

# Analysis of equations of state for neutron star modeling

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# Chapter 1

## Introduction

This chapter aims to introduce the motivation for this project, the structure of this report, and the basic tools and techniques used throughout.

### 1.1 Abstract

Neutron stars are complex physical objects often modelled as a hydrodynamical system. Within these models, the two parameters energy density and pressure are related by an equation known as the “equation of state”. This relationship allows us to calculate energy density if we know pressure, and visa versa. The equation of state itself is important because it encodes the information about interactions between the fundamental particles within the star. Furthermore, as neutron stars are extreme examples of gravitation, and not microscopically observable on Earth, the physics within neutron stars are still not well known. By postulating and analyzing different interactions within a neutron star, and therefore different equations of state, we can simulate the observable characteristics of the resulting star and compare them to empirical data to help better understand how neutron stars behave on a fundamental level.

# Chapter 2

## Static Solutions

Within our study of equations of state, we want to see which predictions each unique equation makes about the macroscopic (observable) properties of a neutron star. To do so, we introduce the Tolman-Oppenheimer-Volkoff (TOV) equations, a time independent description of a spherically symmetric neutron star. By solving the TOV equations, we can calculate theoretical observables, such as the total mass and radius of an individual star, and compare them to empirical data gathered from real neutron stars. This chapter will introduce the TOV equations, show how they are solved, and how through analysis we can determine the aforementioned observable quantities of note.

### 2.1 The Tolman-Oppenheimer-Volkoff (TOV) Equations

The Tolman-Oppenheimer-Volkoff (TOV) equations are the below system of two coupled differential equations

$$\frac{dm}{dr} = 4\pi r^2 \epsilon, \quad \frac{dP}{dr} = -\frac{(4\pi r^3 P + m)(\epsilon + P)}{r^2(1 - 2m/r)}. \quad (2.1)$$

Within these equations, there are four variables of note: the *radius*,  $r$ , the *mass*,  $m$ , the *pressure*,  $P$ , and the *energy density*,  $\epsilon$ . Within this model, we consider neutron stars to be spherically symmetric; this means that only the distance from the center of the star is important. Furthermore, the radius  $r$  is the independent variable; therefore, the other variables can be written as functions of this radius:  $m = m(r)$ ,  $P = P(r)$ , and  $\epsilon = \epsilon(r)$ .

The parameter  $m$  is defined as the total amount of energy within a spherical shell of radius  $r$ . Technically, this is not identical to mass; however, due to Einstein's mass-energy equivalence, it is common and convenient to call this parameter mass. This paper will continue this convention.

At this point, we have two variables remaining, yet only one evolution equation. It is here we can finally show the importance of the equation of state within our neutron star calculations. Within this system, the energy density and pressure are related directly by an equation  $\epsilon = \epsilon(P)$  known as “the equation of state” (EOS). This relationship is important because it allows us to determine the current value of  $\epsilon$  if we already know the value of pressure.<sup>1</sup> The EOS will be derived and analyzed in depth in the later sections of this paper;

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<sup>1</sup>Practically, it is also easy to find pressure if we know energy density, however that is not necessary in this calculation.

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at this point, however, it is important to understand that the EOS encodes the interactions between the particles within the neutron star, and based on the model used to describe those interactions, it will change. For this derivation, we leave the EOS as a general function. After including an EOS in our TOV equations, we know just have two variables to evolve:  $m$  and  $P$ .

To determine a solution to the system of equations in (2.1), we need will solve an initial value problem. As we want to know information about the star from its center radially outward, we therefore need initial conditions for both  $m$  and  $P$  at the center of the star,  $r = 0$ . Determining an initial condition for  $m(r = 0)$  is straightforward; as  $m$  represents the total mass contained within a radius  $r$ , at the center of the star, as no mass is enclosed, so  $m(0) = 0$ . We treat the initial condition for  $P$ ,  $P(0)$ , called the *central pressure*, as a free parameter. Every static solution is uniquely specified by a value of the central pressure; thus, we simply choose a reasonable value (typically  $P(0)$  somewhere between  $10^{-6}$  and  $10^{-1}$ ) and begin our integration. These initial conditions are summarized as

$$m(0) = 0, \quad P(0) \in [10^{-6}, 10^{-1}]. \quad (2.2)$$

When beginning our integration, we cannot, however, start directly at  $r = 0$ , as the denominator of  $dP/dr$  in (2.1) would be undefined. Instead, we simply start at a very small value of  $r$ , say  $r \approx 10^{-8}$ . This is effectively  $r = 0$ , and is accurate enough for our purposes.

We want our integration to terminate once we reach the edge of the star, as we are not interested in anything beyond that point. To find this outer edge, we define the total radius of the star,  $R$ , as the radius when

$$P(R) = 0. \quad (2.3)$$

In practice, once we reach a very small pressure,  $P \sim 10^{-12}$ , we can end the integration. Once have found  $R$ , we can determine  $M = m(R)$ , the total mass enclosed at radius  $R$ . The total mass  $M$  and total radius  $R$  are important to our analyses of different equations of state, as they represent experimentally observable properties of real neutron stars. These theoretically calculated properties can be compared to observed properties to gauge the validity of any given equation of state.

By determining a solution to the TOV equations, we calculate something called “static solution,” a time independent image of a neutron star. A solution contains three curves:  $m(r)$ ,  $\epsilon(r)$ , and  $P(r)$ . Of most importance are the pressure and energy density curves; however because they are related by the EOS, we only need to save one curve, as we can simply calculate the other when desired.

## 2.2 Computing Static Solutions

To compute static solutions, we use numerical integration techniques for solving ordinary differential equations (ODEs). In this situation, we use a specific technique known as the fourth-order Runge-Kutta algorithm (RK4). First, we need a differential equation of the form

$$\frac{dy}{dx} = f(x, y),$$

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where  $x$  is the independent variable, and  $y$  is the function whose solution we wish to find. Importantly,  $y$  can be a vector valued function, so we can evolve a system of coupled differential equations using this method. Given a current value of the function,  $y(x)$ , RK4 allows to find an approximate value of the function a small step  $h$  forward in  $x$ :  $y(x + h)$ . Computationally, the algorithm is

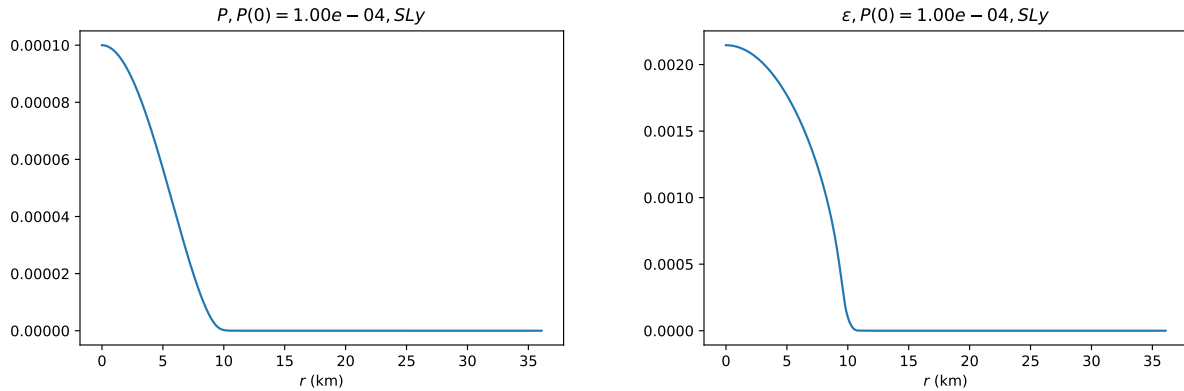
$$y(x_i + h) \cong y(x_i) + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4), \quad (2.4)$$

where

$$\begin{aligned} k_1 &= hf(x_i, y_i), \\ k_2 &= hf(x_i + h/2, y_i + k_1/2), \\ k_3 &= hf(x_i + h/2, y_i + k_2/2), \\ k_4 &= hf(x_i + h, y_i + k_3). \end{aligned}$$

If  $y$  is a vector valued function, then each of the  $k_i$  values will also be vector valued. To determine a solution for  $y$ , we start with an initial condition,  $y(x_0)$ . Then, after choosing a step size  $h$ , we use the initial value of  $y$  to determine  $k_1$  through  $k_4$ . Finally, we can then calculate  $y(x_0 + h)$  using (2.4). At this point, we repeat the process using the newly calculated values of  $y$  as our initial data, allowing us to calculate the value of  $y$  one more step forward. This algorithm is repeated as necessary.

For our system of coupled ODEs in (2.1), the independent variable is  $r$  and our function is the vector  $y = (m, P)$ . Using the initial conditions  $y_0 = (0, P_0)$  from (2.2), we can begin our integration from the center of the star and work outwards with step size  $h = \Delta r$ , a small step in the radial direction. We continually use the newly calculated values of  $y$  at some radius  $r$  to determine the value of  $y(r + \Delta r)$ . As we want the integration to terminate when  $P$  gets too small, every time we calculate new values, we check to make sure that  $P$  is still in an acceptable range, and if it is too small, we terminate.



**Figure 2.1.** The pressure  $P$  (left) and energy density  $\epsilon$  (right) for  $P(0) = 10^{-4}$  with equation of state SLy from [2].

This method allows us to generate static solution curves for various equations of state  $\epsilon(P)$  and central pressures  $P(0)$ . For an equation of state called “SLy” from [2] with a central

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pressure  $P(0) = 10^{-4}$ , an example of the resulting curves for pressure and energy density are shown in Figure 2.1.

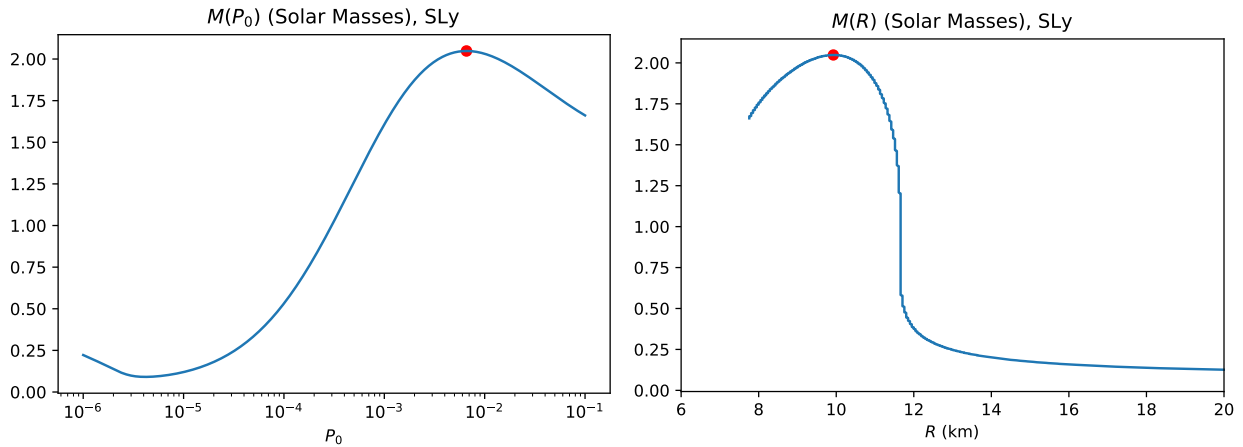
By observation, in Figure 2.1, we can see that the total radius,  $R$ , is about 10 km, as this is when the pressure goes to zero. Using this value, we can determine what the total mass is by plugging it into the mass function (not pictured). These values are vital to the next section.

### 2.3 Analysis of Static Solutions

In Figure 2.1, we looked at one static solution produced using a realistic equation of state called SLy. The decision to use SLy was simply exemplary in this case; we could have used a different equation of state and the static solutions therefore would have been different. To determine the predictions that an individual equation of state will make over a range of values, it is impractical to simply look at static solution curves, such as those in Figure 2.1. Instead, we look at the macroscopic properties,  $M$  and  $R$ , that each predicts over a wide range of central pressure values.

To do this, we create a range of central pressure values, approximately  $P(0) \in [10^{-6}, 10^{-1}]$ . In practice, it is useful to use a logarithmic range of values, such that the values are more densely packed near  $P = 10^{-6}$ ; within our code, we use the numpy function `logspace` to create an array of values with logarithmic spacing. Now, we can iterate through these pressure values and create static solutions for each central pressure. For every static solution, we integrate until we find  $R$ , at which time we also calculate  $M$ . These values are stored, along with the central pressure value from which they were produced.

Using these values, we plot two curves:  $M$  vs.  $P_0$  and  $M$  vs.  $R$ . These are shown in Figure 2.2.



**Figure 2.2.** The curves  $M$  vs.  $P_0$  (left) and  $M$  vs.  $R$  (right) for the SLy equation of state from [2].

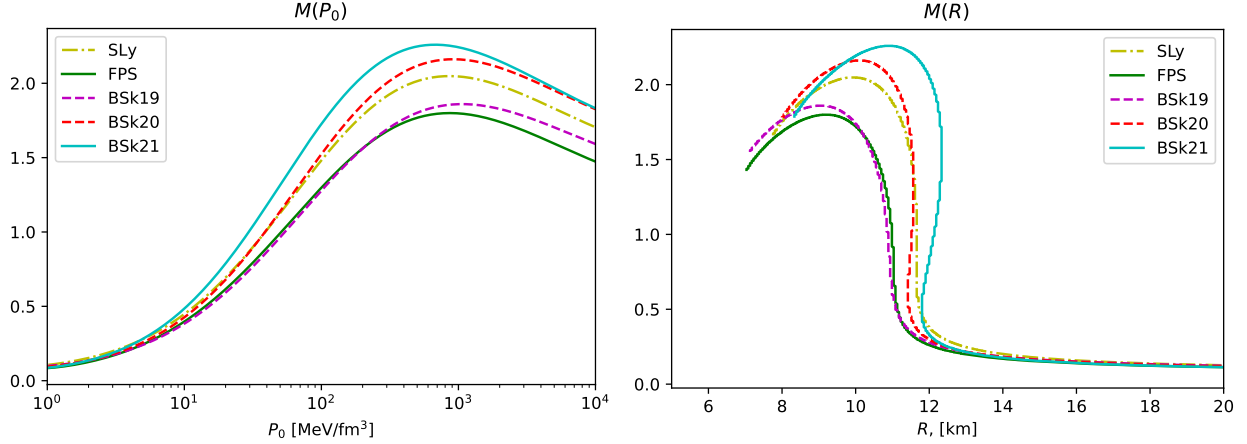
From these curves, we deduce three special values: the *critical pressure*, the *critical radius*, and the *critical mass*. The critical mass is the maximum mass that the EOS predicts, and the critical pressure and critical radius are the pressure and radius, respectively, where the



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critical mass is reached. The critical mass and critical radius are used as measuring sticks for how realistic an equation of state is; if they are on the same order as the largest neutron stars that have been actually observed, the equation of state is considered to be a candidate for a “correct” description of the star. Usually, the critical mass is on the order of about 2 solar masses, while the critical radius is on the order of about 10 km.

In [2], besides the SLy EOS, there is an additional EOS called FPS, derived and implemented in a similar fashion, just with a different parameterization. Furthermore, in another paper by the same authors, [3], there are three additional EOSs of similar form. All of these EOSs are considered realistic equations of state and are computed by creating analytical fits from empirical observational data from neutron stars. From previous work, we have these EOSs implemented, so for demonstration, we have overlaid the mass-radius and mass-pressure curves to demonstrate the differences in predictions of different equations of state. These curves are shown in Figure 2.3.



**Figure 2.3.** <caption>

# Chapter 3

## QHD-I

This chapter will describe the derivation of an equation of state from the QHD-I parameter set, as described within [1].

### 3.1 Introduction

### 3.2 Derivation of equations of motion

The Euler-Lagrange equations, for a Lagrange density  $\mathcal{L}$  over a classical field  $\varphi_\alpha$ , are given by

$$\partial_\nu \left( \frac{\partial \mathcal{L}}{\partial(\partial_\nu \varphi_\alpha)} \right) - \frac{\partial \mathcal{L}}{\partial \varphi_\alpha} = 0. \quad (3.1)$$

From [1, p. 56], we have the Lagrangian density of QHD-I

$$\begin{aligned} \mathcal{L} = & \bar{\psi}[\gamma_\mu(i\partial^\mu - g_s V^\mu) - (M - g_s \phi)]\psi \\ & + \frac{1}{2}(\partial_\mu \phi \partial^\mu \phi - m_s^2 \phi^2) - \frac{1}{4}V_{\mu\nu}V^{\mu\nu} + \frac{1}{2}m_\omega^2 V_\mu V^\mu, \end{aligned} \quad (3.2)$$

where  $V_{\mu\nu} \equiv \partial_\mu V_\nu(x) - \partial_\nu V_\mu(x)$ . To determine the equations of motion for this system, we must apply (3.1) to (3.2) for each unique field in the system

$$\varphi_\alpha = \begin{cases} \phi(x) & : \text{ scalar meson field,} \\ V^\mu(x) & : \text{ vector meson field,} \\ \psi(x) & : \text{ baryon field,} \\ \bar{\psi}(x) & : \text{ Dirac adjoint baryon field,} \end{cases}$$

where  $\bar{\psi}(x) \equiv \psi^\dagger(x)\gamma^0$ , the *Dirac adjoint*.

For the scalar meson field, when  $\varphi_\alpha = \phi(x)$ , the first term in (3.1) gives

$$\partial_\nu \left( \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} \right) = \frac{1}{2} \left[ \frac{\partial}{\partial(\partial_\nu \phi)} (\partial_\mu \partial^\mu \phi) \right] = \partial_\nu \partial^\nu \phi,$$

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while the second term gives

$$\frac{\partial \mathcal{L}}{\partial \phi} = \bar{\psi}[+g_s]\psi + \frac{1}{2}(-m_s(2\phi)) = g_s\bar{\psi}\psi - m_s\phi.$$

Combining, we get the first equation of motion,

$$\partial_\nu \left( \frac{\partial \mathcal{L}}{\partial (\partial_\nu \phi)} \right) - \frac{\partial \mathcal{L}}{\partial \phi} = \partial_\nu \partial^\nu \phi - (g_s\bar{\psi}\psi - m_s\phi) = 0 \quad \Rightarrow \quad \partial_\nu \partial^\nu \phi + m_s\phi = g_s\bar{\psi}\psi. \quad (3.3)$$

This is the form given in [1] (8.1a).

For the vector meson field, when  $\varphi_\alpha = V_\mu$ , the first term in (3.1) gives

$$\partial_\nu \left( \frac{\partial \mathcal{L}}{\partial (\partial_\nu V_\mu)} \right) = \partial_\nu V^{\mu\nu},$$

after many simplifications, including using the definition of  $V^{\mu\nu}$  and the relabelling of indices. The second term gives

$$\frac{\partial \mathcal{L}}{\partial V_\mu} = \frac{\partial}{\partial V_\mu} \bar{\psi} \left[ \gamma_\alpha (-g_v V^\alpha) + \frac{1}{2} m_\omega^2 V^\alpha V_\alpha \right] = -g_v \bar{\psi} \gamma^\mu \psi + m_\omega^2 V^\mu,$$

and combining we have

$$\partial_\nu \left( \frac{\partial \mathcal{L}}{\partial (\partial_\nu V_\mu)} \right) - \frac{\partial \mathcal{L}}{\partial V_\mu} = \partial_\nu V^{\mu\nu} - (-g_v \bar{\psi} \gamma^\mu \psi + m_\omega^2 V^\mu) = 0. \quad (3.4)$$

However, this is not the form given in [1]; to reach his form, we leverage the anti-symmetry of  $V_{\mu\nu}$ , namely

$$V_{\mu\nu} = -V_{\nu\mu}.$$

Therefore, we make the above substitution, multiply (3.4) by  $-1$ , and send  $\mu \leftrightarrow \nu$  to obtain

$$\partial_\mu V^{\mu\nu} + m_\omega^2 V^\nu = g_v \bar{\psi} \gamma^\nu \psi, \quad (3.5)$$

as given in [1] (8.1b).

Next, we have the two equations of motion from the baryon field. For  $\varphi_\alpha = \bar{\psi}$ , applying (3.1) is straight forward, as there is no  $\partial_\nu \bar{\psi}$  dependence in  $\mathcal{L}$ , so the first term in (3.1) is zero. Thus, we obtain

$$\partial_\nu \left( \frac{\partial \mathcal{L}}{\partial (\partial_\nu \bar{\psi})} \right) - \frac{\partial \mathcal{L}}{\partial \bar{\psi}} = [\gamma_\mu (i\partial^\mu - g_v V^\mu) - (M - g_s \phi)] \psi = 0. \quad (3.6)$$

For the final case, when  $\varphi_\alpha = \psi$ , the first term in (3.1) gives

$$\partial_\nu \left( \frac{\partial \mathcal{L}}{\partial (\partial_\nu \psi)} \right) = \partial_\nu \left[ \frac{\partial}{\partial (\partial_\nu \psi)} \bar{\psi} i \gamma_\alpha \partial^\alpha \psi \right] = i \partial_\nu \bar{\psi} \gamma^\nu,$$

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while the second gives

$$\frac{\partial \mathcal{L}}{\partial \psi} = \bar{\psi}[\gamma_\mu(i\partial^\mu - g_v V^\mu) - (M - g_s \phi)].$$

Combining, we get our fourth equation

$$i\partial_\nu \bar{\psi} \gamma^\nu - \bar{\psi}[\gamma_\mu(i\partial^\mu - g_v V^\mu) - (M - g_s \phi)] = 0. \quad (3.7)$$

In summary, here are the four equations of motion for QHD-I:

$$\partial_\nu \partial^\nu \phi + m_s \phi = g_s \bar{\psi} \psi, \quad (3.3)$$

$$\partial_\mu V^{\mu\nu} + m_\omega^2 V^\nu = g_v \bar{\psi} \gamma^\nu \psi, \quad (3.5)$$

$$[\gamma_\mu(i\partial^\mu - g_v V^\mu(x)) - (M - g_s \phi)]\psi = 0, \quad (3.6)$$

$$i\partial_\nu \bar{\psi} \gamma^\nu - \bar{\psi}[\gamma_\mu(i\partial^\mu - g_v V^\mu) - (M - g_s \phi)] = 0. \quad (3.7)$$

The first three are given in [1].

### 3.3 Relativistic Mean Field Simplifications

In [1] §8.3, we find that the equations of motion listed above are very difficult to solve in their current form. To make them more manageable, we approximate them with “relativistic mean field” (RMF) simplifications, where we take each field to be its ground state expectation value. For the meson fields, this simplification yields

$$\phi \Rightarrow \langle \Phi | \phi | \Phi \rangle = \langle \phi \rangle \equiv \phi_0, \quad (3.8)$$

$$V_\mu \Rightarrow \langle \Phi | V_\mu | \Phi \rangle = \langle V_\mu \rangle \equiv \delta_{\mu 0} V_0, \quad (3.9)$$

where  $|\Phi\rangle$  represents the ground state. These results arise from arguing that, in their ground states,  $\phi$  and  $V_\mu$  should be independent of space and time, as the system is both uniform and stationary; therefore,  $\phi_0$  and  $V_0$  are constants. Furthermore, because the system is at rest and the baryon flux,  $\bar{\psi} \gamma^i \psi$ , is zero, the spatial components of the expected value of  $V_\mu$ ,  $\langle V_\mu \rangle$ , must vanish [1].

For the baryon field, a “normal order”, i.e. normalized, expectation value must be taken, as because otherwise, the vacuum would be taken into account and the traditional expectation value would diverge. This “normal ordered” expectation value is denoted with a “:”. Throughout the equations of motion, the

**MORE HERE.**

After evaluating these expectation values, we have

$$\phi_0 = \frac{g_s}{m_s^2} \frac{\gamma}{2\pi^2} \int_0^{k_f} dk \frac{k^2 m^*}{\sqrt{k^2 + m^{*2}}}, \quad (3.10)$$

$$V_0 = \frac{g_v}{m_\omega^2} \rho, \quad (3.11)$$

### 3.4. NUMERICAL GENERATION OF THE EQUATION OF STATE

where once again,  $m^* \equiv M - g_s \phi$ , the *reduced mass*, and  $\rho$  is the nucleon number density. If we assume spherical symmetry, a reasonable condition for the study of star-like systems, we get the following expressions

$$\epsilon = \frac{1}{2}m_s^2\phi_0^2 + \frac{1}{2}m_\omega^2V_0^2 + \frac{\gamma}{2\pi^2} \int_0^{k_f} dk k^2 \sqrt{k^2 + m^{*2}}, \quad (3.12)$$

$$P = -\frac{1}{2}m_s^2\phi_0^2 + \frac{1}{2}m_\omega^2V_0^2 + \frac{1}{3} \left( \frac{\gamma}{2\pi^2} \int_0^{k_f} dk \frac{k^4}{\sqrt{k^2 + m^{*2}}} \right). \quad (3.13)$$

In the next section, we will use these expressions to generate values for this equation of state.

### 3.4 Numerical generation of the equation of state

We now wish to generate tabulated values of the equation of state using the equations for  $\epsilon$ ,  $P$  and  $\phi_0$  at the end of the previous section. Simply, this process requires looping through various values of  $k_f$ ; at each iteration, we find the corresponding value of  $\phi_0$  by using a root finding routine on (3.10), as  $m^*$  depends on  $\phi_0$ , then computing  $V_0$  independently using (3.11), and then finally substituting those values into (3.12) and (3.13) and storing those values. After creating a table of values for  $\epsilon$  and  $P$ , we can verify the validity of the equation of state by solving the TOV equations and comparing the results of static solutions, mass-radius curves, and mass-pressure curves to other, previously calculated equations of state.

- This section is rushed. Need better explanations of TOV equations, etc. in a different section; see Chapter 2, the TOV equations section. This can be referenced here.
- Talk about the parameter set for QHD-1.

To begin, we choose a value of  $k_f$  for the entirety of the iteration; for sake of example, we take  $k_f = 1$ . Then, we find the value of  $\phi_0$  for that  $k_f$  value. We define a function in Python for  $\phi_0$  from (3.10).

```
1 def f_phi0(phi0, kf):
2     mstar = M - g_s * phi0
3     def f(k): # integrand
4         return k**2 * mstar/sqrt(k**2 + mstar**2)
5     integral, err = quad(f,0,kf)
6     return phi0 - (g_s/m_s**2) * (gamma/(2*pi**2)) * integral
```

While most of the function is straight forward, lines 3-5 may be cryptic without context. These lines are responsible for numerically computing the integral at the end of (3.10), given the current  $k_f$  value.  $f(x)$  is simply a function definition for the integrand of that integral, while line 5 uses the SciPy function `quad` to numerically integrate  $f$  from 0 to  $k_f$ . `quad` returns a tuple containing both the numerical value of the integration and the bounds on that value's error, so we unpack it to get the value we want and call it `integral`.

To determine the value of  $\phi_0$ , we must use a root finding routine because in (3.10),  $\phi_0$  appears on both sides of the equation.

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**Unfinished.** Within a tabulated equation of state, we need values for pressure from approximately  $10^{-14}$  to about  $10^{-1}$ . When solving for a static solution using the TOV equations, we terminate the integration when the pressure drops below a certain threshold, which in our case is about  $10^{-11}$ , so we want values of the equation of state below that point in order to ensure that we calculate the solution correctly as it goes to zero.

# Chapter 4

## Advanced QHD Parameter Sets

### 4.1 Equations of Motion

The three equations of motion for the meson fields reduce to, as shown in [1, p. 79]

$$\begin{aligned}\phi_0 &= \frac{g_s}{m_s^2} \left[ \frac{1}{\pi^2} \left( \int_0^{k_p} dk \frac{k^2 m^*}{\sqrt{k^2 + m^{*2}}} + \int_0^{k_n} dk \frac{k^2 m^*}{\sqrt{k^2 + m^{*2}}} \right) - \frac{\kappa}{2} (g_s \phi_0)^2 - \frac{\lambda}{6} (g_s \phi_0)^3 \right] \\ V_0 &= \frac{g_v}{m_s^2} \left[ \rho_p + \rho_n - \frac{\zeta}{6} (g_v V_0)^3 - 2\Lambda_v (g_v V_0) (g_\rho b_0)^2 \right] \\ b_0 &= \frac{g_\rho}{m_\rho^2} \left[ \frac{1}{2} (\rho_p - \rho_n) - 2\Lambda_v (g_v V_0)^2 (g_\rho b_0) \right]\end{aligned}\tag{4.1}$$

### 4.2 Equilibrium Conditions

The number densities of baryons must stay constant; thus, we have

$$\rho = \rho_n + \rho_p,\tag{4.2}$$

where  $\rho_n$  is the number density of the first species, neutrons, while  $\rho_p$  is the number density of the second species, protons. It is important to note now that we use the relation given on [1, p. 90]

$$\rho_x = \frac{k_x^3}{3\pi^2} \iff k_x = \pi^2 (3\pi^2 \rho_x)^{-2/3}$$

to relate the  $x$ th fermi-momenta with its corresponding number density.

For beta-equilibrium to be satisfied, we must have

$$\mu_n = \mu_p + \mu_e,\tag{4.3}$$

where  $\mu_x$  is the  $x$ th *chemical potential*. Using a handy result from [1, p. 90], we can rewrite this more generally as

$$\sqrt{k_n^2 + m^{*2}} = \sqrt{k_p^2 + m^{*2}} + g_\rho b_0 + \sqrt{k_e^2 + m_e^2}.\tag{4.4}$$

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Furthermore, within neutron stars, muon production is probable and often favorable, per [1, p. 90]. Therefore, we have

$$\mu_\mu = \mu_e, \quad \mu_e = \sqrt{k_e^2 + m_e^2}, \quad \mu_\mu = \sqrt{k_\mu^2 + m_\mu^2}. \quad (4.5)$$

Because charge must be conserved, we need to have

$$\rho_p = \rho_e + \rho_\mu \quad \Rightarrow \quad k_p = (k_e^3 + k_\mu^3)^{1/3} \quad (4.6)$$



# Appendix A

## Code: TOV Implementation

```
1  # Joe Nyhan, 15 July 2021
2  # File for computing static solutions for fermionic stars via the
   TOV model.
3  ###
4  import numpy as np
5  import pandas as pd
6  from scipy.integrate import solve_ivp
7  from scipy.interpolate import interp1d
8  # from scipy.optimize import minimize, fmin
9  import matplotlib.pyplot as plt
10 # %%
11
12 # EOS
13 eos_UR          = 0
14 eos_polytrope   = 0
15
16 eos_SLy         = 1
17 eos_FPS        = 0
18
19 eos_BSk19       = 0
20 eos_BSk20       = 0
21 eos_BSk21       = 0
22
23 if eos_UR:
24     eos = "UR"
25     Gamma = 1.3
26 if eos_polytrope:
27     eos = "polytrope"
28     Gamma = 2
29     K = 100
30 if eos_SLy:
31     eos = "SLy"
32 if eos_FPS:
33     eos = "FPS"
34 if eos_BSk19:
```

## APPENDIX A. CODE: TOV IMPLEMENTATION

```
35     eos = "BSk19"
36 if eos_BSk20:
37     eos = "BSk20"
38 if eos_BSk21:
39     eos = "BSk21"
40
41 # eos = "Diener"
42
43 # MODES
44 make_static_solution = 0
45 p0_analysis = 1
46
47 # PARAMETERS
48
49 if make_static_solution:
50     p0 = 1e-4
51
52 if p0_analysis:
53     pmin = 1e-6
54     pmax = 1e-1
55     NUM_POINTS = 1000
56     # p0_vals = np.arange(1e-6, 1e-1, 5e-5)
57     p0_vals = np.logspace(round(np.log10(pmin)), round(np.log10(pmax))
58                          ), NUM_POINTS, base=10.0)
59
60 r0 = 0.000001
61 rmax = 200
62 dr = 0.02
63
64 p_tol = 1e-11
65 r_vals = np.arange(r0, rmax, dr)
66
67 m0 = 0
68
69 if make_static_solution: path = "../input"
70 elif p0_analysis: path = "./p0_analysis"
71
72 # FUNCTIONS
73
74 vals_path = ""
75 if eos_UR or eos_polytrope:
76     pass
77 else:
78     vals_path = f"0-{eos}_vals.vals"
79
80     df = pd.read_csv(vals_path)
```

```

81     interp_rhos = df.iloc[:,0].to_numpy()
82     interp_ps = df.iloc[:,1].to_numpy()
83
84
85 rho_from_P_interp = interp1d(interp_ps, interp_rhos, fill_value="
    extrapolate")
86
87 def rho_from_P(p):
88     if eos_UR:
89         return p/(Gamma-1)
90     elif eos_polytrope:
91         return (p/K)**(1/Gamma)
92     else:
93         return rho_from_P_interp(p)
94
95
96 def event(r,y):
97     m, p = y
98
99     return p - p_tol
100
101 event.terminal = True
102 event.direction = 0
103
104 def f_onefluid(r,y):
105     m, p = y
106
107     rho = rho_from_P(p)
108
109     N = 1-2*m/r
110
111     fm = 4*np.pi*r**2*rho
112     frho = -(4*np.pi*r**3*p + m)*(rho+p)/(r**2*N))
113
114     return fm, frho
115
116 # ONE STATIC SOLUTIION
117
118 if make_static_solution:
119     if eos_polytrope:
120         P_path = f"{path}/polytrope_K{K:.1f}_gamma{Gamma:.1f}_p{p0
            :.8f}_dr{dr:.3f}_P.txt"
121         rho_path = f"{path}/polytrope_K{K:.1f}_gamma{Gamma:.1f}_p{p0
            :.8f}_dr{dr:.3f}_rho.txt"
122
123     # if eos_SLy:
124     else:

```

## APPENDIX A. CODE: TOV IMPLEMENTATION

```
125     P_path = f"{path}/{eos}_p{p0:.8f}_dr{dr:.3f}_P.txt"
126     rho_path = f"{path}/{eos}_p{p0:.8f}_dr{dr:.3f}_rho.txt"
127
128     P_out = open(P_path, "w")
129     rho_out = open(rho_path, "w")
130
131     sol = solve_ivp(f_onefluid, [r0, rmax], [m0, p0], method='RK45',
132                     atol = 1e-12, rtol = 1e-12, t_eval=r_vals, events=event)
133
134     m, p = sol.y
135
136     for i in range(len(r_vals)):
137         if i < len(p):
138             P_out.write(f"{p[i]:.16e}\n")
139             rho_out.write(f"{rho_from_P(p[i]):.16e}\n")
140         else:
141             P_out.write(f"{0:.16e}\n")
142             rho_out.write(f"{0:.16e}\n")
143
144     P_out.close()
145     rho_out.close()
146
147     print(f"COMPLETED.")
148
149
150 # P0 ANALYSIS
151
152 if p0_analysis:
153     output = open(f"{path}/{eos},p{pmin:.3e}-p{pmax:.3e}.vals", "w")
154
155     print(f"Number_of_points:{len(p0_vals)};_for_EoS:_{eos}")
156
157     def evolution(p0_vals):
158         for i in range(len(p0_vals)):
159
160             p0 = p0_vals[i]
161             sol = solve_ivp(f_onefluid, [r0, rmax], [m0, p0], method='
162                 RK45', atol = 1e-12, rtol = 1e-12, t_eval=r_vals,
163                 events=event)
164             M = sol.y[0][-1]
165             R = sol.t[-1]
166
167             if M < 0: print(f"M<0"); break
168
169             output.write(f"{p0:.6e},{M:.6e},{R:.6e}\n")
```

```
169         if i % 10 == 0: print(f"Timestep_{i:3d}_reached,_{M=}.")
170
171     evolution(p0_vals)
172
173     output.close()
```

# Appendix B

## Code: QHD-1 Implementation

```
1 # Joe Nyhan, 18 January 2021
2 %%
3 from numpy import sqrt, pi, logspace
4 import matplotlib.pyplot as plt
5
6 from scipy.integrate import quad
7 from scipy.optimize import root
8
9 gamma = 2
10
11 # masses (in GeV)
12 m_s = .520
13 m_omega = .783
14 M = .939
15
16 # other constants
17 g_s = sqrt(109.6)
18 g_v = sqrt(190.4)
19
20
21 def f_phi0(phi0, kf):
22     """see Diener p. 66, eq 8.21a"""
23     mstar = M - g_s * phi0
24     # integral = 1/2 * mstar * (kf * sqrt(kf**2 + mstar**2) - mstar
25     # * log((kf + sqrt(kf**2 + mstar**2))/mstar))
26     def f(k):
27         return k**2 * mstar/sqrt(k**2 + mstar**2)
28     integral, err = quad(f,0,kf)
29
30     return phi0 - (g_s/m_s**2) * (gamma/(2*pi**2)) * integral
31
32
33 def f_V0(kf):
34     """see Diener p. 66, eq 8.21b"""
```

```

35     rho = 2 * (gamma/(6*pi**2))*kf**3
36     # for rho, see Diener p. 67, eq. 8.25; note that gamma = 2 has
        been substituted
37     # print(f'{rho=}')
38
39     return (g_v/m_omega**2) * rho
40
41 def f_eps(phi0, V0, kf):
42     """see Diener p. 66, eq 8.24a"""
43     mstar = M - g_s * phi0
44     # integral = 1/8 * (kf * (2*kf**2 + mstar**2) * sqrt(kf**2 +
        mstar**2) - mstar**4 * log((kf + sqrt(kf**2 + mstar**2))/
        mstar))
45     def f(k):
46         return k**2 * sqrt(k**2 + mstar**2)
47     integral, err = quad(f,0,kf)
48
49     return 1/2 * m_s**2 * phi0**2 + 1/2 * m_omega**2 * V0**2 + (
        gamma/(2*pi**2)) * integral
50
51 def f_P(phi0, V0, kf):
52     """see Diener p. 66, eq 8.24b"""
53     mstar = M - g_s * phi0
54     # integral = 1/8 * (kf * (2*kf**2 - 3*mstar**2) * sqrt(kf**2 +
        mstar**2) + 3*mstar**4 * log((kf + sqrt(kf**2 + mstar**2))/
        mstar))
55
56     def f(k):
57         return k**4 / sqrt(k**2 + mstar**2)
58     integral, err = quad(f,0,kf)
59
60     return -1/2 * m_s**2 * phi0**2 + 1/2 * m_omega**2 * V0**2 +
        (1/3) * (gamma/(2*pi**2)) * integral
61
62 # kfs = [1e0]
63 kfs = logspace(-6,0,12000,base=10)
64
65 energy_densities = []
66 pressures = []
67
68 phi0_guess = 10**(-1)
69 # phi0_guess = 0
70
71 for kf in kfs:
72
73     res = root(f_phi0, phi0_guess, args=kf)
74     # print(res)

```

## APPENDIX B. CODE: QHD-1 IMPLEMENTATION

```
75     phi0 = res.x[0]
76     # print(phi0)
77
78     V0 = f_V0(kf)
79
80     # print(f'{V0=}, {phi0=}')
81
82     eps = f_eps(phi0, V0, kf)
83     p = f_P(phi0, V0, kf)
84
85     energy_densities.append(eps)
86     pressures.append(p)
87
88     phi0_guess = phi0
89
90 with open('./qhd1-vals.txt', 'w') as f:
91     for i in range(len(pressures)):
92         p = pressures[i]
93         eps = energy_densities[i]
94         f.write(f'{eps:.16e},_{p:.16e}\n')
```



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