Analysis and derivation of realistic equations of state for neutron star simulations

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Introduction

This chapter aims to introduce the motivation for this project, the structure of this report, and the basic tools and techniques used throughout.

Chapter 2
Quantum Hadrodynamics

This chapter will describe the derivation of an equation of state from the QHD-I parameter set, as described within [1].

2.1 Introduction

2.2 Derivation of equations of motion

The Euler-Lagrange equations, for a Lagrange density $\mathscr{L}$ over a classical field φ_α, are given by

$$\partial_\nu \left(\frac{\partial \mathscr{L}}{\partial(\partial_\nu \varphi_\alpha)} \right) - \frac{\partial \mathscr{L}}{\partial \varphi_\alpha} = 0. \tag{2.1}$$

From [1, p. 56], we have the Lagrangian density of QHD-I

$$\begin{aligned} \mathscr{L} &= \bar{\psi}[\gamma_\mu(i\partial^\mu - g_\nu V^\mu) - (M - g_s\phi)]\psi \\ &\quad + \frac{1}{2}\left(\partial_\mu\phi\partial^\mu\phi - m_s^2\phi^2\right) - \frac{1}{4}V_{\mu\nu}V^{\mu\nu} + \frac{1}{2}m_\omega^2 V_\mu V^\mu, \end{aligned} \tag{2.2}$$

where $V_{\mu\nu} \equiv \partial_\mu V_\nu(x) - \partial^\nu V_\mu(x)$. To determine the equations of motion for this system, we must apply (2.1) to (2.2) for each unique field in the system

$$\varphi_\alpha = \begin{cases} \phi(x) & : \quad \text{scalar meson field,} \\ V^\mu(x) & : \quad \text{vector meson field,} \\ \psi(x) & : \quad \text{baryon field,} \\ \bar{\psi}(x) & : \quad \text{Dirac adjoint baryon field,} \end{cases}$$

where $\bar{\psi}(x) \equiv \psi^\dagger(x)\gamma^0$, the *Dirac adjoint.*

For the scalar meson field, when $\varphi_\alpha = \phi(x)$, the first term in (2.1) gives

$$\partial_\nu \left(\frac{\partial \mathscr{L}}{\partial(\partial_\nu \phi)} \right) = \frac{1}{2}\left[\frac{\partial}{\partial(\partial_\nu \phi)}(\partial_\mu \partial^\mu \phi) \right] = \partial_\nu \partial^\nu \phi,$$

while the second term gives

$$\frac{\partial \mathcal{L}}{\partial \phi} = \bar{\psi}[+g_s]\psi + \frac{1}{2}(-m_s(2\phi)) = g_s\bar{\psi}\psi - m_s\phi.$$

Combining, we get the first equation of motion,

$$\partial_\nu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)}\right) - \frac{\partial \mathcal{L}}{\partial \phi} = \partial_\nu\partial^\nu\phi - (g_s\bar{\psi}\psi - m_s\phi) = 0 \quad \Rightarrow \quad \partial_\nu\partial^\nu\phi + m_s\phi = g_s\bar{\psi}\psi. \tag{2.3}$$

This is the form given in [1] (8.1a).

For the vector meson field, when $\varphi_\alpha = V_\mu$, the first term in (2.1) gives

$$\partial_\nu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu V_\mu)}\right) = \partial_\nu V^{\mu\nu},$$

after many simplifications, including using the definition of $V^{\mu\nu}$ and the relabelling of indices. The second term gives

$$\frac{\partial \mathcal{L}}{\partial V_\mu} = \frac{\partial}{\partial V_\mu}\bar{\psi}\left[\gamma_\alpha(-g_v V^\alpha) + \frac{1}{2}m_\omega^2 V^\alpha V_\alpha\right] = -g_v\bar{\psi}\gamma^\mu\psi + m_\omega^2 V^\mu,$$

and combining we have

$$\partial_\nu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu V_\mu)}\right) - \frac{\partial \mathcal{L}}{\partial V_\mu} = \partial_\nu V^{\mu\nu} - (-g_v\bar{\psi}\gamma^\mu\psi + m_\omega^2 V^\mu) = 0. \tag{2.4}$$

However, this is not the form given in [1]; to reach his form, we leverage the anti-symmetry of $V_{\mu\nu}$, namely

$$V_{\mu\nu} = -V_{\nu\mu}.$$

Therefore, we make the above substitution, multiply (2.4) by -1, and send $\mu \leftrightarrow \nu$ to obtain

$$\partial_\mu V^{\mu\nu} + m_\omega^2 V^\nu = g_v\bar{\psi}\gamma^\nu\psi, \tag{2.5}$$

as given in [1] (8.1b).

Next, we have the two equations of motion from the baryon field. For $\varphi_\alpha = \bar{\psi}$, applying (2.1) is straight forward, as there is no $\partial_\nu\bar{\psi}$ dependence in $\mathcal{L}$, so the first term in (2.1) is zero. Thus, we obtain

$$\partial_\nu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu \bar{\psi})}\right) - \frac{\partial \mathcal{L}}{\partial \bar{\psi}} = [\gamma_\mu(i\partial^\mu - g_v V^\mu) - (M - g_s\phi)]\psi = 0. \tag{2.6}$$

For the final case, when $\varphi_\alpha = \psi$, the first term in (2.1) gives

$$\partial_\nu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu \psi)}\right) = \partial_\nu\left[\frac{\partial}{\partial(\partial_\nu\psi)}\bar{\psi}i\gamma_\alpha\partial^\alpha\psi\right] = i\partial_\nu\bar{\psi}\gamma^\nu,$$

while the second gives

$$\frac{\partial \mathcal{L}}{\partial \psi} = \bar{\psi}[\gamma_\mu(i\partial^\mu - g_v V^\mu) - (M - g_s\phi)].$$

Combining, we get our fourth equation

$$i\partial_\nu \bar{\psi}\gamma^\nu - \bar{\psi}[\gamma_\mu(i\partial^\mu - g_v V^\mu) - (M - g_s\phi)] = 0. \tag{2.7}$$

In summary, here are the four equations of motion for QHD-I:

$$\partial_\nu \partial^\nu \phi + m_s \phi = g_s \bar{\psi}\psi, \tag{2.3}$$

$$\partial_\mu V^{\mu\nu} + m_\omega^2 V^\nu = g_v \bar{\psi}\gamma^\nu \psi, \tag{2.5}$$

$$[\gamma_\mu(i\partial^\mu - g_v V^\mu(x)) - (M - g_s\phi)]\psi = 0, \tag{2.6}$$

$$i\partial_\nu \bar{\psi}\gamma^\nu - \bar{\psi}[\gamma_\mu(i\partial^\mu - g_v V^\mu) - (M - g_s\phi)] = 0. \tag{2.7}$$

The first three are given in [1].

2.3 Relativistic Mean Field Simplifications

In [1] §8.3, we find that the equations of motion listed above are very difficult to solve in their current form. To make them more manageable, we approximate them with "relativistic mean field" (RMF) simplifications, where we take each field to be its ground state expectation value. For the meson fields, this simplification yields

$$\phi \quad \Rightarrow \quad \langle\Phi|\,\phi\,|\Phi\rangle = \langle\phi\rangle \equiv \phi_0, \tag{2.8}$$

$$V_\mu \quad \Rightarrow \quad \langle\Phi|\,V_\mu\,|\Phi\rangle = \langle V_\mu\rangle \equiv \delta_{\mu 0} V_0, \tag{2.9}$$

where $|\Phi\rangle$ represents the ground state. These results arise from arguing that, in their ground states, ϕ and V_μ should be independent of space and time, as the system is bot uniform and stationary; therefore, ϕ_0 and V_0 are constants. Furthermore, because the system is at rest and the baryon flux, $\bar{\psi}\gamma^i\psi$, is zero, the spatial components of the expected value of V_μ, $\langle V_\mu\rangle$, must vanish [1].

For the baryon field, a "normal order", i.e. normalized, expectation value must be taken, as because otherwise, the vacuum would be taken into account and the traditional expectation value would diverge. This "normal ordered" expectation value is denoted with a ":". Throughout the equations of motion, the

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After evaluating these expectation values, we have

$$\phi_0 = \frac{g_s}{m_s^2}\frac{\gamma}{2\pi^2}\int_0^{k_f} dk\,\frac{k^2 m^*}{\sqrt{k^2 + m^{*2}}}, \tag{2.10}$$

$$V_\mu = \frac{g_v}{m_\omega^2}\rho, \tag{2.11}$$

where once again, $m^* \equiv M - g_s\phi$, the *reduced mass*, and ρ is the nucleon number density. If we assume spherical symmetry, a reasonable condition for the study of star-like systems, we get thee following expressions

$$\epsilon = \frac{1}{2}m_s^2\phi_0^2 + \frac{1}{2}m_\omega^2 V_0^2 + \frac{\gamma}{2\pi^2}\int_0^{k_f} dk\, k^2\sqrt{k^2+m^{*2}}, \tag{2.12}$$

$$P = -\frac{1}{2}m_s^2\phi_0^2 + \frac{1}{2}m_\omega^2 V_0^2 + \frac{1}{3}\left(\frac{\gamma}{2\pi^2}\int_0^{k_f} dk\, \frac{k^4}{\sqrt{k^2+m^{*2}}}\right). \tag{2.13}$$

In the next section, we will use these expressions to generate values for this equation of state.

2.4 Numerical generation of the equation of state

We now wish to generate tabulated values of the equation of state using the equations for ϵ, P and ϕ_0 at the end of the previous section. To do so, we first wish to find analytical forms for the integrals present in each of those equations, respectively. The aforementioned integrals are the following

$$\int_0^{k_f} dk\, \frac{k^2 m^*}{\sqrt{k^2+m^{*2}}}, \quad \int_0^{k_f} dk\, \frac{k^4}{\sqrt{k^2+m^{*2}}}, \quad \int_0^{k_f} dk\, k^2\sqrt{k^2+m^{*2}}. \tag{2.14}$$

We integrate each of these using Mathematica, giving it the assumptions that $k_f > 0$ and that $\mathrm{Re}[m^*] > 0$. This gives these explicit, closed forms

$$\int_0^{k_f} dk\, \frac{k^2 m^*}{\sqrt{k^2+m^{*2}}} = \frac{1}{2}m^*\left(k_f\sqrt{k_f^2+m^{*2}} - m^{*2}\ln\left[\frac{k_f+\sqrt{k_f+m^{*2}}}{k_f}\right]\right) \tag{2.15}$$

$$\int_0^{k_f} dk\, \frac{k^4}{\sqrt{k^2+m^{*2}}} = \frac{1}{8}\left(k_f(2k_f^2-3m^{*2}) - 3m^{*4}\ln\left[\frac{k_f+\sqrt{k_f+m^{*2}}}{k_f}\right]\right) \tag{2.16}$$

$$\int_0^{k_f} dk\, k^2\sqrt{k^2+m^{*2}} = \frac{1}{8}\left(k_f\sqrt{k_f^2+m^{*2}}(2k_f^2+m^{*2}) - m^{*4}\ln\left[\frac{k_f+\sqrt{k_f+m^{*2}}}{k_f}\right]\right) \tag{2.17}$$

Bibliography

[1] Jacobus Petrus William Diener. “Relativistic mean-field theory applied to the study of neutron star properties”. PhD thesis. Stellenbosch University, 2008.