Analysis of equations of state for neutron star modeling

Joseph E. Nyhan, Prof. Ben Kain

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Contents

Chapter 1
Introduction

This chapter aims to introduce the motivation for this project, the structure of this report, and the basic tools and techniques used throughout.

1.1 Abstract

Neutron stars are complex physical objects often modelled as a hydrodynamical system. Within these models, the two parameters energy density and pressure are related by an equation known as the "equation of state". This relationship allows us to calculate energy density if we know pressure, and visa versa. The equation of state itself is important because it encodes the information about interactions between the fundamental particles within the star. Furthermore, as neutron stars are extreme examples of gravitation, and not microscopically observable on Earth, the physics within neutron stars are still not well known. By postulating and analyzing different interactions within a neutron star, and therefore different equations of state, we can simulate the observable characteristics of the resulting star and compare them to empirical data to help better understand how neutron stars behave on a fundamental level.

Chapter 2
Static Solutions

Within our study of equations of state, we want to see which predictions each unique equation makes about the macroscopic (observable) properties of a neutron star. To do so, we introduce the Tolman-Oppenheimer-Volkoff (TOV) equations, a time independent description of a spherically symmetric neutron star. By solving the TOV equations, we can calculate theoretical observables, such as the total mass and radius of an individual star, and compare them to empirical data gathered from real neutron stars. This chapter will introduce the TOV equations, show how they are solved, and how through analysis we can determine the aforementioned observable quantities of note.

2.1 The Tolman-Oppenheimer-Volkoff (TOV) Equations

The Tolman-Oppenheimer-Volkoff (TOV) equations are the below system of two coupled differential equations

$$\frac{dm}{dr} = 4\pi r^2 \epsilon, \quad \frac{dP}{dr} = -\frac{(4\pi r^3 P + m)(\epsilon + P)}{r^2(1 - 2m/r)}. \tag{2.1}$$

Within these equations, there are four variables of note: the *radius*, r, the *mass*, m, the *pressure*, P, and the *energy density*, ϵ. Within this model, we consider neutron stars to be spherically symmetric; this means that only the distance from the center of the star is important. Furthermore, the radius r is the independent variable; therefore, the other variables can be written as functions of this radius: $m = m(r)$, $P = P(r)$, and $\epsilon = \epsilon(r)$.

The parameter m is defined as the total amount of energy within a spherical shell of radius r. Technically, this is not identical to mass; however, due to Einstein's mass-energy equivalence, it is common and convenient to call this parameter mass. This paper will continue this convention.

At this point, we have two variables remaining, yet only one evolution equation. It is here we can finally show the importance of the equation of state within our neutron star calculations. Within this system, the energy density and pressure are related directly by an equation $\epsilon = \epsilon(P)$ known as "the equation of state" (EOS). This relationship is important because it allows us to determine the current value of ϵ if we already know the value of pressure.[1] The EOS will be derived and analyzed in depth in the later sections of this paper;

[1] Practically, it is also easy to find pressure if we know energy density, however that is not necessary in this calculation.

at this point, however, it is important to understand that the EOS encodes the interactions between the particles within the neutron star, and based on the model used to describe those interactions, it will change. For this derivation, we leave the EOS as a general function. After including an EOS in our TOV equations, we know just have two variables to evolve: m and P.

To determine a solution to the system of equations in (2.1), we need will solve an initial value problem. As we want to know information about the star from its center radially outward, we therefore need initial conditions for both m and P at the center of the star, $r = 0$. Determining an initial condition for $m(r = 0)$ is straightforward; as m represents the total mass contained within a radius r, at the center of the star, as no mass is enclosed, so $m(0) = 0$. We treat the initial condition for P, $P(0)$, called the *central pressure*, as a free parameter. Every static solution is uniquely specified by a value of the central pressure; thus, we simply choose a reasonable value (typically $P(0)$ somewhere between 10^{-6} and 10^{-1}) and begin our integration. These initial conditions are summarized as

$$m(0) = 0, \quad P(0) \in [10^{-6}, 10^{-1}]. \tag{2.2}$$

When beginning our integration, we cannot, however, start directly at $r = 0$, as the denominator of dP/dr in (2.1) would be undefined. Instead, we simply start at a very small value of r, say $r \approx 10^{-8}$. This is effectively $r = 0$, and is accurate enough for our purposes.

We want our integration to terminate once we reach the edge of the star, as we are not interested in anything beyond that point. To find this outer edge, we define the total radius of the star, R, as the radius when

$$P(R) = 0. \tag{2.3}$$

In practice, once we reach a very small pressure, $P \sim 10^{-12}$, we can end the integration. Once have found R, we can determine $M = m(R)$, the total mass enclosed at radius R. The total mass M and total radius R are important to our analyses of different equations of state, as they represent experimentally observable properties of real neutron stars. These theoretically calculated properties can be compared to observed properties to gauge the validity of any given equation of state.

By determining a solution to the TOV equations, we calculate something called "static solution," a time independent image of a neutron star. A solution contains three curves: $m(r)$, $\epsilon(r)$, and $P(r)$. Of most importance are the pressure and energy density curves; however because they are related by the EOS, we only need to save one curve, as we can simply calculate the other when desired.

2.2 Computing Static Solutions

To compute static solutions, we use numerical integration techniques for solving ordinary differential equations (ODEs). In this situation, we use a specific technique known as the fourth-order Runge-Kutta algorithm (RK4). First, we need a differential equation of the form

$$\frac{dy}{dx} = f(x, y),$$

where x is the independent variable, and y is the function whose solution we wish to find. Importantly, y can be a vector valued function, so we can evolve a system of coupled differential equations using this method. Given a current value of the function, $y(x)$, RK4 allows to find an approximate value of the function a small step h forward in x: $y(x+h)$. Computationally, the algorithm is

$$y(x_i + h) \cong y(x_i) + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4), \tag{2.4}$$

where

$$\begin{aligned} k_1 &= hf(x_i, y_i), \\ k_2 &= hf(x_i + h/2, y_i + k_1/2), \\ k_3 &= hf(x_i + h/2, y_i + k_2/2), \\ k_4 &= hf(x_i + h, y_i + k_3). \end{aligned}$$

If y is a vector valued function, then each of the k_i values will also be vector valued. To determine a solution for y, we start with an initial condition, $y(x_0)$. Then, after choosing a step size h, we use the initial value of y to determine k_1 through k_4. Finally, we can then calculate $y(x_0 + h)$ using (2.4). At this point, we repeat the process using the newly calculated values of y as our initial data, allowing us to calculate the value of y one more step forward. This algorithm is repeated as necessary.

For our system of coupled ODEs in (2.1), the independent variable is r and our function is the vector is $y = (m, P)$. Using the initial conditions $y_0 = (0, P_0)$ from (2.2), we can begin our integration from the center of the star and work outwards with step size $h = \Delta r$, a small step in the radial direction. We continually use the newly calculated values of y at some radius r to determine the value of $y(r + \Delta r)$. As we want the integration to terminate when P gets too small, every time we calculate new values, we check to make sure that P is still in an acceptable range, and if it is too small, we terminate.

This method allows us to generate static solution curves for various equations of state $\epsilon(P)$ and central pressures $P(0)$. For an equation of state called "SLy" from [2] with a central pressure $P(0) = 10^{-4}$, an example of the resulting curves for pressure and energy density are shown in Figure 2.1.

By observation, in Figure 2.1, we can see that the total radius, R, is about 10 km, as this is when the pressure goes to zero. Using this value, we can determine what the total mass is by plugging it into the mass function (not pictured). These values are vital within section 2.3.

2.3 Analysis of Static Solutions

In Figure 2.1, we looked at one static solution produced using a realistic equation of state called SLy. The decision to use SLy was simply exemplary in this case; we could have used a different equation of state and the static solutions therefore would have been different. To determine the predictions that an individual equation of state will make over a range of values, it is impractical to simply look at static solution curves, such as those in Figure 2.1.

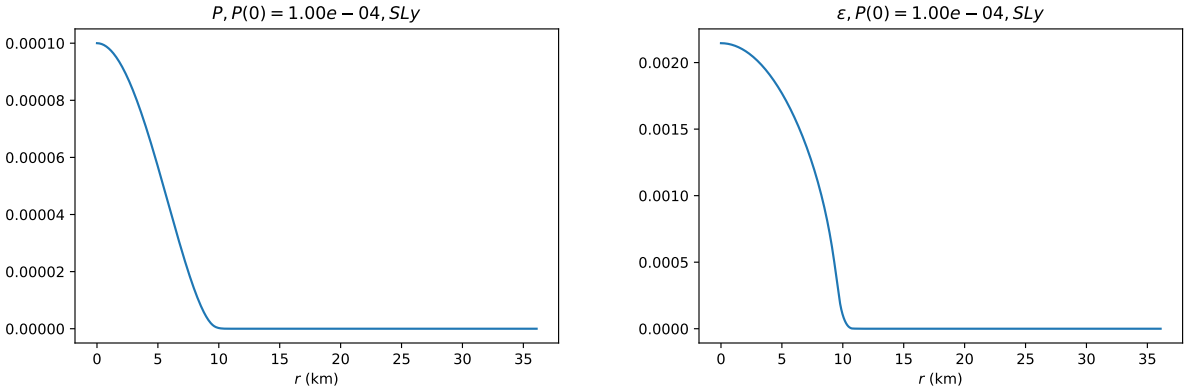

Figure 2.1. The pressure P (left) and energy density ϵ (right) for $P(0) = 10^{-4}$ with equation of state SLy from [2].

Instead, we look at the macroscopic properties, M and R, that each predicts over a wide range of central pressure values.

To do this, we create a range of central pressure values, approximately $P(0) \in [10^{-6}, 10^{-1}]$. In practice, it is useful to use a logarithmic range of values, such that the values are more densely packed near $P = 10^{-6}$; within our code, we use the `numpy` function `logspace` to create an array of values with logarithmic spacing. Now, we can iterate through these pressure values and create static solutions for each central pressure. For every static solution, we integrate until we find R, at which time we also calculate M. These values are stored, along with the central pressure value from which they were produced.

Using these values, we plot two curves: M vs. P_0 and M vs. R. These are shown in Figure 2.2.

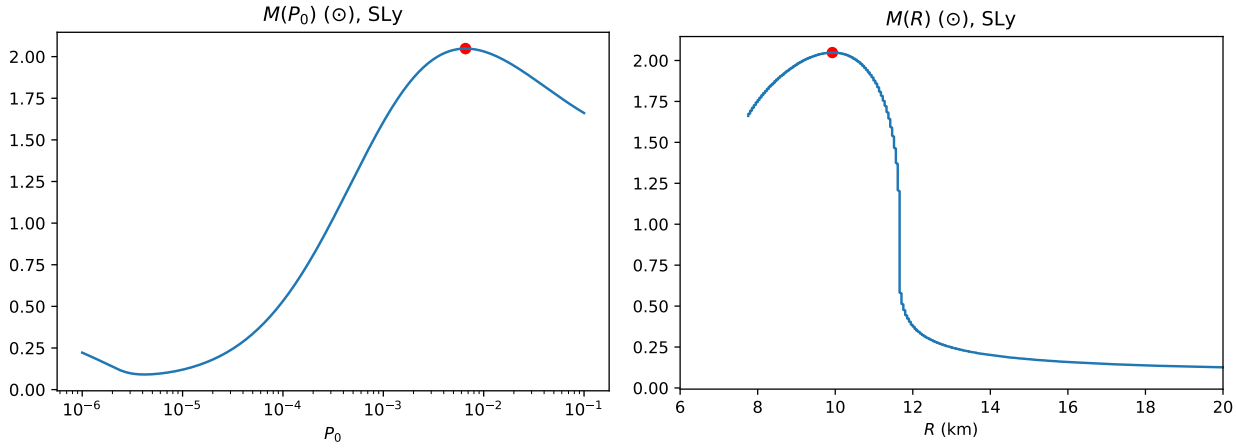

Figure 2.2. The curves M vs. P_0 (left) and M vs. R (right) for the SLy equation of state from [2].

From these curves, we deduce three special values: the *critical pressure*, the *critical radius*, and the *critical mass*. The critical mass is the maximum mass that the EOS predicts, and the critical pressure and critical radius are the pressure and radius, respectively, where the

critical mass is reached. The critical mass and critical radius are used as measuring sticks for how realistic an equation of state is; if they are on the same order as the largest neutron stars that have been actually observed, the equation of state is considered to be a candidate for a "correct" description of the star. Usually, the critical mass is on the order of about 2 solar masses, while the critical radius is on the order of about 10 km. Within the above plots, we have converted the units of radius to km and the units of mass to *solar masses*, denoted $\odot$, where $1\odot = 1.989 \times 10^{30}$ kg.

In [2], besides the SLy EOS, there is an additional EOS called FPS, derived and implemented in a similar fashion, just with a different parameterization. Furthermore, in another paper by the same authors, [3], there are three additional EOSs of similar form. All of these EOSs are considered realistic equations of state and are computed by creating analytical fits from empirical observational data from neutron stars. From previous work, we have these EOSs implemented, so for demonstration, we have overlaid the mass-radius and mass-pressure curves to demonstrate the differences in predictions of different equations of state. These curves are shown in Figure 2.3.

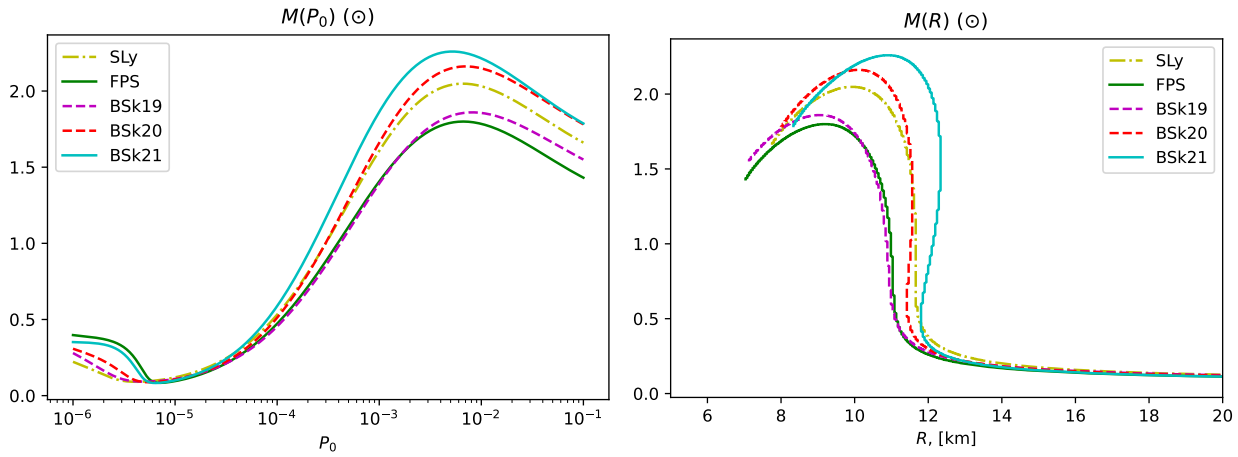

Figure 2.3. The mass-central pressure (left) and mass-radius (right) diagrams for the five equations of state from [2, 3].

As we can see from looking at these curves, these EOSs produce curves that are qualitatively similar yet with slightly different numerical values. For example, the EOS BSk21 predicts a maximum mass of $2.26\odot$ and a critical radius of 10.8 km, while FPS predicts a maximum mass of $1.80\odot$ with a critical radius of 9.2 km, with the predictions of the three other EOSs falling somewhere in between. We see that these five models therefore produce results that are well within the same order of magnitude with similar qualitative features, however their predictions are most definitely the same. As the actual EOS within a neutron star is unknown, these differences are non-trivial and studying them may lead to a better understanding of the real interactions within neutron stars in the future.

2.4 Implementation of Static Solutions and Analyses

During this section, the code given in Appendix A will be referenced extensively. Any line numbers, unless specified, refer to this code

To begin, we wanted to create a program that would allow for an arbitrary equation of state and central pressure to chosen and would then produce the corresponding static solution. In line 42, the boolean flag `make_static_solution` selects this mode, and the central pressure specified in line 48 is the central pressure used as an initial condition in the TOV equations. Furthermore, lines 12 through 39 are used to specify which equation of state is to be used.

The EOSs that we use when solving the TOV equations are not actually analytical equations or functions defined within our code. Instead, we have various `.txt` files that contain tabulated values of both ϵ and P. When a given EOS is chosen, we read in that text file and use the SciPy function `interp1d` to create an "interpolation function." This function takes the P values as independent variables and ϵ values; then, whenever this function is called throughout the remainder of the code, it will use an interpolation routine to predict an approximate value of ϵ for an inputted value of P. The implementation of this procedure is between lines 70 and 90.[2]

We also initialize other important parameters. We want to produce static solutions from a minimum r value ($r = 0$)[3] out to a maximum specified value, in this case, $r = 200$. Furthermore, we want our points spaced at a regular interval, small enough to give us adequate precision, but not too small as to hinder the code's speed too much. Therefore, we choose our radial step size, `dr`, to be 0.02. Then, we create an array of these r values, spaced out at the desired step size. Finally, we set the initial condition of m to 0. This is all given between lines 56 and 64

After the necessary parameters are initialized, we define the functions necessary for integration. First, on line 101, we define the vector funciton `f_onefluid`, a function of both the independent variable r and vector function y. Then, y is unpacked into m and P, the equation of state is used to determine ϵ (or, in the code, called `rho`), and finally the right-hand sides of the equations in (2.1) are evaluated and returned as a tuple. Secondly, we define a function called event, which is also a function of r and y. This function compares the current P value to a parameter called `p_tol`; `p_tol` is the smallest effective pressure value for which we consider the pressure to have gone to zero. On line 61, we define this threshold to be 10^{-11}. This function will be used to terminate the integration once we have reached the edge of the star.

Now, on line 115, we begin the integration for our static solution. We open files, with the current parameters we are using (such as the radial step size and central pressure) in the file name. On line 128, we use a SciPy function called `solve_ivp` ro perform the numerical integration (using the Runge-Kutta method) and return an array containing the

[2]There are two special EOSs called the "ultra-relativisitic equation of state" (UR EOS) and "polytropic equation of state" that are purely analytical. As this is the case, in lines 84-90, we choose the analytical equations of state if either the UR or polytropic EOS is chosen, and otherwise we use the interpolating function as described above.

[3]As mentioned before, we must choose an actual r_0 value that is just slightly bigger than 0, as using $r = 0$ in (2.1) would cause the denominator of dP/dr to be undefined.

solutions for both P and m. We use `solve_ivp` and do not implement RK4 on our own for speed purposes; `solve_ivp` has special optimizations in regards to step size under the hood that make the calculations more efficient when the values are very large and very small. Importantly, we pass the `event` function to `solve_ivp` as a parameter; throughout the integration, `solve_ivp` checks to see if the pressure has dropped below the tolerance, and if it has, it terminates the integration there. Then, we write the static solution to a file, where we fill out any points after the termination but before the $r_{\max}$ value with zeros. These files are then subsequently read into a different program for plotting, producing the curves like those shown in Figure 2.1.

For the analyses (i.e. producing the mass-radius and mass-pressure curves), we use a similar procedure to that described above, instead we iterate over a wide range of pressure values. If the boolean flag `p0_analysis` is set on line 43, then we will enter the loop on line 149. From there, we loop over all P_0 values in the `p0_vals` array, creating a static solution for each and saving the critical mass, critical radius, and critical pressure values to a file. Then after the completion of the loop, we can use an additional program to create a plot. When calculating the critical mass, we use an optimization routine to determine the top of the mass-pressure or mass-radius curve more precisely, and use that calculated critical value to determine the critical radius or pressure, respectively.

For the remainder of this project, this program will be used to produce the plots and analyses for the equations of state that we derive and produce.

Chapter 3
QHD-I

This chapter will describe the derivation of an equation of state from the QHD-I parameter set, as described within [1].

3.1 Introduction

3.2 Derivation of equations of motion

The Euler-Lagrange equations, for a Lagrange density $\mathscr{L}$ over a classical field φ_α, are given by

$$\partial_\nu\left(\frac{\partial\mathscr{L}}{\partial(\partial_\nu\varphi_\alpha)}\right)-\frac{\partial\mathscr{L}}{\partial\varphi_\alpha}=0. \tag{3.1}$$

From [1, p. 56], we have the Lagrangian density of QHD-I

$$\begin{aligned}\mathscr{L}&=\bar{\psi}[\gamma_\mu(i\partial^\mu-g_\nu V^\mu)-(M-g_s\phi)]\psi\\&\quad+\frac{1}{2}\big(\partial_\mu\phi\partial^\mu\phi-m_s^2\phi^2\big)-\frac{1}{4}V_{\mu\nu}V^{\mu\nu}+\frac{1}{2}m_\omega^2V_\mu V^\mu,\end{aligned} \tag{3.2}$$

where $V_{\mu\nu}\equiv\partial_\mu V_\nu(x)-\partial^\nu V_\mu(x)$. To determine the equations of motion for this system, we must apply (3.1) to (3.2) for each unique field in the system

$$\varphi_\alpha=\begin{cases}\phi(x) & : \text{ scalar meson field,}\\ V^\mu(x) & : \text{ vector meson field,}\\ \psi(x) & : \text{ baryon field,}\\ \bar{\psi}(x) & : \text{ Dirac adjoint baryon field,}\end{cases}$$

where $\bar{\psi}(x)\equiv\psi^\dagger(x)\gamma^0$, the *Dirac adjoint.*

For the scalar meson field, when $\varphi_\alpha=\phi(x)$, the first term in (3.1) gives

$$\partial_\nu\left(\frac{\partial\mathscr{L}}{\partial(\partial_\nu\phi)}\right)=\frac{1}{2}\left[\frac{\partial}{\partial(\partial_\nu\phi)}(\partial_\mu\partial^\mu\phi)\right]=\partial_\nu\partial^\nu\phi,$$

while the second term gives

$$\frac{\partial \mathcal{L}}{\partial \phi} = \bar{\psi}[+g_s]\psi + \frac{1}{2}(-m_s(2\phi)) = g_s\bar{\psi}\psi - m_s\phi.$$

Combining, we get the first equation of motion,

$$\partial_\nu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu\phi)}\right) - \frac{\partial \mathcal{L}}{\partial \phi} = \partial_\nu\partial^\nu\phi - (g_s\bar{\psi}\psi - m_s\phi) = 0 \quad \Rightarrow \quad \partial_\nu\partial^\nu\phi + m_s\phi = g_s\bar{\psi}\psi. \tag{3.3}$$

This is the form given in [1] (8.1a).

For the vector meson field, when $\varphi_\alpha = V_\mu$, the first term in (3.1) gives

$$\partial_\nu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu V_\mu)}\right) = \partial_\nu V^{\mu\nu},$$

after many simplifications, including using the definition of $V^{\mu\nu}$ and the relabelling of indices. The second term gives

$$\frac{\partial \mathcal{L}}{\partial V_\mu} = \frac{\partial}{\partial V_\mu}\bar{\psi}\left[\gamma_\alpha(-g_v V^\alpha) + \frac{1}{2}m_\omega^2 V^\alpha V_\alpha\right] = -g_v\bar{\psi}\gamma^\mu\psi + m_\omega^2 V^\mu,$$

and combining we have

$$\partial_\nu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu V_\mu)}\right) - \frac{\partial \mathcal{L}}{\partial V_\mu} = \partial_\nu V^{\mu\nu} - (-g_v\bar{\psi}\gamma^\mu\psi + m_\omega^2 V^\mu) = 0. \tag{3.4}$$

However, this is not the form given in [1]; to reach his form, we leverage the anti-symmetry of $V_{\mu\nu}$, namely

$$V_{\mu\nu} = -V_{\nu\mu}.$$

Therefore, we make the above substitution, multiply (3.4) by -1, and send $\mu \leftrightarrow \nu$ to obtain

$$\partial_\mu V^{\mu\nu} + m_\omega^2 V^\nu = g_v\bar{\psi}\gamma^\nu\psi, \tag{3.5}$$

as given in [1] (8.1b).

Next, we have the two equations of motion from the baryon field. For $\varphi_\alpha = \bar{\psi}$, applying (3.1) is straight forward, as there is no $\partial_\nu\bar{\psi}$ dependence in $\mathcal{L}$, so the first term in (3.1) is zero. Thus, we obtain

$$\partial_\nu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu\bar{\psi})}\right) - \frac{\partial \mathcal{L}}{\partial \bar{\psi}} = [\gamma_\mu(i\partial^\mu - g_v V^\mu) - (M - g_s\phi)]\psi = 0. \tag{3.6}$$

For the final case, when $\varphi_\alpha = \psi$, the first term in (3.1) gives

$$\partial_\nu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu\psi)}\right) = \partial_\nu\left[\frac{\partial}{\partial(\partial_\nu\psi)}\bar{\psi}i\gamma_\alpha\partial^\alpha\psi\right] = i\partial_\nu\bar{\psi}\gamma^\nu,$$

while the second gives

$$\frac{\partial \mathcal{L}}{\partial \bar{\psi}} = \bar{\psi}[\gamma_\mu(i\partial^\mu - g_v V^\mu) - (M - g_s\phi)].$$

Combining, we get our fourth equation

$$i\partial_\nu \bar{\psi}\gamma^\nu - \bar{\psi}[\gamma_\mu(i\partial^\mu - g_v V^\mu) - (M - g_s\phi)] = 0. \tag{3.7}$$

In summary, here are the four equations of motion for QHD-I:

$$\partial_\nu \partial^\nu \phi + m_s \phi = g_s \bar{\psi}\psi, \tag{3.3}$$
$$\partial_\mu V^{\mu\nu} + m_\omega^2 V^\nu = g_v \bar{\psi}\gamma^\nu \psi, \tag{3.5}$$
$$[\gamma_\mu(i\partial^\mu - g_v V^\mu(x)) - (M - g_s\phi)]\psi = 0, \tag{3.6}$$
$$i\partial_\nu \bar{\psi}\gamma^\nu - \bar{\psi}[\gamma_\mu(i\partial^\mu - g_v V^\mu) - (M - g_s\phi)] = 0. \tag{3.7}$$

The first three are given in [1].

3.3 Relativistic Mean Field Simplifications

In [1] §8.3, we find that the equations of motion listed above are very difficult to solve in their current form. To make them more manageable, we approximate them with "relativistic mean field" (RMF) simplifications, where we take each field to be its ground state expectation value. For the meson fields, this simplification yields

$$\phi \quad \Rightarrow \quad \langle\Phi|\,\phi\,|\Phi\rangle = \langle\phi\rangle \equiv \phi_0, \tag{3.8}$$
$$V_\mu \quad \Rightarrow \quad \langle\Phi|\,V_\mu\,|\Phi\rangle = \langle V_\mu\rangle \equiv \delta_{\mu 0} V_0, \tag{3.9}$$

where $|\Phi\rangle$ represents the ground state. These results arise from arguing that, in their ground states, ϕ and V_μ should be independent of space and time, as the system is bot uniform and stationary; therefore, ϕ_0 and V_0 are constants. Furthermore, because the system is at rest and the baryon flux, $\bar{\psi}\gamma^i\psi$, is zero, the spatial components of the expected value of V_μ, $\langle V_\mu\rangle$, must vanish [1].

For the baryon field, a "normal order", i.e. normalized, expectation value must be taken, as because otherwise, the vacuum would be taken into account and the traditional expectation value would diverge. This "normal ordered" expectation value is denoted with a ":". Throughout the equations of motion, the

MORE HERE.

After evaluating these expectation values, we have

$$\phi_0 = \frac{g_s}{m_s^2}\frac{\gamma}{2\pi^2}\int_0^{k_f} dk\, \frac{k^2 m^*}{\sqrt{k^2 + m^{*2}}}, \tag{3.10}$$
$$V_0 = \frac{g_v}{m_\omega^2}\rho, \tag{3.11}$$

where once again, $m^* \equiv M - g_s\phi$, the *reduced mass*, and ρ is the nucleon number density. If we assume spherical symmetry, a reasonable condition for the study of star-like systems, we get thee following expressions

$$\epsilon = \frac{1}{2}m_s^2\phi_0^2 + \frac{1}{2}m_\omega^2 V_0^2 + \frac{\gamma}{2\pi^2}\int_0^{k_f} dk\, k^2\sqrt{k^2 + m^{*2}}, \tag{3.12}$$

$$P = -\frac{1}{2}m_s^2\phi_0^2 + \frac{1}{2}m_\omega^2 V_0^2 + \frac{1}{3}\left(\frac{\gamma}{2\pi^2}\int_0^{k_f} dk\, \frac{k^4}{\sqrt{k^2 + m^{*2}}}\right). \tag{3.13}$$

In the next section, we will use these expressions to generate values for this equation of state.

3.4 Numerical generation of the equation of state

We now wish to generate tabulated values of the equation of state using the equations for ϵ, P and ϕ_0 at the end of the previous section. Simply, this process requires looping through various values of k_f; at each iteration, we find the corresponding value of ϕ_0 by using a root finding routine on (3.10), as m^* depends on ϕ_0, then computing V_0 independently using (3.11), and then finally substituting those values into (3.12) and (3.13) and storing those values. After creating a table of values for ϵ and P, we can verify the validity of the equation of state by solving the TOV equations and comparing the results of static solutions, mass-radius curves, and mass-pressure curves to other, previously calculated equations of state.

- This section is rushed. Need better explanations of TOV equations, etc. in a different section; see Chapter 2, the TOV equations section. This can be referenced here.
- Talk about the parameter set for QHD-1.

To begin, we choose a value of k_f for the entirety of the iteration; for sake of example, we take $k_f = 1$. Then, we find the value of ϕ_0 for that k_f value. We define a function in Python for ϕ_0 from (3.10).

```
1  def f_phi0(phi0, kf):
2      mstar = M - g_s * phi0
3      def f(k): # integrand
4          return k**2 * mstar/sqrt(k**2 + mstar**2)
5      integral, err = quad(f,0,kf)
6      return phi0 - (g_s/m_s**2) * (gamma/(2*pi**2)) * integral
```

While most of the function is straight forward, lines 3-5 may be cryptic without context. These lines are responsible for numerically computing the integral at the end of (3.10), given the current k_f value. `f(x)` is simply a function definition for the integrand of that integral, while line 5 uses the SciPy function `quad` to numerically integrate `f` from 0 to k_f. `quad` returns a tuple containing both the numerical value of the integration and the bounds on that value's error, so we unpack it to get the value we want and call it `integral`.

To determine the value of ϕ_0, we must use a root finding routine because in (3.10), ϕ_0 appears on both sides of the equation.

Unfinished. Within a tabulated equation of state, we need values for pressure from approximately 10^{-14} to about 10^{-1}. When solving for a static solution using the TOV equations, we terminate the integration when the pressure drops below a certain threshold, which in our case is about 10^{-11}, so we want values of the equation of state below that point in order to ensure that we calculate the solution correctly as it goes to zero.

Chapter 4
Advanced QHD Parameter Sets

4.1 Equations of Motion

The three equations of motion for the meson fields reduce to, as shown in [1, p. 79]

$$\begin{aligned}
\phi_0 &= \frac{g_s}{m_s^2}\left[\frac{1}{\pi^2}\left(\int_0^{k_p} dk\,\frac{k^2 m^*}{\sqrt{k^2+m^{*2}}} + \int_0^{k_n} dk\,\frac{k^2 m^*}{\sqrt{k^2+m^{*2}}}\right) - \frac{\kappa}{2}(g_s\phi_0)^2 - \frac{\lambda}{6}(g_s\phi_0)^3\right] \\
V_0 &= \frac{g_v}{m_s^2}\left[\rho_p + \rho_n - \frac{\zeta}{6}(g_v V_0)^3 - 2\Lambda_v (g_v V_0)(g_\rho b_0)^2\right] \\
b_0 &= \frac{g_\rho}{m_\rho^2}\left[\frac{1}{2}(\rho_p - \rho_n) - 2\Lambda_v (g_v V_0)^2 (g_\rho b_0)\right]
\end{aligned} \tag{4.1}$$

4.2 Equilibrium Conditions

The number densities of baryons must stay constant; thus, we have

$$\rho = \rho_n + \rho_p, \tag{4.2}$$

where ρ_n is the number density of the first species, neutrons, while ρ_p is the number density of the second species, protons. It is important to note now that we use the relation given on [1, p. 90]

$$\rho_x = \frac{k_x^3}{3\pi^2} \quad \Longleftrightarrow \quad k_x = \pi^2 (3\pi^2 \rho_x)^{-2/3}$$

to relate the xth fermi-momenta with its corresponding number density.

For beta-equilibrium to be satisfied, we must have

$$\mu_n = \mu_p + \mu_e, \tag{4.3}$$

where μ_x is the xth *chemical potential.* Using a handy result from [1, p. 90], we can rewrite this more generally as

$$\sqrt{k_n^2 + m^{*2}} = \sqrt{k_p^2 + m^{*2}} + g_\rho b_0 + \sqrt{k_e^2 + m_e^2}. \tag{4.4}$$

Furthermore, within neutron stars, muon production is probable and often favorable, per [1, p. 90]. Therefore, we have

$$\mu_\mu = \mu_e, \quad \mu_e = \sqrt{k_e^2 + m_e^2}, \quad \mu_\mu = \sqrt{k_\mu^2 + m_\mu^2}. \tag{4.5}$$

Because charge must be conserved, we need to have

$$\rho_p = \rho_e + \rho_\mu \quad \Rightarrow \quad k_p = \left(k_e^3 + k_\mu^3\right)^{1/3} \tag{4.6}$$

Appendix A
Code: TOV Implementation

```
# Joe Nyhan, 15 July 2021
# File for computing static solutions for fermionic stars via the
   TOV model.
#%%
import numpy as np
import pandas as pd
from scipy.integrate import solve_ivp
from scipy.interpolate import interp1d
# from scipy.optimize import minimize, fmin
import matplotlib.pyplot as plt
# %%

# EOS
eos_UR        = 0
eos_polytrope = 0

eos_SLy       = 1
eos_FPS       = 0

eos_BSk19     = 0
eos_BSk20     = 0
eos_BSk21     = 0

if eos_UR:
    eos = "UR"
    Gamma = 1.3
if eos_polytrope:
    eos = "polytrope"
    Gamma = 2
    K = 100
if eos_SLy:
    eos = "SLy"
if eos_FPS:
    eos = "FPS"
if eos_BSk19:
```

```
35      eos = "BSk19"
36  if eos_BSk20:
37      eos = "BSk20"
38  if eos_BSk21:
39      eos = "BSk21"
40
41  # MODES
42  make_static_solution = 0
43  p0_analysis = 1
44
45  # PARAMETERS
46
47  if make_static_solution:
48      p0 = 1e-4
49
50  if p0_analysis:
51      pmin = 1e-6
52      pmax = 1e-1
53      NUM_POINTS = 1000
54      p0_vals = np.logspace(round(np.log10(pmin)),round(np.log10(pmax)
            ),NUM_POINTS, base=10.0)
55
56  r0 = 0.000001
57  rmax = 200
58  dr = 0.02
59
60
61  p_tol = 1e-11
62  r_vals = np.arange(r0,rmax,dr)
63
64  m0 = 0
65
66  if make_static_solution: path = "../input"
67  elif p0_analysis: path = "./p0_analysis"
68
69  # FUNCTIONS
70
71  vals_path = ""
72  if eos_UR or eos_polytrope:
73      pass
74  else:
75      vals_path = f"0-{eos}_vals.vals"
76
77      df = pd.read_csv(vals_path)
78      interp_rhos = df.iloc[:,0].to_numpy()
79      interp_ps = df.iloc[:,1].to_numpy()
80
```

```
81
82 rho_from_P_interp = interp1d(interp_ps, interp_rhos, fill_value="
      extrapolate")
83
84 def rho_from_P(p):
85     if eos_UR:
86         return p/(Gamma-1)
87     elif eos_polytrope:
88         return (p/K)**(1/Gamma)
89     else:
90         return rho_from_P_interp(p)
91
92
93 def event(r,y):
94     m, p = y
95
96     return p - p_tol
97
98 event.terminal = True
99 event.direction = 0
100
101 def f_onefluid(r,y):
102     m, p = y
103
104     rho = rho_from_P(p)
105
106     N = 1-2*m/r
107
108     fm = 4*np.pi*r**2*rho
109     frho = (-(4*np.pi*r**3*p + m)*(rho+p)/(r**2*N))
110
111     return fm, frho
112
113 # ONE STATIC SOLUTIION
114
115 if make_static_solution:
116     if eos_polytrope:
117         P_path = f"{path}/polytrope_K{K:.1f}_gamma{Gamma:.1f}_p{p0
              :.8f}_dr{dr:.3f}_P.txt"
118         rho_path = f"{path}/polytrope_K{K:.1f}_gamma{Gamma:.1f}_p{p0
              :.8f}_dr{dr:.3f}_rho.txt"
119
120     # if eos_SLy:
121     else:
122         P_path = f"{path}/{eos}_p{p0:.8f}_dr{dr:.3f}_P.txt"
123         rho_path = f"{path}/{eos}_p{p0:.8f}_dr{dr:.3f}_rho.txt"
124
```

```
125     P_out = open(P_path,"w")
126     rho_out = open(rho_path,"w")
127 
128     sol = solve_ivp(f_onefluid,[r0,rmax], [m0,p0], method='RK45',
           atol = 1e-12, rtol = 1e-12, t_eval=r_vals, events=event)
129 
130     m, p = sol.y
131 
132     for i in range(len(r_vals)):
133         if i < len(p):
134             P_out.write(f"{p[i]:.16e}\n")
135             rho_out.write(f"{rho_from_P(p[i]):.16e}\n")
136         else:
137             P_out.write(f"{0:.16e}\n")
138             rho_out.write(f"{0:.16e}\n")
139 
140     P_out.close()
141     rho_out.close()
142 
143     print(f"COMPLETED.")
144 
145 
146 
147 # P0 ANALYSIS
148 
149 if p0_analysis:
150     output = open(f"{path}/{eos},p{pmin:.3e}-p{pmax:.3e}.vals","w")
151 
152     print(f"Number␣of␣points:{len(p0_vals)};␣for␣EoS:␣{eos}")
153 
154     def evolution(p0_vals):
155         for i in range(len(p0_vals)):
156 
157             p0 = p0_vals[i]
158             sol = solve_ivp(f_onefluid,[r0,rmax], [m0,p0], method='
                   RK45', atol = 1e-12, rtol = 1e-12, t_eval=r_vals,
                   events=event)
159             M = sol.y[0][-1]
160             R = sol.t[-1]
161 
162             if M < 0: print(f"M␣<0"); break
163 
164             output.write(f"{p0:.6e},{M:.6e},{R:.6e}\n")
165 
166             if i % 10 == 0: print(f"Timestep␣{i:3d}␣reached,␣{M=}.")
167 
168     evolution(p0_vals)
```

```
169
170     output.close()
```

Appendix B
Code: QHD-1 Implementation

```
# Joe Nyhan, 18 January 2021
#%%
from numpy import sqrt, pi, logspace
import matplotlib.pyplot as plt

from scipy.integrate import quad
from scipy.optimize import root

gamma = 2

# masses (in GeV)
m_s = .520
m_omega = .783
M = .939

# other constants
g_s = sqrt(109.6)
g_v = sqrt(190.4)


def f_phi0(phi0, kf):
    """see Diener p. 66, eq 8.21a"""
    mstar = M - g_s * phi0
    # integral = 1/2* mstar * (kf * sqrt(kf**2 + mstar**2) - mstar
       **2 * log((kf + sqrt(kf**2 + mstar**2))/mstar))
    def f(k):
        return k**2 * mstar/sqrt(k**2 + mstar**2)

    integral, err = quad(f,0,kf)


    return phi0 - (g_s/m_s**2) * (gamma/(2*pi**2)) * integral

def f_V0(kf):
    """see Diener p. 66, eq 8.21b"""
```

```
35      rho = 2 * (gamma/(6*pi**2))*kf**3
36      # for rho, see Diener p. 67, eq. 8.25; note that gamma = 2 has
           been substituted
37      # print(f'{rho=}')
38
39      return (g_v/m_omega**2) * rho
40
41  def f_eps(phi0, V0, kf):
42      """see Diener p. 66, eq 8.24a"""
43      mstar = M - g_s * phi0
44      # integral = 1/8 * (kf * (2*kf**2 + mstar**2) * sqrt(kf**2 +
           mstar**2) - mstar**4 * log((kf + sqrt(kf**2 + mstar**2))/
           mstar))
45      def f(k):
46          return k**2 * sqrt(k**2 + mstar**2)
47      integral, err = quad(f,0,kf)
48
49      return 1/2 * m_s**2 * phi0**2 + 1/2 * m_omega**2 * V0**2 + (
           gamma/(2*pi**2)) * integral
50
51  def f_P(phi0, V0, kf):
52      """see Diener p. 66, eq 8.24b"""
53      mstar = M - g_s * phi0
54      # integral = 1/8 * (kf * (2*kf**2 - 3*mstar**2) * sqrt(kf**2 +
           mstar**2) + 3*mstar**4 * log((kf + sqrt(kf**2 + mstar**2))/
           mstar))
55
56      def f(k):
57          return k**4 / sqrt(k**2 + mstar**2)
58      integral, err = quad(f,0,kf)
59
60      return -1/2 * m_s**2 * phi0**2 + 1/2 * m_omega**2 * V0**2 +
           (1/3) * (gamma/(2*pi**2)) * integral
61
62  # kfs = [1e0]
63  kfs = logspace(-6,0,12000,base=10)
64
65  energy_densities = []
66  pressures = []
67
68  phi0_guess = 10**(-1)
69  # phi0_guess = 0
70
71  for kf in kfs:
72
73      res = root(f_phi0, phi0_guess, args=kf)
74      # print(res)
```

```
75      phi0 = res.x[0]
76      # print(phi0)
77
78      V0 = f_V0(kf)
79
80      # print(f'{V0=}, {phi0=}')
81
82      eps = f_eps(phi0, V0, kf)
83      p = f_P(phi0, V0, kf)
84
85      energy_densities.append(eps)
86      pressures.append(p)
87
88      phi0_guess = phi0
89
90  with open('./qhd1-vals.txt', 'w') as f:
91      for i in range(len(pressures)):
92          p = pressures[i]
93          eps = energy_densities[i]
94          f.write(f'{eps:.16e}, {p:.16e}\n')
```

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