Analysis of equations of state for neutron star modeling

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Chapter 1
Introduction

This chapter aims to introduce the motivation for this project, the structure of this report, and the basic tools and techniques used throughout.

1.1 Abstract

Neutron stars are complex physical objects often modelled as a hydrodynamical system. Within these models, the two parameters energy density and pressure are related by an equation known as the "equation of state". This relationship allows us to calculate energy density if we know pressure, and visa versa. The equation of state itself is important because it encodes the information about interactions between the fundamental particles within the star. Furthermore, as neutron stars are extreme examples of gravitation, and not microscopically observable on Earth, the physics within neutron stars are still not well known. By postulating and analyzing different interactions within a neutron star, and therefore different equations of state, we can simulate the observable characteristics of the resulting star and compare them to empirical data to help better understand how neutron stars behave on a fundamental level.

Chapter 2
Static Solutions

Within our study of equations of state, we want to see which predictions each unique equation makes about the macroscopic (observable) properties of a neutron star. To do so, we introduce the Tolman-Oppenheimer-Volkoff (TOV) equations, a time independent description of a spherically symmetric neutron star. By solving the TOV equations, we can calculate theoretical observables, such as the total mass and radius of an individual star, and compare them to empirical data gathered from real neutron stars. This chapter will introduce the TOV equations, show how they are solved, and how through analysis we can determine the aforementioned observable quantities of note.

2.1 The Tolman-Oppenheimer-Volkoff (TOV) Equations

The Tolman-Oppenheimer-Volkoff (TOV) equations are the below system of two coupled differential equations

$$\frac{dm}{dr} = 4\pi r^2 \epsilon, \quad \frac{dP}{dr} = -\frac{(4\pi r^3 P + m)(\epsilon + P)}{r^2(1-2m/r)}. \tag{2.1}$$

Within these equations, there are four variables of note: the *radius*, r, the *mass*, m, the *pressure*, P, and the *energy density*, ϵ. Within this model, we consider neutron stars to be spherically symmetric; this means that only the distance from the center of the star is important. Furthermore, the radius r is the independent variable; therefore, the other variables can be written as functions of this radius: $m = m(r)$, $P = P(r)$, and $\epsilon = \epsilon(r)$.

The parameter m is defined as the total amount of energy within a spherical shell of radius r. Technically, this is not identical to mass; however, due to Einstein's mass-energy equivalence, it is common and convenient to call this parameter mass. This paper will continue this convention.

At this point, we have two variables remaining, yet only one evolution equation. It is here we can finally show the importance of the equation of state within our neutron star calculations. Within this system, the energy density and pressure are related directly by an equation $\epsilon = \epsilon(P)$ known as "the equation of state" (EOS). This relationship is important because it allows us to determine the current value of ϵ if we already know the value of pressure.[1] The EOS will be derived and analyzed in depth in the later sections of this paper;

[1] Practically, it is also easy to find pressure if we know energy density, however that is not necessary in this calculation.

at this point, however, it is important to understand that the EOS encodes the interactions between the particles within the neutron star, and based on the model used to describe those interactions, it will change. For this derivation, we leave the EOS as a general function. After including an EOS in our TOV equations, we know just have two variables to evolve: m and P.

To determine a solution to the system of equations in (2.1), we need will solve an initial value problem. As we want to know information about the star from its center radially outward, we therefore need initial conditions for both m and P at the center of the star, $r = 0$. Determining an initial condition for $m(r = 0)$ is straightforward; as m represents the total mass contained within a radius r, at the center of the star, as no mass is enclosed, so $m(0) = 0$. We treat the initial condition for P, $P(0)$, called the *central pressure*, as a free parameter. Every static solution is uniquely specified by a value of the central pressure; thus, we simply choose a reasonable value (typically $P(0)$ somewhere between 10^{-6} and 10^{-1}) and begin our integration. These initial conditions are summarized as

$$m(0) = 0, \quad P(0) \in [10^{-6}, 10^{-1}]. \tag{2.2}$$

When beginning our integration, we cannot, however, start directly at $r = 0$, as the denominator of dP/dr in (2.1) would be undefined. Instead, we simply start at a very small value of r, say $r \approx 10^{-8}$. This is effectively $r = 0$, and is accurate enough for our purposes.

We want our integration to terminate once we reach the edge of the star, as we are not interested in anything beyond that point. To find this outer edge, we define the total radius of the star, R, as the radius when

$$P(R) = 0. \tag{2.3}$$

In practice, once we reach a very small pressure, $P \sim 10^{-12}$, we can end the integration. Once have found R, we can determine $M = m(R)$, the total mass enclosed at radius R. The total mass M and total radius R are important to our analyses of different equations of state, as they represent experimentally observable properties of real neutron stars. These theoretically calculated properties can be compared to observed properties to gauge the validity of any given equation of state.

By determining a solution to the TOV equations, we calculate something called "static solution," a time independent image of a neutron star. A solution contains three curves: $m(r)$, $\epsilon(r)$, and $P(r)$. Of most importance are the pressure and energy density curves; however because they are related by the EOS, we only need to save one curve, as we can simply calculate the other when desired.

2.2 Computing Static Solutions

To compute static solutions, we use numerical integration techniques for solving ordinary differential equations (ODEs). Throughout this project, we use a technique known as the fourth-order Runge-Kutta algorithm (RK4). First, we need a differential equation of the form

$$\frac{dy}{dx} = f(x, y),$$

where x is the independent variable, and y is the function whose solution we wish to find. Importantly, y can be a vector valued function, so we can evolve a system of coupled differential equations using this method. Given a current value of the function, $y(x)$, RK4 allows to find an approximate value of the function a small step h forward in x: $y(x+h)$. Computationally, the algorithm is

$$y(x_i + h) \cong y(x_i) + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4), \tag{2.4}$$

where

$$\begin{aligned}
k_1 &= hf(x_i, y_i), \\
k_2 &= hf(x_i + h/2, y_i + k_1/2), \\
k_3 &= hf(x_i + h/2, y_i + k_2/2), \\
k_4 &= hf(x_i + h, y_i + k_3).
\end{aligned}$$

If y is a vector valued function, then each of the k_i values will also be vector valued.

For our system of coupled ODEs in (2.1), the independent variable is r and our function is the vector is $y = (m, P)$. Using the initial conditions $y_0 = (0, P_0)$ from (2.2), we can begin our integration from the center of the star and work outwards with step size $h = \Delta r$, a small step in the radial direction. We continually use the newly calculate values of y at some radius r to determine the value of $y(r + \Delta r)$. As we want the integration to terminate when P gets too small, every time we calculate new values, we check to make sure that P is still in an acceptable range, and if it is too small, we terminate.

2.3 Analysis of Results

Chapter 3
QHD-I

This chapter will describe the derivation of an equation of state from the QHD-I parameter set, as described within [1].

3.1 Introduction

3.2 Derivation of equations of motion

The Euler-Lagrange equations, for a Lagrange density $\mathscr{L}$ over a classical field φ_α, are given by

$$\partial_\nu \left(\frac{\partial \mathscr{L}}{\partial(\partial_\nu \varphi_\alpha)} \right) - \frac{\partial \mathscr{L}}{\partial \varphi_\alpha} = 0. \tag{3.1}$$

From [1, p. 56], we have the Lagrangian density of QHD-I

$$\begin{aligned} \mathscr{L} = &\bar{\psi}[\gamma_\mu(i\partial^\mu - g_\nu V^\mu) - (M - g_s\phi)]\psi \\ &+ \frac{1}{2}\left(\partial_\mu\phi\partial^\mu\phi - m_s^2\phi^2\right) - \frac{1}{4}V_{\mu\nu}V^{\mu\nu} + \frac{1}{2}m_\omega^2 V_\mu V^\mu, \end{aligned} \tag{3.2}$$

where $V_{\mu\nu} \equiv \partial_\mu V_\nu(x) - \partial^\nu V_\mu(x)$. To determine the equations of motion for this system, we must apply (3.1) to (3.2) for each unique field in the system

$$\varphi_\alpha = \begin{cases} \phi(x) & : \quad \text{scalar meson field,} \\ V^\mu(x) & : \quad \text{vector meson field,} \\ \psi(x) & : \quad \text{baryon field,} \\ \bar{\psi}(x) & : \quad \text{Dirac adjoint baryon field,} \end{cases}$$

where $\bar{\psi}(x) \equiv \psi^\dagger(x)\gamma^0$, the *Dirac adjoint.*

For the scalar meson field, when $\varphi_\alpha = \phi(x)$, the first term in (3.1) gives

$$\partial_\nu \left(\frac{\partial \mathscr{L}}{\partial(\partial_\nu \phi)} \right) = \frac{1}{2}\left[\frac{\partial}{\partial(\partial_\nu\phi)}(\partial_\mu\partial^\mu\phi) \right] = \partial_\nu\partial^\nu\phi,$$

while the second term gives

$$\frac{\partial \mathcal{L}}{\partial \phi} = \bar{\psi}[+g_s]\psi + \frac{1}{2}(-m_s(2\phi)) = g_s\bar{\psi}\psi - m_s\phi.$$

Combining, we get the first equation of motion,

$$\partial_\nu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} \right) - \frac{\partial \mathcal{L}}{\partial \phi} = \partial_\nu \partial^\nu \phi - (g_s\bar{\psi}\psi - m_s\phi) = 0 \quad \Rightarrow \quad \partial_\nu \partial^\nu \phi + m_s\phi = g_s\bar{\psi}\psi. \tag{3.3}$$

This is the form given in [1] (8.1a).

For the vector meson field, when $\varphi_\alpha = V_\mu$, the first term in (3.1) gives

$$\partial_\nu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu V_\mu)} \right) = \partial_\nu V^{\mu\nu},$$

after many simplifications, including using the definition of $V^{\mu\nu}$ and the relabelling of indices. The second term gives

$$\frac{\partial \mathcal{L}}{\partial V_\mu} = \frac{\partial}{\partial V_\mu}\bar{\psi}\left[\gamma_\alpha(-g_v V^\alpha) + \frac{1}{2}m_\omega^2 V^\alpha V_\alpha\right] = -g_v\bar{\psi}\gamma^\mu\psi + m_\omega^2 V^\mu,$$

and combining we have

$$\partial_\nu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu V_\mu)} \right) - \frac{\partial \mathcal{L}}{\partial V_\mu} = \partial_\nu V^{\mu\nu} - (-g_v\bar{\psi}\gamma^\mu\psi + m_\omega^2 V^\mu) = 0. \tag{3.4}$$

However, this is not the form given in [1]; to reach his form, we leverage the anti-symmetry of $V_{\mu\nu}$, namely

$$V_{\mu\nu} = -V_{\nu\mu}.$$

Therefore, we make the above substitution, multiply (3.4) by -1, and send $\mu \leftrightarrow \nu$ to obtain

$$\partial_\mu V^{\mu\nu} + m_\omega^2 V^\nu = g_v\bar{\psi}\gamma^\nu\psi, \tag{3.5}$$

as given in [1] (8.1b).

Next, we have the two equations of motion from the baryon field. For $\varphi_\alpha = \bar{\psi}$, applying (3.1) is straight forward, as there is no $\partial_\nu\bar{\psi}$ dependence in $\mathcal{L}$, so the first term in (3.1) is zero. Thus, we obtain

$$\partial_\nu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu \bar{\psi})} \right) - \frac{\partial \mathcal{L}}{\partial \bar{\psi}} = [\gamma_\mu(i\partial^\mu - g_v V^\mu) - (M - g_s\phi)]\psi = 0. \tag{3.6}$$

For the final case, when $\varphi_\alpha = \psi$, the first term in (3.1) gives

$$\partial_\nu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu \psi)} \right) = \partial_\nu \left[\frac{\partial}{\partial(\partial_\nu \psi)} \bar{\psi} i\gamma_\alpha \partial^\alpha \psi \right] = i\partial_\nu\bar{\psi}\gamma^\nu,$$

while the second gives

$$\frac{\partial \mathcal{L}}{\partial \psi} = \bar{\psi}[\gamma_\mu(i\partial^\mu - g_v V^\mu) - (M - g_s\phi)].$$

Combining, we get our fourth equation

$$i\partial_\nu \bar{\psi}\gamma^\nu - \bar{\psi}[\gamma_\mu(i\partial^\mu - g_v V^\mu) - (M - g_s\phi)] = 0. \tag{3.7}$$

In summary, here are the four equations of motion for QHD-I:

$$\begin{align} \partial_\nu\partial^\nu\phi + m_s\phi &= g_s\bar{\psi}\psi, \tag{3.3}\\ \partial_\mu V^{\mu\nu} + m_\omega^2 V^\nu &= g_v\bar{\psi}\gamma^\nu\psi, \tag{3.5}\\ [\gamma_\mu(i\partial^\mu - g_v V^\mu(x)) - (M - g_s\phi)]\psi &= 0, \tag{3.6}\\ i\partial_\nu\bar{\psi}\gamma^\nu - \bar{\psi}[\gamma_\mu(i\partial^\mu - g_v V^\mu) - (M - g_s\phi)] &= 0. \tag{3.7} \end{align}$$

The first three are given in [1].

3.3 Relativistic Mean Field Simplifications

In [1] §8.3, we find that the equations of motion listed above are very difficult to solve in their current form. To make them more manageable, we approximate them with "relativistic mean field" (RMF) simplifications, where we take each field to be its ground state expectation value. For the meson fields, this simplification yields

$$\begin{align} \phi \quad &\Rightarrow \quad \langle\Phi|\,\phi\,|\Phi\rangle = \langle\phi\rangle \equiv \phi_0, \tag{3.8}\\ V_\mu \quad &\Rightarrow \quad \langle\Phi|\,V_\mu\,|\Phi\rangle = \langle V_\mu\rangle \equiv \delta_{\mu 0}V_0, \tag{3.9} \end{align}$$

where $|\Phi\rangle$ represents the ground state. These results arise from arguing that, in their ground states, ϕ and V_μ should be independent of space and time, as the system is bot uniform and stationary; therefore, ϕ_0 and V_0 are constants. Furthermore, because the system is at rest and the baryon flux, $\bar{\psi}\gamma^i\psi$, is zero, the spatial components of the expected value of V_μ, $\langle V_\mu\rangle$, must vanish [1].

For the baryon field, a "normal order", i.e. normalized, expectation value must be taken, as because otherwise, the vacuum would be taken into account and the traditional expectation value would diverge. This "normal ordered" expectation value is denoted with a ":". Throughout the equations of motion, the

MORE HERE.

After evaluating these expectation values, we have

$$\begin{align} \phi_0 &= \frac{g_s}{m_s^2}\frac{\gamma}{2\pi^2}\int_0^{k_f} dk\,\frac{k^2 m^*}{\sqrt{k^2 + m^{*2}}}, \tag{3.10}\\ V_0 &= \frac{g_v}{m_\omega^2}\rho, \tag{3.11} \end{align}$$

where once again, $m^* \equiv M - g_s\phi$, the *reduced mass*, and ρ is the nucleon number density. If we assume spherical symmetry, a reasonable condition for the study of star-like systems, we get thee following expressions

$$\epsilon = \frac{1}{2}m_s^2\phi_0^2 + \frac{1}{2}m_\omega^2 V_0^2 + \frac{\gamma}{2\pi^2}\int_0^{k_f} dk\, k^2\sqrt{k^2+m^{*2}}, \tag{3.12}$$

$$P = -\frac{1}{2}m_s^2\phi_0^2 + \frac{1}{2}m_\omega^2 V_0^2 + \frac{1}{3}\left(\frac{\gamma}{2\pi^2}\int_0^{k_f} dk\, \frac{k^4}{\sqrt{k^2+m^{*2}}}\right). \tag{3.13}$$

In the next section, we will use these expressions to generate values for this equation of state.

3.4 Numerical generation of the equation of state

We now wish to generate tabulated values of the equation of state using the equations for ϵ, P and ϕ_0 at the end of the previous section. Simply, this process requires looping through various values of k_f; at each iteration, we find the corresponding value of ϕ_0 by using a root finding routine on (3.10), as m^* depends on ϕ_0, then computing V_0 independently using (3.11), and then finally substituting those values into (3.12) and (3.13) and storing those values. After creating a table of values for ϵ and P, we can verify the validity of the equation of state by solving the TOV equations and comparing the results of static solutions, mass-radius curves, and mass-pressure curves to other, previously calculated equations of state.

- This section is rushed. Need better explanations of TOV equations, etc. in a different section; see Chapter 2, the TOV equations section. This can be referenced here.
- Talk about the parameter set for QHD-1.

To begin, we choose a value of k_f for the entirety of the iteration; for sake of example, we take $k_f = 1$. Then, we find the value of ϕ_0 for that k_f value. We define a function in Python for ϕ_0 from (3.10).

```
1 def f_phi0(phi0, kf):
2     mstar = M - g_s * phi0
3     def f(k): # integrand
4         return k**2 * mstar/sqrt(k**2 + mstar**2)
5     integral, err = quad(f,0,kf)
6     return phi0 - (g_s/m_s**2) * (gamma/(2*pi**2)) * integral
```

While most of the function is straight forward, lines 3-5 may be cryptic without context. These lines are responsible for numerically computing the integral at the end of (3.10), given the current k_f value. `f(x)` is simply a function definition for the integrand of that integral, while line 5 uses the SciPy function `quad` to numerically integrate `f` from 0 to k_f. `quad` returns a tuple containing both the numerical value of the integration and the bounds on that value's error, so we unpack it to get the value we want and call it `integral`.

To determine the value of ϕ_0, we must use a root finding routine because in (3.10), ϕ_0 appears on both sides of the equation.

Unfinished. Within a tabulated equation of state, we need values for pressure from approximately 10^{-14} to about 10^{-1}. When solving for a static solution using the TOV equations, we terminate the integration when the pressure drops below a certain threshold, which in our case is about 10^{-11}, so we want values of the equation of state below that point in order to ensure that we calculate the solution correctly as it goes to zero.

Chapter 4
Advanced QHD Parameter Sets

4.1 Equations of Motion

The three equations of motion for the meson fields reduce to, as shown in [1, p. 79]

$$\begin{aligned}
\phi_0 &= \frac{g_s}{m_s^2}\left[\frac{1}{\pi^2}\left(\int_0^{k_p} dk\,\frac{k^2 m^*}{\sqrt{k^2+m^{*2}}} + \int_0^{k_n} dk\,\frac{k^2 m^*}{\sqrt{k^2+m^{*2}}}\right) - \frac{\kappa}{2}(g_s\phi_0)^2 - \frac{\lambda}{6}(g_s\phi_0)^3\right] \\
V_0 &= \frac{g_v}{m_s^2}\left[\rho_p + \rho_n - \frac{\zeta}{6}(g_v V_0)^3 - 2\Lambda_v (g_v V_0)(g_\rho b_0)^2\right] \\
b_0 &= \frac{g_\rho}{m_\rho^2}\left[\frac{1}{2}(\rho_p - \rho_n) - 2\Lambda_v (g_v V_0)^2 (g_\rho b_0)\right]
\end{aligned} \tag{4.1}$$

4.2 Equilibrium Conditions

The number densities of baryons must stay constant; thus, we have

$$\rho = \rho_n + \rho_p, \tag{4.2}$$

where ρ_n is the number density of the first species, neutrons, while ρ_p is the number density of the second species, protons. It is important to note now that we use the relation given on [1, p. 90]

$$\rho_x = \frac{k_x^3}{3\pi^2} \quad \Longleftrightarrow \quad k_x = \pi^2(3\pi^2\rho_x)^{-2/3}$$

to relate the xth fermi-momenta with its corresponding number density.

For beta-equilibrium to be satisfied, we must have

$$\mu_n = \mu_p + \mu_e, \tag{4.3}$$

where μ_x is the xth *chemical potential.* Using a handy result from [1, p. 90], we can rewrite this more generally as

$$\sqrt{k_n^2 + m^{*2}} = \sqrt{k_p^2 + m^{*2}} + g_\rho b_0 + \sqrt{k_e^2 + m_e^2}. \tag{4.4}$$

Furthermore, within neutron stars, muon production is probable and often favorable, per [1, p. 90]. Therefore, we have

$$\mu_\mu = \mu_e, \quad \mu_e = \sqrt{k_e^2 + m_e^2}, \quad \mu_\mu = \sqrt{k_\mu^2 + m_\mu^2}. \tag{4.5}$$

Because charge must be conserved, we need to have

$$\rho_p = \rho_e + \rho_\mu \quad \Rightarrow \quad k_p = \left(k_e^3 + k_\mu^3\right)^{1/3} \tag{4.6}$$

Bibliography

[1] Jacobus Petrus William Diener. "Relativistic mean-field theory applied to the study of neutron star properties". PhD thesis. Stellenbosch University, 2008.