Analysis of Equations of State for Neutron Star Modeling

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Outline

- What is an equation of state (EoS)? How do they fit into our model of a neutron star?
- How can we use an EoS to make macroscopic predictions about neutron stars?
- An example and its predictions:
  - QHD-I

Equation of State (EoS)

What is an equation of state?

- A relationship between *energy density* (denoted ϵ) and pressure (denoted P)
  - $\epsilon = \epsilon(P) \quad \Leftrightarrow \quad P = P(\epsilon)$
- Encodes the fundamental interparticle interactions within a neutron star
- True EoS within a neutron star is unknown; multitude of candidates, each based on a slightly different model
  - Models can be very complicated; often simplifications must be made

The Tolman-Oppenheimer-Volkoff (TOV) Equations

Used to describe a static (time independent) spherically symmetric neutron star. Given by

$$\frac{dm}{dr} = 4\pi r^2 \epsilon, \quad \frac{dP}{dr} = -\frac{(4\pi r^3 P + m)(\epsilon + P)}{r^2(1 - 2m/r)}.$$

where ϵ is energy density, P is pressure, and m is mass. Use EoS to determine $\epsilon = \epsilon(P)$. Initial conditions:

$$m(r = 0) = 0, \quad P(r = 0) \equiv P_0 = \text{const}.$$

Each solution uniquely specified by P_0. Outer conditions: *radius* of the star, R, and *mass* of the star, M, defined by

$$P(R) = 0, \quad M = m(R).$$

Solve an *initial value problem*. A solution to the above system is called a *static solution*.

Static Solutions

Creating a static solution:

1. specify a P_0 value, the *central pressure*
2. use a numerical integration technique to determine a solution (e.g. Runge-Kutta, Euler's method). Terminate once the outer condition is satisfied. Use EoS at each step to determine ϵ.

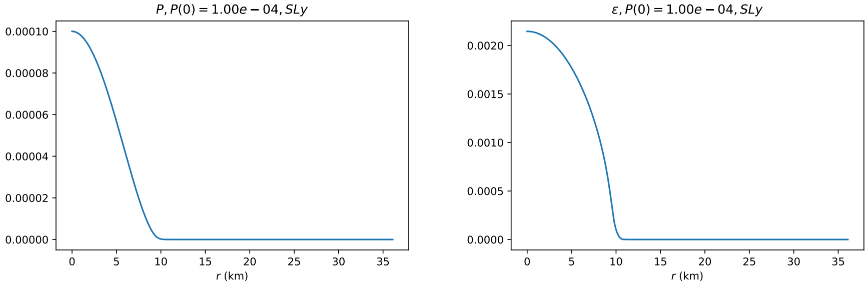


Figure: Example static solution for $P_0 = 10^{-4}$ for an EoS called "SLy."

Static Solutions: $M(R)$ and $M(P_0)$ diagrams

Use static solutions to make predictions using an EoS:

1. Create static solutions for a range of P_0 values: $P_0 \in [10^{-6}, 10^{-1}]$.
2. Find and store the mass M and radius R for each value of P_0.
3. Create $M(R)$ and $M(P_0)$ curves

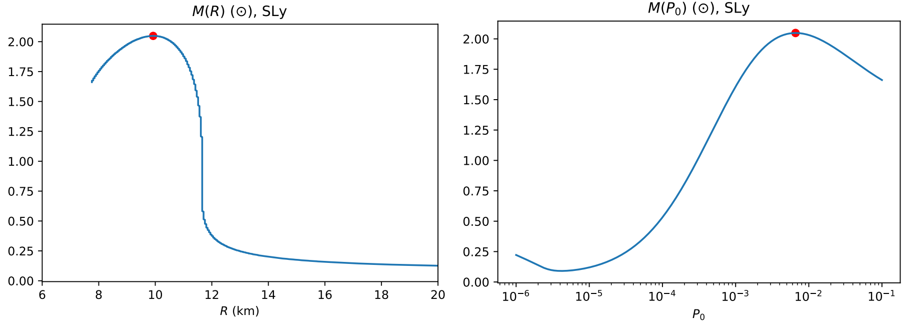
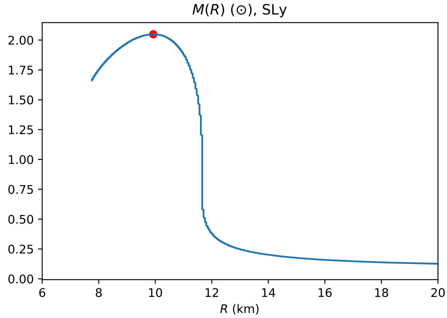
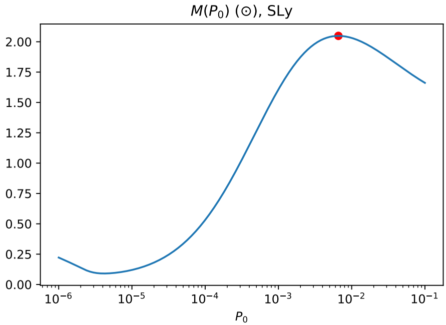


Figure: Example curves for EoS "SLy." $1\odot = 1.989 \times 10^{30}$ kg (solar mass)

Critical Values of P, R, and M





Three important values: *critical pressure*, *critical mass*, and *critical radius*.

- Determined by "peaks" of graph
- Maximum mass and radius predicted by EoS
- Largest "stable" pressure
- For SLy, $M_{\max} = 2.05\,\odot$, $R_{\max} = 9.93\,\text{km}$, and $P_{\text{crit}} = 6.59 \times 10^{-3}$.

Quantum Hadrodynamics

A theory of the quantum mechanical, interparticle interactions within a neutron star.

- Formulation of nuclear interactions between *baryons* by the exchange of *mesons*
  - *baryons* are particles containing three quarks (e.g. protons, neutrons)
  - *mesons* are quark/anti-quark pairs
- Requires experimental input for constraint; implemented using *coupling constants*
  - Multiple *parameter sets* have been developed by fitting observed nuclear properties of nuclear matter
- Considered quite complicated to solve; we introduce some simplifications in the QHD-I model

Quantum Hadrodynamics I (QHD-I)

Models nuclear interaction by the exchange of neutral scalar and vector mesons

- Scalar meson field ϕ, with mass m_ϕ
- Vector meson field V, with mass m_v
- Baryon field ψ, with mass M (nucleon mass; mass of proton or neutron)
- Coupling constants: g_v and g_ϕ

We form the *Lagrangian* (encodes information about the energy in the system):

$$\begin{aligned}\mathscr{L} = \bar{\psi}[\gamma_\mu(i\partial^\mu - g_v V^\mu) - (M - g_\phi \phi)]\psi \\ + \frac{1}{2}\left(\partial_\mu \phi \partial^\mu \phi - m_\phi^2 \phi^2\right) - \frac{1}{4} V_{\mu\nu} V^{\mu\nu} + \frac{1}{2} m_v^2 V_\mu V^\mu.\end{aligned}$$

From $\mathscr{L}$, we can determine ϵ and P, the EoS we desire.

RMF Simplifications

INCLUDE MOTIVATION FOR RMF SIMPLIFICATIONS HERE.

This allows us to simplify $\mathscr{L}$ considerably:

$$\mathscr{L}_{\mathrm{RMF}} = \bar{\psi}[i\gamma_\mu \partial^\mu - g_v \gamma_0 V_0 - (M - g_s \phi_0)]\psi - \frac{1}{2} m_s^2 \phi_0^2 + \frac{1}{2} m_\omega^2 V_0^2,$$

Resulting Equations

From $\mathscr{L}_{\text{RMF}}$ and the other RMF simplifications, we obtain the following equations:

$$
\begin{aligned}
\phi_0 &= \frac{g_\phi}{m_\phi^2}\frac{\gamma}{2\pi^2}\int_0^{k_f} dk\,\frac{k^2 m^*}{\sqrt{k^2+m^{*2}}},\\
V_0 &= \frac{g_v}{m_v^2}\frac{k_f^3}{3\pi^2},\\
\epsilon &= \frac{1}{2}m_\phi^2\phi_0^2+\frac{1}{2}m_v^2 V_0^2+\frac{1}{\pi^2}\int_0^{k_f} dk\,k^2\sqrt{k^2+m^{*2}},\\
P &= -\frac{1}{2}m_\phi^2\phi_0^2+\frac{1}{2}m_v^2 V_0^2+\frac{1}{3}\left(\frac{1}{\pi^2}\int_0^{k_f} dk\,\frac{k^4}{\sqrt{k^2+m^{*2}}}\right).
\end{aligned}
$$

where $m^* = (M - g_\phi \phi)$, the *reduced mass*.

Producing the Equation of State

- Equations have one *free parameter*, k_f; loop through values
- During each iteration, calculate ϕ_0 and V_0; use *rootfinding* for ϕ_0
- Using those values, calculate P and ϵ and store in a table

$M(R)$ and $M(P_0)$ Curves for QHD-I

We use the tabulated values of P and ϵ to create an *interpolated function*; then, we use the TOV equations.

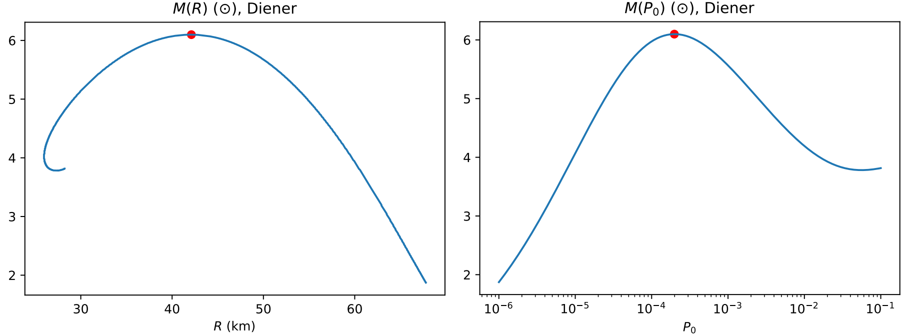


Figure: $M(R)$ and $M(P_0)$ curves for QHD-I EoS.

These curves give

$$M_{\max} = 6.1\,\odot, \quad R_{\max} = 42.1\,\text{km}, \quad P_{\text{crit}} = 1.98 \times 10^{-4}.$$

Conclusion

- An equation of state is a relationship between energy density and pressure within a neutron star
- We use the TOV equations to predict the maximum mass and radius that a given EoS will produce
- We use the QHD-I parameter set and RMF simplifications to solve a system of equations and generate an equation of state

Thanks

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