a_k and b_l preserves equality.

10. (a) The left child of each node is obtained by appending '1'. The right child is obtained by increasing the rightmost digit; this child exists if and only if the parent node ends with unequal digits. All partitions of n appear on level n in lexicographic order.

(b) The left child is obtained by changing '11' to '2'; it exists if and only if the parent node contains at least two 1s. The right child is obtained by deleting a 1 and increasing the smallest part that exceeds 1; it exists if and only if there is at least one 1 and the smallest larger part appears exactly once. All partitions of n into m parts appear on level $n-m$ in lexicographic order; preorder of the entire tree gives lexicographic order of the whole. [T. I. Fenner and G. Loizou, *Comp. J.* **23** (1980), 332–337.]

11. $[z^{100}]\,1/((1-z)(1-z^2)(1-z^5)(1-z^{10})(1-z^{20})(1-z^{50})(1-z^{100})) = 4563$; and $[z^{100}]\,(1+z+z^2)(1+z^2+z^4)\ldots(1+z^{100}+z^{200}) = 7$. [See G. Pólya, *AMM* **63** (1956), 689–697.] In the infinite series $\prod_{k\geq 1}(1+z^k+z^{2k})(1+z^{2k}+z^{4k})(1+z^{5k}+z^{10k})$, the coefficient of z^{10^n} is $2^{n+1}-1$, and the coefficient of z^{10^n-1} is 2^n.

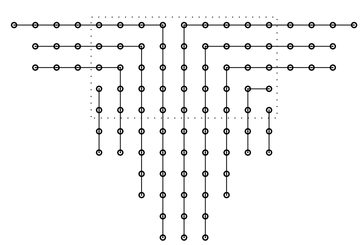
12. To prove that $(1+z)(1+z^2)(1+z^3)\ldots = 1/((1-z)(1-z^3)(1-z^5)\ldots)$, write the left-hand side as

$$\frac{(1-z^2)}{(1-z)}\,\frac{(1-z^4)}{(1-z^2)}\,\frac{(1-z^6)}{(1-z^3)}\ldots$$

and cancel common factors from numerator and denominator. Alternatively, replace z by z^1, z^3, z^5, … in the identity $(1+z)(1+z^2)(1+z^4)(1+z^8)\ldots = 1/(1-z)$ and multiply the results together. [*Novi Comment. Acad. Sci. Pet.* **3** (1750), 125–169, §47.]

13. Map the partition $c_1{\cdot}1 + c_2{\cdot}2 + \cdots$ into $\lfloor c_1/2\rfloor{\cdot}2 + \lfloor c_2/2\rfloor{\cdot}4 + \cdots + r_1{\cdot}1 + r_3{\cdot}3 + \cdots$, where $r_m = (c_m \bmod 2) + 2(c_{2m} \bmod 2) + 4(c_{4m} \bmod 2) + \cdots$. [*Johns Hopkins Univ. Circular* **2** (1882), 72.]

14. Sylvester's correspondence is best understood as a diagram in which the dots of the odd permutation are centered and divided into disjoint hooks. For example, the partition $17+15+15+9+9+9+9+5+5+3+3$, having five different odd parts, corresponds via the diagram



to the all-distinct partition $19+18+16+13+12+9+5+4+3$ with four gaps.

Conversely, a partition into $2t$ distinct nonnegative parts can be written uniquely in the form $(a_1{+}b_1{-}1)+(a_1{+}b_2{-}2)+(a_2{+}b_2{-}3)+(a_2{+}b_3{-}4)+\cdots+(a_{t-1}{+}b_t{-}2t{+}2)+(a_t{+}b_t{-}2t{+}1)+(a_t{+}b_{t+1}{-}2t)$ where $a_1 \geq a_2 \geq \cdots \geq a_t \geq t$ and $b_1 \geq b_2 \geq \cdots \geq b_t \geq b_{t+1} = t$. It corresponds to $(2a_1{-}1)+\cdots+(2a_t-1)+(2A_1-1)+\cdots+(2A_r-1)$, where $A_1+\cdots+A_r$ is the conjugate of $(b_1{-}t)+\cdots+(b_t{-}t)$. The value of t is essentially the size of a "Durfee rectangle."

The relevant odd-parts partitions when $n = 10$ are $9+1$, $7+3$, $7+1+1+1$, $5+5$, $5+3+1+1$, $5+1+1+1+1+1$, $3+3+3+1$, $3+3+1+1+1+1$, $3+1+\cdots+1$, $1+\cdots+1$, corresponding respectively to the distinct-parts partitions $6+4$, $5+4+1$,

$7+3$, $4+3+2+1$, $6+3+1$, $8+2$, $5+3+2$, $7+2+1$, $9+1$, 10. [See Sylvester's remarkable paper in *Amer. J. Math.* **5** (1882), 251–330; **6** (1883), 334–336.]

15. Every self-conjugate partition of trace k corresponds to a partition of n into k distinct odd parts ("hooks"). Therefore we can write the generating function either as the product $(1+z)(1+z^3)(1+z^5)\ldots$ or as the sum $1+z^1/(1-z^2)+z^4/((1-z^2)(1-z^4))+z^9/((1-z^2)(1-z^4)(1-z^6))+\cdots$. [*Johns Hopkins Univ. Circular* **3** (1883), 42–43.]

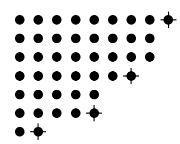
16. The Durfee square contains k^2 dots, and the remaining dots correspond to two independent partitions with largest part $\leq k$. Thus, if we use w to count parts and z to count dots, we find

$$\prod_{m=1}^{\infty}\frac{1}{1-wz^m} \;=\; \sum_{k=0}^{\infty}\frac{w^k z^{k^2}}{(1-z)(1-z^2)\ldots(1-z^k)(1-wz)(1-wz^2)\ldots(1-wz^k)}.$$

[This impressive-looking formula turns out to be just the special case $x=y=0$ of the even more impressive identity of exercise 19.]

17. (a) $\bigl((1+uvz)(1+uvz^2)(1+uvz^3)\ldots\bigr)/\bigl((1-uz)(1-uz^2)(1-uz^3)\ldots\bigr)$.

(b) A joint partition can be represented by a generalized Ferrers diagram in which we merge all the parts together, putting a_i above b_j if $a_i \geq b_j$, then mark the rightmost dot of each b_j. For example, the joint partition $(8,8,5;\ 9,7,5,2)$ has the diagram illustrated here, with marked dots shown as '✦'. Marks appear only in corners; thus the transposed diagram corresponds to another joint partition, which in this case is $(7,6,6,4,3;\ 7,6,4,1)$. [See J. T. Joichi and D. Stanton, *Pacific J. Math.* **127** (1987), 103–120; S. Corteel and J. Lovejoy, *Trans. Amer. Math. Soc.* **356** (2004), 1623–1635; Igor Pak, "Partition bijections, a survey," to appear in *The Ramanujan Journal*.)



Every joint partition with $t>0$ parts corresponds in this way to a "conjugate" in which the largest part is t. And the generating function for such joint partitions is $\bigl((1+vz)\ldots(1+vz^{t-1})\bigr)/\bigl((1-z)\ldots(1-z^t)\bigr)$ times (vz^t+z^t), where vz^t corresponds to the case that $b_1=t$, and z^t corresponds to the case that $r=0$ or $b_1<t$).

(c) Thus we obtain a form of the general z-nomial theorem in answer 1.2.6–58:

$$\frac{(1+uvz)}{(1-uz)}\frac{(1+uvz^2)}{(1-uz^2)}\frac{(1+uvz^3)}{(1-uz^3)}\ldots \;=\; \sum_{t=0}^{\infty}\frac{(1+v)}{(1-z)}\frac{(1+vz)}{(1-z^2)}\ldots\frac{(1+vz^{t-1})}{(1-z^t)}\,u^t z^t.$$

18. The equations obviously determine the a's and b's when the c's and d's are given, so we want to show that the c's and d's are uniquely determined from the a's and b's. The following algorithm determines the c's and d's from right to left:

A1. [Initialize.] Set $i \leftarrow r$, $j \leftarrow s$, $k \leftarrow 0$, and $a_0 \leftarrow b_0 \leftarrow \infty$.

A2. [Branch.] Stop if $i+j=0$. Otherwise go to A4 if $a_i \geq b_j - k$.

A3. [Absorb a_i.] Set $c_{i+j} \leftarrow a_i$, $d_{i+j} \leftarrow 0$, $i \leftarrow i-1$, $k \leftarrow k+1$, and return to A2.

A4. [Absorb b_j.] Set $c_{i+j} \leftarrow b_j - k$, $d_{i+j} \leftarrow 1$, $j \leftarrow j-1$, $k \leftarrow k+1$, and return to A2. ▮

There's also a left-to-right method:

B1. [Initialize.] Set $i \leftarrow 1$, $j \leftarrow 1$, $k \leftarrow r+s$, and $a_{r+1} \leftarrow b_{s+1} \leftarrow -\infty$.

B2. [Branch.] Stop if $k=0$. Otherwise set $k \leftarrow k-1$, then go to B4 if $a_i \leq b_j - k$.

B3. [Absorb a_i.] Set $c_{i+j-1} \leftarrow a_i$, $d_{i+j-1} \leftarrow 0$, $i \leftarrow i+1$, and return to B2.

B4. [Absorb b_j.] Set $c_{i+j-1} \leftarrow b_j - k$, $d_{i+j-1} \leftarrow 1$, $j \leftarrow j+1$, and return to B2. ▮

In both cases the branching is forced and the resulting sequence satisfies $c_1 \geq \cdots \geq c_{r+s}$. Notice that $c_{r+s} = \min(a_r, b_s)$ and $c_1 = \max(a_1, b_1{-}r{-}s{+}1)$.

We have thereby proved the identity of exercise 17(c) in a different way. Extensions of this idea lead to a combinatorial proof of Ramanujan's "remarkable formula with many parameters,"

$$\sum_{n=-\infty}^{\infty} w^n \prod_{k=0}^{\infty} \frac{1-bz^{k+n}}{1-az^{k+n}} = \prod_{k=0}^{\infty} \frac{(1-a^{-1}bz^k)(1-a^{-1}w^{-1}z^{k+1})(1-awz^k)(1-z^{k+1})}{(1-a^{-1}bw^{-1}z^k)(1-a^{-1}z^{k+1})(1-az^k)(1-wz^k)}.$$

[*References:* G. H. Hardy, *Ramanujan* (1940), Eq. (12.12.2); D. Zeilberger, *Europ. J. Combinatorics* **8** (1987), 461–463; A. J. Yee, *J. Comb. Theory* **A105** (2004), 63–77.]

19. [*Crelle* **34** (1847), 285–328.] By exercise 17(c), the hinted sum over k is

$$\Bigl(\sum_{l\geq 0} v^l \frac{(z-bz)\ldots(z-bz^l)}{(1-z)\ldots(1-z^l)} \frac{(1-uz)\ldots(1-uz^l)}{(1-auz)\ldots(1-auz^l)}\Bigr) \cdot \prod_{m=1}^{\infty} \frac{1-auz^m}{1-uz^m};$$

and the sum over l is similar but with $u \leftrightarrow v$, $a \leftrightarrow b$, $k \leftrightarrow l$. Furthermore the sum over both k and l reduces to

$$\prod_{m=1}^{\infty} \frac{(1-uvz^{m+1})(1-auz^m)}{(1-uz^m)(1-vz^m)}$$

when $b = auz$. Now let $u = wxy$, $v = 1/(yz)$, $a = 1/x$, and $b = wyz$; equate this infinite product to the sum over l.

20. To get $p(n)$ we need to add or subtract approximately $\sqrt{8n/3}$ of the previous entries, and most of those entries are $\Theta(\sqrt{n}\,)$ bits long. Therefore $p(n)$ is computed in $\Theta(n)$ steps and the total time is $\Theta(n^2)$.

(A straightforward use of (17) would take $\Theta(n^{5/2})$ steps.)

21. Since $\sum_{n=0}^{\infty} q(n)z^n = (1+z)(1+z^2)\ldots$ is equal to $(1-z^2)(1-z^4)\ldots P(z) = (1-z^2-z^4+z^{10}+z^{14}-z^{24}-\cdots)P(z)$, we have

$$q(n) = p(n) - p(n-2) - p(n-4) + p(n-10) + p(n-14) - p(n-24) - \cdots.$$

[There is also a "pure recurrence" in the q's alone, analogous to the recurrence for $\sigma(n)$ in the next exercise.]

22. From (21) we have $\sum_{n=1}^{\infty} \sigma(n)z^n = \sum_{m,n\geq 1} mz^{mn} = z\frac{d}{dz}\ln P(z) = (z{+}2z^2{-}5z^5{-}7z^7+\cdots)/(1-z-z^2+z^5+z^7+\cdots)$. [*Bibliothèque Impartiale* **3** (1751), 10–31.]

23. Set $u = w$ and $v = z/w$ to get

$$\begin{aligned}\prod_{k=1}^{\infty}(1-z^kw)(1-z^k/w)(1-z^k) &= \sum_{n=-\infty}^{\infty} (-1)^n w^n z^{n(n+1)/2}/(1-w)\\ &= \sum_{n=0}^{\infty}(-1)^n(w^{-n}-w^{n+1})z^{n(n+1)/2}/(1-w)\\ &= \sum_{n=0}^{\infty}(-1)^n(w^{-n}+\cdots+w^n)z^{n(n+1)/2}.\end{aligned}$$

These manipulations are legitimate when $|z| < 1$ and w is near 1. Now set $w = 1$.

[See §57 of Sylvester's paper cited in answer 14. Jacobi's proof is in §66 of his monograph *Fundamenta Nova Theoriæ Functionum Ellipticarum* (1829).]

24. (a) By (18) and exercise 23, $[z^n]\,A(z)=\sum(-1)^{j+k}(2k+1)[3j^2+j+k^2+k=2n]$, summed over all integers j and k. When $n \bmod 5 = 4$, the contributions all have $j \bmod 5 = 4$ and $k \bmod 5 = 2$; but then $(2k+1) \bmod 5 = 0$.

(b) $B(z)^p \equiv B(z^p)$ (modulo p) when p is prime, by Eq. 4.6.2–(5).

(c) Take $B(z) = P(z)$, since $A(z) = P(z)^{-4}$. [*Proc. Cambridge Philos. Soc.* **19** (1919), 207–210. A similar proof shows that $p(n)$ is a multiple of 7 when $n \bmod 7 = 5$. Ramanujan went on to obtain the beautiful formulas $p(5n+4)/5 = [z^n]\,P(z)^6/P(z^5)$; $p(7n+5)/7 = [z^n]\bigl(P(z)^4/P(z^7)^3 + 7zP(z)^8/P(z^7)^7\bigr)$. Atkin and Swinnerton-Dyer, in *Proc. London Math. Soc.* (3) **4** (1953), 84–106, showed that the partitions of $5n+4$ and $7n+5$ can be divided into equal-size classes according to the respective values of (largest part − number of parts) mod 5 or mod 7, as conjectured by F. Dyson. A slightly more complicated combinatorial statistic proves also that $p(n) \bmod 11 = 0$ when $n \bmod 11 = 6$; see F. G. Garvan, *Trans. Amer. Math. Soc.* **305** (1988), 47–77.]

25. [The hint can be proved by differentiating both sides of the stated identity. It is the special case $y = 1-x$ of a beautiful formula discovered by N. H. Abel in 1826:

$$\mathrm{Li}_2(x)+\mathrm{Li}_2(y)=\mathrm{Li}_2\Bigl(\frac{x}{1-y}\Bigr)+\mathrm{Li}_2\Bigl(\frac{y}{1-x}\Bigr)-\mathrm{Li}_2\Bigl(\frac{xy}{(1-x)(1-y)}\Bigr)-\ln(1-x)\ln(1-y).$$

See Abel's *Œuvres Complètes* **2** (Christiania: Grøndahl, 1881), 189–193.]

(a) Let $f(x)=\ln(1/(1-e^{-xt}))$. Then $\int_1^x f(x)\,dx = -\mathrm{Li}_2(e^{-tx})/t$ and $f^{(n)}(x) = (-t)^n e^{tx}\sum_k \bigl\langle {n-1 \atop k} \bigr\rangle e^{ktx}/(e^{tx}-1)^n$, so Euler's summation formula gives $\mathrm{Li}_2(e^{-t})/t + \frac12\ln(1/(1-e^{-t})) + O(1) = (\zeta(2)+t\ln(1-e^{-t})-\mathrm{Li}_2(1-e^{-t}))/t - \frac12\ln t + O(1) = \zeta(2)/t + \frac12\ln t + O(1)$, as $t\to 0$.

(b) We have $\sum_{m,n\ge1} e^{-mnt}/n = \frac{1}{2\pi i}\sum_{m,n\ge1}\int_{1-i\infty}^{1+i\infty}(mnt)^{-z}\Gamma(z)\,dz/n$, which sums to $\frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty}\zeta(z+1)\zeta(z)t^{-z}\Gamma(z)\,dz$. The pole at $z=1$ gives $\zeta(2)/t$; the double pole at $z=0$ gives $-\zeta(0)\ln t + \zeta'(0) = \frac12\ln t - \frac12\ln 2\pi$; the pole at $z=-1$ gives $-\zeta(-1)\zeta(0)t = B_2B_1t = -t/24$. Zeros of $\zeta(z+1)\zeta(z)$ cancel the other poles of $\Gamma(z)$, so the result is $\ln P(e^{-t}) = \zeta(2)/t + \frac12\ln(t/2\pi) - t/24 + O(t^M)$ for arbitrarily large M.

26. Let $F(n)=\sum_{k=1}^{\infty} e^{-k^2/n}$. We can use (25) either with $f(x)=e^{-x^2/n}[x>0]+\frac12\delta_{x0}$, or with $f(x)=e^{-x^2/n}$ for all x because $2F(n)+1=\sum_{k=-\infty}^{\infty}e^{-k^2/n}$. Let's choose the latter alternative; then the right-hand side of (25), for $\theta=0$, is the rapidly convergent

$$\lim_{M\to\infty}\sum_{m=-M}^{M}\int_{-\infty}^{\infty}e^{-2\pi miy-y^2/n}\,dy \;=\; \sum_{m=-\infty}^{\infty}e^{-\pi^2m^2n^2}\int_{-\infty}^{\infty}e^{-u^2/n}\,du$$

if we substitute $u=y+\pi mni$; and the integral is $\sqrt{\pi n}$. [This result is formula (15) on page 420 of Poisson's original paper.]

27. Let $g_n = \sqrt{\pi/6t}\,e^{-n^2\pi^2/6t}\cos\frac{n\pi}{6}$. Then $\int_{-\infty}^{\infty} f(y)\cos 2\pi my\,dy = g_{2m+1}+g_{2m-1}$, so we have

$$\frac{e^{-t/24}}{P(e^{-t})} \;=\; g_1+g_{-1}+2\sum_{m=1}^{\infty}(g_{2m+1}+g_{2m-1}) \;=\; 2\sum_{m=-\infty}^{\infty}g_{2m+1}.$$

The terms g_{6n+1} and g_{-6n-1} combine to give the nth term of (30). [See M. I. Knopp, *Modular Functions in Analytic Number Theory* (1970), Chapter 3.]

28. (a,b,c,d) See *Trans. Amer. Math. Soc.* **43** (1938), 271–295. In fact, Lehmer found explicit formulas for $A_{p^e}(n)$, in terms of the Jacobi symbol of exercise 4.5.4–23:

$$A_{2^e}(n) = (-1)^e\Big(\frac{-1}{m}\Big)2^{e/2}\sin\frac{4\pi m}{2^{e+3}}, \qquad \text{if } (3m)^2 \equiv 1-24n \text{ (modulo } 2^{e+3});$$

$$A_{3^e}(n) = (-1)^{e+1}\Big(\frac{m}{3}\Big)\frac{2}{\sqrt{3}}3^{e/2}\sin\frac{4\pi m}{3^{e+1}}, \quad \text{if } (8m)^2 \equiv 1-24n \text{ (modulo } 3^{e+1});$$

$$A_{p^e}(n) = \begin{cases} 2\Big(\dfrac{3}{p^e}\Big)p^{e/2}\cos\dfrac{4\pi m}{p^e}, & \text{if } (24m)^2 \equiv 1-24n \text{ (modulo } p^e),\ p \ge 5, \\ & \quad\text{and } 24n \bmod p \ne 1; \\ \Big(\dfrac{3}{p^e}\Big)p^{e/2}[e=1], & \text{if } 24n \bmod p = 1 \text{ and } p \ge 5. \end{cases}$$

(e) If $n = 2^a 3^b p_1^{e_1}\dots p_t^{e_t}$ for $3 < p_1 < \cdots < p_t$ and $e_1\dots e_t \ne 0$, the probability that $A_k(n) \ne 0$ is $2^{-t}(1+(-1)^{[e_1=1]}/p_1)\dots(1+(-1)^{[e_t=1]}/p_t)$.

29. $z_1z_2\dots z_m/((1-z_1)(1-z_1z_2)\dots(1-z_1z_2\dots z_m))$.

30. (a) $\left|{n+1 \atop m}\right|$ and (b) $\left|{m+n \atop m}\right|$, by (39).

31. *First solution* [Marshall Hall, Jr., *Combinatorial Theory* (1967), §4.1]: From the recurrence (39), we can show directly that, for $0 \le r < k!$, there is a polynomial $f_{k,r}(n) = n^{k-1}/(k!(k-1)!) + O(n^{k-2})$ such that $\left|{n \atop k}\right| = f_{n,n \bmod k!}(n)$.

Second solution: Since $(1-z)\dots(1-z^m) = \prod_{p\perp q}(1-e^{2\pi ip/q}z)^{\lfloor m/q\rfloor}$, where the product is over all reduced fractions p/q with $0 \le p < q$, the coefficient of z^n in (41) can be expressed as a sum of roots of unity times polynomials in n, namely as $\sum_{p\perp q} e^{2\pi ipn/q} f_{pq}(n)$ where $f_{pq}(n)$ is a polynomial of degree less than m/q. Thus there exist constants such that $\left|{n \atop 2}\right| = a_1n + a_2 + (-1)^n a_3$; $\left|{n \atop 3}\right| = b_1n^2 + b_2n + b_3 + (-1)^n b_4 + \omega^n b_5 + \omega^{-n} b_6$, where $\omega = e^{2\pi i/3}$; etc. The constants are determined by the values for small n, and the first two cases are

$$\left|{n \atop 2}\right| = \frac{1}{2}n - \frac{1}{4} + \frac{1}{4}(-1)^n; \qquad \left|{n \atop 3}\right| = \frac{1}{12}n^2 - \frac{7}{72} - \frac{1}{8}(-1)^n + \frac{1}{9}\omega^n + \frac{1}{9}\omega^{-n}.$$

It follows that $\left|{n \atop 3}\right|$ is the nearest integer to $n^2/12$. Similarly, $\left|{n \atop 4}\right|$ is the nearest integer to $(n^3 + 3n^2 - 9n\,[n \text{ odd}])/144$.

[Exact formulas for $\left|{n \atop 2}\right|$, $\left|{n \atop 3}\right|$, and $\left|{n \atop 4}\right|$, without the simplification of floor functions, were first found by G. F. Malfatti, *Memorie di Mat. e Fis. Società Italiana* **3** (1786), 571–663. W. J. A. Colman, in *Fibonacci Quarterly* **21** (1983), 272–284, showed that $\left|{n \atop 5}\right|$ is the nearest integer to $(n^4+10n^3+10n^2-75n-45n(-1)^n)/2880$, and gave similar formulas for $\left|{n \atop 6}\right|$ and $\left|{n \atop 7}\right|$.]

32. Since $\left|{m+n \atop m}\right| \le p(n)$, with equality if and only if $m \ge n$, we have $\left|{n \atop m}\right| \le p(n-m)$ with equality if and only if $2m \ge n$.

33. A partition into m parts corresponds to at most $m!$ compositions; hence $\binom{n-1}{m-1} \le m!\left|{n \atop m}\right|$. Consequently $p(n) \ge (n-1)!/((n-m)!\,m!\,(m-1)!)$, and when $m = \sqrt{n}$ Stirling's approximation proves that $\ln p(n) \ge 2\sqrt{n} - \ln n - \frac{1}{2} - \ln 2\pi$.

34. $a_1 > a_2 > \cdots > a_m > 0$ if and only if $a_1-m+1 \ge a_2-m+2 \ge \cdots \ge a_m \ge 1$. And partitions into m distinct parts correspond to $m!$ compositions. Thus, by the previous answer, we have

$$\frac{1}{m!}\binom{n-1}{m-1} \le \left|{n \atop m}\right| \le \frac{1}{m!}\binom{n+m(m-1)/2}{m-1}.$$

[See H. Gupta, *Proc. Indian Acad. Sci.* **A16** (1942), 101–102. A detailed asymptotic formula for $\left|{n \atop m}\right|$ when $n = \Theta(m^3)$ appears in exercise 3.3.2–30.]

35. (a) $x = \frac{1}{C}\ln\frac{1}{C} \approx -0.194$.

(b) $x = \frac{1}{C}\ln\frac{1}{C} - \frac{1}{C}\ln\ln 2 \approx 0.092$; in general we have $x = \frac{1}{C}\bigl(\ln\frac{1}{C} - \ln\ln\frac{1}{F(x)}\bigr)$.

(c) $\int_{-\infty}^{\infty} x\,dF(x) = \int_0^\infty (Cu)^{-2}(\ln u)e^{-1/(Cu)}\,du = -\frac{1}{C}\int_0^\infty(\ln C + \ln v)e^{-v}\,dv = (\gamma - \ln C)/C \approx 0.256$.

(d) Similarly, $\int_{-\infty}^{\infty} x^2 e^{-Cx}\exp(-e^{-Cx}/C)\,dx = (\gamma^2 + \zeta(2) - 2\gamma\ln C + (\ln C)^2)/C^2 \approx 1.0656$. So the variance is $\zeta(2)/C^2 = 1$, exactly(!).

[The probability distribution $e^{-e^{(a-x)/b}}$ is commonly called the Fisher–Tippett distribution; see *Proc. Cambridge Phil. Soc.* **24** (1928), 180–190.]

36. The sum over $j_r - (m+r-1) \ge \cdots \ge j_2 - (m+1) \ge j_1 - m \ge 1$ gives

$$\begin{aligned}\Sigma_r &= \sum_t \left|{t - rm - r(r-1)/2 \atop r}\right| \frac{p(n-t)}{p(n)}\\ &= \frac{\alpha}{1-\alpha}\frac{\alpha^2}{1-\alpha^2}\cdots\frac{\alpha^r}{1-\alpha^r}\alpha^{rm}\bigl(1 + O(n^{-1/2+2\epsilon})\bigr) + E\\ &= \frac{n^{-1/2}}{\alpha^{-1}-1}\frac{n^{-1/2}}{\alpha^{-2}-1}\cdots\frac{n^{-1/2}}{\alpha^{-r}-1}\exp\bigl(-Crx + O(rn^{-1/2+2\epsilon})\bigr) + E,\end{aligned}$$

where E is an error term that accounts for the cases $t > n^{1/2+\epsilon}$. The leading factor $n^{-1/2}/(\alpha^{-j}-1)$ is $\frac{1}{jC}\bigl(1+O(jn^{-1/2})\bigr)$. And it is easy to verify that $E = O(n^{\log n}e^{-Cn^\epsilon})$, even if we use the crude upper bound $\left|{t-rm-r(r-1)/2 \atop r}\right| \le t^r$, because

$$\sum_{t \ge xN} t^r e^{-t/N} = O\Bigl(\int_{xN}^\infty t^r e^{-t/N}\,dt\Bigr) = O\bigl(N^{r+1}x^r e^{-x}/(1 - r/x)\bigr),$$

where $N = \Theta(\sqrt{n})$, $x = \Theta(n^\epsilon)$, $r = O(\log n)$.

37. Such a partition is counted once in Σ_0, q times in Σ_1, $\binom{q}{2}$ times in Σ_2, ...; so it is counted exactly $\sum_{j=0}^r(-1)^j\binom{q}{j} = (-1)^r\binom{q-1}{r}$ times in the partial sum that ends with $(-1)^r\Sigma_r$. This count is at most δ_{q0} when r is odd, at least δ_{q0} when r is even. [A similar argument shows that the generalized principle of exercise 1.3.3–26 also has this bracketing property. *Reference:* C. Bonferroni, *Pubblicazioni del Reale Istituto Superiore de Scienze Economiche e Commerciale di Firenze* **8** (1936), 3–62.]

38. $z^{l+m-1}\binom{l+m-2}{m-1}_z = z^{l+m-1}(1-z^l)\ldots(1-z^{l+m-2})/((1-z)\ldots(1-z^{m-1}))$.

39. If $\alpha = a_1\ldots a_m$ is a partition with at most m parts, let $f(\alpha) = \infty$ if $a_1 \le l$, otherwise $f(\alpha) = \min\{j \mid a_1 > l + a_{j+1}\}$. Let g_k be the generating function for partitions with $f(\alpha) > k$. Partitions with $f(\alpha) = k < \infty$ are characterized by the inequalities

$$a_1 \ge a_2 \ge \cdots \ge a_k \ge a_1 - l > a_{k+1} \ge \cdots \ge a_{m+1} = 0.$$

Thus $a_1a_2\ldots a_m = (b_k+l+1)(b_1+1)\ldots(b_{k-1}+1)b_{k+1}\ldots b_m$, where $f(b_1\ldots b_m) \ge k$; and the converse is also true. It follows that $g_k = g_{k-1} - z^{l+k}g_{k-1}$.

[See *American J. Math.* **5** (1882), 254–257.]

40. $z^{m(m+1)/2}\binom{l}{m}_z = (z-z^l)(z^2-z^l)\ldots(z^m-z^l)/((1-z)(1-z^2)\ldots(1-z^m))$. This formula is essentially the z-nomial theorem of exercise 1.2.6–58.

41. See G. Almkvist and G. E. Andrews, *J. Number Theory* **38** (1991), 135–144.

42. A. Vershik [*Functional Anal. Applic.* **30** (1996), 90–105, Theorem 4.7] has stated the formula
$$\frac{1-e^{-c\varphi}}{1-e^{-c(\theta+\varphi)}}e^{-ck/\sqrt{n}} + \frac{1-e^{-c\theta}}{1-e^{-c(\theta+\varphi)}}e^{-ca_k/\sqrt{n}} \approx 1,$$
where the constant c must be chosen as a function of θ and φ so that the area of the shape is n. This constant c is negative if $\theta\varphi < 2$, positive if $\theta\varphi > 2$; the shape reduces to a straight line
$$\frac{k}{\theta\sqrt{n}} + \frac{a_k}{\varphi\sqrt{n}} \approx 1$$
when $\theta\varphi = 2$. If $\varphi = \infty$ we have $c = \sqrt{\mathrm{Li}_2(t)}$ where t satisfies $\theta = (\ln\frac{1}{1-t})/\sqrt{\mathrm{Li}_2(t)}$.

43. We have $a_1 > a_2 > \cdots > a_k$ if and only if the conjugate partition includes each of the the parts 1, 2, $\ldots$, $k-1$ at least once. The number of such partitions is $p(n-k(k-1)/2)$; this total includes $\left|{n-(k-1)(k-2)/2 \atop k-1}\right|$ cases with $a_k = 0$.

44. Assume that $n > 0$. The number with smallest parts *unequal* (or with only one part) is $p(n+1)-p(n)$, the number of partitions of $n+1$ that don't end in 1, because we get the former from the latter by changing the smallest part. Therefore the answer is $2p(n)-p(n+1)$. [See R. J. Boscovich, *Giornale de' Letterati* (Rome, 1748), 15. The number of partitions whose smallest *three* parts are equal is $3p(n)-p(n+1)-2p(n+2)+p(n+3)$; similar formulas can be derived for other constraints on the smallest parts.]

45. By Eq. (37) we have $p(n-j)/p(n) = 1 - Cjn^{-1/2} + (C^2j^2+2j)/(2n) - (8C^3j^3 + 60Cj^2 + Cj + 12C^{-1}j)/(48n^{3/2}) + O(j^4n^{-2})$.

46. If $n > 1$, $T_2'(n) = p(n-1) - p(n-2) \le p(n) - p(n-1) = T_2''(n)$, because $p(n)-p(n-1)$ is the number of partitions of n that don't end in 1; every such partition of $n-1$ yields one for n if we increase the largest part. But the difference is rather small: $\bigl(T_2''(n) - T_2'(n)\bigr)/p(n) = C^2/n + O(n^{-3/2})$.

47. The identity in the hint follows by differentiating (21); see exercise 22. The probability of obtaining the part-counts $c_1 \ldots c_n$ when $c_1 + 2c_2 + \cdots + nc_n = n$ is
$$\begin{aligned}\Pr(c_1\ldots c_n) &= \sum_{k=1}^{n}\sum_{j=1}^{c_k}\frac{kp(n-jk)}{np(n)}\Pr(c_1\ldots c_{k-1}(c_k{-}j)c_{k+1}\ldots c_n)\\ &= \sum_{k=1}^{n}\sum_{j=1}^{c_k}\frac{k}{np(n)} = \frac{1}{p(n)},\end{aligned}$$
by induction on n. [*Combinatorial Algorithms* (Academic Press, 1975), Chapter 10.]

48. The probability that j has a particular fixed value in step N5 is $6/(\pi^2j^2) + O(n^{-1/2})$, and the average value of jk is order $\sqrt{n}$. The average time spent in step N4 is $\Theta(n)$, so the average running time is of order $n^{3/2}$. (A more precise analysis would be desirable.)

49. (a) We have $F(z) = \sum_{k=1}^{\infty} F_k(z)$, where $F_k(z)$ is the generating function for all partitions whose smallest part is $\ge k$, namely $1/((1-z^k)(1-z^{k+1})\ldots) - 1$.

(b) Let $f_k(n) = [z^n]\,F_k(z)/p(n)$. Then $f_1(n) = 1$; $f_2(n) = 1 - p(n{-}1)/p(n) = Cn^{-1/2} + O(n^{-1})$; $f_3(n) = \bigl(p(n) - p(n-1) - p(n-2) + p(n-3)\bigr)/p(n) = 2C^2n^{-1} + O(n^{-3/2})$; and $f_4(n) = 6C^3n^{-3/2} + O(n^{-2})$. (See exercise 45.) It turns out that $f_{k+1}(n) = k!\,C^kn^{-k/2} + O(n^{-(k+1)/2})$; in particular, $f_5(n) = O(n^{-2})$. Hence $f_5(n) + \cdots + f_n(n) = O(n^{-1})$, because $f_{k+1}(n) \le f_k(n)$.

Adding everything up yields $[z^n]\,F(z) = p(n)\bigl(1 + C/\sqrt{n} + O(n^{-1})\bigr)$.

50. (a) $c_m(m+k) = c_{m-1}(m-1+k) + c_m(k) = m-1-k+c(k)+1$ by induction when $0 \le k < m$.

(b) Because $\left|{m+k \atop m}\right| = p(k)$ for $0 \le k \le m$.

(c) When $n = 2m$, Algorithm H essentially generates the partitions of m, and we know that $j-1$ is the second-smallest part in the conjugate of the partition just generated — except when $j-1 = m$, just after the partition $1\ldots1$ whose conjugate has only one part.

(d) If all parts of α exceed k, let $\alpha\, k^{q+1} j$ correspond to $\alpha\,(k{+}1)$.

(e) The generating function $G_k(z)$ for all partitions whose second-smallest part is $\ge k$ is $(z+\cdots+z^{k-1})F_k(z)+F_k(z)-z^k/(1-z) = F_{k+1}(z)/(1-z)$, where $F_k(z)$ is defined in the previous exercise. Consequently $C(z) = \bigl(F(z)-F_1(z)\bigr)/(1-z) + z/(1-z)^2$.

(f) We can show as in the previous exercise that $[z^n]\,G_k(n)/p(n) = O(n^{-k/2})$ for $k \le 5$; hence $c(m)/p(m) = 1 + O(m^{-1/2})$. The ratios $(c(m)+1)/p(m)$, which are readily computed for small m, reach a maximum of 2.6 at $m = 7$ and decrease steadily thereafter. So a rigorous attention to asymptotic error bounds will complete the proof.

Note: B. Fristedt [*Trans. Amer. Math. Soc.* **337** (1993), 703–735] has proved, among other things, that the number of k's in a random partition of n is greater than $Cx\sqrt{n}$ with asymptotic probability e^{-x}.

52. In lexicographic order, $\left|{64+13 \atop 13}\right|$ partitions of 64 have $a_1 \le 13$; $\left|{50+10 \atop 10}\right|$ of them have $a_1 = 14$ and $a_2 \le 10$; etc. Therefore, by the hint, the partition $14\ 11\ 9\ 6\ 4\ 3\ 2\ 1^{15}$ is preceded by exactly $p(64) - 1000000$ partitions in lexicographic order, making it the millionth in *reverse* lexicographic order.

53. As in the previous answer, $\left|{80 \atop 12}\right|$ partitions of 100 have $a_1 = 32$ and $a_2 \le 12$, etc.; so the lexicographically millionth partition in which $a_1 = 32$ is $32\ 13\ 12\ 8\ 7\ 6\ 5\ 5\ 1^{12}$. Algorithm H produces its conjugate, namely $20\ 8\ 8\ 8\ 6\ 5\ 4\ 3\ 3\ 3\ 3\ 3\ 2\ 1^{19}$.

54. (a) Obviously true. This question was just a warmup.

(b) True, but not so obvious. If $\alpha^T = a'_1 a'_2 \ldots$ we have

$$a_1 + \cdots + a_k + a'_1 + \cdots + a'_k \;\le\; n - kl \qquad \text{when } k \le a'_l$$

by considering the Ferrers diagram, with equality when $k = a'_l$. Thus if $\alpha \succeq \beta$ and $a'_1+\cdots+a'_l > b'_1+\cdots+b'_l$ for some l, with l minimum, we have $n-kl = b_1+\cdots+b_k + b'_1+\cdots+b'_l < a_1+\cdots+a_k+a'_1+\cdots+a'_l \le n-kl$ when $k = b'_l$, a contradiction.

(c) The recurrence $c_k = \min(a_1+\cdots+a_k,\ b_1+\cdots+b_k) - (c_1+\cdots+c_{k-1})$ clearly defines a greatest lower bound, if $c_1c_2\ldots$ is a partition. And it is; for if $c_1+\cdots+c_k = a_1+\cdots+a_k$ we have $0 \le \min(a_{k+1}, b_{k+1}) \le c_{k+1} \le a_{k+1} \le a_k = c_k - (c_1+\cdots+c_{k-1}) - (a_1+\cdots+a_{k-1}) \le c_k$.

(d) $\alpha \vee \beta = (\alpha^T \wedge \beta^T)^T$. (Double conjugation is needed because a max-oriented recurrence analogous to the one in part (c) can fail.)

(e) $\alpha \wedge \beta$ has $\max(l,m)$ parts and $\alpha \vee \beta$ has $\min(l,m)$ parts. (Consider the first components of their conjugates.)

(f) True for $\alpha \wedge \beta$, by the derivation in part (c). False for $\alpha \vee \beta$ (although true in Fig. 32); for example, $(17\,16\,5\,4\,3\,2) \vee (17\,9\,8\,7\,6) = (17\,16\,5\,5\,5\,4)$.

Reference: T. Brylawski, *Discrete Mathematics* **6** (1973), 201–219.

55. (a) If $\alpha \vdash \beta$ and $\alpha \succeq \gamma \succeq \beta$, where $\gamma = c_1c_2\ldots$, we have $a_1+\cdots+a_k = c_1+\cdots+c_k = b_1+\cdots+b_k$ for all k except $k = l$ and $k = l+1$; thus α covers β. Therefore β^T covers α^T.

Conversely, if $\alpha \succeq \beta$ and $\alpha \ne \beta$ we can find $\gamma \succeq \beta$ such that $\alpha \vdash \gamma$ or $\gamma^T \vdash \alpha^T$, as follows: Find the smallest k with $a_k > b_k$, and the smallest l with $a_k > a_{l+1}$. If

$a_l > a_{l+1}+1$, define $\gamma = c_1c_2\ldots$ by $c_k = a_k-[k=l]+[k=l+1]$. If $a_l = a_{l+1}+1$, find the smallest l' with $a_{l+1} > a_{l'+1}$ and let $c_k = a_k-[k=l']+[k=l'+1]$ if $a_{l'} > a_{l'+1}+1$, otherwise $c_k = a_k-[k=l]+[k=l'+1]$.

(b) Consider α and β to be strings of n 0s and n 1s, as in (15). Then $\alpha \succ \beta$ if and only if $\alpha \to \beta$, and $\beta^T \succ \alpha^T$ if and only if $\alpha \Rightarrow \beta$, where '$\to$' denotes replacing a substring of the form 011^q10 by 101^q01 and '$\Rightarrow$' denotes replacing a substring of the form 010^q10 by 100^q01, for some $q \ge 0$.

(c) A partition covers at most $[a_1 > a_2] + \cdots + [a_{m-1} > a_m] + [a_m \ge 2]$ others. The partition $\alpha = (n_2+n_1-1)(n_2-2)(n_2-3)\ldots 21$ maximizes this quantity in the case $a_m = 1$; cases with $a_m \ge 2$ give no improvement. (The conjugate partition, namely $(n_2-1)(n_2-2)\ldots 21^{n_1+1}$, is just as good. Therefore both α and α^T are also *covered by* the maximum number of others.)

(d) Equivalently, consecutive parts of μ differ by at most 1, and the smallest part is 1; the rim representation has no consecutive 1s.

(e) Use rim representations and replace $\succ$ by the relation $\to$. If $\alpha \to \alpha_1$ and $\alpha \to \alpha_1'$ we can easily show the existence of a string β such that $\alpha_1 \to \beta$ and $\alpha_1' \to \beta$; for example,

$$\begin{array}{ccccc} & \nearrow & 101^q0111^r10 & \searrow & \\ 011^q1011^r10 & & & & 101^q1011^r01. \\ & \searrow & 011^q1101^r01 & \nearrow & \end{array}$$

Let $\beta = \beta_2 \succ \cdots \succ \beta_m$ where β_m is minimal. Then, by induction on $\max(k,k')$, we have $k = m$ and $\alpha_k = \beta_m$; also $k' = m$ and $\alpha'_{k'} = \beta_m$.

(f) Set $\beta \leftarrow \alpha^T$; then repeatedly set $\beta \leftarrow \beta'$ until β is minimal, using any convenient partition β' such that $\beta \succ \beta'$. The desired partition is β^T.

Proof: Let $\mu(\alpha)$ be the common value $\alpha_k = \alpha'_{k'}$ in part (e); we must prove that $\alpha \succeq \beta$ implies $\mu(\alpha) \succeq \mu(\beta)$. There is a sequence $\alpha = \alpha_0, \ldots, \alpha_k = \beta$ where $\alpha_j \to \alpha_{j+1}$ or $\alpha_j \Rightarrow \alpha_{j+1}$ for $0 \le j < k$. If $\alpha_0 \to \alpha_1$ we have $\mu(\alpha) = \mu(\alpha_1)$; thus it suffices to prove that $\alpha \Rightarrow \beta$ and $\alpha \to \alpha'$ implies $\alpha' \succeq \mu(\beta)$. But we have, for example,

$$\begin{array}{ccccc} & \nearrow\!\!\!\!\nearrow & 100^q0111^r10 & \searrow & \\ 010^q1011^r10 & & & & 100^q1011^r01 \\ & \searrow & 010^q1101^r01 \to 010^{q-1}10011^r01 & \nearrow\!\!\!\!\nearrow & \end{array}$$

because we may assume that $q > 0$; and the other cases are similar.

(g) The parts of λ_n are $a_k = n_2 + [k \le n_1] - k$ for $1 \le k < n_2$; the parts of λ_n^T are $b_k = n_2 - k + [n_2 - k < n_1]$ for $1 \le k \le n_2$. The algorithm of (f) reaches λ_n^T from n^1 after $\binom{n_2+1}{3} - \binom{n_2-n_1}{2}$ steps, because each step increases $\sum kb_k = \sum \binom{a_k+1}{2}$ by 1.

(h) The path n, $(n-1)1$, $(n-2)2$, $(n-2)11$, $(n-3)21$, $\ldots$, 321^{n-5}, 31^{n-3}, 221^{n-4}, 21^{n-2}, 1^n, of length $2n-4$ when $n \ge 3$, is shortest.

It can be shown that the longest path has $m = 2\binom{n_2}{3} + n_1(n_2-1)$ steps. One such path has the form $\alpha_0, \ldots, \alpha_k, \ldots, \alpha_l, \ldots, \alpha_m$ where $\alpha_0 = n^1$; $\alpha_k = \lambda_n$; $\alpha_l = \lambda_n^T$; $\alpha_j \succ \alpha_{j+1}$ for $0 \le j < l$; and $\alpha_{j+1}^T \succ \alpha_j^T$ for $k \le j < m$.

Reference: C. Greene and D. J. Kleitman, *Europ. J. Combinatorics* **7** (1986), 1–10.

56. Suppose $\lambda = u_1\ldots u_m$ and $\mu = v_1\ldots v_m$. The following (unoptimized) algorithm applies the theory of exercise 54 to generate the partitions in colex order, maintaining $\alpha = a_1a_2\ldots a_m \preceq \mu$ as well as $\alpha^T = b_1b_2\ldots b_l \preceq \lambda^T$. To find the successor of α, we first find the largest j such that b_j can be increased. Then we have

$\beta = b_1 \ldots b_{j-1}(b_j{+}1)1\ldots 1 \preceq \lambda^T$, hence the desired successor is $\beta^T \wedge \mu$. The algorithm maintains auxiliary tables $r_j = b_j + \cdots + b_l$, $s_j = v_1 + \cdots + v_j$, and $t_j = w_j + w_{j+1} + \cdots$, where $\lambda^T = w_1 w_2 \ldots$.

M1. [Initialize.] Set $q \leftarrow 0$, $k \leftarrow u_1$. For $j = 1, \ldots, m$, while $u_{j+1} < k$ set $t_k \leftarrow q \leftarrow q + j$ and $k \leftarrow k - 1$. Then set $q \leftarrow 0$ again, and for $j = 1, \ldots, m$ set $a_j \leftarrow v_j$, $s_j \leftarrow q \leftarrow q + a_j$. Then set $q \leftarrow 0$ yet again, and $k \leftarrow l \leftarrow a_1$. For $j = 1, \ldots, m$, while $a_{j+1} < k$ set $b_k \leftarrow j$, $r_k \leftarrow q \leftarrow q + j$, and $k \leftarrow k - 1$. Finally, set $t_1 \leftarrow 0$, $b_0 \leftarrow 0$, $b_{-1} \leftarrow -1$.

M2. [Visit.] Visit the partition $a_1 \ldots a_m$ and/or its conjugate $b_1 \ldots b_l$.

M3. [Find j.] Let j be the largest integer $< l$ such that $r_{j+1} > t_{j+1}$ and $b_j \neq b_{j-1}$. Terminate the algorithm if $j = 0$.

M4. [Increase b_j.] Set $x \leftarrow r_{j+1} - 1$, $k \leftarrow b_j$, $b_j \leftarrow k + 1$, and $a_{k+1} \leftarrow j$. (The previous value of a_{k+1} was $j - 1$. Now we're going to update $a_1 \ldots a_k$ using essentially the method of exercise 54(c) to distribute x dots into columns $j + 1$, $j + 2$, ...)

M5. [Majorize.] Set $z \leftarrow 0$ and then do the following for $i = 1, \ldots, k$: Set $x \leftarrow x + j$, $y \leftarrow \min(x, s_i)$, $a_i \leftarrow y - z$, $z \leftarrow y$; if $i = 1$ set $l \leftarrow p \leftarrow a_1$ and $q \leftarrow 0$; if $i > 1$ while $p > a_i$ set $b_p \leftarrow i - 1$, $r_p \leftarrow q \leftarrow q + i - 1$, $p \leftarrow p - 1$. Finally, while $p > j$ set $b_p \leftarrow k$, $r_p \leftarrow q \leftarrow q + k$, $p \leftarrow p - 1$. Return to M2. ▌

57. If $\lambda = \mu^T$ there obviously is only one such matrix, essentially the Ferrers diagram of λ. And the condition $\lambda \preceq \mu^T$ is necessary, for if $\mu^T = b_1 b_2 \ldots$ we have $b_1 + \cdots + b_k = \min(c_1, k) + \min(c_2, k) + \cdots$, and this quantity must not be less than the number of 1s in the first k rows. Finally, if there is a matrix for λ and μ and if λ covers α, we can readily construct a matrix for α and μ by moving a 1 from any specified row to another that has fewer 1s.

Notes: This result is often called the Gale–Ryser theorem, because of well-known papers by D. Gale [*Pacific J. Math.* **7** (1957), 1073–1082] and H. J. Ryser [*Canadian J. Math.* **9** (1957), 371–377]. But the number of 0–1 matrices with row sums λ and column sums μ is the coefficient of the monomial symmetric function $\sum x_{i_1}^{c_1} x_{i_2}^{c_2} \ldots$ in the product of elementary symmetric functions $e_{r_1} e_{r_2} \ldots$, where

$$e_r \;=\; [z^r]\,(1 + x_1 z)(1 + x_2 z)(1 + x_3 z)\ldots.$$

In this context the result has been known at least since the 1930s; see D. E. Littlewood's formula for $\prod_{m,n \geq 0}(1 + x_m y_n)$ in *Proc. London Math. Soc.* (2) **40** (1936), 40–70. [Cayley had shown much earlier, in *Philosophical Trans.* **147** (1857), 489–499, that the lexicographic condition $\lambda \leq \mu^T$ is necessary.]

58. [R. F. Muirhead, *Proc. Edinburgh Math. Soc.* **21** (1903), 144-157.] The condition $\alpha \succeq \beta$ is necessary, because we can set $x_1 = \cdots = x_k = x$ and $x_{k+1} = \cdots = x_n = 1$ and let $x \to \infty$. It is sufficient because we need only prove it when α covers β. Then if, say, parts (a_1, a_2) become $(a_1 - 1, a_2 + 1)$, the left-hand side is the right-hand side plus the nonnegative quantity

$$\frac{1}{2m!} \sum x_{p_1}^{a_2} x_{p_2}^{a_2} \ldots x_{p_m}^{a_m} (x_{p_1}^{a_1 - a_2 - 1} - x_{p_2}^{a_1 - a_2 - 1})(x_{p_1} - x_{p_2}).$$

[*Historical notes:* Muirhead's paper is the earliest known appearance of the concept now known as majorization; shortly afterward, an equivalent definition was given by M. O. Lorenz, *Quarterly Publ. Amer. Stat. Assoc.* **9** (1905), 209–219, who was interested in measuring nonuniform distribution of wealth. Yet another equivalent concept was formulated by I. Schur in *Sitzungsberichte Berliner Math. Gesellschaft* **22** (1923), 9–20. "Majorization" was named by Hardy, Littlewood, and Pólya, who established its most basic properties in *Messenger of Math.* **58** (1929), 145–152; see exercise 2.3.4.5–17. An excellent book, *Inequalities* by A. W. Marshall and I. Olkin (Academic Press, 1979), is entirely devoted to the subject.]

59. The unique paths for $n = 0$, 1, 2, 3, 4, and 6 must have the stated symmetry. There is one such path for $n = 5$, namely 11111, 2111, 221, 311, 32, 41, 5. And there are four for $n = 7$:

$$1111111, 211111, 22111, 2221, 322, 3211, 31111, 4111, 511, 421, 331, 43, 52, 61, 7;$$
$$1111111, 211111, 22111, 2221, 322, 421, 511, 4111, 31111, 3211, 331, 43, 52, 61, 7;$$
$$1111111, 211111, 31111, 22111, 2221, 322, 3211, 4111, 421, 331, 43, 52, 511, 61, 7;$$
$$1111111, 211111, 31111, 22111, 2221, 322, 421, 4111, 3211, 331, 43, 52, 511, 61, 7.$$

There are no others, because at least two self-conjugate partitions exist for all $n \geq 8$ (see exercise 16).

60. For $L(6,6)$, use (59); otherwise use $L'(4,6)$ and $L'(3,5)$ everywhere.
In $M(4,18)$, insert 444222, 4442211 between 443322 and 4432221.
In $M(5,11)$, insert 52211, 5222 between 62111 and 6221.
In $M(5,20)$, insert 5542211, 554222 between 5552111 and 555221.
In $M(6,13)$, insert 72211, 7222 between 62221 and 6322.
In $L(4,14)$, insert 44222, 442211 between 43322 and 432221.
In $L(5,15)$, insert 542211, 54222 between 552111 and 55221.
In $L(7,12)$, insert 62211, 6222 between 72111 and 7221.

62. The statement holds for $n = 7$, 8, and 9, except in two cases: $n = 8$, $m = 3$, $\alpha = 3221$; $n = 9$, $m = 4$, $\alpha = 432$.

64. If $n = 2^k q$ where q is odd, let ω_n denote the partition $(2^k)^q$, namely q parts equal to 2^k. The recursive rule

$$B(n) \quad = \quad B(n-1)^R 1,\ 2 \times B(n/2)$$

for $n > 0$, where $2\times B(n/2)$ denotes doubling all parts of $B(n/2)$ (or the empty sequence if n is odd), defines a pleasant Gray path that begins with $\omega_{n-1}1$ and ends with ω_n, if we let $B(0)$ be the unique partition of 0. Thus,

$$B(1) = 1; \quad B(2) = 11, 2; \quad B(3) = 21, 111; \quad B(4) = 1111, 211, 22, 4.$$

Among the remarkable properties satisfied by this sequence is the fact that

$$B(n) = (2 \times B(0))1^n,\ (2 \times B(1))1^{n-2},\ (2 \times B(2))1^{n-4},\ \ldots,\ (2 \times B(n/2))1^0,$$

when n is even; for example,

$$B(8) = 11111111, 2111111, 221111, 41111, 4211, 22211, 2222, 422, 44, 8.$$

The following algorithm generates $B(n)$ looplessly when $n \geq 2$:

K1. [Initialize.] Set $c_0 \leftarrow p_0 \leftarrow 0$, $p_1 \leftarrow 1$. If n is even, set $c_1 \leftarrow n$, $t \leftarrow 1$; otherwise let $n - 1 = 2^k q$ where q is odd and set $c_1 \leftarrow 1$, $c_2 \leftarrow q$, $p_2 \leftarrow 2^k$, $t \leftarrow 2$.

K2. [Even visit.] Visit the partition $p_t^{c_t} \ldots p_1^{c_1}$. (Now $c_t + \cdots + c_1$ is even.)

K3. [Change the largest part.] If $c_t = 1$, split the largest part: If $p_t \neq 2p_{t-1}$, set $c_t \leftarrow 2$, $p_t \leftarrow p_t/2$, otherwise set $c_{t-1} \leftarrow c_{t-1} + 2$, $t \leftarrow t - 1$. But if $c_t > 1$, merge two of the largest parts: If $c_t = 2$, set $c_t \leftarrow 1$, $p_t \leftarrow 2p_t$, otherwise set $c_t \leftarrow c_t - 2$, $c_{t+1} \leftarrow 1$, $p_{t+1} \leftarrow 2p_t$, $t \leftarrow t + 1$.

K4. [Odd visit.] Visit the partition $p_t^{c_t} \dots p_1^{c_1}$. (Now $c_t + \cdots + c_1$ is odd.)

K5. [Change the next-largest part.] Now we wish to apply the following transformation: "Remove $c_t - [t \text{ is even}]$ of the largest parts temporarily, then apply step K3, then restore the removed parts." More precisely, there are nine cases: (1a) If c_t is odd and $t = 1$, terminate. (1b1) If c_t is odd, $c_{t-1} = 1$, and $p_{t-1} = 2p_{t-2}$, set $c_{t-2} \leftarrow c_{t-2} + 2$, $c_{t-1} \leftarrow c_t$, $p_{t-1} \leftarrow p_t$, $t \leftarrow t - 1$. (1b2) If c_t is odd, $c_{t-1} = 1$, and $p_{t-1} \neq 2p_{t-2}$, set $c_{t-1} \leftarrow 2$, $p_{t-1} \leftarrow p_{t-1}/2$. (1c1) If c_t is odd, $c_{t-1} = 2$, and $p_t = 2p_{t-1}$, set $c_{t-1} \leftarrow c_t + 1$, $p_{t-1} \leftarrow p_t$, $t \leftarrow t - 1$. (1c2) If c_t is odd, $c_{t-1} = 2$, and $p_t \neq 2p_{t-1}$, set $c_{t-1} \leftarrow 1$, $p_{t-1} \leftarrow 2p_{t-1}$. (1d1) If c_t is odd, $c_{t-1} > 2$, and $p_t = 2p_{t-1}$, set $c_{t-1} \leftarrow c_{t-1} - 2$, $c_t \leftarrow c_t + 1$. (1d2) If c_t is odd, $c_{t-1} > 2$, and $p_t \neq 2p_{t-1}$, set $c_{t+1} \leftarrow c_t$, $p_{t+1} \leftarrow p_t$, $c_t \leftarrow 1$, $p_t \leftarrow 2p_{t-1}$, $c_{t-1} \leftarrow c_{t-1} - 2$, $t \leftarrow t + 1$. (2a) If c_t is even and $p_t = 2p_{t-1}$, set $c_t \leftarrow c_t - 1$, $c_{t-1} \leftarrow c_{t-1} + 2$. (2b) If c_t is even and $p_t \neq 2p_{t-1}$, set $c_{t+1} \leftarrow c_t - 1$, $p_{t+1} \leftarrow p_t$, $c_t \leftarrow 2$, $p_t \leftarrow p_t/2$, $t \leftarrow t + 1$. Return to K2. ▌

[The transformations in K3 and K5 undo themselves when performed twice in a row. This construction is due to T. Colthurst and M. Kleber, "A Gray path on binary partitions," to appear. Euler considered the number of such partitions in §50 of his paper in 1750.]

65. If $p_1^{e_1} \dots p_r^{e_r}$ is the prime factorization of m, the number of such factorizations is $p(e_1) \dots p(e_r)$, and we can let $n = \max(e_1, \dots, e_r)$. Indeed, for each r-tuple $(x_1, \dots, x_r)$ with $0 \leq x_k < p(e_k)$ we can let $m_j = p_1^{a_{1j}} \dots p_r^{a_{rj}}$, where $a_{k1} \dots a_{kn}$ is the $(x_k + 1)$st partition of e_k. Thus we can use a reflected Gray code for r-tuples together with a Gray code for partitions.

66. Let $a_1 \dots a_m$ be an m-tuple that satisfies the specified inequalities. We can sort it into nonincreasing order $a_{x_1} \geq \cdots \geq a_{x_m}$, where the permutation $x_1 \dots x_m$ is uniquely determined if we require the sorting to be *stable*; see Eq. 5–(2).

If $j \prec k$, we have $a_j \geq a_k$, hence j appears to the left of k in the permutation $x_1 \dots x_m$. Therefore $x_1 \dots x_m$ is one of the permutations output by Algorithm 7.2.1.2V. Moreover, j will be left of k also when $a_j = a_k$ and $j < k$, by stability. Hence a_{x_i} is strictly greater than $a_{x_{i+1}}$ when $x_i > x_{i+1}$ is a "descent."

To generate all the relevant partitions of n, take each topological permutation $x_1 \dots x_m$ and generate the partitions $y_1 \dots y_m$ of $n - t$ where t is the *index* of $x_1 \dots x_m$ (see Section 5.1.1). For $1 \leq j \leq m$ set $a_{x_j} \leftarrow y_j + t_j$, where t_j is the number of descents to the right of x_j in $x_1 \dots x_m$.

For example, if $x_1 \dots x_m = 314592687$ we want to generate all cases with $a_3 > a_1 \geq a_4 \geq a_5 \geq a_9 > a_2 \geq a_6 \geq a_8 > a_7$. In this case $t = 1 + 5 + 8 = 14$; so we set $a_1 \leftarrow y_2 + 2$, $a_2 \leftarrow y_6 + 1$, $a_3 \leftarrow y_1 + 3$, $a_4 \leftarrow y_3 + 2$, $a_5 \leftarrow y_4 + 2$, $a_6 \leftarrow y_7 + 1$, $a_7 \leftarrow y_9$, $a_8 \leftarrow y_8 + 1$, and $a_9 \leftarrow y_5 + 2$. The generalized generating function $\sum z_1^{a_1} \dots z_9^{a_9}$ in the sense of exercise 29 is

$$\frac{z_1^2 z_2 z_3^3 z_4^2 z_5^2 z_6 z_8 z_9^2}{(1 - z_3)(1 - z_3 z_1)(1 - z_3 z_1 z_4)(1 - z_3 z_1 z_4 z_5) \dots (1 - z_3 z_1 z_4 z_5 z_9 z_2 z_6 z_8 z_7)}.$$

When $\prec$ is any given partial ordering, the ordinary generating function for all such partitions of n is therefore $\sum z^{\operatorname{ind} \alpha}/((1 - z)(1 - z^2) \dots (1 - z^m))$, where the sum is over all outputs α of Algorithm 7.2.1.2V.

[See R. P. Stanley, *Memoirs Amer. Math. Soc.* **119** (1972), for significant extensions and applications of these ideas. See also L. Carlitz, *Studies in Foundations and Combinatorics* (New York: Academic Press, 1978), 101–129, for information about up-down partitions.]

67. If $n+1 = q_1 \ldots q_r$, where the factors $q_1, \ldots, q_r$ are all ≥ 2, we get a perfect partition $\{(q_1{-}1)\cdot 1, (q_2{-}1)\cdot q_1, (q_3{-}1)\cdot q_1q_2, \ldots, (q_r{-}1)\cdot q_1 \ldots q_{r-1}\}$ that corresponds in an obvious way to mixed radix notation. (The order of the factors q_j is significant.)

Conversely, all perfect partitions arise in this way. Suppose the multiset $M = \{k_1 \cdot p_1, \ldots, k_m \cdot p_m\}$ is a perfect partition, where $p_1 < \cdots < p_m$; then we must have $p_j = (k_1{+}1)\ldots(k_{j-1}{+}1)$ for $1 \leq j \leq m$, because p_j is the smallest sum of a submultiset of M that is not a submultiset of $\{k_1 \cdot p_1, \ldots, k_{j-1} \cdot p_{j-1}\}$.

The perfect partitions of n with fewest elements occur if and only if the q_j are all prime, because $pq - 1 > (p{-}1) + (q{-}1)$ whenever $p > 1$ and $q > 1$. Thus, for example, the minimal perfect partitions of 11 correspond to the ordered factorizations $2\cdot 2\cdot 3$, $2\cdot 3\cdot 2$, and $3\cdot 2\cdot 2$. *Reference: Quarterly Journal of Mathematics* **21** (1886), 367–373.

68. (a) If $a_i + 1 \leq a_j - 1$ for some i and j we can change $\{a_i, a_j\}$ to $\{a_i{+}1, a_j{-}1\}$, thereby increasing the product by $a_j - a_i - 1 > 0$. Thus the optimum occurs only in the optimally balanced partition of exercise 3. [L. Oettinger and J. Derbès, *Nouv. Ann. Math.* **18** (1859), 442; **19** (1860), 117–118.]

(b) No part is 1; and if $a_j \geq 4$ we can change it to $2 + (a_j{-}2)$ without decreasing the product. Thus we can assume that all parts are 2 or 3. We get an improvement by changing $2+2+2$ to $3+3$, hence there are at most two 2s. The optimum therefore is $3^{n/3}$ when $n \bmod 3$ is 0; $4\cdot 3^{(n-4)/3} = 3^{(n-4)/3}\cdot 2\cdot 2 = (4/3^{4/3})3^{n/3}$ when $n \bmod 3$ is 1; $3^{(n-2)/3}\cdot 2 = (2/3^{2/3})3^{n/3}$ when $n \bmod 3$ is 2. [O. Meißner, *Mathematisch-naturwissenschaftliche Blätter* **4** (1907), 85.]

69. All $n > 2$ have the solution $(n, 2, 1, \ldots, 1)$. We can "sieve out" the other cases $\leq N$ by starting with $s_2 \ldots s_N \leftarrow 1 \ldots 1$ and then setting $s_{ak-b} \leftarrow 0$ whenever $ak - b \leq N$, where $a = x_1 \ldots x_t - 1$, $b = x_1 + \cdots + x_t - t - 1$, $k \geq x_1 \geq \cdots \geq x_t$, and $a > 1$, because $k + x_1 + \cdots + x_t + (ak - b - t - 1) = kx_1 \ldots x_t$. The sequence $(x_1, \ldots, x_t)$ needs to be considered only when $(x_1 \ldots x_t - 1)x_1 - (x_1 + \cdots + x_t) < N - t$; we can also continue to decrease N so that $s_N = 1$. In this way only $(32766, 1486539, 254887, 1511, 937, 478, 4)$ sequences $(x_1, \ldots, x_t)$ need to be tried when N is initially 2^{30}, and the only survivors turn out to be 2, 3, 4, 6, 24, 114, 174, and 444. [See E. Trost, *Elemente der Math.* **11** (1956), 135; M. Misiurewicz, *Elemente der Math.* **21** (1966), 90.]

Notes: No new survivors are likely as $N \to \infty$, but a new idea will be needed to rule them out. The simplest sequences $(x_1, \ldots, x_t) = (3)$ and $(2,2)$ already exclude all $n > 5$ with $n \bmod 6 \neq 0$; this fact can be used to speed up the computation by a factor of 6. The sequences (6) and $(3,2)$ exclude 40% of the remainder (namely all n of the forms $5k-4$ and $5k-2$); the sequences (8), $(4,2)$, and $(2,2,2)$ exclude $3/7$ of the remainder; the sequences with $t = 1$ imply that $n - 1$ must be prime; the sequences in which $x_1 \ldots x_t = 2^r$ exclude about $p(r)$ residues of $n \bmod (2^r - 1)$; sequences in which $x_1 \ldots x_t$ is the product of r distinct primes will exclude about ϖ_r residues of $n \bmod (x_1 \ldots x_t{-}1)$.

70. Each step takes one partition of n into another, so we must eventually reach a repeating cycle. Many partitions simply perform a cyclic shift on each northeast-to-southwest diagonal of the Ferrers diagram, changing it

$$\text{from}\quad \begin{matrix} x_1 & x_2 & x_4 & x_7 & x_{11} & x_{16}\ldots \\ x_3 & x_5 & x_8 & x_{12} & x_{17} & x_{23}\ldots \\ x_6 & x_9 & x_{13} & x_{18} & x_{24} & x_{31}\ldots \\ x_{10} & x_{14} & x_{19} & x_{25} & x_{32} & x_{40}\ldots \\ x_{15} & x_{20} & x_{26} & x_{33} & x_{41} & x_{50}\ldots \\ x_{21} & x_{27} & x_{34} & x_{42} & x_{51} & x_{61}\ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{matrix} \qquad \text{to}\quad \begin{matrix} x_1 & x_3 & x_6 & x_{10} & x_{15} & x_{21}\ldots \\ x_2 & x_4 & x_7 & x_{11} & x_{16} & x_{22}\ldots \\ x_5 & x_8 & x_{12} & x_{17} & x_{23} & x_{30}\ldots \\ x_9 & x_{13} & x_{18} & x_{24} & x_{31} & x_{39}\ldots \\ x_{14} & x_{19} & x_{25} & x_{32} & x_{40} & x_{49}\ldots \\ x_{20} & x_{26} & x_{33} & x_{41} & x_{50} & x_{60}\ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{matrix}\;;$$

in other words, they apply the permutation $\rho = (1)(2\,3)(4\,5\,6)(7\,8\,9\,10)\ldots$ to the cells. Exceptions occur only when ρ introduces an empty cell above a dot; for example, x_{10} might be empty when x_{11} isn't. But we can get the correct new diagram by moving the top row down, sorting it into its proper place after applying ρ in such cases. Such a move always reduces the number of occupied diagonals, so it cannot be part of a cycle. Thus every cycle consists entirely of permutations by ρ.

If any element of a diagonal is empty in a cyclic partition, all elements of the next diagonal must be empty. For if, say, x_5 is empty, repeated application of ρ will make x_5 adjacent to each of the cells x_7, x_8, x_9, x_{10} of the next diagonal. Therefore if $n = \binom{n_2}{2} + \binom{n_1}{1}$ with $n_2 > n_1 \geq 0$ the cyclic states are precisely those with $n_2 - 1$ completely filled diagonals and n_1 dots in the next. [This result is due to J. Brandt, *Proc. Amer. Math. Soc.* **85** (1982), 483–486. The origin of the problem is unknown; see Martin Gardner, *The Last Recreations* (1997), Chapter 2.]

71. When $n = 1 + \cdots + m > 1$, the starting partition $(m{-}1)(m{-}1)(m{-}2)\ldots 211$ has distance $m(m-1)$ from the cyclic state, and this is maximum. [K. Igusa, *Math. Magazine* **58** (1985), 259–271; G. Etienne, *J. Combin. Theory* **A58** (1991), 181–197.] In the general case, Griggs and Ho [*Advances in Appl. Math.* **21** (1998), 205–227] have conjectured that the maximum distance to a cycle is $\max(2n{+}2{-}n_1(n_2{+}1), n{+}n_2{+}1, n_1(n_2{+}1)) - 2n_2$ for all $n > 1$; their conjecture has been verified for $n \leq 100$. Moreover, the worst-case starting partition appears to be unique when $n_2 = 2n_1 + \{-1, 0, 2\}$.

72. (a) Swap the jth occurrence of k in the partition $n = j \cdot k + \alpha$ with the kth occurrence of j in $k \cdot j + \alpha$, for every partition α of $n - jk$. For example, when $n = 6$ the swaps are

```
6, 51, 42, 411, 33, 321, 3111, 222, 2211, 21111, 111111.
a  b1  fg  c1g  hi  jk1  d1kh  n2i  m21n  e1mjf  1edcba
```

(b) $p(n{-}k) + p(n{-}2k) + p(n{-}3k) + \cdots$. [A. H. M. Hoare, *AMM* **93** (1986), 475–476.]

SECTION 7.2.1.5

1. Whenever m is set equal to r in step H6, change it back to $r - 1$.

2. **L1.** [Initialize.] Set $l_j \leftarrow j - 1$ and $a_j \leftarrow 0$ for $1 \leq j \leq n$. Also set $h_1 \leftarrow n$, $t \leftarrow 1$, and set l_0 to any convenient nonzero value.

L2. [Visit.] Visit the t-block partition represented by $l_1 \ldots l_n$ and $h_1 \ldots h_t$. (The restricted growth string corresponding to this partition is $a_1 \ldots a_n$.)

L3. [Find j.] Set $j \leftarrow n$; then, while $l_j = 0$, set $j \leftarrow j - 1$ and $t \leftarrow t - 1$.

L4. [Move j to the next block.] Terminate if $j = 0$. Otherwise set $k \leftarrow a_j + 1$, $h_k \leftarrow l_j$, $a_j \leftarrow k$. If $k = t$, set $t \leftarrow t + 1$ and $l_j \leftarrow 0$; otherwise set $l_j \leftarrow h_{k+1}$. Finally set $h_{k+1} \leftarrow j$.

L5. [Move $j + 1$, ..., n to block 1.] While $j < n$, set $j \leftarrow j + 1$, $l_j \leftarrow h_1$, $a_j \leftarrow 0$, and $h_1 \leftarrow j$. Return to L2. ▮

3. Let $\tau(k, n)$ be the number of strings $a_1 \ldots a_n$ that satisfy the condition $0 \leq a_j \leq 1 + \max(k{-}1, a_1, \ldots, a_{j-1})$ for $1 \leq j \leq n$; thus $\tau(k, 0) = 1$, $\tau(0, n) = \varpi_n$, and $\tau(k, n) = k\tau(k, n{-}1) + \tau(k{+}1, n{-}1)$. [S. G. Williamson has called $\tau(k, n)$ a "tail coefficient"; see *SICOMP* **5** (1976), 602–617.] The number of strings that are generated by Algorithm H before a given restricted growth string $a_1 \ldots a_n$ is $\sum_{j=1}^{n} a_j \tau(b_j, n - j)$, where $b_j = 1 + \max(a_1, \ldots, a_{j-1})$. Working backwards with the help of a precomputed table of the tail coefficients, we find that this formula yields 999999 when $a_1 \ldots a_{12} = 010220345041$.

4. The most common representatives of each type, subscripted by the number of corresponding occurrences in the GraphBase, are $\texttt{zzzzz}_0$, $\texttt{ooooh}_0$, $\texttt{xxxix}_0$, $\texttt{xxxii}_0$, $\texttt{ooops}_0$, $\texttt{llull}_0$, $\texttt{llala}_0$, $\texttt{eeler}_0$, $\texttt{iitti}_0$, $\texttt{xxiii}_0$, $\texttt{ccxxv}_0$, $\texttt{eerie}_1$, $\texttt{llama}_1$, $\texttt{xxvii}_0$, $\texttt{oozed}_5$, $\texttt{uhuuu}_0$, $\texttt{mamma}_1$, $\texttt{puppy}_{28}$, $\texttt{anana}_0$, $\texttt{hehee}_0$, $\texttt{vivid}_{15}$, $\texttt{rarer}_3$, $\texttt{etext}_1$, $\texttt{amass}_2$, $\texttt{again}_{137}$, $\texttt{ahhaa}_0$, $\texttt{esses}_1$, $\texttt{teeth}_{25}$, $\texttt{yaaay}_0$, $\texttt{ahhhh}_2$, $\texttt{pssst}_2$, $\texttt{seems}_7$, $\texttt{added}_6$, $\texttt{lxxii}_0$, $\texttt{books}_{184}$, $\texttt{swiss}_3$, $\texttt{sense}_{10}$, $\texttt{ended}_3$, $\texttt{check}_{160}$, $\texttt{level}_{18}$, $\texttt{tepee}_4$, $\texttt{slyly}_5$, $\texttt{never}_{154}$, $\texttt{sells}_6$, $\texttt{motto}_{21}$, $\texttt{whooo}_2$, $\texttt{trees}_{384}$, $\texttt{going}_{307}$, $\texttt{which}_{151}$, $\texttt{there}_{174}$, $\texttt{three}_{100}$, $\texttt{their}_{3834}$. (See S. Golomb, *Math. Mag.* **53** (1980), 219–221. Words with only two distinct letters are, of course, rare. The 18 representatives listed here with subscript 0 can be found in larger dictionaries or in English-language pages of the Internet.)

5. (a) $112 = \rho(0225)$. The sequence is $r(0)$, $r(1)$, $r(4)$, $r(9)$, $r(16)$, $\ldots$, where $r(n)$ is obtained by expressing n in decimal notation (with one or more leading zeros), applying the ρ function of exercise 4, then deleting the leading zeros. Notice that $n/9 \le r(n) \le n$.

(b) $1012 = r(45^2)$. The sequence is the same as (a), but sorted into order and with duplicates removed. (Who knew that $88^2 = 7744$, $212^2 = 44944$, and $264^2 = 69696$?)

6. Use the topological sorting approach of Algorithm 7.2.1.2V, with an appropriate partial ordering: Include c_j chains of length j, with their least elements ordered. For example, if $n = 20$, $c_2 = 3$, and $c_3 = c_4 = 2$, we use that algorithm to find all permutations $a_1 \ldots a_{20}$ of $\{1, \ldots, 20\}$ such that $1 \prec 2$, $3 \prec 4$, $5 \prec 6$, $1 \prec 3 \prec 5$, $7 \prec 8 \prec 9$, $10 \prec 11 \prec 12$, $7 \prec 10$, $13 \prec 14 \prec 15 \prec 16$, $17 \prec 18 \prec 19 \prec 20$, $13 \prec 17$, forming the restricted growth strings $\rho(f(a_1) \ldots f(a_{20}))$, where ρ is defined in exercise 4 and $(f(1), \ldots, f(20)) = (1,1,2,2,3,3,4,4,4,5,5,5,6,6,6,6,7,7,7,7)$. The total number of outputs is, of course, given by (48).

7. Exactly ϖ_n. They are the permutations we get by reversing the left-right order of the blocks in (2) and dropping the '|' symbols: 1234, 4123, 3124, 3412, $\ldots$, 4321. [See A. Claesson, *European J. Combinatorics* **22** (2001), 961–971. S. Kitaev, in "Partially ordered generalized patterns," *Discrete Math.*, to appear, has discovered a far-reaching generalization: Let π be a permutation of $\{0, \ldots, r\}$, let g_n be the number of permutations $a_1 \ldots a_n$ of $\{1, \ldots, n\}$ such that $a_{k-0\pi} > a_{k-1\pi} > \cdots > a_{k-r\pi} > a_j$ implies $j > k$, and let f_n be the number of permutations $a_1 \ldots a_n$ for which the pattern $a_{k-0\pi} > a_{k-1\pi} > \cdots > a_{k-r\pi}$ is avoided altogether for $r < k \le n$. Then $\sum_{n\ge 0} g_n z^n/n! = \exp\bigl(\sum_{n\ge 1} f_{n-1} z^n/n!\bigr)$.]

8. For each partition of $\{1, \ldots, n\}$ into m blocks, arrange the blocks in decreasing order of their smallest elements, and permute the non-smallest block elements in all possible ways. If $n = 9$ and $m = 3$, for example, the partition 126|38|4579 would yield 457938126 and eleven other cases obtained by permuting $\{5, 7, 9\}$ and $\{2, 6\}$ among themselves. (Essentially the same method generates all permutations that have exactly k cycles; see the "unusual correspondence" of Section 1.3.3.)

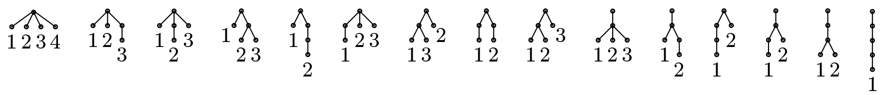
9. Among the permutations of the multiset $\{k_0 \cdot 0, k_1 \cdot 1, \ldots, k_{n-1} \cdot (n-1)\}$, exactly

$$\binom{k_0 + k_1 + \cdots + k_{n-1}}{k_0,\ k_1,\ \ldots,\ k_{n-1}} \frac{k_0}{(k_0 + k_1 + \cdots + k_{n-1})} \frac{k_1}{(k_1 + \cdots + k_{n-1})} \cdots \frac{k_{n-1}}{k_{n-1}}$$

have restricted growth, since $k_j/(k_j + \cdots + k_{n-1})$ is the probability that j precedes $\{j+1, \ldots, n-1\}$.

The average number of 0s, if $n > 0$, is $1 + (n-1)\varpi_{n-1}/\varpi_n = \Theta(\log n)$, because the total number of 0s among all ϖ_n cases is $\sum_{k=1}^{n} k\binom{n-1}{k-1}\varpi_{n-k} = \varpi_n + (n-1)\varpi_{n-1}$.

10. Given a partition of $\{1, \ldots, n\}$, construct an oriented tree on $\{0, 1, \ldots, n\}$ by letting $j-1$ be the parent of all members of a block whose least member is j. Then relabel the leaves, preserving order, and erase the other labels. For example, the 15 partitions in (2) correspond respectively to



To reverse the process, take a semilabeled tree and assign new numbers to its nodes by considering the nodes first encountered on the path from the root to the smallest leaf, then on the path from the root to the second-smallest leaf, etc. The number of leaves is $n+1$ minus the number of blocks. [This construction is closely related to exercise 2.3.4.4–18 and to many enumerations in that section. See P. L. Erdős and L. A. Székely, *Advances in Applied Math.* **10** (1989), 488–496.]

11. We get pure alphametics from 900 of the 64855 set partitions into at most 10 blocks for which $\rho(a_1 \ldots a_{13}) = \rho(a_5 \ldots a_8 a_1 \ldots a_4 a_9 \ldots a_{13})$, and from 563,527 of the 13,788,536 for which $\rho(a_1 \ldots a_{13}) < \rho(a_5 \ldots a_8 a_1 \ldots a_4 a_9 \ldots a_{13})$. The first examples are `aaaa` + `aaaa` = `baaac`, `aaaa` + `aaaa` = `bbbbc`, and `aaaa` + `aaab` = `baaac`; the last are `abcd` + `efgd` = `dceab` (`goat` + `newt` = `tango`) and `abcd` + `efgd` = `dceaf` (`clad` + `nerd` = `dance`). [The idea of hooking a partition generator to an alphametic solver is due to Alan Sutcliffe.]

12. (a) Form $\rho\bigl((a_1a_1')\ldots(a_na_n')\bigr)$, where ρ is defined in exercise 4, since we have $x \equiv y$ (modulo $\Pi \vee \Pi'$) if and only if $x \equiv y$ (modulo Π) and $x \equiv y$ (modulo Π').

(b) Represent Π by links as in exercise 2; represent Π' as in Algorithm 2.3.3E; and use that algorithm to make $j \equiv l_j$ whenever $l_j \neq 0$. (For efficiency, we can assume that Π has at least as many blocks as Π'.)

(c) When one block of Π has been split into two parts; that is, when two blocks of Π' have been merged together.

(d) $\binom{t}{2}$; (e) $(2^{s_1-1}-1)+\cdots+(2^{s_t-1}-1)$.

(f) True: Let $\Pi \vee \Pi'$ have blocks $B_1|B_2|\cdots|B_t$, where $\Pi = B_1B_2|B_3|\cdots|B_t$. Then Π' is essentially a partition of $\{B_1,\ldots,B_t\}$ with $B_1 \not\equiv B_2$, and $\Pi \wedge \Pi'$ is obtained by merging the block of Π' that contains B_1 with the block that contains B_2. [A finite lattice that satisfies this condition is called *lower semimodular*; see G. Birkhoff, *Lattice Theory* (1940), §I.8. The majorization lattice of exercise 7.2.1.4–54 does not have this property when, for example, $\alpha = 4111$ and $\alpha' = 331$.]

(g) False: For example, let $\Pi = 0011$, $\Pi' = 0101$.

(h) The blocks of Π and Π' are unions of the blocks of $\Pi \vee \Pi'$, so we can assume that $\Pi \vee \Pi' = \{1,\ldots,t\}$. As in part (b), merge j with l_j to get Π in r steps, when Π has $t-r$ blocks. These merges applied to Π' will each reduce the number of blocks by 0 or 1. Hence $b(\Pi') - b(\Pi \wedge \Pi') \le r = b(\Pi \vee \Pi') - b(\Pi)$.

[In *Algebra Universalis* **10** (1980), 74–95, P. Pudlák and J. Tůma proved that *every* finite lattice is a sublattice of the partition lattice of $\{1,\ldots,n\}$, for suitably large n.]

13. [See *Advances in Math.* **26** (1977), 290–305.] If the j largest elements of a t-block partition appear in singleton blocks, but the next element $n-j$ does not, let us say that the partition has order $t-j$. Define the "Stirling string" Σ_{nt} to be the sequence of orders of the t-block partitions Π_1, Π_2, …; for example, $\Sigma_{43} = 122333$. Then $\Sigma_{tt} = 0$, and we get $\Sigma_{(n+1)t}$ from Σ_{nt} by replacing each digit d in the latter by the string $d^d(d{+}1)^{d+1}\ldots t^t$ of length $\binom{t+1}{2} - \binom{d}{2}$; for example,

$$\Sigma_{53} = 12233\underset{\wedge}{3}22333\underset{\wedge}{2}2333\underset{\wedge}{3}33\underset{\wedge}{3}33\underset{\wedge}{3}333.$$

The basic idea is to consider the lexicographic generation process of Algorithm H. Suppose $\Pi = a_1 \ldots a_n$ is a t-block partition of order j; then it is the lexicographically smallest t-block partition whose restricted growth string begins with $a_1 \ldots a_{n-t+j}$. The partitions covered by Π are, in lexicographic order, Π_{12}, Π_{13}, Π_{23}, Π_{14}, Π_{24}, Π_{34}, $\ldots$, $\Pi_{(t-1)t}$, where Π_{rs} means "coalesce blocks r and s of Π" (that is, "change all occurrences of $s-1$ to $r-1$ and then apply ρ to get a restricted growth string"). If Π' is any of the last $\binom{t}{2} - \binom{j}{2}$ of these, from $\Pi_{1(j+1)}$ onwards, then Π is the smallest t-block partition following Π'. For example, if $\Pi = 001012034$, then $n = 9$, $t = 5$, $j = 3$, and the relevant partitions Π' are $\rho(001012004)$, $\rho(001012014)$, $\rho(001012024)$, $\rho(001012030)$, $\rho(001012031)$, $\rho(001012032)$, $\rho(001012033)$.

Therefore $f_{nt}(N) = f_{nt}(N-1) + \binom{t}{2} - \binom{j}{2}$, where j is the Nth digit of Σ_{nt}.

14. **E1.** [Initialize.] Set $a_j \leftarrow 0$ and $b_j \leftarrow d_j \leftarrow 1$ for $1 \le j \le n$.

E2. [Visit.] Visit the restricted growth string $a_1 \ldots a_n$.

E3. [Find j.] Set $j \leftarrow n$; then, while $a_j = d_j$, set $d_j \leftarrow 1 - d_j$ and $j \leftarrow j - 1$.

E4. [Done?] Terminate if $j = 1$. Otherwise go to E6 if $d_j = 0$.

E5. [Move down.] If $a_j = 0$, set $a_j \leftarrow b_j$, $m \leftarrow a_j + 1$, and go to E7. Otherwise if $a_j = b_j$, set $a_j \leftarrow b_j - 1$, $m \leftarrow b_j$, and go to E7. Otherwise set $a_j \leftarrow a_j - 1$ and return to E2.

E6. [Move up.] If $a_j = b_j - 1$, set $a_j \leftarrow b_j$, $m \leftarrow a_j + 1$, and go to E7. Otherwise if $a_j = b_j$, set $a_j \leftarrow 0$, $m \leftarrow b_j$, and go to E7. Otherwise set $a_j \leftarrow a_j + 1$ and return to E2.

E7. [Fix $b_{j+1} \ldots b_n$.] Set $b_k \leftarrow m$ for $k = j+1, \ldots, n$. Return to E2. ▌

[This algorithm can be extensively optimized because, as in Algorithm H, j is almost always equal to n.]

15. It corresponds to the first n digits of the infinite binary string $01011011011\ldots$, because ϖ_{n-1} is even if and only if $n \bmod 3 = 0$ (see exercise 23).

16. 00012, 01012, 01112, 00112, 00102, 01102, 01002, 01202, 01212, 01222, 01022, 01122, 00122, 00121, 01121, 01021, 01221, 01211, 01201, 01200, 01210, 01220, 01020, 01120, 00120.

17. The following solution uses two mutually recursive procedures, $f(\mu,\nu,\sigma)$ and $b(\mu,\nu,\sigma)$, for "forward" and "backward" generation of $A_{\mu\nu}$ when $\sigma = 0$ and of $A'_{\mu\nu}$ when $\sigma = 1$. To start the process, assuming that $1 < m < n$, first set $a_j \leftarrow 0$ for $1 \le j \le n - m$ and $a_{n-m+j} \leftarrow j - 1$ for $1 \le j \le m$, then call $f(m,n,0)$.

Procedure $f(\mu,\nu,\sigma)$: If $\mu = 2$, visit $a_1 \ldots a_n$; otherwise call $f(\mu-1, \nu-1, (\mu{+}\sigma) \bmod 2)$. Then, if $\nu = \mu + 1$, do the following: Change a_μ from 0 to $\mu - 1$, and visit $a_1 \ldots a_n$; repeatedly set $a_\nu \leftarrow a_\nu - 1$ and visit $a_1 \ldots a_n$, until $a_\nu = 0$. But if $\nu > \mu + 1$, change $a_{\nu-1}$ (if $\mu{+}\sigma$ is odd) or a_μ (if $\mu{+}\sigma$ is even) from 0 to $\mu - 1$; then call $b(\mu,\nu{-}1,0)$ if $a_\nu + \sigma$ is odd, $f(\mu,\nu{-}1,0)$ if $a_\nu + \sigma$ is even; and while $a_\nu > 0$, set $a_\nu \leftarrow a_\nu - 1$ and call $b(\mu,\nu{-}1,0)$ or $f(\mu,\nu{-}1,0)$ again in the same way until $a_\nu = 0$.

Procedure $b(\mu,\nu,\sigma)$: If $\nu = \mu + 1$, first do the following: Repeatedly visit $a_1 \ldots a_n$ and set $a_\nu \leftarrow a_\nu + 1$, until $a_\nu = \mu - 1$; then visit $a_1 \ldots a_n$ and change a_μ from $\mu - 1$ to 0. But if $\nu > \mu + 1$, call $f(\mu,\nu{-}1,0)$ if $a_\nu + \sigma$ is odd, $b(\mu,\nu{-}1,0)$ if $a_\nu + \sigma$ is even; then while $a_\nu < \mu - 1$, set $a_\nu \leftarrow a_\nu + 1$ and call $f(\mu,\nu{-}1,0)$ or $b(\mu,\nu{-}1,0)$ again in the same way until $a_\nu = \mu - 1$; finally change $a_{\nu-1}$ (if $\mu{+}\sigma$ is odd) or a_μ (if $\mu{+}\sigma$ is even) from $\mu - 1$ to 0. And finally, in both cases, if $\mu = 2$ visit $a_1 \ldots a_n$, otherwise call $b(\mu-1, \nu-1, (\mu{+}\sigma) \bmod 2)$.

Most of the running time is actually spent handing the case $\mu = 2$; faster routines based on Gray binary code (and deviating from Ruskey's actual sequences) could be substituted for this case. A streamlined procedure could also be used when $\mu = \nu - 1$.

18. The sequence must begin (or end) with $01\ldots(n{-}1)$. By exercise 32, no such Gray code can exist when $0 \neq \delta_n \neq (1)^{0+1+\cdots+(n-1)}$, namely when $n \bmod 12$ is 4, 6, 7, or 9.

The cases $n = 1, 2, 3$, are easily solved; and 1,927,683,326 solutions exist when $n = 5$. Thus there probably are zillions of solutions for all $n \geq 8$ except for the cases already excluded. Indeed, we can probably find such a Gray path through all ϖ_{nk} of the strings considered in answer 28(e) below, except when $n \equiv 2k + (2,4,5,7)$ (modulo 12).

Note: The generalized Stirling number ${n \brace m}_{-1}$ in exercise 30 exceeds 1 for $2 < m < n$, so there can be no such Gray code for the partitions of $\{1,\ldots,n\}$ into m blocks.

19. (a) Change (6) to the pattern $0, 2, \ldots, m, \ldots, 3, 1$ or its reverse, as in endo-order (7.2.1.3–(45)).

(b) We can generalize (8) and (9) to obtain sequences $A_{mn\alpha}$ and A'_{mna} that begin with $0^{n-m}01\ldots(m{-}1)$ and end with $01\ldots(m{-}1)\alpha$ and $0^{n-m-1}01\ldots(m{-}1)a$, respectively, where $0 \leq a \leq m-2$ and α is any string $a_1\ldots a_{n-m}$ with $0 \leq a_j \leq m-2$. When $2 < m < n$ the new rules are

$$A_{m(n+1)(\alpha a)} = \begin{cases} A_{(m-1)n(b\beta)}x_1, A^R_{mn\beta}x_1, A_{mn\alpha}x_2, \ldots, A_{mn\alpha}x_m, & \text{if } m \text{ is even;} \\ A'_{(m-1)nb}x_1, A_{mn\alpha}x_1, A^R_{mn\alpha}x_2, \ldots, A_{mn\alpha}x_m, & \text{if } m \text{ is odd;} \end{cases}$$

$$A'_{m(n+1)a} = \begin{cases} A'_{(m-1)nb}x_1, A_{mn\beta}x_1, A^R_{mn\beta}x_2, \ldots, A^R_{mn\beta}x_m, & \text{if } m \text{ is even;} \\ A_{(m-1)n(b\beta)}x_1, A^R_{mn\beta}x_1, A_{mn\beta}x_2, \ldots, A^R_{mn\beta}x_m, & \text{if } m \text{ is odd;} \end{cases}$$

here $b = m-3$, $\beta = b^{n-m}$, and $(x_1,\ldots,x_m)$ is a path from $x_1 = m-1$ to $x_m = a$.

20. 012323212122; in general $(a_1\ldots a_n)^T = \rho(a_n\ldots a_1)$, in the notation of exercise 4.

21. The numbers $\langle s_0, s_1, s_2, \ldots\rangle = \langle 1, 1, 2, 3, 7, 12, 31, 59, 164, 339, 999, \ldots\rangle$ satisfy the recurrences $s_{2n+1} = \sum_k \binom{n}{k} s_{2n-2k}$, $s_{2n+2} = \sum_k \binom{n}{k}(2^k+1)s_{2n-2k}$, because of the way the middle elements relate to the others. Therefore $s_{2n} = n!\,[z^n]\,\exp\bigl((e^{2z}-1)/2+e^z-1\bigr)$ and $s_{2n+1} = n!\,[z^n]\,\exp\bigl((e^{2z}-1)/2+e^z+z-1\bigr)$. By considering set partitions on the first half we also have $s_{2n} = \sum_k {n \brace k} x_k$ and $s_{2n+1} = \sum_k {n+1 \brace k} x_{k-1}$, where $x_n = 2x_{n-1} + (n-1)x_{n-2} = n!\,[z^n]\,\exp(2z+z^2/2)$. [T. S. Motzkin considered the sequence $\langle s_{2n}\rangle$ in *Proc. Symp. Pure Math.* **19** (1971), 173.]

22. (a) $\sum_{k=0}^{\infty} k^n \Pr(X{=}k) = e^{-1}\sum_{k=0}^{\infty} k^n/k! = \varpi_n$ by (16). (b) $\sum_{k=0}^{\infty} k^n \Pr(X{=}k) = \sum_{k=0}^{\infty} k^n \sum_{j=0}^{m}\binom{j}{k}(-1)^{j-k}/j!$, and we can extend the inner sum to $j = \infty$ because $\sum_k \binom{j}{k}(-1)^k k^n = 0$ when $j > n$. Thus we get $\sum_{k=0}^{\infty}(k^n/k!)\sum_{l=0}^{\infty}(-1)^l/l! = \varpi_n$. [See J. O. Irwin, *J. Royal Stat. Soc.* **A118** (1955), 389–404; J. Pitman, *AMM* **104** (1997), 201–209.]

23. (a) The formula holds whenever $f(x) = x^n$, by (14), so it holds in general. (Thus we also have $\sum_{k=0}^{\infty} f(k)/k! = e\,f(\varpi)$, by (16).)

(b) Suppose we have proved the relation for k, and let $h(x) = (x-1)^{\underline{k}} f(x)$, $g(x) = f(x{+}1)$. Then $f(\varpi+k+1) = g(\varpi+k) = \varpi^{\underline{k}} g(\varpi) = h(\varpi+1) = \varpi h(\varpi) = \varpi^{\underline{k+1}} f(\varpi)$. [See J. Touchard, *Ann. Soc. Sci. Bruxelles* **53** (1933), 21–31. This symbolic "umbral calculus," invented by John Blissard in *Quart. J. Pure and Applied Math.* **4** (1861), 279–305, is quite useful; but it must be handled carefully because $f(\varpi) = g(\varpi)$ does not imply that $f(\varpi)h(\varpi) = g(\varpi)h(\varpi)$.]

(c) The hint is a special case of exercise 4.6.2–16(c). Setting $f(x) = x^n$ and $k = p$ in (b) then yields $\varpi_n \equiv \varpi_{p+n} - \varpi_{1+n}$.

(d) Modulo p, the polynomial $x^N - 1$ is divisible by $g(x) = x^p - x - 1$, because $x^{p^k} \equiv x + k$ and $x^N \equiv x^{\bar{p}} \equiv x^{\underline{p}} \equiv x^p - x \equiv 1$ (modulo $g(x)$ and p). Thus if $h(x) = (x^N - 1)x^n/g(x)$ we have $h(\varpi) \equiv h(\varpi + p) = \varpi^{\underline{p}} h(\varpi) \equiv (\varpi^p - \varpi)h(\varpi)$; and $0 \equiv g(\varpi)h(\varpi) = \varpi^{N+n} - \varpi^n$ (modulo p).

24. The hint follows by induction on e, because $x^{\underline{p^e}} = \prod_{k=0}^{p-1}(x - kp^{e-1})^{\underline{p^{e-1}}}$. We can also prove by induction on n that $x^n \equiv r_n(x)$ (modulo $g_1(x)$ and p) implies

$$x^{p^{e-1}n} \equiv r_n(x)^{p^{e-1}} \text{ (modulo } g_e(x),\ pg_{e-1}(x),\ \ldots,\ p^{e-1}g_1(x), \text{ and } p^e).$$

Hence $x^{p^{e-1}N} = 1 + h_0(x)g_e(x) + ph_1(x)g_{e-1}(x) + \cdots + p^{e-1}h_{e-1}(x)g_1(x) + p^e h_e(x)$ for certain polynomials $h_k(x)$ with integer coefficients. Modulo p^e we have $h_0(\varpi)\varpi^n \equiv h_0(\varpi + p^e)(\varpi + p^e)^n = \varpi^{\underline{p^e}} h_0(\varpi)\varpi^n \equiv \bigl(g_e(\varpi) + 1\bigr)h_0(\varpi)\varpi^n$; hence

$$\varpi^{p^{e-1}N+n} \;=\; \varpi^n + h_0(\varpi)g_e(\varpi)\varpi^n + ph_1(\varpi)g_{e-1}(\varpi)\varpi^n + \cdots \;\equiv\; \varpi^n.$$

[A similar derivation applies when $p = 2$, but we let $g_{j+1}(x) = g_j(x)^2 + 2[j = 2]$, and we obtain $\varpi_n \equiv \varpi_{n+3\cdot 2^e}$ (modulo 2^e). These results are due to Marshall Hall; see *Bull. Amer. Math. Soc.* **40** (1934), 387; *Amer. J. Math.* **70** (1948), 387–388. For further information see W. F. Lunnon, P. A. B. Pleasants, and N. M. Stephens, *Acta Arith.* **35** (1979), 1–16.]

25. The first inequality follows by applying a much more general principle to the tree of restricted growth strings: In any tree for which $\deg(p) \ge \deg(\mathrm{parent}(p))$ for all non-root nodes p, we have $w_k/w_{k-1} \le w_{k+1}/w_k$ when w_k is the total number of nodes on level k. For if the $m = w_{k-1}$ nodes on level $k-1$ have respectively $a_1, \ldots, a_m$ children, they have at least $a_1^2 + \cdots + a_m^2$ grandchildren; hence $w_{k-1}w_{k+1} \ge m(a_1^2 + \cdots + a_m^2) \ge (a_1 + \cdots + a_m)^2 = w_k^2$.

For the second inequality, note that $\varpi_{n+1} - \varpi_n = \sum_{k=0}^{n}\bigl(\binom{n}{k} - \binom{n-1}{k-1}\bigr)\varpi_{n-k}$; thus

$$\frac{\varpi_{n+1}}{\varpi_n} - 1 \;=\; \sum_{k=0}^{n-1}\binom{n-1}{k}\frac{\varpi_{n-k}}{\varpi_n} \;\le\; \sum_{k=0}^{n-1}\binom{n-1}{k}\frac{\varpi_{n-k-1}}{\varpi_{n-1}} \;=\; \frac{\varpi_n}{\varpi_{n-1}}$$

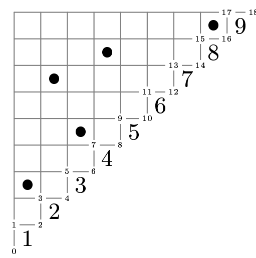
because, for example, $\varpi_{n-3}/\varpi_n = (\varpi_{n-3}/\varpi_{n-2})(\varpi_{n-2}/\varpi_{n-1})(\varpi_{n-1}/\varpi_n)$ is less than or equal to $(\varpi_{n-4}/\varpi_{n-3})(\varpi_{n-3}/\varpi_{n-2})(\varpi_{n-2}/\varpi_{n-1}) = \varpi_{n-4}/\varpi_{n-1}$.

26. There are $\binom{n-1}{n-t}$ rightward paths from $(n1)$ to (tt); we can represent them by 0s and 1s, where 0 means "go right," 1 means "go up," and the positions of the 1s tell us which $n-t$ of the elements are in the block with 1. The next step, if $t > 1$, is to another vertex at the far left; so we continue with a path that defines a partition on the remaining $t-1$ elements. For example, the partition 14|2|3 corresponds to the path 0010 under these conventions, where the respective bits mean that $1 \not\equiv 2$, $1 \not\equiv 3$, $1 \equiv 4$, $2 \not\equiv 3$. [Many other interpretations are possible. The convention suggested here shows that ϖ_{nk} enumerates partitions with $1 \not\equiv 2, \ldots, 1 \not\equiv k$, a combinatorial property discovered by H. W. Becker; see *AMM* **51** (1944), 47, and *Mathematics Magazine* **22** (1948), 23–26.]

27. (a) In general, $\lambda_0 = \lambda_1 = \lambda_{2n-1} = \lambda_{2n} = 0$. The following list shows also the restricted growth strings that correspond to each loop via the algorithm of part (b):

0, 0, 0, 0, 0, 0, 0, 0, 0	0123	0, 0, 1, 0, 0, 0, 0, 0, 0	0012	0, 0, 1, 1, 1, 0, 0, 0, 0	0102
0, 0, 0, 0, 0, 0, 1, 0, 0	0122	0, 0, 1, 0, 0, 0, 1, 0, 0	0011	0, 0, 1, 1, 1, 0, 1, 0, 0	0100
0, 0, 0, 0, 1, 0, 0, 0, 0	0112	0, 0, 1, 0, 1, 0, 0, 0, 0	0001	0, 0, 1, 1, 1, 1, 1, 0, 0	0120
0, 0, 0, 0, 1, 0, 1, 0, 0	0111	0, 0, 1, 0, 1, 0, 1, 0, 0	0000	0, 0, 1, 1, 11, 1, 1, 0, 0	0101
0, 0, 0, 0, 1, 1, 1, 0, 0	0121	0, 0, 1, 0, 1, 1, 1, 0, 0	0010	0, 0, 1, 1, 2, 1, 1, 0, 0	0110

(b) The name "tableau" suggests a connection to Section 5.1.4, and indeed the theory developed there leads to an interesting one-to-one correspondence. We can represent set partitions on a triangular chessboard by putting a rook in column l_j of row $n+1-j$ whenever $l_j \neq 0$ in the linked list representation of exercise 2 (see the answer to exercise 5.1.3–19). For example, the rook representation of 135|27|489|6 is shown here. Equivalently, the nonzero links can be specified in a two-line array, such as $\binom{1\,2\,3\,4\,8}{3\,7\,5\,8\,9}$; see 5.1.4–(11).



Consider the path of length $2n$ that begins at the lower left corner of this triangular diagram and follows the right boundary edges, ending at the upper right corner: The points of this path are $z_k = (\lfloor k/2 \rfloor, \lceil k/2 \rceil)$ for $0 \le k \le 2n$. Moreover, the rectangle above and to the left of z_k contains precisely the rooks that contribute coordinate pairs $\frac{i}{j}$ to the two-line array when $i \le \lfloor k/2 \rfloor$ and $j > \lceil k/2 \rceil$; in our example, there are just two such rooks when $9 \le k \le 12$, namely $\binom{2\,4}{7\,8}$. Theorem 5.1.4A tells us that such two-line arrays are equivalent to tableaux (P_k, Q_k), where the elements of P_k come from the lower line and the elements of Q_k come from the upper line, and where both P_k and Q_k have the same shape. It is advantageous to use decreasing order in the P tableaux but increasing order in the Q tableaux, so that in our example they are respectively

k	P_k	Q_k	k	P_k	Q_k	k	P_k	Q_k
2	$\begin{array}{\|c\|}\hline 3\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 1\\\hline\end{array}$	7	$\begin{array}{\|c\|c\|}\hline 7&5\\\hline\end{array}$	$\begin{array}{\|c\|c\|}\hline 2&3\\\hline\end{array}$	12	$\begin{array}{\|c\|}\hline 8\\\hline 7\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 2\\\hline 4\\\hline\end{array}$
3	$\begin{array}{\|c\|}\hline 3\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 1\\\hline\end{array}$	8	$\begin{array}{\|c\|c\|}\hline 8&5\\\hline 7\\\cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|}\hline 2&3\\\hline 4\\\cline{1-1}\end{array}$	13	$\begin{array}{\|c\|}\hline 8\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 4\\\hline\end{array}$
4	$\begin{array}{\|c\|}\hline 7\\\hline 3\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 1\\\hline 2\\\hline\end{array}$	9	$\begin{array}{\|c\|}\hline 8\\\hline 7\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 2\\\hline 4\\\hline\end{array}$	14	$\begin{array}{\|c\|}\hline 8\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 4\\\hline\end{array}$
5	$\begin{array}{\|c\|}\hline 7\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 2\\\hline\end{array}$	10	$\begin{array}{\|c\|}\hline 8\\\hline 7\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 2\\\hline 4\\\hline\end{array}$	15	·	·
6	$\begin{array}{\|c\|c\|}\hline 7&5\\\hline\end{array}$	$\begin{array}{\|c\|c\|}\hline 2&3\\\hline\end{array}$	11	$\begin{array}{\|c\|}\hline 8\\\hline 7\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 2\\\hline 4\\\hline\end{array}$	16	$\begin{array}{\|c\|}\hline 9\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 8\\\hline\end{array}$

while P_k and Q_k are empty for $k = 0$, 1, 17, and 18.

In this way every set partition leads to a vacillating tableau loop $\lambda_0, \lambda_1, \ldots, \lambda_{2n}$, if we let λ_k be the integer partition that specifies the common shape of P_k and Q_k. (The loop is 0, 0, 1, 1, 11, 1, 2, 2, 21, 11, 11, 11, 11, 1, 1, 0, 1, 0, 0 in our example.) Moreover, $t_{2k-1} = 0$ if and only if row $n+1-k$ contains no rook, if and only if k is smallest in its block.

Conversely, the elements of P_k and Q_k can be uniquely reconstructed from the sequence of shapes λ_k. Namely, $Q_k = Q_{k-1}$ if $t_k = 0$. Otherwise, if k is even, Q_k is Q_{k-1}

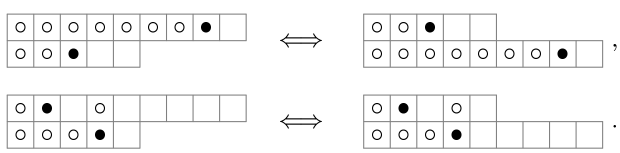
with the number $k/2$ placed in a new cell at the right of row t_k; if k is odd, Q_k is obtained from Q_{k-1} by using Algorithm 5.1.4D to delete the rightmost entry of row t_k. A similar procedure defines P_k from the values of P_{k+1} and t_{k+1}, so we can work back from P_{2n} to P_0. Thus the sequence of shapes λ_k is enough to tell us where to place the rooks.

Vacillating tableau loops were introduced in the paper "Crossings and nestings of matchings and partitions" by W. Y. C. Chen, E. Y. P. Deng, R. R. X. Du, R. P. Stanley, and C. H. Yan (preprint, 2005), who showed that the construction has significant (and surprising) consequences. For example, if the set partition Π corresponds to the vacillating tableau loop λ_0, λ_1, $\ldots$, λ_{2n}, let's say that its *dual* Π^D is the set partition that corresponds to the sequence of transposed shapes λ_0^T, λ_1^T, $\ldots$, λ_{2n}^T. Then, by exercise 5.1.4–7, Π contains a "k-crossing at l," namely a sequence of indices with $i_1 < \cdots < i_k \le l < j_1 < \cdots < j_k$ and $i_1 \equiv j_1$, $\ldots$, $i_k \equiv j_k$ (modulo Π), if and only if Π^D contains a "k-nesting at l," which is a sequence of indices with $i'_1 < \cdots < i'_k \le l < j'_k < \cdots < j'_1$ and $i'_1 \equiv j'_1$, $\ldots$, $i'_k \equiv j'_k$ (modulo Π^D). Notice also that an involution is essentially a set partition in which all blocks have size 1 or 2; the dual of an involution is an involution having the same singleton sets. In particular, the dual of a perfect matching (when there are no singleton sets) is a perfect matching.

Furthermore, an analogous construction applies to rook placements in *any* Ferrers diagram, not only in the stairstep shapes that correspond to set partitions. Given a Ferrers diagram that has at most m parts, all of size $\le n$, we simply consider the path $z_0 = (0,0)$, $z_1, \ldots, z_{m+n} = (n,m)$ that hugs the right edge of the diagram, and stipulate that $\lambda_k = \lambda_{k-1} + e_{t_k}$ when $z_k = z_{k-1} + (1,0)$, $\lambda_k = \lambda_{k-1} - e_{t_k}$ when $z_k = z_{k-1} + (0,1)$. The proof we gave for stairstep shapes shows also that every placement of rooks in the Ferrers diagram, with at most one rook in each row and at most one in each column, corresponds to a unique tableau loop of this kind.

[And much more is true, besides! See S. Fomin, *J. Combin. Theory* **A72** (1995), 277–292; M. van Leeuwen, *Electronic J. Combinatorics* **3**, 2 (1996), paper #R15.]

28. (a) Define a one-to-one correspondence between rook placements, by interchanging the positions of rooks in rows j and $j+1$ if and only if there's a rook in the "panhandle" of the longer row:



(b) This relation is obvious from the definition, by transposing all the rooks.

(c) Suppose $a_1 \ge a_2 \ge \cdots$ and $a_k > a_{k+1}$. Then we have

$$R(a_1, a_2, \ldots) = x R(a_1-1, \ldots, a_{k-1}-1, a_{k+1}, \ldots) + y R(a_1, \ldots, a_{k-1}, a_k-1, a_{k+1}, \ldots)$$

because the first term counts cases where a rook is in row k and column a_k. Also $R(0) = 1$ because of the empty placement. From these recurrences we find

$$R(1) = x + y;\;\; R(2) = R(1,1) = x + xy + y^2;\;\; R(3) = R(1,1,1) = x + xy + xy^2 + y^3;$$
$$R(2,1) = x^2 + 2xy + xy^2 + y^3;$$
$$R(3,1) = R(2,2) = R(2,1,1) = x^2 + x^2y + xy + 2xy^2 + xy^3 + y^4;$$
$$R(3,1,1) = R(3,2) = R(2,2,1) = x^2 + 2x^2y + x^2y^2 + 2xy^2 + 2xy^3 + xy^4 + y^5;$$
$$R(3,2,1) = x^3 + 3x^2y + 3x^2y^2 + x^2y^3 + 3xy^3 + 2xy^4 + xy^5 + y^6.$$

(d) For example, the formula $\varpi_{73}(x,y) = x\varpi_{63}(x,y) + y\varpi_{74}(x,y)$ is equivalent to $R(5,4,4,3,2,1) = xR(4,3,3,2,1) + yR(5,4,3,3,2,1)$, a special case of (c); and $\varpi_{nn}(x,y) = R(n-2,\ldots,0)$ is obviously equal to $\varpi_{(n-1)1}(x,y) = R(n-2,\ldots,1)$.

(e) In fact $y^{k-1}\varpi_{nk}(x,y)$ is the stated sum over all restricted growth strings $a_1\ldots a_n$ for which $a_2>0, \ldots, a_k>0$.

29. (a) If the rooks are respectively in columns $(c_1,\ldots,c_n)$, the number of free cells is the number of inversions of the permutation $(n{+}1{-}c_1)\ldots(n{+}1{-}c_n)$. [Rotate the right-hand example of Fig. 35 by 180° and compare the result to the illustration following Eq. 5.1.1–(5).]

(b) Each $r\times r$ configuration can be placed in, say, rows $i_1<\cdots<i_r$ and columns $j_1<\cdots<j_r$, yielding $(m-r)(n-r)$ free cells in the unchosen rows and columns; there are $(i_2{-}i_1{+}1)+2(i_3{-}i_2{-}1)+\cdots+(r{-}1)(i_r{-}i_{r-1}{-}1)+r(m{-}i_r)$ in the unchosen rows and chosen columns, and a similar number in the chosen rows and unchosen columns. Furthermore

$$\sum_{1\le i_1<\cdots<i_r\le m} y^{(i_2-i_1+1)+2(i_3-i_2-1)+\cdots+(r-1)(i_r-i_{r-1}-1)+r(m-i_r)}$$

may be regarded as the sum of $y^{a_1+a_2+\cdots+a_{m-r}}$ over all partitions $r\ge a_1\ge a_2\ge\cdots\ge a_{m-r}\ge 0$, so it is $\binom{m}{r}_y$ by Theorem C. The polynomial $r!_y$ generates free cells for the chosen rows and columns, by (a). Therefore the answer is $y^{(m-r)(n-r)}\binom{m}{r}_y\binom{n}{r}_y r!_y = y^{(m-r)(n-r)}m!_y n!_y/((m-r)!_y(n-r)!_y r!_y)$.

(c) The left-hand side is the generating function $R_m(t+a_1,\ldots,t+a_m)$ for the Ferrers diagram with t additional columns of height m. For there are $t+a_m$ ways to put a rook in row m, yielding $1+y+\cdots+y^{t+a_m-1}=(1-y^{t+a_m})/(1-y)$ free cells with respect to those choices; then there are $t+a_{m-1}-1$ available cells in row $m-1$, etc.

The right-hand side, likewise, equals $R_m(t+a_1,\ldots,t+a_m)$. For if $m-k$ rooks are placed into columns $>t$, we must put k rooks into columns $\le t$ of the k unused rows; and we have seen that $t!_y/(t-k)!_y$ is the generating function for free cells when k rooks are placed on a $k\times t$ board.

[The formula proved here can be regarded as a polynomial identity in the variables y and y^t; therefore it is valid for arbitrary t, although our proof assumed that t is a nonnegative integer. This result was discovered in the case $y=1$ by J. Goldman, J. Joichi, and D. White, *Proc. Amer. Math. Soc.* **52** (1975), 485–492. The general case was established by A. M. Garsia and J. B. Remmel, *J. Combinatorial Theory* **A41** (1986), 246–275, who used a similar argument to prove the additional formula

$$\sum_{t=0}^{\infty} z^t \prod_{j=1}^{m}\frac{1-y^{a_j+m-j+t}}{1-y} = \sum_{k=0}^{n} k!_y\left(\frac{z}{1-yz}\right)\ldots\left(\frac{z}{1-y^k z}\right)R_{m-k}(a_1,\ldots,a_m).$$

(d) This statement, which follows immediately from (c), also implies that we have $R(a_1,\ldots,a_m)=R(a'_1,\ldots,a'_m)$ if and only if equality holds for all x and for any nonzero value of y. The Peirce polynomial $\varpi_{nk}(x,y)$ of exercise 28(d) is the rook polynomial for $\binom{n-1}{k-1}$ different Ferrers diagrams; for example, $\varpi_{63}(x,y)$ enumerates rook placements for the shapes 43321, 44221, 44311, 4432, 53221, 53311, 5332, 54211, 5422, and 5431.

30. (a) We have $\varpi_n(x,y)=\sum_m x^{n-m}A_{mn}$, where $A_{mn}=R_{n-m}(n{-}1,\ldots,1)$ satisfies a simple law: If we don't place a rook in row 1 of the shape $(n-1,\ldots,1)$, that row has $m-1$ free cells because of the $n-m$ rooks in other rows. But if we do put a rook there, we leave 0 or 1 or $\cdots$ or $m-1$ of its cells free. Hence $A_{mn} = y^{m-1}A_{(m-1)(n-1)} + (1+y+\cdots+y^{m-1})A_{m(n-1)}$, and it follows by induction that $A_{mn} = y^{m(m-1)/2}\left\{{n \atop m}\right\}_y$.

(b) The formula $\varpi_{n+1}(x,y) = \sum_k \binom{n}{k} x^{n-k} y^k \varpi_k(x,y)$ yields

$$A_{m(n+1)} = \sum_k \binom{n}{k} y^k A_{(m-1)k}.$$

(c) From (a) and (b) we have

$$\frac{z^n}{(1-z)(1-(1+q)z)\ldots(1-(1+q+\cdots+q^{n-1})z)} = \sum_k \left\{{k \atop n}\right\}_q z^k;$$

$$\sum_k \binom{n}{k}_q (-1)^k q^{\binom{k}{2}} e^{(1+q+\cdots+q^{n-k-1})z} = q^{\binom{n}{2}}\, n!_q \sum_k \left\{{k \atop n}\right\}_q \frac{z^k}{k!}.$$

[The second formula is proved by induction on n, because both sides satisfy the differential equation $G'_{n+1}(z) = (1+q+\cdots+q^n)e^z G_n(qz)$; exercise 1.2.6–58 proves equality when $z=0$.]

Historical note: Leonard Carlitz introduced q-Stirling numbers in *Transactions of the Amer. Math. Soc.* **33** (1933), 127–129. Then in *Duke Math. J.* **15** (1948), 987–1000, he derived (among other things) an appropriate generalization of Eq. 1.2.6–(45):

$$(1+q+\cdots+q^{m-1})^n \;=\; \sum_k \left\{{n \atop k}\right\}_q q^{\binom{k}{2}} \frac{m!_q}{(m-k)!_q}.$$

31. $\exp(e^{w+z}+w-1)$; therefore $\varpi_{nk} = (\varpi+1)^{n-k}\varpi^{k-1} = \varpi^{n+1-k}(\varpi-1)^{k-1}$ in the umbral notation of exercise 23. [L. Moser and M. Wyman, *Trans. Royal Soc. Canada* (3) **43** (1954), Section 3, 31–37.] In fact, the numbers $\varpi_{nk}(x,1)$ of exercise 28(d) are generated by $\exp\bigl((e^{xw+xz}-1)/x+xw\bigr)$.

32. We have $\delta_n = \varpi_n(1,-1)$, and a simple pattern is easily perceived in the generalized Peirce triangle of exercise 28(d) when $x=1$ and $y=-1$: We have $|\varpi_{nk}(1,-1)| \le 1$ and $\varpi_{n(k+1)}(1,-1) \equiv \varpi_{nk}(1,-1) + (-1)^n$ (modulo 3) for $1 \le k < n$. [In *JACM* **20** (1973), 512–513, Gideon Ehrlich gave a combinatorial proof of an equivalent result.]

33. Representing set partitions by rook placements as in answer 27 leads to the answer ϖ_{nk}, by setting $x=y=1$ in exercise 28(d). [The case $k=n$ was discovered by H. Prodinger, *Fibonacci Quarterly* **19** (1981), 463–465.]

34. (a) Guittone's *Sonetti* included 149 of scheme 01010101232323, 64 of scheme 01010101234234, two of scheme 01010101234342, seven with schemes used only once (like 01100110234432), and 29 poems that we would no longer consider to be sonnets because they do not have 14 lines.

(b) Petrarch's *Canzoniere* included 115 sonnets of scheme 01100110234234, 109 of scheme 01100110232323, 66 of scheme 01100110234324, 7 of scheme 01100110232232, and 20 others of schemes like 01010101232323 used at most three times each.

(c) In Spenser's *Amoretti*, 88 of 89 sonnets used the scheme 01011212232344; the exception (number 8) was "Shakespearean."

(d) Shakespeare's 154 sonnets all used the rather easy scheme 01012323454566, except that two of them (99 and 126) didn't have 14 lines.

(e) Browning's 44 *Sonnets From the Portuguese* obeyed the Petrarchan scheme 01100110232323.

Sometimes the lines would rhyme (by chance?) even when they didn't need to; for example, Browning's final sonnet actually had the scheme 01100110121212.

Incidentally, the lengthy cantos in Dante's *Divine Comedy* used an interlocking scheme of rhymes in which $1 \equiv 3$ and $3n-1 \equiv 3n+1 \equiv 3n+3$ for $n = 1, 2, \ldots$.

35. Every incomplete n-line rhyme scheme Π corresponds to a singleton-free partition of $\{1,\ldots,n{+}1\}$ in which $(n{+}1)$ is grouped with all of Π's singletons. [H. W. Becker gave an algebraic proof in *AMM* **48** (1941), 702. Notice that $\varpi'_n = \sum_k \binom{n}{k}(-1)^{n-k}\varpi_k$, by the principle of inclusion and exclusion, and $\varpi_n = \sum_k \binom{n}{k}\varpi'_k$; we can in fact write $\varpi' = \varpi - 1$ in the umbral notation of exercise 23. J. O. Shallit has suggested extending Peirce's triangle by setting $\varpi_{n(n+1)} = \varpi'_n$; see exercises 38(e) and 33. In fact, ϖ_{nk} is the number of partitions of $\{1,\ldots,n\}$ with the property that $1, \ldots, k-1$ are not singletons; see H. W. Becker, *Bull. Amer. Math. Soc.* **58** (1954), 63.]

36. $\exp(e^z - 1 - z)$. (In general, if ϑ_n is the number of partitions of $\{1,\ldots,n\}$ into subsets of allowable sizes $s_1 < s_2 < \cdots$, the exponential generating function $\sum_n \vartheta_n z^n/n!$ is $\exp(z^{s_1}/s_1! + z^{s_2}/s_2! + \cdots)$, because $(z^{s_1}/s_1! + z^{s_2}/s_2! + \cdots)^k$ is the exponential generating function for partitions into exactly k parts.)

37. There are $\sum_k \binom{n}{k}\varpi'_k\varpi'_{n-k}$ possibilities of length n, hence 784,071,966 when $n = 14$. (But Pushkin's scheme is hard to beat.)

38. (a) Imagine starting with $x_1x_2\ldots x_n = 0\,1\ldots(n{-}1)$, then successively removing some element b_j and placing it at the left, for $j = 1, 2, \ldots, n$. Then x_k will be the kth most recently moved element, for $1 \le k \le |\{b_1,\ldots,b_n\}|$; see exercise 5.2.3–36. Consequently the array $x_1\ldots x_n$ will return to its original state if and only if $b_n\ldots b_1$ is a restricted growth string. [Robbins and Bolker, *Æquat. Math.* **22** (1981), 281–282.]

In other words, let $a_1\ldots a_n$ be a restricted growth string. Set $b_{-j} \leftarrow j$ and $b_{j+1} \leftarrow a_{n-j}$ for $0 \le j < n$. Then for $1 \le j \le n$, define k_j by the rule that b_j is the k_jth distinct element of the sequence $b_{j-1}, b_{j-2}, \ldots$. For example, the string $a_1\ldots a_{16} =$ 0123032303456745 corresponds in this way to the σ-cycle 6688448628232384.

(b) Such paths correspond to restricted growth strings with $\max(a_1,\ldots,a_n) \le m$, so the answer is ${n \brace 0} + {n \brace 1} + \cdots + {n \brace m}$.

(c) We may assume that $i = 1$, because the sequence $k_2\ldots k_nk_1$ is a σ-cycle whenever $k_1k_2\ldots k_n$ is. Thus the answer is the number of restricted growth strings with $a_n = j-1$, namely ${n-1 \brace j-1} + {n-1 \brace j} + {n-1 \brace j+1} + \cdots$.

(d) If the answer is f_n we must have $\sum_k \binom{n}{k} f_k = \varpi_n$, since σ_1 is the identity permutation. Therefore $f_n = \varpi'_n$, the number of set partitions without singletons (exercise 35).

(e) Again ϖ'_n, by (a) and (d). [Consequently $\varpi'_p \bmod p = 1$ when p is prime.]

39. Set $u = t^{p+1}$ to obtain $\frac{1}{p+1}\int_0^\infty e^{-u}u^{(q-p)/(p+1)}\,du = \frac{1}{p+1}\Gamma\bigl(\frac{q+1}{p+1}\bigr)$.

40. We have $g(z) = cz - n\ln z$, so the saddle point occurs at n/c. The rectangular path now has corners at $\pm n/c \pm mi/c$; and $\exp g(n/c + it) = (e^nc^n/n^n)\exp(-t^2c^2/(2n) + it^3c^3/(3n^2) + \cdots)$. The final result is $e^n(c/n)^{n-1}/\sqrt{2\pi n}$ times $1 + n/12 + O(n^{-2})$.

(Of course we could have obtained this result more quickly by letting $w = cz$ in the integral. But the answer given here applies the saddle point method mechanically, without attempting to be clever.)

41. Again the net result is just to multiply (21) by c^{n-1}; but in this case the *left* edge of the rectangular path is significant instead of the right edge. (Incidentally, when $c = -1$ we cannot derive an analog of (22) using Hankel's contour when x is real and positive, because the integral on that path diverges. But with the usual definition of z^x, a suitable path of integration does yield the formula $-(\cos \pi x)/\Gamma(x)$ when $n = x > 0$.)

42. We have $\oint e^{z^2} dz/z^n = 0$ when n is even. Otherwise both left and right edges of the rectangle with corners $\pm\sqrt{n/2} \pm in$ contribute approximately

$$\frac{e^{n/2}}{2\pi(n/2)^{n/2}} \int_{-\infty}^{\infty} \exp\Bigl(-2t^2 - \frac{(-it)^3}{3}\frac{2^{3/2}}{n^{1/2}} + \frac{(it)^4}{n} - \cdots\Bigr)\, dt,$$

when n is large. We can restrict $|t| \le n^{\epsilon}$ to show that this integral is $I_0 + (I_4 - \frac{4}{9}I_6)/n$ with relative error $O(n^{9\epsilon-3/2})$, where $I_k = \int_{-\infty}^{\infty} e^{-2t^2} t^k\, dt$. As before, the relative error is actually $O(n^{-2})$; we deduce the answer

$$\frac{1}{((n-1)/2)!} \;=\; \frac{e^{n/2}}{\sqrt{2\pi}(n/2)^{n/2}}\Bigl(1 + \frac{1}{12n} + O\Bigl(\frac{1}{n^2}\Bigr)\Bigr), \qquad n \text{ odd}.$$

(The analog of (22) is $(\sin \frac{\pi x}{2})^2/\Gamma((x-1)/2)$ when $n = x > 0$.)

43. Let $f(z) = e^{e^z}/z^n$. When $z = -n + it$ we have $|f(z)| < en^{-n}$; when $z = t + 2\pi in + i\pi/2$ we have $|f(z)| = |z|^{-n} < (2\pi n)^{-n}$. So the integral is negligible except on a path $z = \xi + it$; and on that path $|f|$ decreases as $|t|$ increases from 0 to π. Already when $t = n^{\epsilon-1/2}$ we have $|f(z)|/f(\xi) = O(\exp(-n^{2\epsilon}/(\log n)^2))$. And when $|t| > \pi$ we have $|f(z)|/f(\xi) < 1/|1 + i\pi/\xi|^n = \exp(-\frac{n}{2}\ln(1 + \pi^2/\xi^2))$.

44. Set $u = na_2t^2$ in (25) to obtain $\Re\int_0^{\infty} e^{-u}\exp(n^{-1/2}c_3(-u)^{3/2} + n^{-1}c_4(-u)^2 + n^{-3/2}c_5(-u)^{5/2} + \cdots)\, du/\sqrt{na_2u}$ where $c_k = (2/(\xi+1))^{k/2}(\xi^{k-1} + (-1)^k(k-1)!)/k! = a_k/a_2^{k/2}$. This expression leads to

$$b_l = \sum_{\substack{k_1+2k_2+3k_3+\cdots=2l\\ k_1+k_2+k_3+\cdots=m\\ k_1,k_2,k_3,\ldots\ge 0}} \Bigl(-\frac{1}{2}\Bigr)^{\underline{l+m}} \frac{c_3^{k_1}}{k_1!}\frac{c_4^{k_2}}{k_2!}\frac{c_5^{k_3}}{k_3!}\cdots,$$

a sum over partitions of $2l$. For example, $b_1 = \frac{3}{4}c_4 - \frac{15}{16}c_3^2$.

45. To get $\varpi_n/n!$ we replace $g(z)$ by $e^z - (n+1)\ln z$ in the derivation of (26). This change multiplies the integrand in the previous answer by $1/(1 + it/\xi)$, which is $1/(1 - n^{-1/2}a(-u)^{1/2})$ where $a = -\sqrt{2/(\xi+1)}$. Thus we get

$$b'_l = \sum_{\substack{k+k_1+2k_2+3k_3+\cdots=2l\\ k_1+k_2+k_3+\cdots=m\\ k,k_1,k_2,k_3,\ldots\ge 0}} \Bigl(-\frac{1}{2}\Bigr)^{\underline{l+m}} a^k \frac{c_3^{k_1}}{k_1!}\frac{c_4^{k_2}}{k_2!}\frac{c_5^{k_3}}{k_3!}\cdots,$$

a sum of $p(2l) + p(2l-1) + \cdots + p(0)$ terms; $b'_1 = \frac{3}{4}c_4 - \frac{15}{16}c_3^2 + \frac{3}{4}ac_3 - \frac{1}{2}a^2$. [The coefficient b'_1 was obtained in a different way by L. Moser and M. Wyman, *Trans. Royal Soc. Canada* (3) **49**, Section 3 (1955), 49–54, who were the first to deduce an asymptotic series for ϖ_n. Their approximation is slightly less accurate than the result of (26) with n changed to $n+1$, because it doesn't pass exactly through the saddle point. Formula (26) is due to I. J. Good, *Iranian J. Science and Tech.* **4** (1975), 77–83.]

46. Eqs. (13) and (31) show that $\varpi_{nk} = (1 - \xi/n)^k\varpi_n(1 + O(n^{-1}))$ for fixed k as $n \to \infty$. And this approximation also holds when $k = n$, but with relative error $O((\log n)^2/n)$.

47. Steps (H1, ..., H6) are performed respectively $(1, \varpi_n, \varpi_n - \varpi_{n-1}, \varpi_{n-1}, \varpi_{n-1}, \varpi_{n-1} - 1)$ times. The loop in H4 sets $j \leftarrow j - 1$ a total of $\varpi_{n-2} + \varpi_{n-3} + \cdots + \varpi_1$ times; the loop in H6 sets $b_j \leftarrow m$ a total of $(\varpi_{n-2} - 1) + \cdots + (\varpi_1 - 1)$ times. The ratio ϖ_{n-1}/ϖ_n is approximately $(\ln n)/n$, and $(\varpi_{n-2} + \cdots + \varpi_1)/\varpi_n \approx (\ln n)^2/n^2$.

48. We can easily verify the interchange of summation and integration in

$$\frac{e\varpi_x}{\Gamma(x+1)} = \frac{1}{2\pi i}\oint \frac{e^{e^z}}{z^{x+1}}\,dz = \frac{1}{2\pi i}\oint \sum_{k=0}^{\infty}\frac{e^{kx}}{k!\,z^{x+1}}\,dz$$
$$= \sum_{k=0}^{\infty}\frac{1}{k!}\frac{1}{2\pi i}\oint \frac{e^{kz}}{z^{x+1}}\,dz = \sum_{k=0}^{\infty}\frac{1}{k!}\frac{k^x}{\Gamma(x+1)}.$$

49. If $\xi = \ln n - \ln\ln n + x$, we have $\beta = 1 - e^{-x} - \alpha x$. Therefore by Lagrange's inversion formula (exercise 4.7–8),

$$x \;=\; \sum_{k=1}^{\infty}\frac{\beta^k}{k}\,[t^{k-1}]\left(\frac{f(t)}{1-\alpha f(t)}\right)^k \;=\; \sum_{k=1}^{\infty}\sum_{j=0}^{\infty}\frac{\beta^k}{k}\alpha^j\binom{k+j-1}{j}[t^{k-1}]\,f(t)^{j+k},$$

where $f(t) = t/(1-e^{-t})$. So the result follows from the handy identity

$$\left(\frac{z}{1-e^{-z}}\right)^m \;=\; \sum_{n=0}^{\infty}\left[{m \atop m-n}\right]\frac{z^n}{(m-1)(m-2)\ldots(m-n)}.$$

(This identity should be interpreted carefully when $n \ge m$; the coefficient of z^n is a polynomial in m of degree n, as explained in *CMath* equation (7.59).)

The formula in this exercise is due to L. Comtet, *Comptes Rendus Acad. Sci.* (A) **270** (Paris, 1970), 1085–1088, who identified the coefficients previously computed by N. G. de Bruijn, *Asymptotic Methods in Analysis* (1958), 25–28. Convergence for $n \ge e$ was shown by Jeffrey, Corless, Hare, and Knuth, *Comptes Rendus Acad. Sci.* (I) **320** (1995), 1449–1452, who also derived a formula that converges somewhat faster.

(The equation $\xi e^{\xi} = n$ has complex roots as well. We can obtain them all by using $\ln n + 2\pi i m$ in place of $\ln n$ in the formula of this exercise; the sum converges rapidly when $m \ne 0$. See Corless, Gonnet, Hare, Jeffrey, and Knuth, *Advances in Computational Math.* **5** (1996), 347–350.)

50. Let $\xi = \xi(n)$. Then $\xi'(n) = \xi/((\xi+1)n)$, and the Taylor series

$$\xi(n+k) \;=\; \xi + k\xi'(n) + \frac{k^2}{2}\xi''(n) + \cdots$$

can be shown to converge for $|k| < n + 1/e$.

Indeed, much more is true, because the function $\xi(n) = -T(-n)$ is obtained from the tree function $T(z)$ by analytic continuation to the negative real axis. (The tree function has a quadratic singularity at $z = e^{-1}$; after going around this singularity we encounter a logarithmic singularity at $z = 0$, as part of an interesting multi-level Riemann surface on which the quadratic singularity appears only at level 0.) The derivatives of the tree function satisfy $z^k T^{(k)}(z) = R(z)^k p_k(R(z))$, where $R(z) = T(z)/(1-T(z))$ and $p_k(x)$ is the polynomial of degree $k-1$ defined by $p_{k+1}(x) = (1+x)^2 p_k'(x) + k(2+x)p_k(x)$. For example,

$$p_1(x) = 1,\;\; p_2(x) = 2 + x,\;\; p_3(x) = 9 + 10x + 3x^2,\;\; p_4(x) = 64 + 113x + 70x^2 + 15x^3.$$

(The coefficients of $p_k(x)$, incidentally, enumerate certain phylogenetic trees called Greg trees: $[x^j]\,p_k(x)$ is the number of oriented trees with j unlabeled nodes and k labeled nodes, where leaves must be labeled and unlabeled nodes must have at least two children. See J. Felsenstein, *Systematic Zoology* **27** (1978), 27–33; L. R. Foulds and R. W. Robinson, *Lecture Notes in Math.* **829** (1980), 110–126; C. Flight, *Manuscripta* **34** (1990), 122–128.) If $q_k(x) = p_k(-x)$, we can prove by induction that $(-1)^m q_k^{(m)}(x) \ge 0$ for $0 \le x \le 1$. Therefore $q_k(x)$ decreases monotonically from k^{k-1} to $(k-1)!$ as x goes from 0 to 1, for all $k, m \ge 1$. It follows that

$$\xi(n+k) = \xi + \frac{kx}{n} - \left(\frac{kx}{n}\right)^2 \frac{q_2(x)}{2!} + \left(\frac{kx}{n}\right)^3 \frac{q_3(x)}{3!} - \cdots, \qquad x = \frac{\xi}{\xi+1},$$

where the partial sums alternately overshoot and undershoot the correct value if $k > 0$.

51. There are two saddle points, $\sigma = \sqrt{n+5/4} - 1/2$ and $\sigma' = -1-\sigma$. Integration on a rectangular path with corners at $\sigma \pm im$ and $\sigma' \pm im$ shows that only σ is relevant as $n \to \infty$ (although σ' contributes a relative error of roughly $e^{-\sqrt{n}}$, which can be significant when n is small). Arguing almost as in (25), but with $g(z) = z + z^2/2 - (n+1)\ln z$, we find that t_n is well approximated by

$$\frac{n!}{2\pi} \int_{-n^\epsilon}^{n^\epsilon} e^{g(\sigma) - a_2 t^2 + a_3 i t^3 + \cdots + a_l (-it)^l + O(n^{(l+1)\epsilon - (l-1)/2})}\, dt, \qquad a_k = \frac{\sigma+1}{k\sigma^{k-1}} + \frac{[k=2]}{2}.$$

The integral expands as in exercise 44 to

$$\frac{n!\, e^{(n+\sigma)/2}}{2\sigma^{n+1}\sqrt{\pi a_2}} \bigl(1 + b_1 + b_2 + \cdots + b_m + O(n^{-m-1})\bigr).$$

This time $c_k = (\sigma+1)\sigma^{1-k}(1 + 1/(2\sigma))^{-k/2}/k$ for $k \ge 3$, hence $(2\sigma+1)^{3k}\sigma^k b_k$ is a polynomial in σ of degree $2k$; for example,

$$b_1 \;=\; \frac{3}{4}c_4 - \frac{15}{16}c_3^2 \;=\; \frac{8\sigma^2 + 7\sigma - 1}{12\sigma(2\sigma+1)^3}.$$

In particular, Stirling's approximation and the b_1 term yield

$$t_n = \frac{1}{\sqrt{2}} n^{n/2} e^{-n/2 + \sqrt{n} - 1/4} \left(1 + \frac{7}{24} n^{-1/2} - \frac{119}{1152} n^{-1} - \frac{7933}{414720} n^{-3/2} + O(n^{-2})\right)$$

after we plug in the formula for σ — a result substantially more accurate than equation 5.1.4–(53), and obtained with considerably less labor.

52. Let $G(z) = \sum_k \Pr(X = k) z^k$, so that the jth cumulant κ_j is $j!\,[t^j]\ln G(e^t)$. In case (a) we have $G(z) = e^{e^{\xi z} - e^{\xi}}$; hence

$$\ln G(e^t) = e^{\xi e^t} - e^{\xi} = e^{\xi}(e^{\xi(e^t-1)} - 1) = e^{\xi} \sum_{k=1}^{\infty} (e^t - 1)^k \frac{\xi^k}{k!}, \quad \kappa_j = e^{\xi} \sum_k \left\{ {k \atop j} \right\} \xi^k [j \ne 0].$$

Case (b) is sort of a dual situation: Here $\kappa = j = \varpi_j\,[j \ne 0]$ because

$$G(z) = e^{e^{-1}-1} \sum_{j,k} \left\{ {k \atop j} \right\} e^{-j} \frac{z^k}{k!} = e^{e^{-1}-1} \sum_j \frac{(e^{z-1} - e^{-1})^j}{j!} = e^{e^{z-1}-1}.$$

[If $\xi e^{\xi} = 1$ in case (a) we have $\kappa_j = e\varpi\,[j \ne 0]$. But if $\xi e^{\xi} = n$ in that case, the mean is $\kappa_1 = n$ and the variance σ^2 is $(\xi+1)n$. Thus, the formula in exercise 45 states that the mean value n occurs with approximate probability $1/\sqrt{2\pi\sigma}$ and relative error $O(1/n)$. This observation leads to another way to prove that formula.]

53. We can write $\ln G(e^t) = \mu t + \sigma^2 t^2/2 + \kappa_3 t^3/3! + \cdots$ as in Eq. 1.2.10–(23), and there is a positive constant δ such that $\sum_{j=3}^{\infty} |\kappa_j|\,|t|^j/j! < \sigma^2 t^2/6$ when $|t| \le \delta$. Hence, if $0 < \epsilon < 1/2$, we can prove that

$$\begin{aligned}[z^{\mu n+r}]\,G(z)^n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{G(e^{it})^n\,dt}{e^{it(\mu n+r)}}\\ &= \frac{1}{2\pi}\int_{-n^{\epsilon-1/2}}^{n^{\epsilon-1/2}} \exp\Bigl(-irt - \frac{\sigma^2 t^2 n}{2} + O(n^{3\epsilon-1/2})\Bigr)\,dt + O(e^{-cn^{2\epsilon}})\end{aligned}$$

as $n \to \infty$, for some constant $c > 0$: The integrand for $n^{\epsilon-1/2} \le |t| \le \delta$ is bounded in absolute value by $\exp(-\sigma^2 n^{2\epsilon}/3)$; and when $\delta \le |t| \le \pi$ its magnitude is at most α^n, where $\alpha = \max|G(e^{it})|$ is less than 1 because the individual terms $p_k e^{kit}$ don't all lie on a straight line by our assumption. Thus

$$\begin{aligned}[z^{\mu n+r}]\,G(z)^n &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\Bigl(-irt - \frac{\sigma^2 t^2 n}{2} + O(n^{3\epsilon-1/2})\Bigr)\,dt + O(e^{-cn^{2\epsilon}})\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\Bigl(-\frac{\sigma^2 n}{2}\Bigl(t + \frac{ir}{\sigma^2 n}\Bigr)^2 - \frac{r^2}{2\sigma^2 n} + O(n^{3\epsilon-1/2})\Bigr)\,dt + O(e^{-cn^{2\epsilon}})\\ &= \frac{e^{-r^2/(2\sigma^2 n)}}{\sigma\sqrt{2\pi n}} + O(n^{3\epsilon-1}).\end{aligned}$$

By taking account of κ_3, κ_4, … in a similar way we can refine the estimate to $O(n^{-m})$ for arbitrarily large m; thus the result is valid also for $\epsilon = 0$. [In fact, such refinements lead to the "Edgeworth expansion," according to which $[z^{\mu n+r}]\,G(z)^n$ is asymptotic to

$$\frac{e^{-r^2/(2\sigma^2 n)}}{\sigma\sqrt{2\pi n}} \sum_{\substack{k_1+2k_2+3k_3+\cdots=m\\ k_1+k_2+k_3+\cdots=l\\ k_1,k_2,k_3,\ldots\ge 0\\ 0\le s\le l+m/2}} \frac{(-1)^s(2l+m)^{\underline{2s}}}{\sigma^{4l+2m-2s}2^s s!}\,\frac{r^{2l+m-2s}}{n^{l+m-s}}\,\frac{1}{k_1!\,k_2!\,\ldots}\Bigl(\frac{\kappa_3}{3!}\Bigr)^{k_1}\Bigl(\frac{\kappa_4}{4!}\Bigr)^{k_2}\ldots;$$

the absolute error is $O(n^{-p/2})$, where the constant hidden in the O depends only on p and G but not on r or n, if we restrict the sum to cases with $m < p-1$. For example, when $p = 3$ we get

$$[z^{\mu n+r}]\,G(z)^n = \frac{e^{-r^2/(2\sigma^2 n)}}{\sigma\sqrt{2\pi n}}\Bigl(1 - \frac{\kappa_3}{2\sigma^4}\Bigl(\frac{r}{n}\Bigr) + \frac{\kappa_3}{6\sigma^6}\Bigl(\frac{r^3}{n^2}\Bigr)\Bigr) + O\Bigl(\frac{1}{n^{3/2}}\Bigr),$$

and there are seven more terms when $p = 4$. See P. L. Chebyshev, *Zapiski Imp. Akad. Nauk* **55** (1887), No. 6, 1–16; *Acta Math.* **14** (1890), 305–315; F. Y. Edgeworth, *Trans. Cambridge Phil. Soc.* **20** (1905), 36–65, 113–141; H. Cramér, *Skandinavisk Aktuarietidsskrift* **11** (1928), 13–74, 141–180.]

54. Formula (40) is equivalent to $\alpha = s\coth s + s$, $\beta = s\coth s - s$.

55. Let $c = \alpha e^{-\alpha}$. The Newtonian iteration $\beta_0 = c$, $\beta_{k+1} = (1-\beta_k)ce^{\beta_k}/(1-ce^{-\beta_k})$ rises rapidly to the correct value, unless α is extremely close to 1. For example, β_7 differs from $\ln 2$ by less than 10^{-75} when $\alpha = \ln 4$.

56. (a) By induction on n, $g^{(n+1)}(z) = (-1)^n\Bigl(\dfrac{\sum_{k=0}^{n}\left\langle {n \atop k} \right\rangle e^{(n-k)z}}{\alpha(e^z-1)^{n+1}} - \dfrac{n!}{z^{n+1}}\Bigr)$.

(b) $\sum_{k=0}^{n} \left\langle {n \atop k} \right\rangle e^{k\sigma}/n! = \int_0^1 \ldots \int_0^1 \exp(\lfloor u_1 + \cdots + u_n \rfloor \sigma)\, du_1 \ldots du_n$
$< \int_0^1 \ldots \int_0^1 \exp((u_1 + \cdots + u_n)\sigma)\, du_1 \ldots du_n = (e^\sigma - 1)^n/\sigma^n.$
The lower bound is similar, since $\lfloor u_1 + \cdots + u_n \rfloor > u_1 + \cdots + u_n - 1$.

(c) Thus $n!\,(1-\beta/\alpha) < (-\sigma)^n g^{(n+1)}(\sigma) < 0$, and we need only verify that $1-\beta/\alpha < 2(1-\beta)$, namely that $2\alpha\beta < \alpha + \beta$. But $\alpha\beta < 1$ and $\alpha + \beta > 2$, by exercise 54.

57. (a) $n + 1 - m = (n+1)(1 - 1/\alpha) < (n+1)(1 - \beta/\alpha) = (n+1)\sigma/\alpha \le 2N$ as in answer 56(c). (b) The quantity $\alpha + \alpha\beta$ increases as α increases, because its derivative with respect to α is $1 + \beta + \beta(1-\alpha)/(1-\beta) = (1-\alpha\beta)/(1-\beta) + \beta > 0$. Therefore $1 - \beta < 2(1 - 1/\alpha)$.

58. (a) The derivative of $|e^{\sigma+it} - 1|^2/|\sigma + it|^2 = (e^{\sigma+it} - 1)(e^{\sigma-it} - 1)/(\sigma^2 + t^2)$ with respect to t is $(\sigma^2 + t^2)\sin t - t(2\sin\frac{t}{2})^2 - (2\sinh\frac{\sigma}{2})^2 t$ times a positive function. This derivative is always negative for $0 < t \le 2\pi$, because it is less than $t^2 \sin t - t(2\sin\frac{t}{2})^2 = 8u\sin u\cos u(u - \tan u)$ where $t = 2u$.

Let $s = 2\sinh\frac{\sigma}{2}$. When $\sigma \ge \pi$ and $2\pi \le t \le 4\pi$, the derivative is still negative, because we have $t \le 4\pi \le s^2 - \sigma^2/(2\pi) \le s^2 - \sigma^2/t$. Similarly, when $\sigma \ge 2\pi$ the derivative remains negative for $4\pi \le t \le 168\pi$; the proof gets easier and easier.

(b) Let $t = u\sigma/\sqrt{N}$. Then (41) and (42) prove that

$$\int_{-\tau}^{\tau} e^{(n+1)g(\sigma+it)}\, dt =$$
$$\frac{(e^\sigma - 1)^m}{\sigma^n\sqrt{N}} \int_{-N^\epsilon}^{N^\epsilon} \exp\Bigl(-\frac{u^2}{2} + \frac{(-iu)^3 a_3}{N^{1/2}} + \cdots + \frac{(-iu)^l a_l}{N^{l/2-1}} + O(N^{(l+1)\epsilon-(l-1)/2})\Bigr)\, du,$$

where $(1-\beta)a_k$ is a polynomial of degree $k-1$ in α and β, with $0 \le a_k \le 2/k$. (For example, $6a_3 = (2 - \beta(\alpha+\beta))/(1-\beta)$ and $24a_4 = (6 - \beta(\alpha^2 + 4\alpha\beta + \beta^2))/(1-\beta)$.) The monotonicity of the integrand shows that the integral over the rest of the range is negligible. Now trade tails, extend the integral over $-\infty < u < \infty$, and use the formula of answer 44 with $c_k = 2^{k/2}a_k$ to define $b_1, b_2, \ldots$.

(c) We will prove that $|e^z - 1|^m \sigma^{n+1}/((e^\sigma - 1)^m |z|^{n+1})$ is exponentially small on those three paths. If $\sigma \le 1$, this quantity is less than $1/(2\pi)^{n+1}$ (because, for example, $e^\sigma - 1 > \sigma$). If $\sigma > 1$, we have $\sigma < 2|z|$ and $|e^z - 1| \le e^\sigma - 1$.

59. In this extreme case, $\alpha = 1 + n^{-1}$ and $\beta = 1 - n^{-1} + \frac{2}{3}n^{-2} + O(n^{-3})$; hence $N = 1 + \frac{1}{3}n^{-1} + O(n^{-2})$. The leading term $\beta^{-n}/\sqrt{2\pi N}$ is $e/\sqrt{2\pi}$ times $1 - \frac{1}{3}n^{-1} + O(n^{-2})$. (Notice that $e/\sqrt{2\pi} \approx 1.0844$.) The quantity a_k in answer 58(b) turns out to be $1/k + O(n^{-1})$. So the correction terms, to first order, are

$$\frac{b_j}{N^j} = [z^j]\exp\Bigl(-\sum_{k=1}^{\infty} \frac{B_{2k} z^{2k-1}}{2k(2k-1)}\Bigr) + O\Bigl(\frac{1}{n}\Bigr),$$

namely the terms in the (divergent) series corresponding to Stirling's approximation

$$\frac{1}{1!} \sim \frac{e}{\sqrt{2\pi}}\Bigl(1 - \frac{1}{12} + \frac{1}{288} + \frac{139}{51840} - \frac{571}{2488320} - \cdots\Bigr).$$

60. (a) The number of m-ary strings of length n in which all m digits appear is $m!\left\{ {n \atop m} \right\}$, and the inclusion-exclusion principle expresses this quantity as $\binom{m}{0}m^n - \binom{m}{1}(m-1)^n + \cdots$. Now see exercise 7.2.1.4–37.

(b) We have $(m-1)^n/(m-1)! = (m^n/m!)m\exp(n\ln(1-1/m))$, and $\ln(1-1/m)$ is less than $-n^{\epsilon-1}$.

(c) In this case $\alpha > n^\epsilon$ and $\beta = \alpha e^{-\alpha}e^\beta < \alpha e^{1-\alpha}$. Therefore $1 < (1-\beta/\alpha)^{m-n} < \exp(nO(e^{-\alpha}))$; and $1 > e^{-\beta m} = e^{-(n+1)\beta/\alpha} > \exp(-nO(e^{-\alpha}))$. So (45) becomes $(m^n/m!)\bigl(1+O(n^{-1})+O(ne^{-n^\epsilon})\bigr)$.

61. Now $\alpha = 1+\frac{r}{n}+O(n^{2\epsilon-2})$ and $\beta = 1-\frac{r}{n}+O(n^{2\epsilon-2})$. Thus $N = r+O(n^{2\epsilon-1})$, and the case $l=0$ of Eq. (43) reduces to

$$n^{\underline{r}}\left(\frac{n}{2}\right)^r\frac{e^r}{r^r\sqrt{2\pi r}}\left(1+O(n^{2\epsilon-1})+O\left(\frac{1}{r}\right)\right).$$

(This approximation meshes well with identities such as $\left\{{n \atop n-1}\right\} = \binom{n}{2}$ and $\left\{{n \atop n-2}\right\} = 2\binom{n}{4}+\binom{n+1}{4}$; indeed, we have

$$\left\{{n \atop n-r}\right\} = \frac{n^{2r}}{2^r r!}\left(1+O\left(\frac{1}{n}\right)\right) \qquad \text{as } n\to\infty$$

when r is constant, according to formulas (6.42) and (6.43) of *CMath*.)

62. The assertion is true for $1 \le n \le 10000$ (with $m = \lfloor e^\xi - 1\rfloor$ in 5648 of those cases). E. R. Canfield and C. Pomerance, in a paper that nicely surveys previous work on related problems, have shown that the statement holds for all sufficiently large n, and that the maximum occurs in *both* cases only if $e^\xi \bmod 1$ is extremely close to $\frac{1}{2}$. [*Integers* **2** (2002), A1, 1–13.]

63. (a) The result holds when $p_1 = \cdots = p_n = p$, because $a_{k-1}/a_k = \bigl(k/(n+1-k)\bigr)\times\bigl((n-\mu)/\mu\bigr) \le (n-\mu)/(n+1-\mu) < 1$. It is also true by induction when $p_n = 0$ or 1. For the general case, consider the minimum of $a_k - a_{k-1}$ over all choices of $(p_1,\ldots,p_n)$ with $p_1+\cdots+p_n = \mu$: If $0 < p_1 < p_2 < 1$, let $p_1' = p_1-\delta$ and $p_2' = p_2+\delta$, and notice that $a_k' - a_{k-1}' = a_k - a_{k-1} + \delta(p_1-p_2-\delta)\alpha$ for some α depending only on $p_3,\ldots,p_n$. At a minimum point we must have $\alpha = 0$; thus we can choose δ so that either $p_1' = 0$ or $p_2'{=}1$. The minimum can therefore be achieved when all p_j have one of three values $\{0,1,p\}$. But we have proved that $a_k - a_{k-1} > 0$ in such cases.

(b) Changing each p_j to $1-p_j$ changes μ to $n-\mu$ and a_k to a_{n-k}.

(c) No roots of $f(x)$ are positive. Hence $f(z)/f(1)$ has the form in (a) and (b).

(d) Let $C(f)$ be the number of sign changes in the sequence of coefficients of f; we want to show that $C((1-x)^2 f) = 2$. In fact, $C((1-x)^m f) = m$ for all $m \ge 0$. For $C((1-x)^m) = m$, and $C((a+bx)f) \le C(f)$ when a and b are positive; hence $C((1-x)^m f) \le m$. And if $f(x)$ is any nonzero polynomial whatsoever, $C((1-x)f) > C(f)$; hence $C((1-x)^m f) \ge m$.

(e) Since $\sum_k \left[{n \atop k}\right] x^k = x(x{+}1)\ldots(x{+}n{-}1)$, part (c) applies directly with $\mu = H_n$. And for the polynomials $f_n(x) = \sum_k \left\{{n \atop k}\right\} x^k$, we can use part (c) with $\mu = \varpi_{n+1}/\varpi_n - 1$, if $f_n(x)$ has n real roots. The latter statement follows by induction because $f_{n+1}(x) = x(f_n(x)+f_n'(x))$: If $a > 0$ and if $f(x)$ has n real roots, so does the function $g(x) = e^{ax}f(x)$. And $g(x) \to 0$ as $x \to -\infty$; hence $g'(x) = e^{ax}(af(x)+f'(x))$ also has n real roots (namely, one at the far left, and $n-1$ between the roots of $g(x)$).

[See E. Laguerre, *J. de Math.* (3) **9** (1883), 99–146; W. Hoeffding, *Annals Math. Stat.* **27** (1956), 713–721; J. N. Darroch, *Annals Math. Stat.* **35** (1964), 1317–1321; J. Pitman, *J. Combinatorial Theory* **A77** (1997), 297–303.]

64. We need only use computer algebra to subtract $\ln\varpi_n$ from $\ln\varpi_{n-k}$.

65. It is ϖ_n^{-1} times the number of occurrences of k-blocks plus the number of occurrences of ordered pairs of k-blocks in the list of all set partitions, namely $\bigl(\binom{n}{k}\varpi_{n-k} + \binom{n}{k}\binom{n-k}{k}\varpi_{n-2k}\bigr)/\varpi_n$, minus the square of (49). Asymptotically, $(\xi^k/k!)(1+O(n^{4\epsilon-1}))$.

66. (The maximum of (48) when $n = 100$ is achieved for the partitions $7^1 6^2 5^4 4^6 3^7 2^6 1^4$ and $7^1 6^2 5^4 4^6 3^8 2^5 1^3$.)

67. The expected value of M^k is ϖ_{n+k}/ϖ_n. By (50), the mean is therefore $\varpi_{n+1}/\varpi_n = n/\xi + \xi/(2(\xi+1)^2) + O(n^{-1})$, and the variance is

$$\frac{\varpi_{n+2}}{\varpi_n} - \frac{\varpi_{n+1}^2}{\varpi_n^2} \;=\; \Bigl(\frac{n}{\xi}\Bigr)^2\Bigl(1+\frac{\xi(2\xi+1)}{(\xi+1)^2 n} - 1 - \frac{\xi^2}{(\xi+1)^2 n} + O\Bigl(\frac{1}{n^2}\Bigr)\Bigr) \;=\; \frac{n}{\xi(\xi+1)} + O(1).$$

68. The maximum number of nonzero components in all parts of a partition is $n = n_1 + \cdots + n_m$; it occurs if and only if all component parts are 0 or 1. The maximum level is also equal to n.

69. At the beginning of step M3, if $k > b$ and $l = r-1$, go to M5. In step M5, if $j = a$ and $(v_j - 1)(r - l) < u_j$, go to M6 instead of decreasing v_j.

70. (a) $\left|{n-1 \atop r-1}\right| + \left|{n-2 \atop r-1}\right| + \cdots + \left|{r-1 \atop r-1}\right|$, since $\left|{n-k \atop r-1}\right|$ contain the block $\{0, \ldots, 0, 1\}$ with k 0s. The total, also known as $p(n-1, 1)$, is $p(n-1) + \cdots + p(1) + p(0)$.

(b) Exactly $N = \left\{{n-1 \atop r}\right\} + \left\{{n-2 \atop r-2}\right\}$ of the r-block partitions of $\{1, \ldots, n-1, n\}$ are the same if we interchange $n-1 \leftrightarrow n$. So the answer is $N + \frac{1}{2}(\left\{{n \atop r}\right\} - N) = \frac{1}{2}(\left\{{n \atop r}\right\} + N)$, which is also the number of restricted growth strings $a_1 \ldots a_n$ with $\max(a_1, \ldots, a_n) = r-1$ and $a_{n-1} \le a_n$. And the total is $\frac{1}{2}(\varpi_n + \varpi_{n-1} + \varpi_{n-2})$.

71. $\lfloor \frac{1}{2}(n_1+1)\ldots(n_m+1) - \frac{1}{2} \rfloor$, because there are $(n_1+1)\ldots(n_m+1) - 2$ *compositions* into two parts, and half of those compositions fail to be in lexicographic order unless all n_j are even. (See exercise 7.2.1.4–31. Formulas for up to 5 parts have been worked out by E. M. Wright, *Proc. London Math. Soc.* (3) **11** (1961), 499–510.)

72. Yes. The following algorithm computes $a_{jk} = p(j, k)$ for $0 \le j, k \le n$ in $\Theta(n^4)$ steps: Start with $a_{jk} \leftarrow 1$ for all j and k. Then for $l = 0, 1, \ldots, n$ and $m = 0, 1, \ldots, n$ (in any order), if $l + m > 1$ set $a_{jk} \leftarrow a_{jk} + a_{(j-l)(k-m)}$ for $j = l, \ldots, n$ and $k = m, \ldots, n$ (in increasing order).

(See Table A-1. A similar method computes $p(n_1, \ldots, n_m)$ in $O(n_1 \ldots n_m)^2$ steps. Cheema and Motzkin, in the cited paper, have derived the recurrence relation

$$n_1 p(n_1, \ldots, n_m) \;=\; \sum_{l=1}^{\infty} \sum_{k_1, \ldots, k_m \ge 0} k_1 p(n_1 - k_1 l, \ldots, n_m - k_m l),$$

but this interesting formula is helpful for computation only in certain cases.)

Table A-1

MULTIPARTITION NUMBERS

n	0	1	2	3	4	5	6
$p(0,n)$	1	1	2	3	5	7	11
$p(1,n)$	1	2	4	7	12	19	30
$p(2,n)$	2	4	9	16	29	47	77
$p(3,n)$	3	7	16	31	57	97	162
$p(4,n)$	5	12	29	57	109	189	323
$p(5,n)$	7	19	47	97	189	339	589

n	0	1	2	3	4	5
$P(0,n)$	1	2	9	66	712	10457
$P(1,n)$	1	4	26	249	3274	56135
$P(2,n)$	2	11	92	1075	16601	325269
$P(3,n)$	5	36	371	5133	91226	2014321
$P(4,n)$	15	135	1663	26683	537813	13241402
$P(5,n)$	52	566	8155	149410	3376696	91914202

73. Yes. Let $P(m,n) = p(1,\ldots,1,2,\ldots,2)$ when there are m 1s and n 2s; then $P(m,0) = \varpi_m$, and we can use the recurrence

$$2P(m,n+1) \;=\; P(m+2,n) + P(m+1,n) + \sum_k \binom{n}{k} P(m,k).$$

This recurrence can be proved by considering what happens when we replace a pair of x's in the multiset for $P(m,n+1)$ by two distinct elements x and x'. We get $2P(m,n+1)$ partitions, representing $P(m+2,n)$, except in the $P(m+1,n)$ cases where x and x' belong to the same block, or in $\binom{n}{k}P(m,n-k)$ cases where the blocks containing x and x' are identical and have k additional elements.

Notes: See Table A-1. Another recurrence, less useful for computation, is

$$P(m+1,n) \;=\; \sum_{j,k} \binom{n}{k}\binom{n-k+m}{j} P(j,k).$$

The sequence $P(0,n)$ was first investigated by E. K. Lloyd, *Proc. Cambridge Philos. Soc.* **103** (1988), 277–284, and by G. Labelle, *Discrete Math.* **217** (2000), 237–248, who computed it by completely different methods. Exercise 70(b) showed that $P(m,1) = (\varpi_m + \varpi_{m+1} + \varpi_{m+2})/2$; in general $P(m,n)$ can be written in the umbral notation $\varpi^m q_n(\varpi)$, where $q_n(x)$ is a polynomial of degree $2n$ defined by the generating function $\sum_{n=0}^{\infty} q_n(x)z^n/n! = \exp\bigl((e^z + (x+x^2)z - 1)/2\bigr)$. Thus, by exercise 31,

$$\sum_{n=0}^{\infty} P(m,n)\frac{z^n}{n!} \;=\; e^{(e^z-1)/2} \sum_{k=0}^{\infty} \frac{\varpi_{(2k+m+1)(k+m+1)}}{2^k}\,\frac{z^k}{k!}.$$

Labelle proved, as a special case of much more general results, that the number of partitions of $\{1,1,\ldots,n,n\}$ into exactly r blocks is

$$n!\,[x^r z^n]\, e^{-x+x^2(e^z-1)/2} \sum_{k=0}^{\infty} e^{zk(k+1)/2}\,\frac{x^k}{k!}.$$

75. The saddle point method yields $Ce^{An^{2/3}+Bn^{1/3}}/n^{55/36}$, where $A = 3\zeta(3)^{1/3}$, $B = \pi^2\zeta(3)^{-1/3}/2$, and $C = \zeta(3)^{19/36}(2\pi)^{-5/6}3^{-1/2}\exp(1/3 + B^2/4 + \zeta'(2)/(2\pi^2) - \gamma/12)$. [F. C. Auluck, *Proc. Cambridge Philos. Soc.* **49** (1953), 72–83; E. M. Wright, *American J. Math.* **80** (1958), 643–658.]

76. Using the fact that $p(n_1,n_2,n_3,\ldots) \ge p(n_1+n_2,n_3,\ldots)$, hence $P(m+2,n) \ge P(m,n+1)$, one can prove by induction that $P(m,n+1) \ge (m+n+1)P(m,n)$. Thus

$$2P(m,n) \;\le\; P(m+2,n-1) + P(m+1,n-1) + eP(m,n-1).$$

Iterating this inequality shows that $2^nP(0,n) = (\varpi^2+\varpi)^n + O(n(\varpi^2+\varpi)^{n-1}) = (n\varpi_{2n-1}+\varpi_{2n})(1+O((\log n)^3/n))$. (A more precise asymptotic formula can be obtained from the generating function in the answer to exercise 75.)

78. 3 3 3 3 2 1 0 0 0
1 0 0 0 2 2 3 2 0 (because the encoded partitions
2 2 1 0 0 2 1 0 2 must all be (000000000))
2 1 0 2 2 0 0 1 3

79. There are 432 such cycles. But they yield only 304 different cycles of set partitions, since different cycles might describe the same sequence of partitions. For example, (000012022332321) and (000012022112123) are partitionwise equivalent.

80. [See F. Chung, P. Diaconis, and R. Graham, *Discrete Mathematics* **110** (1992), 52–55.] Construct a digraph with ϖ_{n-1} vertices and ϖ_n arcs; each restricted growth string $a_1 \ldots a_n$ defines an arc from vertex $a_1 \ldots a_{n-1}$ to vertex $\rho(a_2 \ldots a_n)$, where ρ is the function of exercise 4. (For example, arc 01001213 runs from 0100121 to 0110203.) Every universal cycle defines an Eulerian trail in this digraph; conversely, every Eulerian trail can be used to define one or more universal sequences of restricted growth on the elements $\{0, 1, \ldots, n-1\}$.

An Eulerian trail exists by the method of Section 2.3.4.2, if we let the last exit from every nonzero vertex $a_1 \ldots a_{n-1}$ be through arc $a_1 \ldots a_{n-1} a_{n-1}$. The sequence might not be cyclic, however. For example, no universal cycle exists when $n < 4$; and when $n = 4$ the universal sequence 000012030110100222 defines a cycle of set partitions that does not correspond to any universal cycle.

The existence of a cycle can be proved for $n \geq 6$ if we start with an Eulerian trail that begins $0^n xyx^{n-3}u(uv)^{\lfloor (n-2)/2 \rfloor} u^{[n \text{ odd}]}$ for some distinct elements $\{u, v, x, y\}$. This pattern is possible if we alter the last exit of $0^k 121^{n-3-k}$ from $0^{k-1}121^{n-2-k}$ to $0^{k-1}121^{n-3-k}2$ for $2 \leq k \leq n-4$, and let the last exits of 0121^{n-4} and $01^{n-3}2$ be respectively $010^{n-4}1$ and $0^{n-3}10$. Now if we choose numbers of the cycle *backwards*, thereby determining u and v, we can let x and y be the smallest elements distinct from $\{0, u, v\}$.

We can conclude in fact that the number of universal cycles having this extremely special type is huge — at least

$$\Bigl(\prod_{k=2}^{n-1} (k!\,(n-k))^{\left\{ {n-1 \atop k} \right\}}\Bigr) \Big/ \bigl((n-1)!\,(n-2)^3 3^{2n-5} 2^2\bigr), \qquad \text{when } n \geq 6.$$

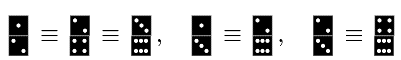
Yet none of them are known to be readily decodable. See below for the case $n = 5$.

81. Noting that $\varpi_5 = 52$, we use a universal cycle for $\{1, 2, 3, 4, 5\}$ in which the elements are 13 clubs, 13 diamonds, 13 hearts, 12 spades, and a joker. One such cycle, found by trial and error using Eulerian trails as in the previous answer, is

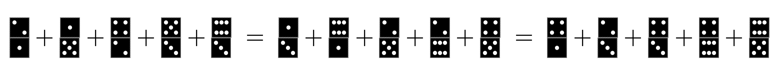
(♠♠♠♠♠♣♢♡J♣♡♢♡♠♣♣♢♡♣♢♣♡♢♢♣♡♣♡♣♠♣♣♢♣♢♢♣♢♠♠♢♡♡♡♠♡♡♡♠♠♢♢).

(In fact, there are essentially 114,056 such cycles if we branch to $a_k = a_{k-1}$ as a last resort and if we introduce the joker as soon as possible.) The trick still works with probability $\frac{47}{52}$ if we call the joker a spade.

82. There are 13644 solutions, although this number reduces to 1981 if we regard



The smallest common sum is $5/2$, and the largest is $25/2$; the remarkable solution



is one of only two essentially distinct ways to get the common sum $118/15$. [This problem was posed by B. A. Kordemsky in *Matematicheskaîâ Smekalka* (1954); it is number 78 in the English translation, *The Moscow Puzzles* (1972).]

INDEX AND GLOSSARY

When an index entry refers to a page containing a relevant exercise, see also the *answer* to that exercise for further information. An answer page is not indexed here unless it refers to a topic not included in the statement of the exercise.