

Analysis of Equations of State for Neutron Star Modeling and Simulation

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- What is a neutron star?
- What is an equation of state (EoS)? How does it fit into our model of a neutron star?
- How can we use an EoS to make macroscopic predictions about neutron stars?
- What is the importance and role of an equation of state in a temporal simulation of a neutron star?
- A derivation of an EoS and its predictions:
 - ▶ Quantum Hydrodynamics and the QHD-I parameter set

Introduction

Neutron Stars

- Dense core left behind after a supernovae explosion
- Made mostly of neutrons, protons, and electrons; overall, is neutral
- Radius: ~ 10 km; Mass: $\sim 1 \odot$.
- Approximately the density of atomic nuclei ($\sim 10^{17}$ kg/m³)
- Core held together by intense gravitational attraction
 - ▶ Gravitational acceleration on Earth's Surface: ≈ 10 m/s²
 - ▶ Neutron star: $\approx 10^{12}$ m/s² (escape velocity $\sim 100\,000$ km/s = $c/3$)
- Why are they interesting?
 - ▶ Smallest, densest observed stellar objects
 - ▶ Exotic physics

Equation of State (EoS)

What is an equation of state?

- A relationship between *energy density* (denoted ϵ) and pressure (denoted P)
 - ▶ $\epsilon = \epsilon(P) \Leftrightarrow P = P(\epsilon)$
- Encodes the fundamental interparticle interactions within a neutron star
- True EoS within a neutron star is unknown; multitude of candidates, each based on a slightly different model and fit of empirical data
 - ▶ Models can be very complicated; often simplifications must be made to be solved practically
- Often a tabulated list of P and ϵ values; however, in simulation work, analytical fits may be required

Using an Equation of State to Make Predictions

Using an EoS to Make Predictions

We want a way to understand the effects of an EoS on the observable properties of a star

- e.g. total mass, total radius

We create *static solutions*; “images” of neutron star

- solve the Tolman-Oppenheimer-Volkoff (TOV) equations
- extract information about maximum mass and radius allowed by the EoS

The Tolman-Oppenheimer-Volkoff (TOV) Equations

Used to describe a static (time independent) spherically symmetric star.
Given by

$$\frac{dm}{dr} = 4\pi r^2 \epsilon, \quad \frac{dP}{dr} = -\frac{(4\pi r^3 P + m)(\epsilon + P)}{r^2(1 - 2m/r)}.$$

where ϵ is energy density, P is pressure, and m is “mass.” Use EoS to determine $\epsilon = \epsilon(P)$.

- Initial conditions:

$$m(r=0) = 0, \quad P(r=0) \equiv P_0 = \text{const.}$$

Each solution is uniquely defined by P_0 , the *central pressure*.

- Outer conditions: Let R, M to be the total radius and total mass of the star, respectively. Defined by:

$$P(R) = 0, \quad M = m(R).$$

TOV Equations: Computing a Solution

$$\frac{dm}{dr} = 4\pi r^2 \epsilon, \quad \frac{dP}{dr} = -\frac{(4\pi r^3 P + m)(\epsilon + P)}{r^2(1 - 2m/r)}.$$

- Specify a central pressure $P(r=0) = P_0$
- Begin at very small $r \approx 0$; (10^{-8})
- Use a numerical integration technique
 - ▶ The Runge-Kutta 4 Algorithm
 - ▶ In practice, use Scipy `solve_ivp`; faster due to optimized step size
- Integrate outwards until $P = 0$; use to define R , calculate M
- Store curves for $m(r)$, $P(r)$, use to calculate $\epsilon(r)$
- *Note*: works well with tabulated EoSs; an interpolating function is often used

Static Solution: Example

Use an EoS called “SLy” from [3]. A realistic equation of state from an analytical fit of empirical neutron star data.

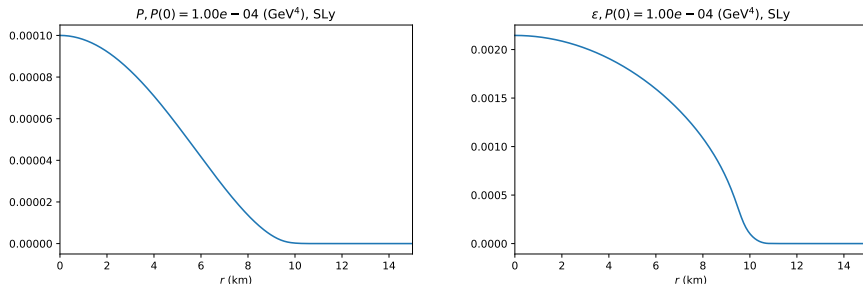


Figure: Example static solution for $P_0 = 10^{-4} \text{ GeV}^4$ for an EoS called “SLy.”

Static Solutions: $M(R)$ and $M(P_0)$ diagrams

Single solutions don't tell much about star as a whole; instead, look at trends over lots of solutions

- 1 Solve TOV equations for lots of P_0 values: $P_0 \in [10^{-6}, 10^{-1}](\text{GeV}^4)$.
- 2 Calculate the total mass M and total radius R for each value of P_0
- 3 Plot $M(R)$ and $M(P_0)$

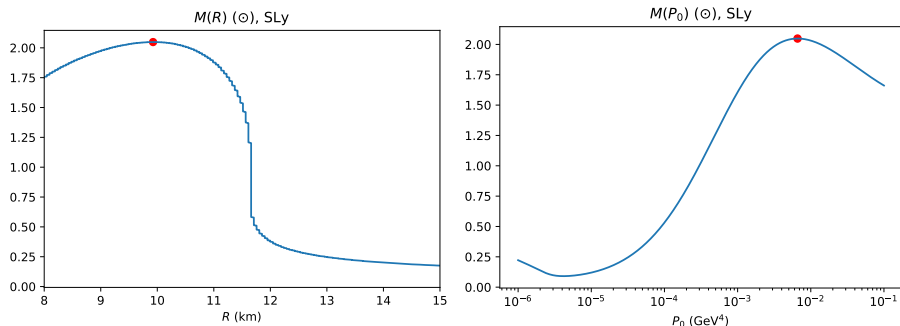
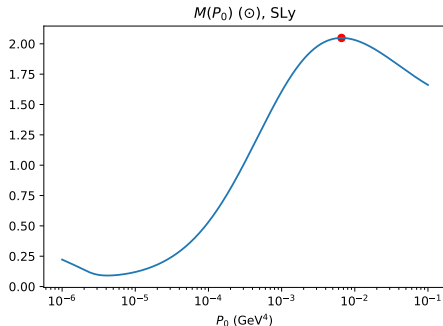
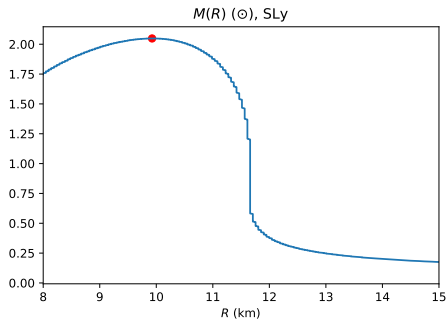


Figure: Example curves for EoS “SLy.” $1 \odot = 1.989 \times 10^{30} \text{ kg}$ (solar mass)

Critical Values of P , R , and M



Three important values: *critical pressure*, *critical mass*, and *critical radius*.

- Determined by “peaks” of graph; calculated using an optimization routine
- Maximum mass and radius predicted by EoS
- Largest “stable” pressure
- SLy: $M_{\text{max}} = 2.05 \odot$, $R_{\text{max}} = 9.93 \text{ km}$, and $P_{\text{crit}} = 6.59 \times 10^{-3} \text{ GeV}^4$.

Predictions of Other Equations of State

There are other realistic, analytical fit EoSs of a similar form SLy given in [3, 5]: FPS, BSk19, BSk20, BSk21

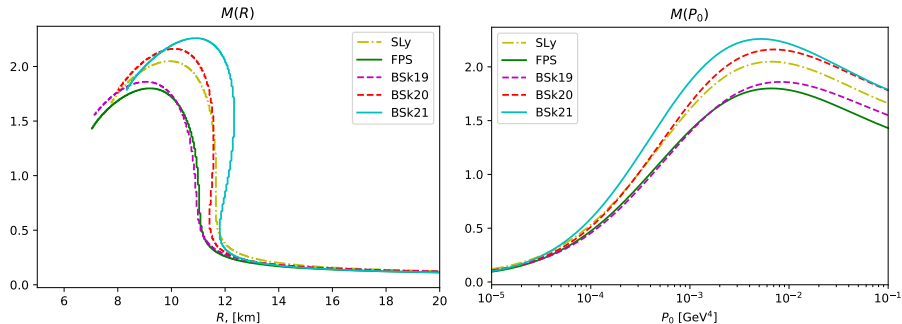


Figure: $M(R)$ and $M(P_0)$ for SLy family EoSs. M in \odot .

Critical values of the extreme EoSs:

- FPS: $R_{\text{max}} = 9.20$ km, $M_{\text{max}} = 1.80 \odot$, $P_{\text{crit}} = 6.67 \times 10^{-3} \text{ GeV}^4$.
- BSk21: $R_{\text{max}} = 10.9$ km, $M_{\text{max}} = 2.26 \odot$, $P_{\text{crit}} = 5.17 \times 10^{-3} \text{ GeV}^4$.

Temporal Simulations of Neutron Stars

Temporal Simulations of Neutron Stars: Background

- Neutron star as a spherically symmetric hydrodynamical system
- Three *primitive* variables: pressure P , energy density ϵ , and velocity v
- ϵ and P related by an EoS
- Define *conservative* variables Π, Φ in terms of primitive variables

$$\Pi = \frac{\epsilon + P}{1 - v} - P, \quad \Phi = \frac{\epsilon + P}{1 + v} - P.$$

- Π, Φ obey a conservation equation

$$\partial_t \vec{u} = -\frac{1}{r^2} \partial_r \left(r^2 \frac{\alpha}{a} \vec{f}^{(1)} \right) - \partial_r \left(\frac{\alpha}{a} \vec{f}^{(2)} \right) + \vec{s}, \quad \vec{u} = \begin{bmatrix} \Pi \\ \Phi \end{bmatrix},$$

where a, α are the *gravity* variables.

- $\vec{f}^{(1)} = \vec{f}^{(1)}(\vec{u}, v)$, $\vec{f}^{(2)} = \vec{f}^{(2)}(P)$, $\vec{s} = \vec{s}(\vec{u}, P, \epsilon, v, a, \alpha)$.
- Separate evolution equations for a, α

Temporal Simulations of Neutron Stars: Background

- Evolve a set of discrete spatial gridpoints \rightarrow advanced numerical techniques
 - ▶ Finite differencing (for spatial derivatives) and the method of lines
 - ▶ High-resolution shock-capturing methods
 - ▶ Evolve through time using numerical integration (Runge-Kutta 3, Modified Euler's Method)
- Use EoSs that are analytical fits for numerical stability and root-finding abilities
 - ▶ Determine P, ϵ, ν numerically from Π, Φ ; need to be able to differentiate (e.g. Newton-Raphson Method)
 - ▶ Extensive studies of realistic, analytical EoSs from [3, 5]; SLy family

Temporal Simulations of Neutron Stars

- Use static solutions as initial data for temporal simulation
 - ▶ Above *critical pressure*: unstable; below: stable
- Stable solutions exhibit *radial oscillations*
 - ▶ Evolve out to large t and perform a Fourier transform

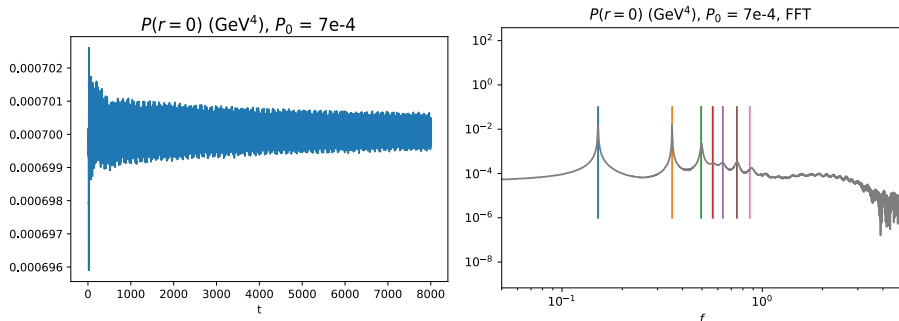


Figure: Plots of $P(r=0)$ for EoS “SLy” and initial $P_0 = 7 \times 10^{-4} \text{ GeV}^4$. Colored lines on FFT plot represent predicted frequencies.

- Radial oscillations differ by EoS; they could soon be measurable!

Simulations of Neutron Stars: Nearing Publication

“Dynamical evolution of fermion-boson stars with realistic equations of state” under Prof. Ben Kain, College of the Holy Cross

- Simulate presence of dark matter as a boson; superimpose a complex scalar field
- First temporal simulations of these “mixed” stars using *realistic* equations of state; hope is to see how dark matter could affect observable properties of neutron stars
- Interesting finding: there is a range of unstable static solutions that *migrate* to stable solutions

Derivation and Computation of an Equation of State

Computing an EoS: Quantum Hadrodynamics

A theory of the quantum mechanical, interparticle interactions within a neutron star.

- Formulation of nuclear interactions between *baryons* by the exchange of *mesons*
 - ▶ *baryons* are particles containing three quarks (e.g. protons, neutrons)
 - ▶ *mesons* are quark/anti-quark pairs
- Requires experimental input for constraint; implemented using *coupling constants*
 - ▶ Models the strength of the interactions between particles
 - ▶ Multiple *parameter sets* have been developed by fitting observed nuclear properties of nuclear matter
- Considered quite complicated to solve; we introduce some simplifications in the QHD-I model

Quantum Hadrodynamics I (QHD-I)

We form the Lagrange Density for QHD-I:

$$\begin{aligned}\mathcal{L} = & \bar{\psi}[\gamma_{\mu}(i\partial^{\mu} - g_v V^{\mu}) - (M - g_{\phi}\phi)]\psi \\ & + \frac{1}{2}(\partial_{\mu}\phi\partial^{\mu}\phi - m_{\phi}^2\phi^2) - \frac{1}{4}V_{\mu\nu}V^{\mu\nu} + \frac{1}{2}m_v^2V_{\mu}V^{\mu},\end{aligned}$$

where $V_{\mu\nu} \equiv \partial_{\mu}V_{\nu} - \partial_{\nu}V_{\mu}$, $\partial_{\mu} \equiv \partial/\partial x^{\mu}$. The fields:

- Baryon field (protons and neutrons) $\psi(x^{\mu})$, with mass M
- Scalar meson field: $\phi(x^{\mu})$, with mass m_{ϕ}
- Vector meson field: $V^{\mu}(x^{\mu})$, with mass m_v
- Experimental coupling constants: g_v and g_{ϕ}

From \mathcal{L} , we can determine ϵ and P , the EoS we desire.

QHD-I: Derivation of Equations of Motion

$$\begin{aligned}\mathcal{L} = & \bar{\psi}[\gamma_{\mu}(i\partial^{\mu} - g_v V^{\mu}) - (M - g_{\phi}\phi)]\psi \\ & + \frac{1}{2}(\partial_{\mu}\phi\partial^{\mu}\phi - m_{\phi}^2\phi^2) - \frac{1}{4}V_{\mu\nu}V^{\mu\nu} + \frac{1}{2}m_v^2 V_{\mu}V^{\mu},\end{aligned}$$

Applying the Euler-Lagrange equations for \mathcal{L} over a classical field

$$\partial_{\nu}\left(\frac{\partial\mathcal{L}}{\partial(\partial_{\nu}\varphi_{\alpha})}\right) - \frac{\partial\mathcal{L}}{\partial\varphi_{\alpha}} = 0,$$

for $\varphi_{\alpha} \in \{\phi, V^{\mu}, \psi\}$, we obtain the equations of motion:

$$\begin{aligned}\partial_{\nu}\partial^{\nu}\phi + m_s^2\phi &= g_s\bar{\psi}\psi, \\ \partial_{\mu}V^{\mu\nu} + m_{\omega}^2V^{\nu} &= g_v\bar{\psi}\gamma^{\nu}\psi, \\ [\gamma_{\mu}(i\partial^{\mu} - g_v V^{\mu}) - (M - g_s\phi)]\psi &= 0,\end{aligned}$$

QHD-I: RMF Simplifications

We introduce the *Relativistic Mean Field* (RMF) simplifications. We treat the interactions (exchange of mesons) as their average values:

$$\phi \rightarrow \langle \phi \rangle = \phi_0, \quad V_\mu \rightarrow \langle V_\mu \rangle = V_0, \quad \bar{\psi}\psi \rightarrow \langle \bar{\psi}\psi \rangle, \quad \bar{\psi}\gamma^\mu\psi \rightarrow \langle \bar{\psi}\gamma^0\psi \rangle,$$

where ϕ_0 and V_0 are constants. This allows us to simplify \mathcal{L} considerably:

$$\mathcal{L}_{\text{RMF}} = \bar{\psi}[i\gamma_\mu\partial^\mu - g_v\gamma_0 V_0 - (M - g_s\phi_0)]\psi - \frac{1}{2}m_s^2\phi_0^2 + \frac{1}{2}m_\omega^2 V_0^2,$$

Applying the same simplifications to the equations of motions gives:

$$m_s^2\phi_0^2 = g_s \langle \bar{\psi}\psi \rangle$$

$$m_\omega^2 V_0 = g_v \langle \bar{\psi}\gamma^0\psi \rangle$$

$$[i\gamma_\mu\partial^\mu - g_v\gamma_0 V_0 - (M - g_s\phi_0)]\psi = 0$$

QHD-I: Closed forms for ϵ and P

We can now find closed forms of ϵ and P . From [1], we have

$$\epsilon = \langle T^{00} \rangle, \quad P = \frac{1}{3} \langle T^{ii} \rangle,$$

where $T^{\mu\nu}$ is the energy momentum tensor, given by

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_\alpha)} \partial^\nu \varphi_\alpha - \mathcal{L} \eta^{\mu\nu}.$$

Using \mathcal{L}_{RMF} , we obtain

$$T_{\text{RMF}}^{\mu\nu} = i\bar{\psi}\gamma^\mu \partial^\nu \psi - \eta^{\mu\nu} \left(-\frac{1}{2} m_s^2 \phi_0^2 + \frac{1}{2} m_\omega^2 V_0^2 \right).$$

This gives

$$\begin{aligned} \epsilon &= \langle i\bar{\psi}\gamma^0 \partial^0 \psi \rangle + \frac{1}{2} m_s^2 \phi_0^2 - \frac{1}{2} m_\omega^2 V_0^2, \\ P &= \langle i\bar{\psi}\gamma^i \partial^i \psi \rangle - \frac{1}{2} m_s^2 \phi_0^2 + \frac{1}{2} m_\omega^2 V_0^2. \end{aligned}$$

The above expectation values are non-trivial and are derived in [1].

QHD-I: Resulting Equations

After calculation the expectation values, we obtain the following equations:

$$\phi_0 = \frac{g_\phi}{m_\phi^2} \frac{1}{\pi^2} \int_0^{k_f} dk \frac{(M - g_\phi \phi_0) k^2}{\sqrt{k^2 + (M - g_\phi \phi_0)}},$$

$$V_0 = \frac{g_v}{m_v^2} \frac{k_f^3}{3\pi^2},$$

$$\epsilon = \frac{1}{2} m_\phi^2 \phi_0^2 + \frac{1}{2} m_v^2 V_0^2 + \frac{1}{\pi^2} \int_0^{k_f} dk k^2 \sqrt{k^2 + m^{*2}},$$

$$P = -\frac{1}{2} m_\phi^2 \phi_0^2 + \frac{1}{2} m_v^2 V_0^2 + \frac{1}{3} \left(\frac{1}{\pi^2} \int_0^{k_f} dk \frac{k^4}{\sqrt{k^2 + m^{*2}}} \right).$$

where $m^* = (M - g_\phi \phi)$, the *reduced mass*. k_f , the Fermi wavenumber, is a free parameter.

Resulting Equations

Goal: create a list of values that show us $\epsilon(P)$; each value of k_f gives us a different ϵ and P .

To produce the EoS: (repeat the following)

- Choose a k_f value
- calculate ϕ_0 and V_0 ; use *rootfinding* for ϕ_0
- Using those values, calculate P and ϵ and store in a table

We loop through k_f values until we have a large range of P values

$$P \in [10^{-20}, 10^{-1}](\text{GeV}^4).$$

$M(R)$ and $M(P_0)$ Curves for QHD-I

We use the tabulated values of P and ϵ to solve the TOV equations:

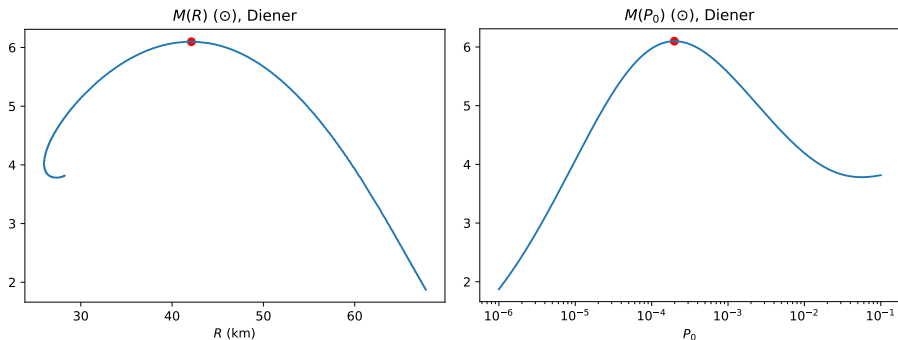


Figure: $M(R)$ and $M(P_0)$ curves for QHD-I EoS.

These curves give

$$M_{\text{max}} = 6.1 \odot, \quad R_{\text{max}} = 42.1 \text{ km}, \quad P_{\text{crit}} = 1.98 \times 10^{-4} \text{ GeV}^4.$$

Conclusion

- An equation of state is a relationship between energy density and pressure within a neutron star
- We use the TOV equations to predict the maximum mass and radius that a given EoS will produce
- Within a temporal simulation of a neutron star:
 - ▶ Static solutions are used as initial data
 - ▶ We can predict radial oscillation frequencies of neutron stars, which could soon be measurable
- We use the QHD-I parameter set and RMF simplifications to solve a system of equations and generate an equation of state

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