Pre-Algebra Notes

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MATH

Number sets

When we talk about "numbers," we probably immediately think about these:

$$1, 2, 3, 4, 5, \ \dots$$

or, we might think of these:

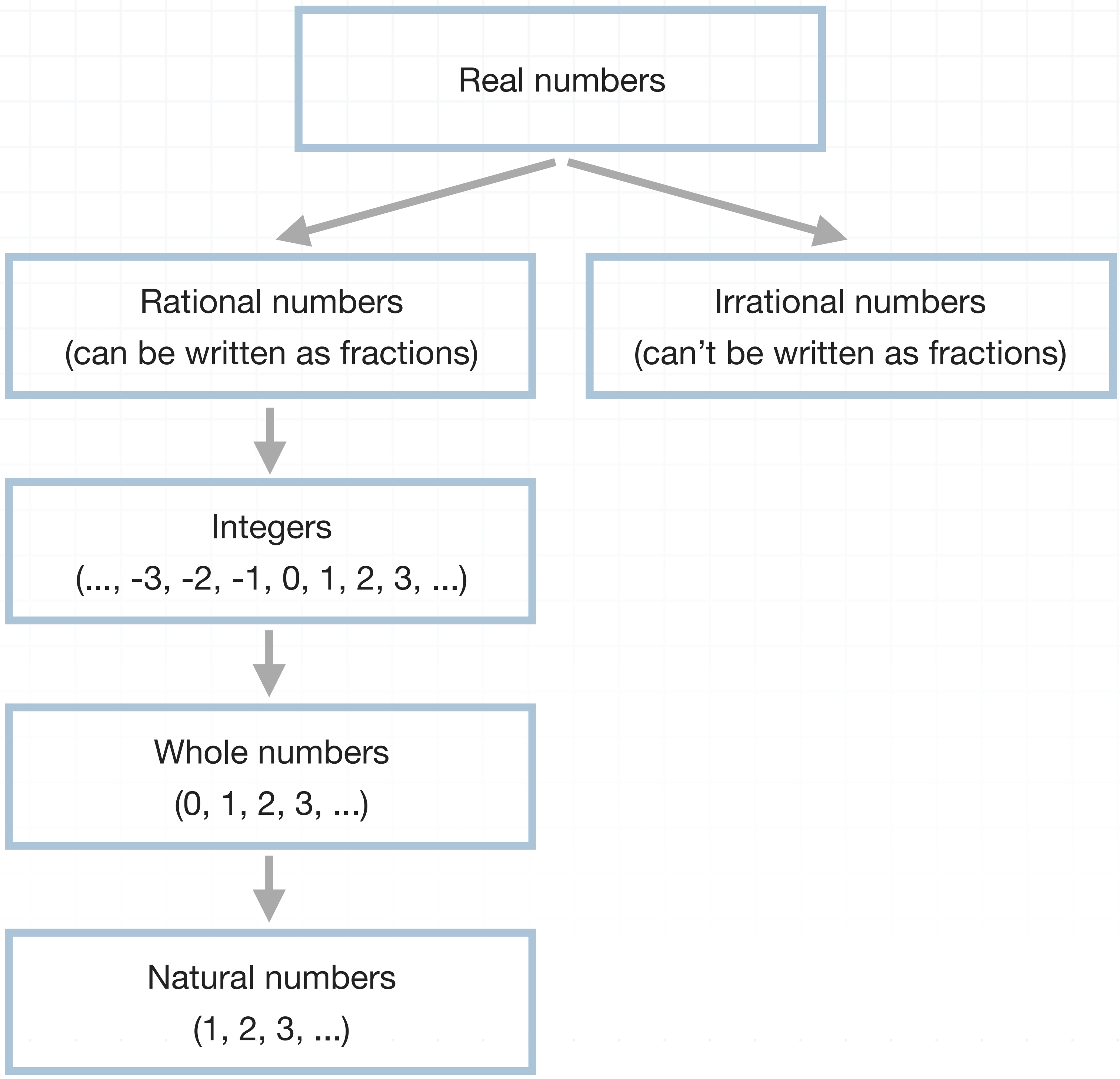
$$0, 1, 2, 3, 4, 5, \ \dots$$

But when most people think about numbers in general, they don't usually think about the other kinds of numbers that we'll talk about in this course, like negative numbers, fractions and decimals, or radical numbers.

Without going into too much detail in this lesson, we just want to take this lesson to get a little preview of different kinds of number sets. Since we'll be covering each of these kinds of numbers later on, right now we really just want to define each of the different number sets.

Real numbers

The vast majority of the numbers we'll use in most math classes are called **real numbers**, and the whole universe of real numbers is what makes up the **Real Number System**. Let's start with a diagram.



Real numbers include rational numbers and irrational numbers. **Rational numbers** are fractions, which look like this:

$$\frac{1}{4},\ \frac{3}{2},\ \ -\frac{1}{8},\ \frac{6}{7},\ \ -\frac{2}{3},\ \ \dots$$

Decimal numbers that terminate or repeat are also rational numbers and can look like this:

$$-0.25,\ 0.333,\ 5.7,\ \dots$$

Irrational numbers are all numbers that can't be written as a fraction. These are irrational roots like $\sqrt{2}$ and decimal numbers like π that go on forever.

$$\sqrt{2} = 1.41421356237...$$

$$\pi = 3.14159265359...$$

Again, we'll go into more detail about these later.

Integers are a special kind of rational number. They're made up of all the numbers we normally think about, like $0, 1, 2, 3, 4, 5, \ ...$, plus negative numbers as well. So the set of integers looks like this:

$$\ldots \ -5, \ -4, \ -3, \ -2, \ -1, 0, 1, 2, 3, 4, 5, \ \ldots$$

Within the set of integer numbers is a set of numbers that we call **whole numbers**, which is made up of all positive integers, plus 0. So the set of whole numbers is

$$0, 1, 2, 3, 4, 5, \ \ldots$$

And within the set of whole numbers, we define a set called the **natural numbers**, which is only the set of all positive integers, without 0. So the set of natural numbers is

$$1, 2, 3, 4, 5, \ \ldots$$

We often refer to natural numbers as the "**counting numbers**," since $1, 2, 3, 4, 5, \ \ldots$ is how we learn to count when we're young.

Number set symbols

Each of these number sets is indicated with a symbol. We use the symbol as a short-hand way of referring to the values in the set.

$\mathbb{R}$	represents the set of	real numbers
$\mathbb{Q}$	represents the set of	rational numbers
$\mathbb{Z}$	represents the set of	integers
$\mathbb{W}$	represents the set of	whole numbers
$\mathbb{N}$	represents the set of	natural numbers

Because irrational numbers is all real numbers, except all of the rational numbers (which includes rationals, integers, whole numbers and natural numbers), we usually express irrational numbers as $\mathbb{R}$-$\mathbb{Q}$, or $\mathbb{R}\backslash\mathbb{Q}$.

$\mathbb{R}$-$\mathbb{Q}$	represents the set of	irrational numbers

Set notation

Here we've talked about number sets, but we can also think about sets of things other than numbers. For example, we could describe the people in a family as

$$\text{Family} = \{\text{Father}, \text{Mother}, \text{Sister}, \text{ Brother}\}$$

or countries in North America as

$$\text{North American countries} = \{\text{Canada}, \text{United States}, \text{Mexico}\}$$

We usually use those curly braces to enclose the members of the set. So using the symbols we learned for number sets, in set notation we could write the set of all natural numbers as

$$N = \{1, 2, 3, 4, 5, \ldots\}$$

Identity numbers

Identity numbers are numbers that don't change the "identity" of the original value.

The identity for **addition** is 0.

The identity for **multiplication** is 1.

The reason is that

we can **add** 0 to any number and it doesn't change the original value.

we can **multiply** any number by 1 and it doesn't change the original value.

Let's look at an example with 0, the identity number for addition.

Example

What is $17 + 0$?

Without even thinking about this in terms of identity numbers, we should already know that $17 + 0 = 17$, because if we have 17 and we add nothing to it, we still have 17.

If we think about this more technically in terms of identity numbers, we know that 0 is the identity number for addition. Since we are adding 0, and because 0 is the identity number for addition, we know that adding 0 to 17 won't change the identity of 17, so $17+0$ will just be 17.

Let's look at an example with the identity number for multiplication.

Example

What is 4×1?

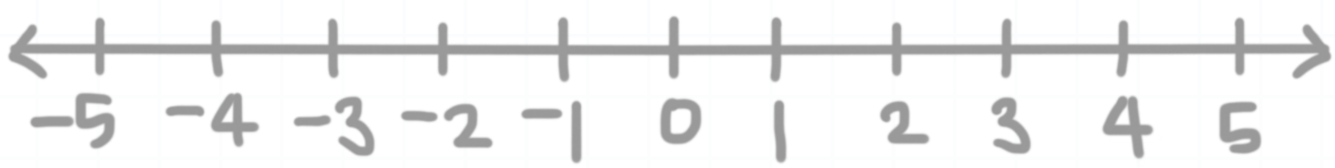
In this problem we're multiplying 4 by 1. We should already know that 4 times 1 is just 4, and we don't really need identity numbers to tell us this.

But the identity number concept confirms that this is true. We know that 1 is the identity number for multiplication. Since we are multiplying by 1, and because 1 is the identity number for multiplication, we know that multiplying 4 by 1 won't change the identity of 4, so 4×1 will just be 4.

Opposite of a number

When we talk about the “opposite of a number,” we’re specifically talking about the positive and negative versions of the same number. Now that’s not a technical definition by any means, so let’s talk more about what we mean here.

But before we get into that, let’s take a step back and talk about negative numbers. The easiest way to illustrate a negative number is by using the number line.



On this number line, we can see the number 0 in the center. The numbers to the right of 0 are always positive numbers; the numbers to the left of 0 are always negative numbers, and this is always true when we’re talking about a number line like this.

The other important thing to realize about this number line is that the numbers 1 and -1 are the same distance away from 0. We could say they’re both “one unit” away from 0, in fact. In the same way, 4 and -4 are the same distance from 0; they’re both “four units” away from 0.

And this brings us back to the concept of “opposite” numbers. Opposite numbers are numbers which are the same distance away from 0. And that’s why 1 and -1 are opposites (because they are the same distance from 0), and why 4 and -4 are opposites (because they are also the same distance from 0).

On the other hand, 2 and 4 are not opposites of each other, because the number 2 is two units away from 0, whereas the number 4 is four units away from 0. Two numbers are **not** opposites of each other if they are different distances away from 0.

The easiest way to remember how to find the opposite of a number is to add a negative sign if the number is positive, or to take away the negative sign if the number is negative.

Example

What is the opposite of 7?

To find the opposite of the positive number 7, we simply add a negative sign to get -7. Both 7 and -7 are seven units from 0 on the number line, so 7 and -7 are opposites of each other.

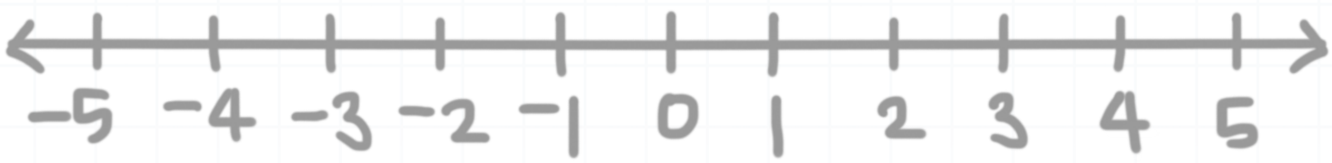
Let's try an example with a negative number.

Example

What is the opposite of -10?

To find the opposite of the negative number -10, we simply take away the negative sign to get 10. Both 10 and -10 are ten units from 0 on the number line, so 10 and -10 are opposites of each other.

How about 0? Does 0 have an opposite? We know from looking at the number line from earlier



that we can find the opposite of a number by measuring out the same distance from 0, but just on the opposite side of 0. So since 5 is five units away from 0 to the right of 0, then the opposite of 5 is the number that's five units away from 0 to the left of 0, which is -5.

But 0 itself is "zero units" away from 0 (keep in mind that it's the ONLY number that's "zero units" away from 0). 0 isn't a positive number or a negative number. So the opposite of 0 is actually 0. In other words, 0 is its own opposite!

Absolute value

The idea of absolute value is really just the idea of “distance from 0.” Going back to the idea of a number line,

we can say that 2 is a distance of two units away from 0, but also that -2, the opposite of 2 (and on the other side of 0), is two units away from 0. So both 2 and -2 are two units away from 0. The number 4 is four units away from 0, and -5 is five units away from 0.

So notice that, if we translate both positive and negative numbers into just their distance from 0, we get

Number	Distance from 0
2	2
-2	2
4	4
-5	5

Interestingly, all these numbers, whether they started out as positive or negative, become positive numbers when we think only about their distance from 0.

In fact, this is exactly what the absolute value operation does for us. It turns each number into its distance from the origin, essentially turning all positive and negative numbers into only positive numbers. (Also, it "turns" 0 into 0, because the number 0 is zero units away from itself.)

We indicate the absolute value of a number by enclosing it in a pair of vertical bars (which we'll call "absolute value bars"). Here's an example of what the absolute value operation looks like:

$$|-3|$$

This means "take the absolute value of -3." We already know that the absolute value of -3 is 3, since -3 is a distance of three units away from 0. Here are some more examples:

$$|-1| \quad \to \quad 1$$

$$|1| \quad \to \quad 1$$

$$|-6| \quad \to \quad 6$$

$$|6| \quad \to \quad 6$$

$$|-100| \quad \to \quad 100$$

$$|100| \quad \to \quad 100$$

Adding and subtracting signed numbers

Think about signed numbers just as positive and negative numbers. Positive numbers have positive signs (even though we often write positive numbers without actually putting a positive sign in front of them), whereas negative numbers have negative signs. So 3, 7, and 11 are all positive numbers, and -2, -6, and -9 are all negative numbers.

When it comes to adding and subtracting signed numbers, let's break down the three possible combinations we could face when we have two signed numbers:

1. Two positive numbers

2. Two negative numbers

3. One positive number and one negative number

Let's tackle **adding** numbers for these three combinations.

Positive $+$ **Positive** $=$ **Positive.** The trick is: add the numbers, keeping the sign positive.

$$3 + 4 = 7$$

$$10 + 1 = 11$$

Negative $+$ **Negative** $=$ **Negative.** The trick is: add the numbers as if they were both positive, but make the sign negative.

$$-3 + (-4) = -7$$

$$-10+(-1)=-11$$

Positive + Negative, Negative + Positive. When we add one positive number and one negative number, we want to start by considering their absolute values. For instance, if we're trying to add -7 and 4, we want to instead first consider $|-7|=7$ and $|4|=4$. The result will be **positive if the absolute value of** the **positive number is larger, but negative if the absolute value of** the **negative number is larger.**

So 7 is larger than 4, but 7 was originally the negative number, so the absolute value of the negative number is larger, in this case. The trick is to find the absolute value of both numbers and subtract the absolute value of the smaller number from the absolute value of the larger number. The sign of the answer will be the original sign of the number whose absolute value is larger.

$$3+(-4)=-1$$

Here, the negative number is -4 and the positive number is 3. 4 is larger than 3, so the absolute value of the negative number is larger, which means the answer will be negative. So we subtract 3 from 4 and get 1. The sign needs to be negative, so we get -1.

$$10+(-1)=9$$

Here, the negative number is -1, and the positive number is 10. 10 is larger than 1, so the absolute value of the positive number is larger, which means the answer will be positive. So we subtract 1 from 10 and get 9. The sign needs to be positive, so we get 9.

If the positive number and the negative number are opposites, the answer is 0.

$$3 + (-3) = 0$$

$$-10 + 10 = 0$$

Let's tackle **subtracting** numbers for the same combinations we considered for addition.

Positive – **Positive.** When we subtract one positive number from another, the result will be **positive if the first number is larger, but negative if the second number is larger.**

$$3 - 4 = 3 + (-4) = -1$$

Here, the first number is 3 and the second number is 4. Since $4 > 3$, the second number is larger so the result is negative.

$$10 - 1 = 10 + (-1) = 9$$

Here, the first number is 10 and the second number is 1. Since $10 > 1$, the first number is larger so the sign of the result is positive.

If the two positive numbers are equal, the result is 0.

$$3 - 3 = 3 + (-3) = 0$$

Negative − Negative. When we subtract one negative number from another, the result will be **positive if the first number is larger, but negative if the second number is larger.** In this context, the "larger" number refers to the number further right on a number line. For instance, the number -2 is to the right of -6 on the number line, so -2 is the larger number.

$$-3-(-4)=-3+4=1$$

Here, the first number is -3 and the second number is -4. Since $-3>-4$, the first number is larger so the result is positive.

$$-10-(-1)=-10+1=-9$$

Here, the first number is -10 and the second number is -1. Since $-1>-10$, the second number is larger so the sign of the result is negative.

If the two negative numbers are equal, the result is 0.

$$-3-(-3)=-3+3=0$$

Notice that the effect of subtracting a negative number is that the two negative signs cancel.

Positive − Negative = Positive. When we subtract a negative number from a positive number, the result will always be positive, because of the fact that the negative signs will cancel, leaving just the addition of two positive numbers.

$$3 - (-4) = 3 + 4 = 7$$

$$10 - (-1) = 10 + 1 = 11$$

Negative − Positive = Negative. When we subtract a positive number from a negative number, the result will always be negative.

$$-3 - 4 = -3 + (-4) = -7$$

$$-10 - 1 = -10 + (-1) = -11$$

Here's a summary of our findings:

Positive + Positive	Positive
Negative + Negative	Negative
Positive + Negative	Positive if the absolute value of the positive number is larger than the absolute value of the negative number
	Negative if the absolute value of the negative number is larger than the absolute value of the positive number
	0 if the numbers are equal
Negative + Positive	Positive if the absolute value of the positive number is larger than the absolute value of the negative number

	Negative if the absolute value of the negative number is larger than the absolute value of the positive number
	0 if the numbers are equal
Positive – Positive	Positive if the first number is larger
	Negative if the second number is larger
	0 if the numbers are equal
Negative – Negative	Positive if the first number is larger
	Negative if the second number is larger
	0 if the numbers are equal
Positive – Negative	Positive
Negative – Positive	Negative

Keep in mind that when we add signed numbers, the order of the numbers doesn't make a difference.

$$3 + 4 = 7 = 4 + 3$$

$$-10 + (-1) = -11 = -1 + (-10)$$

$$-3 + (-4) = -7 = -4 + (-3)$$

$$10 + (-1) = 9 = -1 + 10$$

$$3 + (-3) = 0 = -3 + 3$$

But if we subtract signed numbers, the order of the numbers always matters.

$3 - 4 = 3 + (-4) = -1$ but $4 - 3 = 4 + (-3) = 1$

$-10 - (-1) = -10 + 1 = -9$ but $-1 - (-10) = -1 + 10 = 9$

$-3 - (-4) = -3 + 4 = 1$ but $-4 - (-3) = -4 + 3 = -1$

$10 - (-1) = 10 + 1 = 11$ but $-1 - 10 = -1 + (-10) = -11$

Multiplying signed numbers

As a reminder, signed numbers are positive and negative numbers. When we multiply signed numbers, therefore, there are three possible combinations. We could be multiplying

1. Two positive numbers

2. Two negative numbers

3. One positive number and one negative number

For each of these combinations, let's talk about what happens when we multiply.

When we multiply two positive numbers, the result will always be positive.

$$3 \times 4 = 12$$

$$10 \times 1 = 10$$

When we multiply two negative numbers, the result will always be positive.

$$(-3) \times (-4) = 12$$

$$(-10) \times (-1) = 10$$

When we multiply a positive number by a negative number, the result will always be negative.

$$3 \times (-4) = -12$$

$$(-10) \times 1 = -10$$

Here's a summary of our findings:

Positive × Positive	Positive
Negative × Negative	Positive
Positive × Negative	Negative
Negative × Positive	Negative

In other words, if the signs are the same, the product will be positive. But if the signs are different, the product will be negative.

Even though 0 isn't a signed number, it's important to know that the result of multiplying any number by 0 is 0.

$$5 \times 0 = 0$$

$$0 \times -3 = 0$$

The result of a multiplication doesn't depend on the order of the numbers.

$$4 \times 5 = 20 = 5 \times 4$$

$$3 \times -2 = -6 = -2 \times 3$$

$$-5 \times 6 = -30 = 6 \times -5$$

$$-3 \times -7 = 21 = -7 \times -3$$

Up to now, we've usually used the times symbol $\times$ to indicate multiplication. But as we go forward in math, we'll start to use the dot symbol $\cdot$ more and more for multiplication. So 6×2 and $6 \cdot 2$ mean the same thing.

The dot is a little bit raised, so we won't confuse it with a decimal point. The reason we use this dot symbol is because it's easy to confuse $\times$ with the variable x.

When we're multiplying negative numbers, it's good to get into the habit of enclosing negative numbers in parentheses, especially when the negative number is the second number, but all of these are acceptable:

$$6 \times -2 = -12$$

$$(-5) \cdot 3 = -15$$

$$(-7) \cdot (-4) = 28$$

$$(-8) \times (6) = -48$$

It's also acceptable to indicate multiplication using only parentheses, instead of the $\times$ or $\cdot$ symbols. We can put just one number in parentheses or both numbers in parentheses. All of these express multiplication:

$$3(4) = 12$$

$$-2(5) = -10$$

$$(-7)(3) = -21$$

$$(-6)(-7) = 42$$

Dividing signed numbers

As a reminder, signed numbers are positive and negative numbers. When we divide signed numbers, therefore, there are three possible combinations. We could be dividing

1. Two positive numbers

2. Two negative numbers

3. One positive number and one negative number

For each of these combinations, let's talk about what happens when we divide.

When we divide two positive numbers, the result will always be positive.

$$12 \div 3 = 4$$

When we divide two negative numbers, the result will always be positive.

$$(-12) \div (-3) = 4$$

When we divide a positive number by a negative number, or a negative number by a positive number, the result will always be negative.

$$12 \div (-3) = -4$$

$$(-12) \div 3 = -4$$

Here’s a summary of our findings:

Positive ÷ Positive	Positive
Negative ÷ Negative	Positive
Positive ÷ Negative	Negative
Negative ÷ Positive	Negative

In other words, if the signs are the same, the result will be positive. But if the signs are different, the result will be negative.

Even though 0 isn’t a signed number, it’s very important to keep in mind that we can’t divide any number by 0. Also, if we divide 0 by any signed number, the result is 0.

$$0 \div 3 = 0$$

$$0 \div -4 = 0$$

Division is often presented “horizontally” with the division symbol, like $6 \div 2 = 3$. Division can also be presented “vertically”, like

$$\frac{6}{2} = 3$$

It’s also good to get in the habit of enclosing negative numbers in parentheses when we’re dividing. We can do this no matter how we write the division, but it’s especially important to do it when we’re writing the division horizontally.

$$18 \div (-3) = -6$$

$$(-10) \div 5 = -2$$

$$\frac{35}{(-7)} = -5$$

$$\frac{(-8)}{(-4)} = 2$$

In most cases, there's probably no good reason to enclose a positive number in parentheses, but it would be okay to do so.

Absolute value of an expression

We aren't limited to taking the absolute value of just a number. In fact, we can take the absolute value of any mathematical expression (e.g., an expression for addition, subtraction, multiplication, or division, or an expression that includes two or more of those operations), and we enclose the entire expression in a pair of absolute value bars.

While it's true that the absolute value bars make whatever's inside them positive (unless the value of the expression is 0, in which case it makes the absolute value 0), there's one important thing we need to say. We MUST evaluate the expression inside the absolute value bars before taking the absolute value. As an example, consider

$$|-3-4|$$

We might be tempted to think that the absolute value operation just takes away the negative sign that goes with the 3, and we would probably get 1 as the answer:

$$|-3-4|$$

$$|3-4|$$

$$|-1|$$

$$1$$

But this is wrong, because we would have performed the absolute value operation first, by taking away the first negative sign inside the absolute value bars. What we need to do instead is evaluate the expression inside the absolute value bars first, and THEN take the absolute value. This is how we should solve it:

$$|-3-4|$$

$$|-7|$$

$$7$$

Notice how we first dealt with the subtraction problem inside the absolute value bars, by evaluating $-3-4$ as -7. Only then, once we had evaluated that expression, did we take the absolute value.

Example

Simplify the expression.

$$2+|5-9|-|-3|$$

Whenever we have an expression like this one, we have to deal with everything inside the absolute value bars first, and then do the rest of the operations.

$$2+|5-9|-|-3|$$

$$2+|-4|-|-3|$$

Absolute value bars tell us that we need to find the distance from 0 of whatever's inside the absolute value bars. Since -4 is 4 units from 0, we get

$$2 + 4 - |-3|$$

Since -3 is 3 units from 0, we get

$$2 + 4 - 3$$

The $|-3|$ was simplified to 3, but we still have the subtraction from the previous expression that was in front of the absolute value.

$$6 - 3$$

$$3$$

Divisibility

When we talk about the "divisibility" of a whole number, we're just talking about the whole numbers that divide evenly into it. As an example, 5 divides evenly into 15 three times, since $15 \div 5 = 3$, which means we can say that 15 is "divisible" by 5.

Or put another way, 15 can be cut evenly into three equal pieces of size 5. When we can cut some X into equally sized pieces of size Y, using up the entire X without any remainder, then X is divisible by Y.

If we want to be technical, 15 is divisible by 5 because when we do the division $15 \div 5$, the result 3 is a whole number. That's the technical definition of divisibility: The result of the division must be a whole number.

Another way to say this is that we must get a remainder of 0. Since we often think of 0 as "nothing," we sometimes say that there's **no** remainder when the remainder is 0. So a third way to say that one whole number is divisible by another is that we don't get a remainder when we do the division.

Example

Is 56 divisible by 8?

If we do the division, we get $56 \div 8 = 7$. Since 7 is a whole number, we can say that 56 is divisible by 8.

The counterexample

Here's a counterexample. Is 9 divisible by 4? If we divide 9 by 4, we know 4 goes into 9 two times, and that gets us up to 8, but then we have a remainder of 1. In other words, because we have a remainder (other than 0), our answer isn't a whole number.

We get a whole number as the answer for $8 \div 4$ (the answer is the whole number 2), and we get a whole number as the answer for $12 \div 4$ (the answer is the whole number 3), but we don't get a whole number as the answer for $9 \div 4$. Therefore, 9 is not divisible by 4. But as we just saw, 8 and 12 are both divisible by 4.

Divisibility rules

The following list of divisibility rules are a shorthand way of determining if a particular integer is divisible by another, without actually performing the division.

Divisible by 2 if the last digit is $0, 2, 4, 6, 8$

Ex: 24 divisible by 2 because the last digit is 4.

Divisible by 3 if the sum of the digits is divisible by 3

Ex: 123 divisible by 3 because the sum of the digits is $1+2+3=6$. Since 6 is divisible by 3, 123 is also divisible by 3.

Divisible by 4 if the last two digits are divisible by 4

Ex: $3{,}448$ divisible by 4 because the last two digits form the number 48. Since 48 is divisible by 4, $3{,}448$ is also divisible by 4.

Divisible by 5 if the last digit is 0, 5

Ex: $2{,}360$ divisible by 5 because the last digit is 0.

Divisible by 6 if divisible by 2 and 3

Ex: 330 divisible by 6 because 330 is divisible by 2, because the last digit is 0. The sum of the digits is $3+3+0=6$. Since 6 is divisible by 3, 330 is also divisible by 3.

Divisible by 7 if $5 \times$ last digit + rest of the number is divisible by 7, or if subtracting twice the last digit from the rest of the number is divisible by 7

Ex: 256 is not divisible by 7 because multiplying the last digit 6 by 5 gives 30. When we take the 6 off of 256, we're left with 25. Adding 30 to 25 gives 55, which is not divisible by 7.

Ex: 256 is not divisible by 7 because the last digit is 6. Multiplying 6 by 2 gives $6 \times 2 = 12$, and when we take the 6 off of 256, we're left with 25. Subtracting 12 from 25 gives 13, which is not divisible by 7.

Divisible by 8 if the last three digits are divisible by 8

Ex: $34{,}256$ is divisible by 8 because the last three digits are 256, and 256 is divisible by 8.

Divisible by 9 if the sum of the digits is divisible by 9

Ex: $3{,}254$ is not divisible by 9 because the sum of the digits is $3+2+5+4=14$. Since 14 is not divisible by 9, $3{,}254$ is not divisible by 9.

Divisible by 10 if the last digit is 0

Ex: 125 is not divisible by 10 because the last digit is 5, not 0.

Multiples

It's helpful to think about multiples and divisibility as two parts of the same idea. We know that 10 is "divisible" by 5 because when we do the division $10 \div 5$, the result 2 is a whole number. It's the fact that the result is a whole number that proves that 10 is divisible by 5.

To understand the concept of "multiples," we need to turn around our thinking about divisibility. So instead of thinking of $10 \div 5 = 2$, let's think about $2 \times 5 = 10$. This multiplication problem tells us that 10 is a "multiple" of both 2 and 5.

A multiple of a number is what we get when we multiply that number by a whole number. Here's how we get some of the multiples of 2:

$$2 \times 1 = 2$$

$$2 \times 2 = 4$$

$$2 \times 3 = 6$$

$$2 \times 4 = 8$$

$$2 \times 5 = 10$$

So 2, 4, 6, 8, and 10 are multiples of 2. And here's how we get some of the multiples of 5:

$$5 \times 1 = 5$$

$$5 \times 2 = 10$$

$$5 \times 3 = 15$$

$$5 \times 4 = 20$$

$$5 \times 5 = 25$$

So 5, 10, 15, 20, and 25 are multiples of 5.

The multiples of 2 that we've listed here aren't the **only** multiples of 2, and the multiples of 5 that we've listed here aren't the **only** multiples of 5. They're just a few of the multiples of those numbers, to give us an idea of what a "multiple" is.

To relate multiples to divisibility, we now know two reasons why 20 is a multiple of 5:

1. 20 is a multiple of 5 because 20 is divisible by 5, since $20 \div 5$ gives a whole number answer of 4.

2. 20 is a multiple of 5 because 5 multiplied by a certain whole number, 4, gives $5 \times 4 = 20$.

Let's do an example.

Example

Is 80 a multiple of 10? If so, give two reasons why.

Yes, 80 is a multiple of 10, for two reasons:

1. 80 is a multiple of 10 because 80 is divisible by 10, since $80 \div 10$ is 8, a whole number.

2. 80 is a multiple of 10 because 10 multiplied by a certain whole number, 8, gives $10 \times 8 = 80$.

We could think of 0 as a multiple of **any** number, because if we multiply any number by 0, we get 0. For example, we could think of the multiples of 2 as 0, 2, 4, 6, ..., which are also known as even numbers. However, there is no number that's divisible by 0, because division by 0 is never allowed!

Prime and composite

We've already learned several ways to classify integers, including as positive, negative, or 0. But now we'll look at another way to classify integers that are greater than 1, in other words, the counting numbers 2, 3, 4, 5, etc.

Any whole number greater than 1 can be classified as either a "prime" number or a "composite" number. A number can be either prime or composite, but it can't be both.

A prime number is a whole number greater than 1 that's divisible by only 1 and itself. We already know that every whole number is divisible by 1, because if we divide any whole number by 1, we'll get the original number as the result. Here are some examples:

$$10 \div 1 = 10$$

$$7 \div 1 = 7$$

$$316 \div 1 = 316$$

And every whole number other than 0 is divisible by itself, because if we divide any whole number other than 0 by itself, we'll get 1 (a whole number) as the result. Here are some examples:

$$10 \div 10 = 1$$

$$7 \div 7 = 1$$

$$316 \div 316 = 1$$

Prime numbers

Prime numbers are divisible by only 1 and themselves. As an example, 7 is a prime number because it's divisible by 1 ($7 \div 1 = 7$) and by itself ($7 \div 7 = 1$), but not by anything else. It's not divisible by 2, 3, 4, 5, or 6, because none of these numbers divides evenly into 7 (we don't get a whole number result when we divide 7 by any number other than 1 or 7):

$$7 \div 2 = 3.5$$

$$7 \div 3 \approx 2.33$$

$$7 \div 4 = 1.75$$

$$7 \div 5 = 1.4$$

$$7 \div 6 \approx 1.17$$

We don't have to check numbers greater than 7 (8, 9, 10, etc.) because dividing 7 by any number greater than 7 will give a decimal number between 0 and 1.

The number 11 is also a prime number, because the only numbers that divide evenly into 11 are 1 and 11.

Because 7 and 11 are divisible only by 1 and themselves, we call them "prime" numbers.

Composite numbers

Contrast this with **composite numbers**, which are numbers greater than 1 that are divisible by something other than just 1 and themselves. For example, 6 is a composite number; it's divisible by 1 ($6 \div 1 = 6$) and by itself ($6 \div 6 = 1$), but it's also divisible by 2 ($6 \div 2 = 3$) and by 3 ($6 \div 3 = 2$). Because 6 is divisible by more than just 1 and itself, we call it a "composite" number.

So to determine whether a number is prime or composite, we only need to determine whether that number is evenly divisible by a number other than 1 and itself. If we find any whole number that divides evenly into the number in question (other than 1 or the number itself), then we determine that the number is composite. Otherwise, the number is prime.

Let's look at an example.

Example

Say whether 21 is prime or composite.

The number 21 is divisible by 1 ($21 \div 1 = 21$) and by itself ($21 \div 21 = 1$), but it's also divisible by 3 ($21 \div 3 = 7$) and by 7 ($21 \div 7 = 3$). Because 21 is divisible by more than just 1 and itself, it's a composite number, not a prime number.

In the next lesson, we'll look at a systematic way to determine whether a number is prime or composite, which will take the guesswork out of trying to find a number that divides evenly into the number we're investigating.

Prime factorization and product of primes

Remember that we learned previously that a prime number is a whole number greater than 1 which is divisible by only 1 and itself. In contrast, a composite number is a whole number greater than 1 which is divisible by 1 and itself, but also by at least one other number.

This lesson is all about prime factorization and product of primes, so we'll define what these mean. Before we do that, however, we need to talk about factors.

Factors and factorization

Factors are just things that get multiplied by one another. In math, when we multiply numbers or expressions together, we call each piece a "factor." On the other hand, when we add numbers or expressions together, we call each piece a "term."

For now, when we talk about factors, we'll think about positive whole numbers. So for now, when we talk about "factorization" of a whole number, we're talking about coming up with the whole numbers that multiply together to give us that original number.

For example, the factors of 12 are 1, 2, 3, 4, 6, and 12, because they're the positive whole numbers that divide evenly into 12.

Product of primes

A “product of primes” is a product in which every factor is a prime number. Of all the factorizations of 12 we could come up with, the only one that's a product of primes is

$$2\times2\times3$$

A “prime factorization" of a composite number is an expression of that number as a product of primes.

The factors in a prime factorization can appear in any order, but we usually list them from smallest to largest, and we group factors together that are the same. For example, we already know that the prime factorization of 12 is $2\times2\times3$, but we’d actually write this more compactly as

$$2^2\times3$$

where the little 2 indicates that there are two factors of 2. Taking another example, if in the prime factorization of some other number, the factor 3 occurs twice, the factor 5 occurs four times, and the factor 13 appears once, we could write its prime factorization as

$$3\times3\times5\times5\times5\times5\times13$$

or more compactly as

$$3^2\times5^4\times13$$

These little numbers are called “exponents” and we’ll learn more about them later on when we get to the exponents section of the course.

Example

Find the prime factorization of 45.

The goal in finding the prime factorization of a composite number is to keep breaking that number down into smaller and smaller factors until all the factors are prime numbers. There are multiple ways that we could do this.

We know that 45 is the product of 5 and 9, so we could say

$$45 = 5 \times 9$$

5 is a prime number, so we can't break that down any further. But 9 can be expressed as the product 3×3, so we could break down the 9 into 3×3 and write 45 as

$$45 = 5 \times 3 \times 3$$

Now we're done because 5, 3, and 3 are all prime numbers, so they can't be broken down any further. We can also write the prime factorization in exponential form as $3^2 \times 5$.

Least common multiple

A common multiple of two positive whole numbers is a number that's divisible by both of them. Their least common multiple (sometimes abbreviated LCM) is the smallest number that's divisible by both of them. For example,

4 is the least common multiple of 2 and 4

6 is the least common multiple of 2 and 3

10 is the least common multiple of 2 and 5

When we're dealing with small numbers, the easiest way to find the least common multiple is to start with the larger number, and test each of its positive multiples, one at a time starting with the smallest one, until we find one that's divisible by the other number.

Remember, the positive multiples of a positive whole number are the values we get when we multiply it by 1, 2, 3, 4, etc. For example, here's how we get the first twelve positive multiples of 3:

$3 \cdot 1 = 3$ $\quad$ $3 \cdot 4 = 12$ $\quad$ $3 \cdot 7 = 21$ $\quad$ $3 \cdot 10 = 30$

$3 \cdot 2 = 6$ $\quad$ $3 \cdot 5 = 15$ $\quad$ $3 \cdot 8 = 24$ $\quad$ $3 \cdot 11 = 33$

$3 \cdot 3 = 9$ $\quad$ $3 \cdot 6 = 18$ $\quad$ $3 \cdot 9 = 27$ $\quad$ $3 \cdot 12 = 36$

Let's try an example with small numbers.

Example

Find the least common multiple of 3 and 4.

Since 4 is the larger number, we'll use its positive multiples ($4 \cdot 1$, $4 \cdot 2$, etc.) to find the smallest one that 3 divides into evenly.

$4 \cdot 1 = 4$ 3 doesn't go evenly into 4, so we have to keep going.

$4 \cdot 2 = 8$ 3 doesn't go evenly into 8, so we have to keep going.

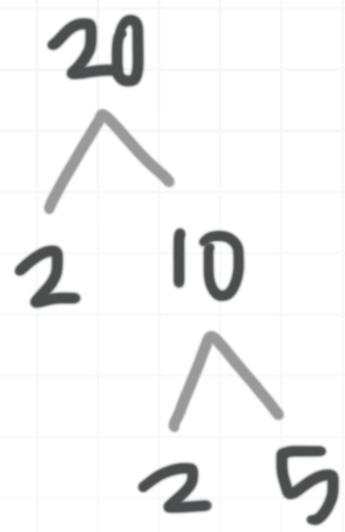
$4 \cdot 3 = 12$ 3 goes evenly into 12, so 12 is the least common multiple of 3 and 4.

Sometimes we need to find the least common multiple of larger numbers, where using this method might be more difficult. In this case, we can use a different method for finding the least common multiple: finding the prime factorization of each number, and then using the prime factors to build the least common multiple.

Example

Find the least common multiple of 20 and 75.

We need to reduce each of these numbers to its prime factors.



The prime factor 2 appears twice, and the prime factor 5 appears once, so the prime factorization of 20 is:

$$2^2 \cdot 5$$

The prime factor 3 appears once, and the prime factor 5 appears twice, so the prime factorization of 75 is:

$$3 \cdot 5^2$$

We found prime factors of 2, 3, and 5. The number of times each of these prime factors appears in the least common multiple of 20 and 75 will be the larger of the number of times it appears in the prime factorization of 20 and the number of times it appears in the prime factorization of 75.

There are two factors of 2 in 20, and none in 75, so there will be two factors of 2 in their least common multiple.

There are no factors of 3 in 20, and one in 75, so there will be one factor of 3 in their least common multiple.

There is one factor of 5 in 20, and there are two in 75, so there will be two factors of 5 in their least common multiple.

Therefore, the least common multiple of 20 and 75 is $2^2 \cdot 3 \cdot 5^2$. Now we'll multiply this out to express the least common multiple as a single number:

$2^2 \cdot 3 \cdot 5^2$

$4 \cdot 3 \cdot 25$

$12 \cdot 25$

300

This tells us that 300 is the least common multiple of 20 and 75.

Remember, we can always double-check our answer by dividing the least common multiple by the original numbers. Using the previous example, we said that 300 was the least common multiple of 20 and 75. If that's true, then 20 and 75 must both go evenly into 300.

$300 \div 20 = 15$

$300 \div 75 = 4$

Because we got whole numbers for answers, we know that 300 is a common multiple of 20 and 75. In fact, 300 is the **least** common multiple of 20 and 75: When we divide 300 by 20 and 75, we get 15 and 4, respectively, and the only number by which both 15 and 4 are divisible is 1.

Greatest common factor

A common factor of two positive whole numbers is a number that divides evenly into both of them. Their greatest common factor (sometimes abbreviated GCF) is the largest number that divides evenly into both of them. Another name that's used for "greatest common factor" is "greatest common divisor" (sometimes abbreviated GCD).

One way to find the greatest common factor of two positive whole numbers is to find the prime factorizations of both numbers and then to look for the factors that appear in both factorizations.

Example

Find the greatest common factor of 14 and 28.

In order to find the greatest common factor, we need to look for the largest number that divides evenly into both 14 and 28.

To do this, we'll break down number into its prime factors.

14	28
$2 \cdot 7$	$2 \cdot 14$
	$2 \cdot 2 \cdot 7$

The factor of 2 appears once in 14 and twice in 28, so we'll have one factor of 2 in the greatest common factor. The factor of 7 appears once in 14 and once in 28, so we'll have one factor of 7 in the greatest common factor. Therefore, the greatest common factor of 14 and 28 is $2 \cdot 7$. Multiplying this out, we get $2 \cdot 7 = 14$, which means 14 is the greatest common factor of 14 and 28.

Let's double-check our answer by making sure that 14 divides evenly into both 14 and 28.

$$14 \div 14 = 1$$

$$28 \div 14 = 2$$

It does, so 14 is a common factor of 14 and 28. In fact, 14 is the **greatest** common factor of 14 and 28: When we divide 14 and 28 by 14, we get 1 and 2, respectively, and the only number that divides evenly into both 1 and 2 is 1.

In that example, the greatest common factor 14 was equal to one of the original numbers, but that won't always be the case.

Fractions

A fraction may look like a completely different kind of number, but the way we need to think about it is just as "part of a whole". A fraction always has two numbers: the top number, which is called the "numerator," and the bottom number, which is called the "denominator." A fraction looks like this:

$$\frac{2}{3}$$

where 2 is the numerator and 3 is the denominator. A fraction also represents the division of the numerator by the denominator. So we can say that $2/3$ is "2 divided by 3" or "2 over 3."

Let's use an example with pizza. Let's say I have a whole pizza that I want to split with my friend. We're going to split the pizza evenly, and I want to use a fraction to express how much of the pizza I get to eat.

Let's just say up front that the denominator of the fraction will be the number of pieces I cut the pizza into, and the numerator of the fraction will be the number of pieces I personally get to eat.

So if I want to split the pizza equally with my friend, I'll cut the whole pizza into two pieces. Because I'm cutting it into two pieces, I put a 2 in the denominator of the fraction:

$$\frac{}{2 \text{ pieces total}}$$

After I cut it into two pieces, I give her one of the pieces, and I personally get to keep the other piece. Since I get to keep one piece, I put a 1 in the numerator of the fraction:

$$\frac{1 \text{ piece for me}}{2 \text{ pieces total}}$$

So I get to eat 1 of the 2 pieces, and "1 of 2" or "1 out of 2" is the fraction

$$\frac{1}{2}$$

If we write a fraction on its own, we write it like we just did, with the numerator above the denominator. But if we write a fraction within a line of text, we write it with a slash as $1/2$.

Example

If a store is 4 miles from my home and I've already walked 3 miles, express my progress as a fraction.

Since I have to walk a total of 4 miles, I put a 4 in the denominator of the fraction.

$$\frac{\phantom{4 \text{ miles total}}}{4 \text{ miles total}}$$

Since I've already walked 3 of those miles, I put a 3 in the numerator of the fraction.

$$\frac{3 \text{ miles I've walked}}{4 \text{ miles total}}$$

So the portion of the walk that I've completed is $3/4$.

Here's a table that summarizes how to describe some simple fractions.

1/2: one-half	1/3: one-third	1/4: one-fourth
2/2: two-halves	2/3: two-thirds	2/4: two-fourths
3/2: three-halves	3/3: three-thirds	3/4: three-fourths
4/2: four-halves	4/3: four-thirds	4/4: four-fourths

Percent

When we hear the word "percent," we should think "divided by 100" in order to turn the percent value into a fraction. So 50%, expressed as a fraction is $50/100$, and 76% is the same as $76/100$.

Simplifying fractions and equivalent fractions

We already understand that a fraction just represents “part of a whole.” If a baseball player gets three “at bats” in a game (gets an opportunity as a hitter three times), and out of those three chances they get a hit two times, then their success rate for that game is

$$\frac{2\text{ hits}}{3\text{ chances}}=\frac{2}{3}$$

Or if I borrow five books from the library but read only two of them, then I’ve read two out of five books, or

$$\frac{2\text{ books read}}{5\text{ books total}}=\frac{2}{5}$$

What we want to be able to do now is learn to simplify fractions. For example, we know that 50 is half of 100, so if we see the fraction

$$\frac{50}{100}$$

then we want to be able to rewrite that as

$$\frac{1}{2}$$

Fractions as relationships

Because remember, a fraction is really just a relationship between the numerator and the denominator. If I’m driving 100 miles to visit my family, and I've already driven 50 miles, then I know that I'm halfway there. It would be simpler for me to express my progress as $1/2$ than as $50/100$, so we need to know how to change $50/100$ into $1/2$.

The reason we want to reduce fractions to lowest terms is that even though a fraction like

$$\frac{630}{945}$$

is actually the same as

$$\frac{2}{3},$$

that isn't obvious to us when we look at it, because the numbers 630 and 945 are so large. But if we simplify that fraction to $2/3$, we'll be able to easily tell that we have "2 out of 3 parts."

Example

Simplify the fraction to lowest terms.

$$\frac{630}{945}$$

There are a couple ways to tackle this, but here's a reliable way to go about simplifying a fraction to lowest terms. We can first find the prime factorizations of the numerator and the denominator.

630 $\qquad\qquad\qquad\qquad\qquad\qquad$ 945

$3 \cdot 210$ $\quad$ $3 \cdot 315$

$3 \cdot 3 \cdot 70$ $\quad$ $3 \cdot 3 \cdot 105$

$3 \cdot 3 \cdot 5 \cdot 14$ $\quad$ $3 \cdot 3 \cdot 5 \cdot 21$

$3 \cdot 3 \cdot 5 \cdot 7 \cdot 2$ $\quad$ $3 \cdot 3 \cdot 5 \cdot 7 \cdot 3$

What we get is

$$\frac{630}{945} = \frac{3 \cdot 3 \cdot 5 \cdot 7 \cdot 2}{3 \cdot 3 \cdot 5 \cdot 7 \cdot 3}$$

Now we'll group together the factors that are common to the numerator and denominator, so our fraction can be expressed as

$$\frac{630}{945} = \frac{(3 \cdot 3 \cdot 5 \cdot 7) \cdot 2}{(3 \cdot 3 \cdot 5 \cdot 7) \cdot 3} = \frac{2}{3}$$

What we want to remember at this point is that, whenever a factor is common to the numerator and denominator of a fraction, that factor in the numerator "cancels against" the matching factor in the denominator. It's as if that factor just disappears from both places. So the $(3 \cdot 3 \cdot 5 \cdot 7)$ cancels in the numerator and denominator, leaving us with just $2/3$.

When the numerator and denominator are equal

However, if the numerator is equal to the denominator, the fraction doesn't disappear (we have to have something there), so the fraction simplifies to 1. For example,

$$\frac{3}{3} = 1$$

$$\frac{10}{10} = 1$$

$$\frac{67}{67} = 1$$

Equivalent fractions

We already know how to simplify a fraction to lowest terms. We just pull out the common factors from the numerator and denominator, cancel those out, and what's left is the fraction simplified to lowest terms, resulting in a fraction that's equivalent to the original. For example, given the fraction

$$\frac{30}{45}$$

we first find the prime factorizations of the numerator and denominator,

$$\frac{3 \cdot 5 \cdot 2}{3 \cdot 5 \cdot 3}$$

then we cancel the 3 in the numerator with one of the 3's in the denominator, and the 5 in the numerator with the 5 in the denominator. The result is $2/3$.

$$\frac{\cancel{3}\cdot\cancel{5}\cdot 2}{\cancel{3}\cdot\cancel{5}\cdot 3}=\frac{2}{3}$$

So we can say that the two fractions $30/45$ and $2/3$ are **equivalent fractions**, which just means that they're equal to each other. They represent the same proportion of the whole.

$$\frac{30}{45}=\frac{2}{3}$$

Now we want to turn this process around, and learn how to express something like $2/3$ in terms of 12ths in the denominator, instead of 3rds in the denominator.

Let's do an example.

Example

Express $3/5$ as an equivalent fraction, but with 10 in the denominator instead of 5.

If we start with the fraction $3/5$, we can say that we have "3 out of 5 parts." Since we're being asked to express this as an equivalent fraction with 10 in the denominator, we're being asked the question "If we have 3 out of every 5 pieces, how many pieces would we have if there were 10 total pieces?"

Mathematically, we can represent this as

$$\frac{3}{5} = \frac{?}{10}$$

Now the question becomes, how did we get from 5 to 10 in the denominator? Well, we had to multiply 5 by 2 in order to get to 10. So in order to figure out what numerator we'll get, we need to multiply the numerator 3 by 2 as well.

$$\frac{3}{5} = \frac{3 \cdot 2}{5 \cdot 2} = \frac{6}{10}$$

This means that $3/5$ and $6/10$ are equivalent fractions.

We can double-check our answer by simplifying $6/10$ to show that we get back to $3/5$.

$$\frac{6}{10} = \frac{3 \cdot 2}{5 \cdot 2} = \frac{3 \cdot \cancel{2}}{5 \cdot \cancel{2}} = \frac{3}{5}$$

Division of zero

A fraction really represents the division of its numerator by its denominator. For example, the fraction

$$\frac{3}{8}$$

means "3 divided by 8," or "3 parts out of 8 equal parts."

One of the important things to remember about fractions is that we can never divide by 0. Since we're always dividing by whatever is in the denominator, this just means that we can't have 0 in the denominator of a fraction. It's just one of those weird things in math that we can't do. And so if we ever have 0 in the denominator, all we can say is that the fraction is "undefined," which means we never want to put 0 in the denominator of a fraction.

Consider the idea of "parts of a whole." The fraction $4/0$ could be read as, "4 parts out of 0 equal parts." But that doesn't make sense. How can we take 4 parts if the whole is 0, or empty? We can't, it's impossible; and this is one explanation of why $4/0$ is undefined.

Keep in mind that this is totally different than having 0 in the numerator of a fraction. If we have 0 in the numerator, the value of the fraction is 0. So something like

$$\frac{0}{6}$$

is equal to 0.

Example

Find the value of $3/0$.

Because we can't have 0 in the denominator of a fraction, all we can say is that this fraction is "undefined." The value isn't 0, it isn't 3, it's just undefined.

Adding and subtracting fractions

Before we can add two fractions or subtract one fraction from another, we need to see whether the fractions have the same denominator or different denominators.

Same denominator, or like fractions

When we want to add or subtract fractions that have the same denominator, we just add or subtract the numerators, and keep the same denominator.

$$\frac{5}{7}+\frac{3}{7}=\frac{5+3}{7}=\frac{8}{7}$$

$$\frac{5}{7}-\frac{3}{7}=\frac{5-3}{7}=\frac{2}{7}$$

Different denominators, or unlike fractions

When the denominators are unequal, we have to find a common denominator before we can add or subtract the fractions.

To find a common denominator, we need to find the least common multiple (LCM) of the denominators. Then we can use the LCM as our common denominator. In fact, we call that the "least common denominator," because it's the smallest common denominator we can use.

Consider the fractions $3/5$ and $1/3$. The LCM of 5 and 3 is 15, so the least common denominator will be 15.

Since $15 = 3 \cdot 5$, we have to multiply 5 (the denominator of $3/5$) by 3 to get the common denominator of 15. And we have to multiply 3 (the denominator of $1/3$) by 5 to get the common denominator 15.

We aren't allowed to change the value of either of the fractions, so we have to multiply the numerator of each fraction by the same number that we multiplied by its denominator. So we have to multiply the numerator and denominator of $3/5$ by 3, and we have to multiply the numerator and denominator of $1/3$ by 5.

Problem	With a common denominator	Result
$\frac{3}{5} + \frac{1}{3}$	$\frac{3 \cdot 3}{5 \cdot 3} + \frac{1 \cdot 5}{3 \cdot 5} = \frac{9}{15} + \frac{5}{15}$	$\frac{14}{15}$
$\frac{3}{5} - \frac{1}{3}$	$\frac{3 \cdot 3}{5 \cdot 3} - \frac{1 \cdot 5}{3 \cdot 5} = \frac{9}{15} - \frac{5}{15}$	$\frac{4}{15}$

Multiplying and dividing fractions

When we multiply fractions, we multiply their numerators to find the numerator of the result, and we multiply their denominators to find the denominator of the result.

$$\frac{3}{4} \times \frac{1}{7} \qquad \frac{3 \times 1}{4 \times 7} \qquad \frac{3}{28}$$

When we divide fractions, we actually turn the division problem into a multiplication problem by turning the divisor (the second fraction) upside down (switching its numerator with its denominator) and changing the division symbol to a multiplication symbol at the same time. We call this process “multiplying by the reciprocal.” The **reciprocal** of a fraction a/b is the fraction b/a (where the numerator and denominator are flipped.

$$\frac{3}{4} \div \frac{1}{7} \qquad \frac{3}{4} \times \frac{7}{1} \qquad \frac{3 \times 7}{4 \times 1} \qquad \frac{21}{4}$$

It’s okay that in this last fraction, the numerator is larger than the denominator. When that’s the case, the fraction is called an “improper” fraction.

Example

Multiply the fractions.

$$\frac{2}{3} \times \frac{4}{11}$$

To multiply the fractions, we multiply the numerators and the denominators separately.

$$\frac{2 \times 4}{3 \times 11}$$

$$\frac{8}{33}$$

Let's do an example with division.

Example

Divide the fractions.

$$\frac{2}{3} \div \frac{4}{11}$$

To do division with fractions, we turn the second fraction upside down and change the division symbol to a multiplication symbol at the same time.

$$\frac{2}{3} \times \frac{11}{4}$$

Then we treat this as a multiplication problem, by multiplying the numerators and the denominators separately.

$$\frac{2 \times 11}{3 \times 4}$$

$$\frac{22}{12}$$

We always like to give our answer in lowest terms, so we'll simplify this fraction by canceling a 2 from the numerator and denominator.

$$\frac{22}{12}=\frac{2\cdot 11}{2\cdot 6}$$

$$\frac{\cancel{2}\cdot 11}{\cancel{2}\cdot 6}$$

$$\frac{11}{6}$$

Fractions and whole numbers

To multiply a fraction and a whole number, we have to convert the whole number number into fraction by rewriting it with a denominator of 1. For example, 5 can be written as $5/1$, and any number x can be rewritten as $x/1$.

Let's do an example where we rewrite the whole number as a fraction before multiplying the two numbers.

Example

Multiply the two numbers.

$$\frac{3}{4}\times 7$$

We have to start by rewriting 7 as $7/1$. Then we can multiply the fractions.

$$\frac{3}{4} \times \frac{7}{1}$$

$$\frac{3 \times 7}{4 \times 1}$$

$$\frac{21}{4}$$

Signs of fractions

Until now, every fraction we've dealt with has had a denominator that's positive, and a numerator that's either positive or 0. Now we're going to deal with fractions involving negative whole numbers as well, so we have to take positive and negative signs into account.

There are three signs associated with every fraction. But this can be hard to remember, because not all of the signs are always visible. Before we talk about signs of fractions, let's talk about signs of integers.

We know that -3 is a negative integer, because the negative sign is present. And we know that 4 is a positive integer, even though we haven't written a positive sign in front of the 4. The fact that we haven't written $+4$ doesn't mean that 4 isn't a positive integer.

Translating this to fractions, if we look at the fraction

$$\frac{3}{4}$$

we know that, even though there aren't any positive signs, this fraction has a positive 3 in the numerator and a positive 4 in the denominator. This gives us our first hint about the three signs that are associated with every fraction. Two of these signs are the sign of the numerator and the sign of the denominator. In the fraction above, the sign of the numerator is positive, and the sign of the denominator is positive, since we have both positive 3 and positive 4.

If we have the fraction

$$\frac{-3}{4}$$

then the sign of the numerator is negative since -3 is a negative integer, and the sign of the denominator is positive since 4 is a positive integer.

So we've talked about the sign of the numerator and the sign of the denominator, but what is the third sign associated with a fraction? That's the fraction's own sign. The numerator always has a sign, the denominator always has a sign, and the fraction always has a sign of its own as well. For example, we'll often see a fraction like

$$-\frac{3}{4}$$

In a fraction like this one, the sign of the numerator is positive since 3 is a positive integer, the sign of the denominator is positive since 4 is a positive integer, and the fraction's own sign is negative since we have a negative sign in front of the fraction.

Remember how we talked before about the fact that, even though there are no positive signs in the fraction

$$\frac{3}{4}$$

we still know that the sign of the numerator and the sign of the denominator are both positive? Well in the same way, even though there's no positive sign in front of this fraction, we still know that the fraction's own sign is positive. In other words, if there's a negative sign in front of a fraction, then the fraction's own sign is negative. If there's no negative sign in front of a fraction (if there's a positive sign in front of the fraction or if there's no sign in front of it), then the fraction's own sign is positive.

Now that we have this foundation, the most important thing to know about signs of fractions is that we can always change exactly two signs of the fraction, and still keep the value of the fraction the same. This works because of the way that every two negative signs cancel each other out.

Example

Write the fraction in at least two other ways.

$$-\frac{4}{7}$$

Before we change anything, we can say that the sign of the numerator is positive since 4 is a positive integer, that the sign of the denominator is positive since 7 is a positive integer, and that the fraction's own sign is negative, since we have a negative sign in front of the fraction.

In order to keep the value of the fraction the same, we have to change two signs at the same time.

Let's change the fraction's own sign and the sign of the numerator.

$$-\frac{4}{7} \quad \text{becomes} \quad \frac{-4}{7}$$

Let's change the fraction's own sign and the sign of the denominator.

$$-\frac{4}{7} \quad \text{becomes} \quad \frac{4}{-7}$$

Or we could change just the sign of the numerator and the sign of the denominator, and keep the fraction's own sign the same.

$$-\frac{4}{7} \quad \text{becomes} \quad -\frac{-4}{-7}$$

So we rewrote the fraction three different ways, which means we can say that the values of all four fractions (the original fraction and the three fractions we found by changing the signs) are equal:

$$-\frac{4}{7} = \frac{-4}{7} = \frac{4}{-7} = -\frac{-4}{-7}$$

Therefore, these four fractions are all equivalent.

The way we add, subtract, multiply, and divide fractions with different signs is similar to the way add, subtract, multiply, and divide integers with different signs.

Reciprocals

Think about it this way: The reciprocal of a fraction is what we get when we turn the fraction upside down. We first saw the reciprocal when we learned about dividing by fractions, because that fraction division process required us to multiply by the reciprocal. In other words, what we get when we switch its numerator with its denominator. So the reciprocal of

$$\frac{3}{4}$$

is

$$\frac{4}{3}$$

Now there are a couple of situations with reciprocals that we should clarify. The first one is that integers have a reciprocal. We have to remember that a number like 2 can be written in fraction form as

$$\frac{2}{1}$$

Because when we divide by 1, it doesn't change the value at all. So $2/1$ is the same as just 2. But then we can take the reciprocal of 2, or $2/1$, and we see that it's

$$\frac{1}{2}$$

The second thing to know is that 0 doesn't have a reciprocal. Because 0 is the same as $0/1$, and the reciprocal of $0/1$ is $1/0$. But in $1/0$, we have a 0 in the denominator, which is undefined. So we can't take the reciprocal of 0.

The third thing we want to say about reciprocals is that whenever we multiply a fraction by its reciprocal, we'll always get 1 as the result. In other words, since $3/4$ and $4/3$ are reciprocals of one another,

$$\frac{3}{4} \times \frac{4}{3} = \frac{3 \times 4}{4 \times 3} = \frac{12}{12} = 1$$

Or, since $2/1$ and $1/2$ are reciprocals of one another,

$$\frac{2}{1} \times \frac{1}{2} = \frac{2 \times 1}{1 \times 2} = \frac{2}{2} = 1$$

Last, we need to know that if we're taking the reciprocal of a fraction that has at least one negative sign associated with it, then we need to include the negative sign(s) in the reciprocal as well. That way, when we multiply the original fraction by its reciprocal, all the negative signs will cancel and the result will be positive 1.

Example

Find the reciprocal, and then double-check the result by making sure that the product of the fraction and the reciprocal is 1.

$$-\frac{2}{9}$$

The reciprocal of $2/9$ is $9/2$, because that's what we get when we flip the fraction upside down. Because the original fraction's own sign is negative, we include that negative sign in the reciprocal as well. So the reciprocal of $-(2/9)$ is $-(9/2)$.

Let's check to make sure we get positive 1 when we multiply the original fraction by its reciprocal.

$$-\frac{2}{9} \times \left(-\frac{9}{2}\right)$$

The negative signs cancel and go away. Then, as always, we multiply the numerators and the denominators separately.

$$\frac{2 \times 9}{9 \times 2}$$

$$\frac{18}{18}$$

$$1$$

Since we get positive 1, we know that we found the correct reciprocal.

Mixed numbers and improper fractions

In this lesson, we're going to focus on particular kinds of fractions. We'll start out by talking about positive fractions, and then we'll deal with negative fractions at the end of the lesson.

Up until now, most of the fractions we've dealt with are what we call "proper" fractions, where the numerator is less than the denominator. Here are some examples of proper fractions:

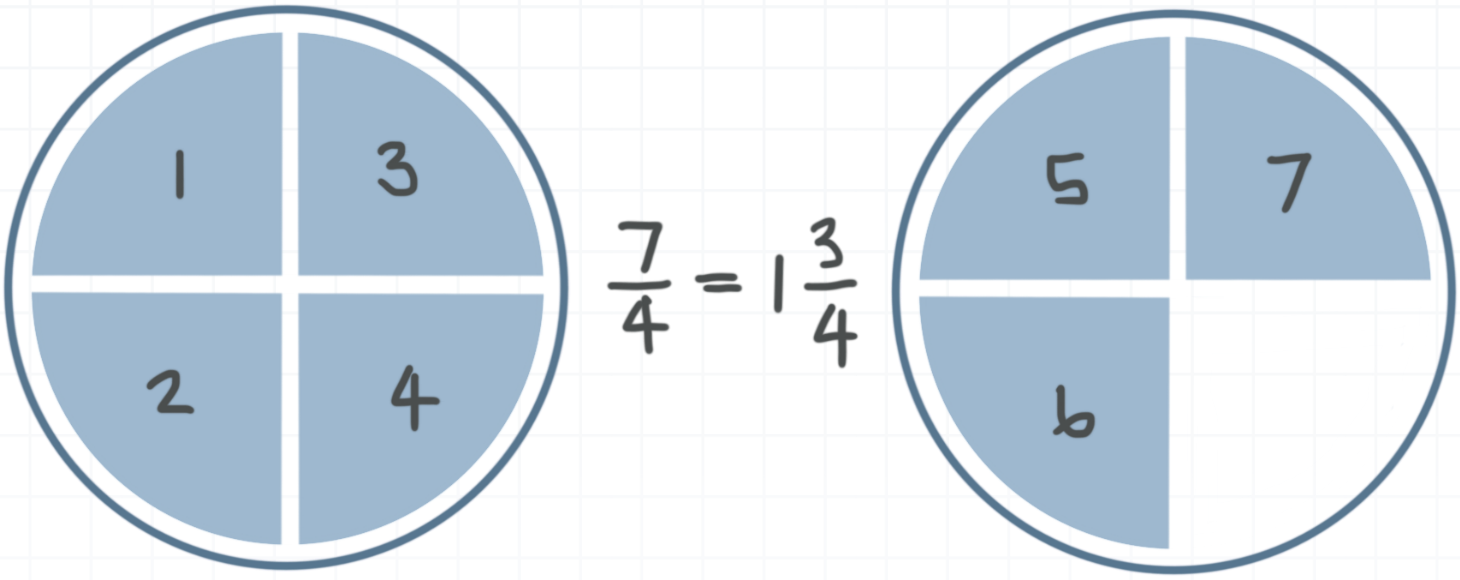
$$\frac{4}{7} \qquad \frac{2}{3} \qquad \frac{2}{27} \qquad \frac{3}{8}$$

Now we're introducing a new kind of fraction, called an "improper" fraction, where the numerator is greater than or equal to the denominator. In other words, improper fractions are "top-heavy." Here are some examples of improper fractions:

$$\frac{7}{4} \qquad \frac{3}{2} \qquad \frac{27}{2} \qquad \frac{8}{3} \qquad \frac{10}{10}$$

Even though "improper" sounds like a bad thing, there's nothing bad about improper fractions. In the same way that a proper fraction represents a number between -1 and 1 (but not equal to -1 or 1), an improper fraction represents a number that's greater than or equal to 1, or less than or equal to -1.

For example, $7/4$ ("seven-fourths") means seven times $1/4$, or $1/4$ seven times. Each of the blue sections below represents $1/4$ of a circle, and we have seven of them.



As we can see, this is the same thing as saying that we have one circle plus three-fourths of a circle, because we have one full circle on the left and $3/4$ of a circle on the right.

Now, we just said that

$$\frac{7}{4}=1\frac{3}{4}$$

We know $7/4$ is an improper fraction. And

$$1\frac{3}{4}$$

("one and three-fourths") is called a mixed number, because it's a mix of the whole number 1 and the fraction $3/4$. 1 is the number of complete circles we have, and $3/4$ is the portion of a circle that's left over.

If we summarize what we know so far, we can say that there are three types of fractions:

Proper fraction	numerator $<$ denominator	$\frac{3}{4}$

Improper fraction	numerator $\geq$ denominator	$\frac{6}{5}$
Mixed number	sum of a whole number and a fraction	$2\frac{3}{4}$

It's important to know that we can always convert improper fractions to mixed numbers, and vice versa. For example, we'd write "two and three-sevenths" as

$$2\frac{3}{7}$$

Remember,

mixed numbers don't indicate multiplication: $2\frac{3}{7} \neq 2 \times \frac{3}{7}$

mixed numbers indicate addition: $2\frac{3}{7} = 2 + \frac{3}{7}$

Converting mixed numbers to improper fractions

If we want to convert a positive mixed number to an improper fraction, we follow these steps:

Multiply the fraction's denominator by the whole number

Add the result to the fraction's numerator

Write that result on top of the original denominator

Example

Convert the mixed number to an improper fraction.

$$2\frac{3}{7}$$

We know that

2 is the whole number

$3/7$ is the fraction

3 is the fraction's numerator

7 is the fraction's denominator

We need to multiply the fraction's denominator (7) by the whole number (2).

$$7 \times 2 = 14$$

Then we add the fraction's numerator (3) to the result of that multiplication (14).

$$14 + 3 = 17$$

This result (17) will be the numerator of our improper fraction, and we'll keep the original denominator (7) to get the final answer:

$$\frac{17}{7}$$

In other words,

$$2\frac{3}{7} = 2 + \frac{3}{7} = \frac{7 \times 2}{7} + \frac{3}{7} = \frac{7 \times 2 + 3}{7} = \frac{14 + 3}{7} = \frac{17}{7}$$

Converting improper fractions to mixed numbers

If we want to convert an improper fraction to a mixed number, we follow these steps:

Divide the numerator of the improper fraction by its denominator.

Write down the whole number part of that result.

Write down any remainder as the numerator in the fraction part, above the original denominator in the fraction part.

If the remainder of the division is 0, that just means we can write the result as a whole number, instead of a mixed number.

Example

Convert the improper fraction to a mixed number.

$$\frac{19}{6}$$

If we look at the positive multiples of 6,

$$6 \cdot 1 = 6$$

$$6 \cdot 2 = 12$$

$$6 \cdot 3 = 18$$

$$6 \cdot 4 = 24$$

we can see that 6 goes into 19 three times, but not four times, since $6 \cdot 3 = 18$ is still less than 19 but $6 \cdot 4 = 24$ isn't.

That means that 3 will be the whole number in our mixed number. Since $6 \cdot 3 = 18$, we have to add 1 to get from 18 to 19, which means the remainder is 1. Therefore, that remainder (1) will be the numerator of the fraction (in our mixed number), and the denominator of the improper fraction (6) will be the denominator of the fraction (in our mixed numbers), so the mixed number that's equivalent to the original improper fraction is

$$3\frac{1}{6}$$

In other words,

$$\frac{19}{6} = \frac{18+1}{6} = \frac{(6 \cdot 3)+1}{6} = \frac{6 \cdot 3}{6} + \frac{1}{6} = 3 + \frac{1}{6} = 3\frac{1}{6}$$

Negative fractions

Just as there are positive proper fractions, positive improper fractions, and positive mixed numbers, there are negative proper fractions, negative improper fractions, and negative mixed numbers.

We have to be careful about signs when we express a negative mixed number as the sum of a whole number and a fraction. Both the whole number and the fraction must be negative. It helps to use grouping symbols (such as parentheses or square brackets) in doing this. For example,

$$-2\frac{3}{7} \text{ means } -\left(2+\frac{3}{7}\right) = -2-\frac{3}{7}, \text{ not } -2+\frac{3}{7}$$

Going back to our examples, we find that

$$-2\frac{3}{7} = -\left(2+\frac{3}{7}\right) = -\left(\frac{7\times 2}{7}+\frac{3}{7}\right) = -\left[\frac{(7\times 2)+3}{7}\right] = -\frac{17}{7}$$

and

$$-\frac{19}{6} = -\left(\frac{18+1}{6}\right) = -\left[\frac{(6\cdot 3)+1}{6}\right] = -\left(\frac{6\cdot 3}{6}+\frac{1}{6}\right) = -\left(3+\frac{1}{6}\right) = -3\frac{1}{6}$$

Adding and subtracting mixed numbers

We can add and subtract mixed numbers, each of which is the sum of a whole number and a fraction.

When we need to add or subtract mixed numbers, we deal with the whole numbers separately from the fractions, and we find a common denominator for the fractions.

Two methods

Essentially, we'll follow these steps to add or subtract mixed numbers:

1. Add or subtract the whole number parts.
2. Check whether the fraction parts have a common denominator, and find one if not.
3. Set up equivalent fractions when needed.
4. Add or subtract the numerators of the fractions while keeping the denominator the same.
5. Simplify the fraction into a mixed number if the result is an improper fraction.

Alternatively, we can add or subtract mixed numbers by first converting the mixed numbers into improper fractions, and then adding or subtracting the improper fractions using the rules we learned earlier for fraction addition and subtraction.

Let's try an example of addition of mixed numbers.

Example

Find the sum.

$$6\frac{3}{7} + 2\frac{1}{4}$$

First, we'll separate the whole numbers from the fractions.

$$(6 + 2) + \left(\frac{3}{7} + \frac{1}{4}\right)$$

Next, we'll find a common denominator for the fractions.

$$(6 + 2) + \left[\frac{3}{7}\left(\frac{4}{4}\right) + \frac{1}{4}\left(\frac{7}{7}\right)\right]$$

$$(6 + 2) + \left(\frac{12}{28} + \frac{7}{28}\right)$$

Now we'll add the whole numbers and the fractions separately.

$$8 + \frac{19}{28}$$

As a mixed number, the answer is

$$8\frac{19}{28}$$

We can also convert this to an improper fraction.

$$8\frac{19}{28} = \frac{(28 \times 8) + 19}{28}$$

$$8\frac{19}{28} = \frac{224 + 19}{28}$$

$$8\frac{19}{28} = \frac{243}{28}$$

Let's try an example of subtraction with mixed numbers.

Example

Find the difference.

$$6\frac{3}{7} - 2\frac{1}{4}$$

First, we'll separate the whole numbers from the fractions.

$$(6 - 2) + \left(\frac{3}{7} - \frac{1}{4}\right)$$

Next, we'll find a common denominator for the fractions.

$$(6-2)+\left[\frac{3}{7}\left(\frac{4}{4}\right)-\frac{1}{4}\left(\frac{7}{7}\right)\right]$$

$$(6-2)+\left(\frac{12}{28}-\frac{7}{28}\right)$$

Now we'll subtract the whole numbers and the fractions separately.

$$4+\frac{5}{28}$$

As a mixed number, the answer is

$$4\frac{5}{28}$$

We can also convert this to an improper fraction.

$$4\frac{5}{28}=\frac{(28\times 4)+5}{28}$$

$$4\frac{5}{28}=\frac{112+5}{28}$$

$$4\frac{5}{28}=\frac{117}{28}$$

Let's try another example of mixed number subtraction.

Example

Simplify the expression.

$$3\frac{1}{7}-1\frac{1}{4}$$

We'll subtract the whole numbers separately from the fractions.

$$3\frac{1}{7}-1\frac{1}{4}$$

$$3-1+\frac{1}{7}-\frac{1}{4}$$

$$2+\frac{1}{7}-\frac{1}{4}$$

To subtract the fractions, we have to find the lowest common denominator (LCD), which is the least common multiple of the denominators.

$$2+\frac{1}{7}\left(\frac{4}{4}\right)-\frac{1}{4}\left(\frac{7}{7}\right)$$

$$2+\frac{4}{28}-\frac{7}{28}$$

$$2-\frac{3}{28}$$

To change this to a mixed number, we need to change the expression so that it's addition instead of subtraction. We can change the 2 into the equivalent $1+1$.

$$1+1-\frac{3}{28}$$

Make a common denominator with the 1 and the fraction.

$$1 + 1\left(\frac{28}{28}\right) - \frac{3}{28}$$

$$1 + \frac{28}{28} - \frac{3}{28}$$

$$1 + \frac{25}{28}$$

$$1\frac{25}{28}$$

Multiplying and dividing mixed numbers

We can multiply and divide mixed numbers. We just change them to improper fractions, and then multiply or divide as usual, and finally convert the answer to a mixed number.

Remember, to convert a mixed number to an improper fraction, we just multiply the denominator of the fraction by the whole number, then add the result to the numerator of the fraction, then put that whole thing over the original denominator.

Let's try an example with multiplication of mixed numbers.

Example

Find the product.

$$2\frac{2}{3} \times 5\frac{3}{7}$$

First, we'll convert both mixed numbers to improper fractions.

$$\left[\frac{(3 \cdot 2) + 2}{3}\right] \times \left[\frac{(7 \cdot 5) + 3}{7}\right]$$

$$\left(\frac{6 + 2}{3}\right) \times \left(\frac{35 + 3}{7}\right)$$

$$\frac{8}{3} \times \frac{38}{7}$$

Then, to multiply the fractions, we multiply the numerators and the denominators separately.

$$\frac{8 \times 38}{3 \times 7}$$

$$\frac{304}{21}$$

We can leave the answer as an improper fraction, but If we want to convert it to a mixed number, we can say that 21 goes into 304 fourteen times, with 10 left over, so

$$\frac{304}{21} = 14\frac{10}{21}$$

Let's try an example with division of mixed numbers.

Example

Find the quotient.

$$4\frac{1}{6} \div 3\frac{1}{3}$$

First, we'll convert both mixed numbers to improper fractions.

$$\left[\frac{(6 \cdot 4) + 1}{6}\right] \div \left[\frac{(3 \cdot 3) + 1}{3}\right]$$

$$\left(\frac{24+1}{6}\right) \div \left(\frac{9+1}{3}\right)$$

$$\frac{25}{6} \div \frac{10}{3}$$

Then, to divide the fractions, we'll invert the divisor (turn the second fraction upside down), and then multiply.

$$\frac{25}{6} \times \frac{3}{10}$$

$$\frac{75}{60}$$

We'll reduce the fraction to lowest terms by dividing by 15, the greatest common factor. This doesn't change the value of the fraction; it just reduces it.

$$\frac{75 \div 15}{60 \div 15}$$

$$\frac{5}{4}$$

We can leave the answer as an improper fraction, but If we want to convert it to a mixed number, we can say that 4 goes into 5 one time, with 1 left over, so

$$\frac{5}{4} = 1\frac{1}{4}$$

Relationships of numbers

This topic can be a little challenging, so let's walk through it one step at a time. We're talking about relationships between numbers.

The first thing we'll deal with is how to determine which of two fractions is greater than the other. If two fractions have the same denominator, the fraction with the greater numerator is the greater one.

For example, the denominators of $2/7$ and $5/7$ are equal, and the numerator 5 is greater than the numerator 2, so $5/7$ is greater than $2/7$.

When the numerators are equivalent, the larger fraction is the one with the smaller denominator.

If the denominators of two fractions are different, we can't compare them directly; we first have to find a common denominator. For example, consider $5/8$ and $2/3$. For a common denominator, we'll use the least common multiple of the denominators 8 and 3, which is 24. So

$$\frac{5}{8}=\frac{5}{8}\left(\frac{3}{3}\right)=\frac{5\times 3}{8\times 3}=\frac{15}{24}$$

$$\frac{2}{3}=\frac{2}{3}\left(\frac{8}{8}\right)=\frac{2\times 8}{3\times 8}=\frac{16}{24}$$

The fractions $15/24$ and $16/24$ are equivalent to $5/8$ and $2/3$, respectively. Since they have the same denominator 24, the fraction with the greater numerator is greater than the fraction with the lesser numerator.

Therefore, $16/24$ is greater than $15/24$, which means that $2/3$ is greater than $5/8$, or we could say $5/8$ is less than $2/3$.

Now let's talk about the relationship between two integers - in particular, how we can find the number that's a fraction of the distance (along the number line) from the smaller integer to the larger one. Let's say we're thinking about the integers 3 and 8. We know that 3 is five units to the left of 8, or that 8 is five units to the right of 3. In other words, they're five units apart, since $8-3=5$.

Now what if we're asked for the number that's two-fifths of the way from 3 to 8? In other words, "If we divide the distance between 3 and 8 into five equal pieces, and then we start from 3 and move toward 8 by two of those five equal pieces, where do we end up?"

Here's how we figure that out. First we find the distance between 3 and 8 by subtracting the smaller number from the bigger number.

$$8-3=5$$

The distance between 3 and 8 is 5. Now, since we're looking to go two-fifths of that distance, we want to first divide the distance 5 into five equal pieces (each of which is one-fifth of that distance), which we do by dividing 5 by 5.

$$\frac{5}{5}=1$$

So one-fifth of the 5 units of distance between 3 and 8 is 1 unit. Since I want two-fifths of the distance between 3 and 8, I need to multiply 1 unit by 2, and I get 2 units, so two-fifths of the distance between 3 and 8 is 2.

This means that if we want to go two-fifths of the way from 3 to 8, we start at 3, and add 2, and we end up at

$$3 + 2 = 5$$

So the number that's two-fifths of the way from 3 to 8 is 5.

In general, when we have some fraction "of" another number, it means we need to multiply the fraction by the other number. For example, $2/3$ of 6 is

$$\frac{2}{3} \cdot 6 = \frac{2}{3} \cdot \frac{6}{1} = \frac{2 \cdot 6}{3 \cdot 1} = \frac{12}{3} = 4$$

Therefore, we say that $2/3$ of 6 is 4, or that two-thirds of 6 is 4.

Now we can also do this with fractions. The process is exactly the same; we're just dealing with fractions instead of integers.

Example

Find a number that's $1/2$ of the way from $1/7$ to $6/11$.

First, we'll find the distance between $1/7$ and $6/11$.

$$\frac{6}{11} - \frac{1}{7}$$

In order to do the subtraction, we have to find a common denominator.

$$\frac{6}{11}\left(\frac{7}{7}\right) - \frac{1}{7}\left(\frac{11}{11}\right)$$

$$\frac{42}{77}-\frac{11}{77}$$

$$\frac{31}{77}$$

Now we want to find $1/2$ of this distance, which means we need to multiply it by $1/2$.

$$\frac{31}{77}\times\frac{1}{2}$$

$$\frac{31}{154}$$

This is half the distance from $1/7$ to $6/11$, and since we want to end up exactly one-half of the way from $1/7$ to $6/11$, we simply add $31/154$ to the smaller fraction, $1/7$.

$$\frac{1}{7}+\frac{31}{154}$$

In order to do the addition, we have to find a common denominator.

$$\frac{1}{7}\left(\frac{22}{22}\right)+\frac{31}{154}$$

$$\frac{22}{154}+\frac{31}{154}$$

$$\frac{53}{154}$$

So $53/154$ is the number that's $1/2$ of the way from $1/7$ to $6/11$.

Adding mixed measures

Mixed measures are measurements like hours, minutes and seconds, or yards, feet, and inches. It's fairly simple to understand the distance

3 yards, 2 feet, 4 inches

If we want to add two sets of mixed measures, the method we use is essentially the same thing we do when we add mixed numbers. First, we add the individual measures, like hours or minutes, or yards or feet, separately. That's the easy part. The trickier part is the second part, which is simplifying the result of the first part.

As a reminder, here are some conversion formulas to use in these kinds of problems:

1 yard = 3 feet = 36 inches

1 hour = 60 minutes = $3{,}600$ seconds

Let's do an example.

Example

Find the sum of the mixed measures.

3 yards, 2 feet, 4 inches

6 yards, 2 feet, 8 inches

First, we'll add the yards, the feet, and the inches separately.

$(3+6)$ yards, $(2+2)$ feet, $(4+8)$ inches

9 yards, 4 feet, 12 inches

We've now converted the sum of the two original sets of mixed measures to a single set of mixed measures. Since 12 inches is equivalent to 1 foot, we want to simplify that result so that the number of inches is less than 12. We do this by working from right to left. We'll first rewrite 12 inches as 1 foot, and then we'll add that 1 foot to the 4 feet we've already found:

9 yards, 4 feet, 12 inches

9 yards, 4 feet, 1 foot

9 yards, $(4+1)$ feet

9 yards, 5 feet

Since there are 3 feet in a yard, we want to simplify that result so that the number of feet is less than 3. We'll first express 5 feet as the sum of 3 feet and 2 feet, then rewrite 3 feet as 1 yard, and add that 1 yard to the 9 yards we've already found:

9 yards, 3 feet $+ 2$ feet

9 yards, 1 yard $+ 2$ feet

$(9+1)$ yards, 2 feet

10 yards, 2 feet

This is the sum of 3 yards, 2 feet, 4 inches and 6 yards, 2 feet, 8 inches.

Place value

When we talk about **place value**, we're talking about the value of the location of a particular digit within a given number (the value of the place where that digit is located within that number). Given a number like 4.321, place value is what allows us to easily say where the 3 is located or where the 1 is located. In math, every digit in a number has its own place value.

Digits to the left of the decimal point

Below is a table of some place values, but we can extend this table further in both directions. The table centers around the **ones place**, or units place, which is the place of all single-digit integers.

1,	0	0	0,	0	0	0	.	0	0	0	0	0	0
Millions	Hundred Thousands	Ten Thousands	Thousands	Hundreds	Tens	Ones (units)		Tenths	Hundredths	Thousandths	Ten-Thousandths	Hundred-Thousandths	Millionths

From the ones place, if we move one place to the left we get to the "tens" place, and if we move one place to the right we get to the "tenths" place.

As we continue to move to the left we have the "hundreds" and "thousands" places, and as we move to the right we have the "hundredths" and "thousandths" places.

The ones place is separated from the tenths place by a symbol that looks like a period, but when we use that symbol in a decimal number we call it a **decimal point**.

Let's do an example where we identify the place value of each digit in a whole number.

Example

Write out the place value of each digit in the number $12,854$.

Working from left to right through $12,854$, we can say that

- there's a 1 in the ten thousands place, which represents $10,000$
- there's a 2 in the thousands place, which represents $2,000$
- there's an 8 in the hundreds, which represents 800
- there's a 5 in the tens place, which represents 50
- there's a 4 in the ones place, which represents 4

If we sum all these values, we see that they form our original number.

$$12,854 = 10,000 + 2,000 + 800 + 50 + 4$$

Digits to the right of the decimal point

Now let's look more at the digits to the right of the decimal point. These digits really just represent fractions with denominators of different powers of ten, where the digits to the left of the decimal point represented whole numbers. Each digit to the right of the decimal point is ten times smaller than the previous one.

For example, in the decimal number 17.546, 17 is a whole number in which 1 is in the tens place with a value of 10 and 7 is in the ones place with a value of 7. The three digits to the right of the decimal point can be expressed as

- 5 in the tenths place to represent 0.5 or $5/10$
- 4 in the hundredths place to represent 0.04 or $4/100$
- 6 in the thousandths place to represent 0.006 or $6/1{,}000$

We also want to know that the number **decimal places** in a decimal number is given by the number of digits to the right of the decimal point. So 47.603 has five digits (4, 7, 6, 0, and 3), but only three decimal places since there are only three digits to the right of its decimal point. The digits in the first, second, and third decimal places are 6, 0, and 3, respectively.

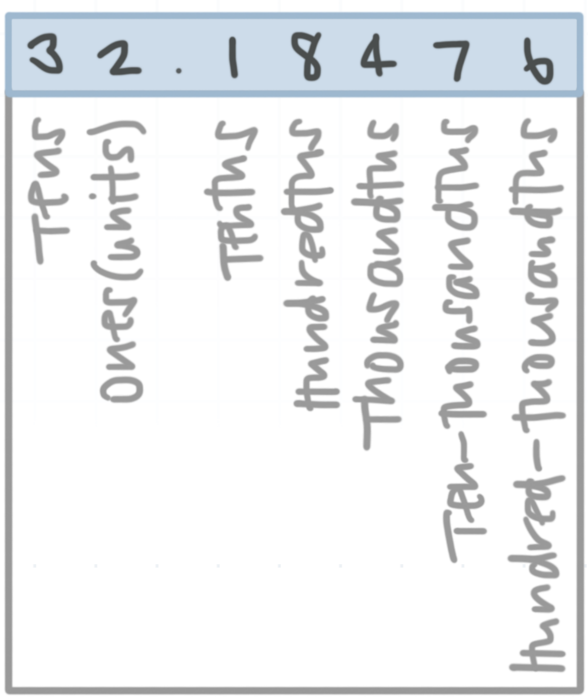
Let's do an example where we identify the place value of one digit in a decimal number.

Example

In which place is the 7 located?

$$32.18476$$

The 7 is four places to the right of the decimal point, which means it’s in the ten-thousandths place. We could even break down each place in this number:



The 7 therefore represents a value of

$$7\left(\frac{1}{10{,}000}\right)=\frac{7}{10{,}000}=0.0007$$

Let’s do one more example with non-zero digits on both sides of the decimal point.

Example

Identify the value represented by each digit of $2,635.487$, then show that those values sum to the original number.

Let's make a table to summarize the value of each digit.

Digit	Place	Value	Description
2	$1,000$	$2 \times 1,000 = 2,000$	two thousand
6	100	$6 \times 100 = 600$	six hundred
3	10	$3 \times 10 = 30$	thirty
5	1	$5 \times 1 = 5$	five
4	0.1	$4 \times 0.1 = \frac{4}{10}$	four-tenths
8	0.01	$8 \times 0.01 = \frac{8}{100}$	eight-hundredths
7	0.001	$7 \times 0.001 = \frac{7}{1,000}$	seven-thousandths

If we sum all these values, we find the original number, $2,635.487$.

$$2,635.487 = 2,000 + 600 + 30 + 5 + 0.4 + 0.08 + 0.007$$

Decimal arithmetic

We can add, subtract, multiply, and divide decimal numbers.

Addition and subtraction of decimal numbers works the same way as whole number addition and subtraction; we just need to make sure that we line up the decimal points.

Let's try an example with addition and subtraction of decimal numbers.

Example

Find the sum and difference.

$$13.16 + 8.74$$

$$13.16 - 8.74$$

To find the sum, we'll line up the decimal points, making sure that they're stacked directly on top of each other.

$$\begin{array}{r} 13.16 \\ +\ \ 8.74 \end{array}$$

Then we'll bring the decimal point straight down and add the numbers as usual, starting with the digits in the hundredths place, carrying anything extra to the tenths place, adding the digits in the tenths place (including anything extra from the addition of the digits in the hundredths place), carrying anything extra to the ones place, etc.

$$\begin{array}{r} 13.16 \\ +\ 8.74 \\ \hline 21.90 \end{array}$$

We can say that the sum is $13.16 + 8.74 = 21.90$.

To find the difference, we'll line up the decimal points, making sure that they're stacked directly on top of each other.

$$\begin{array}{r} 13.16 \\ -\ 8.74 \end{array}$$

Then we'll bring the decimal point straight down and subtract the numbers as usual, starting with the digits in the hundredths place, borrowing from the tenths place if necessary, subtracting the digits in the tenths place (excluding anything we borrowed for the subtraction in the hundredths place), borrowing from the ones place if necessary, etc.

$$\begin{array}{r} 13.16 \\ -\ 8.74 \\ \hline 4.42 \end{array}$$

We can say that the difference is $13.16 - 8.74 = 4.42$.

Let’s try an example with multiplication and division of decimal numbers.

Example

Find the product and quotient.

$$13.1 \times 8.74$$

$$13.1 \div 8.74$$

To find the product, we’ll right-align the decimal numbers.

$$\begin{array}{r} 13.1 \\ \times\ 8.74 \end{array}$$

We’ll ignore the decimal points for now, and multiply the numbers as usual.

$$\begin{array}{r} 13.1 \\ \times\ 8.74 \\ \hline 524 \\ 9170 \\ +104800 \\ \hline 114494 \end{array}$$

Now we’ll count the number of digits to the right of the decimal point in each of the two decimal numbers, and then add. There’s one digit (a 1), after the decimal point in 13.1. There are two digits (a 7 and a 4) after the decimal point in 8.74. That's a total of three digits after the decimal points.

To decide where to place the decimal point in our answer, we start by putting it on the far right of the result, and we get "$114494.$". Then, since we had a total of three digits after the decimal points, we move the decimal point three places to the left to get our final answer. Therefore, $13.1 \times 8.74 = 114.494$.

To find the quotient, we'll do long division, but not until after we determine where to place the decimal point in our answer.

To figure out where it should go, we need to change both numbers into whole numbers. In order to change 8.74 to a whole number, we need to move the decimal point two spots to the right. In order to change 13.1 to a whole number, we need to move the decimal point one spot to the right. But we always have to move the decimal point the same number of places in both numbers.

Moving the decimal point one spot will change 13.1 to a whole number, but will change 8.74 into 87.4, which is still a decimal number. So we need to move the decimal point two spots in both numbers, adding a 0 to the end of 13.1. So 13.1 becomes 1310 and 8.74 becomes 874. Then we can do the long division as if we were doing division with whole numbers, instead of decimal numbers.

$$
\begin{array}{r}
1.4988... \\
874 \overline{)\,1310.00000} \\
\underline{-874} \\
4360 \\
\underline{-3496} \\
8640 \\
\underline{-7866} \\
7740 \\
...
\end{array}
$$

As we can see, 8.74 doesn't divide evenly into 13.1, so we can stop after a few decimal places and just give the estimation as $13.1 \div 8.74 \approx 1.4988$.

If we want to multiply a decimal number by 10, we can easily get the answer by just moving the decimal point one place to the right. If we want to multiply a decimal number by 100, we can just move the decimal point two places to the right; and so on.

Similarly, if we want to divide a decimal number by 10, we can easily get the answer by moving the decimal point one place to the left. If we want to divide a decimal number by 100, we can just move the decimal point two places to the left; and so on.

Repeating decimals

Up to now, we've been dealing with only finite decimal numbers (numbers with a finite number of decimal places). For example, 32.18476 is a finite decimal number, because it ends at the 6. In contrast, there are two kinds of decimal numbers that go on forever and ever. Some decimals that go on forever eventually get to a point where a certain digit (or sequence of digits) repeats infinitely, but some decimal number that go on forever don't repeat.

A decimal number where a digit or sequence of digits repeats infinitely is called a repeating decimal. An example is

$$32.184766666666...$$

The ... means that the 6 repeats forever. We can rewrite a repeating decimal in compact form by writing the repeating digit/sequence just once and putting a bar over it. For example, we can write $32.184766666666...$ as

$$32.1847\overline{6}$$

An example of the other kind of infinite decimal is π, whose decimal representation goes on forever but never repeats. Here are the first 46 digits of π.

$$3.141592653589793238462643383279502884197169399$$

What we want to focus on are decimals that go on forever but eventually repeat. We'll set these non-repeating decimals like π aside for now.

Example

Rewrite the repeating decimal.

$$0.5454545454...$$

What we have in this decimal number is a two-digit sequence, 54, that repeats over and over. Therefore $0.5454545454...$ can be rewritten as

$$0.\overline{54}$$

Rounding

When we're dealing with decimal numbers, rounding is an important tool that we need to know how to use. Let's take an extreme example to see why.

Given the decimal number

$$102.374000000000000000000000000000000000001$$

what we notice is that the 1 hanging way out there on the end makes only a tiny, tiny, tiny, tiny difference in the value of the number as a whole. So writing all of those extra 0's just for the sake of including that 1, doesn't make that much sense. It's much more practical to write the number as 102.374, because it saves so much time and effort not having to write all those extra digits, and 102.374 is so close to the actual number that it doesn't even really make a difference.

We round decimals (meaning we approximate them with fewer than the number of digits in their actual values) because it saves us a lot of time and lets us express really long numbers as shorter, simpler numbers.

When we round, we always round to a certain decimal place. This goes back to what we learned previously about place value. For example, we know that the second digit to the right of the decimal point is the hundredths place. This means that if we're asked to round to the hundredths place, we want the last digit in the number to be the digit in the hundredths place; we don't want to include any digits after that.

We do have to follow certain rules when we round. The rule we need to remember is:

> "If the next digit is less than 5, round the **previous digit down**; if it's 5 or greater, round the **previous digit up**."

To round a digit **down** means to leave it unchanged; to round a digit **up** means to increase it by one unit. For example, if we round down a 3, we leave it unchanged (at 3); if we round up a 3, we increase it to 4.

Example

Round to the nearest hundredth.

$$3.14159$$

Since we've been asked to round to the nearest hundredth, that means we're taking everything that comes after the hundredths place and rounding it into the hundredths place.

In 3.14159, the 4 is in the hundredths place, so we need to round everything that comes after that. The question we need to ask is, "What digit comes right after the 4?" Well, we have a 1 right after the 4 in the thousandths place. Since 1 is less than 5, that means we're rounding the 4 down, so the digit that was originally in the hundredths place (the 4) will be unchanged.

The digits 5 and 9 that come after the 1 that we used in deciding whether to round up or down don't matter. Since we're rounding to the nearest hundredth, we need to consider only the digit that comes right after the hundredths place, which is the thousandths place.

So 3.14159, rounded to the nearest hundredth, is

$$3.14$$

Let's do another example. In this one we'll round up.

Example

Round to the nearest hundred.

$$130{,}874.62$$

Since we've been asked to round to the nearest hundred, that means we're taking everything that comes after the hundreds place and rounding it into the hundreds place.

In $130{,}874.62$, the 8 is in the hundreds place, so we need to round everything that comes after that. The question we need to ask is, "What digit comes right after the 8?" Well, we have a 7 right after the 8, and since 7 is greater than or equal to 5, that means we're rounding the 8 up, so the original digit in the hundreds place (the 8) needs to get bumped up to the next higher digit, 9.

The numbers 4, 6, and 2 that come after the 7 that we used in deciding whether to round up or down don't matter. Since we're rounding to the nearest hundred, we need to consider only the digit that comes right after the hundreds place.

There's another aspect of rounding that we sometimes have to deal with. Whenever we're rounding to a digit that's left of the decimal point, we drop the decimal point and all the digits to the right of it. But we have to have some digit in each place between the decimal point and the place we're rounding to, so we put a 0 in each of those places. In this example, we have to put 0's in the ones and tens places.

So $130{,}874.62$, rounded to the nearest hundred, is

$$130{,}900$$

Now what if, in the last example, we'd had $130{,}974.62$ instead of $130{,}874.62$, and we wanted to round it to the nearest hundred? Well, we'd first look at the 7, and because 7 is greater than or equal to 5, we'd know that we need to round up the 9. But we can't round up the 9 to a 10 - that would be replacing one digit with two, which doesn't work.

What we do in the special case where we're rounding up a 9 is change the 9 to a 0 and increase the digit to the left of that 0 by one unit. So in rounding $130{,}974.62$ to the nearest hundred, we change the 9 to a 0 and increase the digit in the thousands place (the 0 that comes right after the 3) by one unit (we increase it from 0 to 1). Of course, we also have to drop the decimal point (and the digits to the right of it), and put 0's in the ones and tens places. Therefore, $130{,}974.62$ rounded to the nearest hundred, is

$$131{,}000$$

Ratio and proportion

Ratio and proportion is an application of fractions. A ratio is basically just a fraction, but one that emphasizes the relationship of its numerator to its denominator. For example, we say that

$$\frac{2}{3}$$

is "the ratio of 2 to 3." We sometimes write a ratio as $2:3$.

A proportion is an equation of two ratios, such as

$$\frac{2}{3}=\frac{8}{12}$$

Because the two ratios in a proportion are equal to each other, a proportion is an equation that expresses the equivalence of two fractions. We could read this proportion as follows:

"2 is to 3 as 8 is to 12"

What this says is that the relationship of 2 to 3 is the same as the relationship of 8 to 12. We sometimes write a proportion as $2:3::8:12$, with a single colon within each ratio and a sequence of two colons between the two ratios.

Sometimes we're given a proportion such as

$$\frac{4}{5}=\frac{2x}{40}$$

and we're asked to solve for x. The x (which we call "a variable" or "an unknown") stands for a particular number (namely, the number that makes this equation true, which we can also refer to as "the number that satisfies this equation"). To "solve for x" means to find that number.

The expression $2x$ means "2 multiplied by x," so the numerator of the fraction on the right side of the equation is the product of 2 and x. By the way, we can also refer to the left and right sides (of an equation) as "the left-hand side" and "the right-hand side," respectively.

To solve for x, we have to take steps that will allow us to ultimately get an equation in which x is all by itself on one side. When we get that equation, the value of x (the number that makes our original equation true) will be on the other side of it.

Example

Solve for the unknown.

$$\frac{4}{5}=\frac{2x}{40}$$

To solve for the unknown, we simply need to remember the following rule:

> "If we make some change to the expression on one side of an equation, make the same change to the expression on the other side."

In the case of a proportion, the easiest way to do this is to cross multiply, which means we'll multiply both sides by both denominators.

First, we'll multiply both sides by 40, the denominator from the right side. This will cancel out the 40 in the denominator on the right side.

$$40\left(\frac{4}{5}\right)=40\left(\frac{2x}{40}\right)$$

$$40\left(\frac{4}{5}\right)=2x$$

$$\frac{160}{5}=2x$$

$$5\left(\frac{160}{5}\right)=5(2x)$$

$$160=10x$$

Now divide both sides by 10, which will cancel the 10 on the right side.

$$\frac{160}{10}=\frac{10x}{10}$$

$$16=x$$

$$x=16$$

Let's revisit that example, and see what we actually ended up doing when we cross multiplied.

Since each of the denominators canceled out when we multiplied both sides of the equation

$$\frac{4}{5} = \frac{2x}{40}$$

by it, what we eventually got on the left side was the product of the denominator from the right side (40) and the numerator from the left side (4), and what we eventually got on the right side was the product of the denominator from the left side (5) and the numerator from the right side ($2x$).

The arrows (from 40 to 4, and from 5 to $2x$) show which numerator each of the denominator ended up being multiplied by.

$$\frac{4}{5} = \frac{2x}{40}$$

$$40(4) = 5(2x)$$

$$160 = 10x$$

From there, the only thing we had to do (to get the value of x) was to divide both sides of this equation by 10, which gave us $16 = x$.

We can think of the following as a shortcut way to cross multiply: Write the given proportion, and then (immediately below that) write the equation in which the expression on the left side is the product of the denominator from the right side (of the given proportion) and the numerator from the left side (of the given proportion), and the expression on the right side is the product of the denominator from the left side (of the given proportion) and the numerator from the right side (of the given proportion).

If we ever forget this shortcut or feel uncomfortable using it, we can always multiply both sides of a proportion by both denominators.

We can solve for the variable, no matter where it appears in the equation. It can be on the left side or the right side, and it can be in the numerator or the denominator. Let's try an example where the variable appears in the denominator on the left side.

Example

Solve for the variable.

$$\frac{1}{6x} = \frac{3}{20}$$

This time, we'll use the shortcut way to cross multiply (which is the method that most people actually think of as cross multiplying). Our first step in solving for the variable will consist of writing an equation as follows: On the left-hand side, we'll write the product of the denominator from the right-hand side (20) and the numerator from the left-hand side (1). On the right-hand side, we'll write the product of the denominator from the left-hand side ($6x$) and the numerator from the right-hand side (3).

$$20(1) = 6x(3)$$

$$20 = 18x$$

Next, we'll divide both sides of this equation by 18, to get the x all by itself.

$$\frac{20}{18} = \frac{18x}{18}$$

$$\frac{20}{18} = x$$

Finally, we'll reduce the fraction to lowest terms.

$$\frac{10}{9} = x$$

Unit price

When we talk about unit price, what we're really talking about is the "price per unit" of a product (the price per pound of tomatoes or the price per quart of milk). This is the math that helps us compare the prices of things.

For example, when we're buying peanut butter at the grocery store and one jar costs $\$2.00$ for 1 pound and another jar costs $\$1.80$ for 12 ounces, we can use unit price to figure out which one is a better deal.

The way to figure that out is by finding either the price per ounce for both jars, or the price per pound for both jars. That way, we're comparing equivalent values. Whichever jar is cheaper per pound (or per ounce) will be the better value.

We're going to be using proportions to figure this out.

Example

Which jar of peanut butter is a better value?

Jar A costs $\$2.00$ for 1 pound

Jar B costs $\$1.80$ for 12 ounces

There are several ways we could approach this problem. Since we already have the price for 1 pound with Jar A, we'll find the price per pound for Jar B, and then we'll be able to compare prices directly. We want to set up a proportion, so we'll let our unknown x be the price per pound for Jar B.

$$\frac{\$1.80}{12 \text{ ounces}} = \frac{x}{1 \text{ pound}}$$

This proportion is saying "If 12 ounces costs $\$1.80$, at this same cost per unit, how much will 1 pound cost?" In order to solve this proportion, we need to use the same unit of weight on both sides. We know that there are 16 ounces in a pound, so on the right-hand side we'll substitute 16 ounces for 1 pound.

$$\frac{\$1.80}{12 \text{ ounces}} = \frac{x}{16 \text{ ounces}}$$

Now we'll multiply both sides of this equation by 16 ounces, which will cancel out the 16 ounces in the denominator on the right-hand side and leave just x in the numerator.

$$\frac{\$1.80}{12 \text{ ounces}}(16 \text{ ounces}) = x$$

$$\$1.80\left(\frac{16}{12}\right) = x$$

To make the computation a little easier, we'll first reduce the fraction.

$$\$1.80\left(\frac{16 \div 4}{12 \div 4}\right) = x$$

$$\$1.80\left(\frac{4}{3}\right) = x$$

$$\frac{\$1.80(4)}{3} = x$$

$$\frac{\$7.20}{3} = x$$

$$\$2.40 = x$$

What this tells us is that Jar B costs $\$2.40$ per pound. Comparing this to Jar A, which costs $\$2.00$ per pound, we can say that Jar A is a better value than Jar B.

Unit multipliers

Unit multipliers are what we use to convert one set of units to another. A really easy example is using a unit multiplier to convert feet to inches. If we want to write 4 feet in terms of inches, we use a unit multiplier and write

$$4 \text{ feet} \cdot \frac{12 \text{ inches}}{1 \text{ foot}}$$

We cancel the units of "feet," and then simplify in order to get a value that's only in terms of inches.

$$4 \cdot \frac{12 \text{ inches}}{1}$$

$$\frac{48 \text{ inches}}{1}$$

$$48 \text{ inches}$$

Then we can say that a distance of 4 feet is the same as a distance of 48 inches. So what did we just do? Well, the fraction $(12 \text{ inches})/(1 \text{ feet})$ is the unit multiplier. Since we want to change our units from feet to inches, essentially what we're doing is using a fraction to relate feet to inches (we know that there are 12 inches in 1 foot), and then multiplying by that fraction in order to cancel the units we want to get rid of, and keep only the units we want.

When we set up our unit multiplier fraction, we put in the denominator the units we want to cancel and we put in the numerator the units we want to keep. That way, the units we want to get rid of will cancel out, and we'll be left with only the units we want to keep.

Let's do another example so we can start to get the hang of this.

Example

Convert the value from meters to centimeters. Hint: there are 100 centimeters in 1 meter.

$$3.5 \text{ meters}$$

Let's realize first that we want to get rid of the meters (the units we've been given), and we want to end up with only units of centimeters. We've been told that there are 100 centimeters in 1 meter, which means we need to use either the unit multiplier

$$\frac{1 \text{ meter}}{100 \text{ centimeters}}$$

or the unit multiplier

$$\frac{100 \text{ centimeters}}{1 \text{ meter}}$$

Remember that we put on the top (the numerator) the units we want to keep, and we put on the bottom (the denominator) the units we want to get rid of. Since we want to keep centimeters and get rid of meters, we'll put centimeters on the top and meters on the bottom, and use

$$\frac{100 \text{ centimeters}}{1 \text{ meter}}$$

Let's multiply this unit multiplier by the 3.5 meters we were given originally.

$$3.5 \text{ meters} \cdot \frac{100 \text{ centimeters}}{1 \text{ meter}}$$

Cancel the units of "meters," and then simplify in order to get a value that's only in terms of centimeters.

$$3.5 \cdot \frac{100 \text{ centimeters}}{1}$$

$$\frac{350 \text{ centimeters}}{1}$$

$$350 \text{ centimeters}$$

Then we can say that a distance of 3.5 meters is the same as a distance of 350 centimeters.

Let's do another example, but this time we'll have to do multiple conversions.

Example

Convert the value from inches to yards. Hint: there are 12 inches in 1 foot and 3 feet in 1 yard.

$$288 \text{ inches}$$

If we think ahead about our plan a little bit, we realize that what we want to do is convert first from inches to feet, and then from feet to yards. We can do this in two separate steps or all at once.

First, we know we want to get rid of the units of inches and keep the units of feet. So we can use a unit multiplier and write

$$288 \text{ inches} \cdot \frac{1 \text{ foot}}{12 \text{ inches}}$$

This will cancel out inches and leave us with feet. But we also want to convert feet to yards. To do that, we'll multiply by another unit multiplier - one that has feet in the denominator (since that's what we want to cancel) and yards in the numerator (since that's what we want to keep).

$$288 \text{ inches} \cdot \frac{1 \text{ foot}}{12 \text{ inches}} \cdot \frac{1 \text{ yard}}{3 \text{ feet}}$$

Now we'll start canceling units. Let's first cancel out the inches.

$$288 \cdot \frac{1 \text{ foot}}{12} \cdot \frac{1 \text{ yard}}{3 \text{ foot}}$$

Now we'll cancel out the feet.

$$288 \cdot \frac{1}{12} \cdot \frac{1 \text{ yard}}{3}$$

All we have left is yards, which is exactly what we want. So we'll just do the multiplication to simplify.

$$\frac{288 \cdot 1 \cdot 1}{12 \cdot 3} \text{ yards}$$

Combining the 1's with the 288, we have

$$\frac{288}{12 \cdot 3} \text{ yards}$$

To simplify this, we'll first divide 288 and 3 by 3.

$$\frac{288 \div 3}{12 \cdot (3 \div 3)} \text{ yards}$$

$$\frac{96}{12 \cdot 1} \text{ yards}$$

$$\frac{96}{12} \text{ yards}$$

8 yards

Exponents

Exponents are a tool we can use to write numbers in a simpler way. An exponent is a little number that we write above and to the right of another number, like this:

$$3^2$$

When we see an expression like this, the little 2 is the exponent, and the 3 is called the "base." The exponent tells us the number of times to multiply the base by itself. So the expression 3^2 is telling us to multiply 3 by itself 2 times, since the base is 3 and the exponent is 2. Here are some others:

2^3	Multiply 2 by itself 3 times
5^4	Multiply 5 by itself 4 times

Let's expand these examples just to be clear what we really mean. When we say to multiply 3 by itself 2 times, we mean that

$$3^2 = 3 \cdot 3 = 9$$

In the same way, multiplying 2 by itself 3 times means that $2^3 = 2 \cdot 2 \cdot 2 = 8$. And multiplying 5 by itself 4 times means that $5^4 = 5 \cdot 5 \cdot 5 \cdot 5 = 625$.

Exponents are really helpful to us as we go further in math, because if we want to express a multiplication like $7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7$ in a simpler way, we can use an exponent and write it as 7^{11}.

Let's do an example.

Example

Use an exponent to write the expression.

$$4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4$$

In this expression, we're multiplying 4 by itself 6 times. Which means we need the base to be 4 and the exponent to be 6. Therefore, we can write the expression as

$$4^6$$

Let's do another example, but this time we'll take an exponential expression and expand it.

Example

Write the expression in expanded form.

$$6^3$$

This expression tells us to use 6 as a factor 3 times, which means we can rewrite it without an exponent as

$$6 \cdot 6 \cdot 6$$

We don't have to find the result of the multiplication, but we could also do that and say

$$6^3 = 6 \cdot 6 \cdot 6 = 216$$

We can use exponents with variables as well. So if we want to multiply x by itself 3 times, we can write that as x^3:

$$x \cdot x \cdot x = x^3$$

This is also true when the base is negative.

Example

Write the expression in expanded form.

$$(-2)^3$$

This expression tells us to use -2 as a factor 3 times, which means we can rewrite it without an exponent as

$$(-2) \cdot (-2) \cdot (-2)$$

We don't have to find the result of the multiplication, but we could also do that and say

$$(-2)^3 = (-2) \cdot (-2) \cdot (-2) = -8$$

Rules of exponents

When it comes to dealing with exponents, we have to follow certain rules.

Addition and subtraction

When we want to find the sum or difference of two exponential expressions, they must be **like terms**, meaning that they must have the same base and the same exponent; otherwise, we can't add or subtract them.

For example, we can add or subtract $3x^2$ and x^2, because the bases are both x and the exponents are both 2. The 3 is what we call a **coefficient**; it just tells us we have three x^2's added together ($3x^2 = x^2 + x^2 + x^2$), so the sum of $3x^2$ and x^2 is found by adding one x^2 to three x^2's.

$$3x^2 + x^2$$

$$(x^2 + x^2 + x^2) + x^2$$

$$x^2 + x^2 + x^2 + x^2$$

$$4x^2$$

Now let's subtract x^2 from $3x^2$.

$$3x^2 - x^2$$

$$(x^2 + x^2 + x^2) - x^2$$

$$x^2 + x^2 + x^2 - x^2$$

Now take a look at the last two terms in the expression we just found: $x^2 - x^2$. As we might guess, when we have $x^2 - x^2$ (when we want to subtract x^2 from x^2), we get 0. That's because no matter what number the x stands for, the number $-x^2$ is the opposite of x^2.

$$x^2 + x^2 + (x^2 - x^2)$$

$$x^2 + x^2 + 0$$

$$x^2 + x^2$$

$$2x^2$$

Multiplication and division

Multiplication and division of exponential expressions is a little different. For the purposes of multiplication and division, only the bases need to be the same in order for the terms to be alike. We don't need the exponents to be the same.

For example, if we want to multiply x^4 by x^5, we can do it because the bases are the same, even though the exponents are different.

$$x^4 \cdot x^5$$

$$(xxxx) \cdot (xxxxx)$$

$$(xxxxxxxxx)$$

$$x^9$$

From this example, we realize that we're really just adding the exponents when we multiply two terms with the same base. In other words, the **product rule** for multiplication is

$$x^a \cdot x^b = x^{a+b}$$

A related exponent rule tells us that, when two terms are multiplied and the exponents are equal, we can raise the product of the bases to that exponent.

$$a^n b^n = (ab)^n$$

Similarly, if we want to divide x^5 by x^2, we can do it because the bases are the same, even though the exponents are different.

$$\frac{x^5}{x^2}$$

$$\frac{x \cdot x \cdot x \cdot x \cdot x}{x \cdot x}$$

The factor that's common to the numerator and the denominator is $x \cdot x$, so we'll divide top and bottom by $x \cdot x$.

$$\frac{(x \cdot x \cdot x \cdot x \cdot x) \div (x \cdot x)}{(x \cdot x) \div (x \cdot x)}$$

$$\frac{x \cdot x \cdot x}{1}$$

$$x^3$$

From this example, we realize that we're really just subtracting the exponents when we divide two exponential expressions with the same base. In other words, the **quotient rule** for division is

$$\frac{x^a}{x^b} = x^{a-b}$$

Let's do another example.

Example

Use the quotient rule for exponents to simplify the expression.

$$\frac{x^4}{x^3}$$

The base of the expression in the numerator is x, and the base of the expression in the denominator is x, which means that the bases are the same, so we can use the quotient rule for exponents. We'll subtract the exponent in the denominator from the exponent in the numerator, keeping the base the same.

$$\frac{x^4}{x^3} = x^{4-3} = x^1 = x$$

Remember, the quotient rule works only with like bases, so

$$\frac{y^3}{x^2}$$

can't be simplified, because the bases y and x aren't the same, so we can't use the quotient rule.

Power rule for exponents

In this section we're going to dive into the power rule for exponents. Think about this one as the "power to a power" rule. This rule tells us what happens when we raise one exponent to another exponent.

The trick to these problems is to get back to the basics of exponents and remember that the exponent simply tells us how many times to multiply the base by itself. So if we're given

$$(3^2)^3$$

it means that we're supposed to multiply 3^2 by itself 3 times, since 3^2 is the base and 3 is the exponent. So we could rewrite the expression as

$$(3^2)(3^2)(3^2)$$

From here, we remember that when we multiply exponential expressions with the same base, we add the exponents. Since the base of each factor of 3^2 is 3, all our bases are the same, so we just add the exponents and we get

$$3^{2+2+2}$$

$$3^6$$

But this is the long way of expanding the expression $(3^2)^3$. What we actually want to do is use the power rule for exponents. The power rule tells us that when we raise one exponent to another, we can just multiply the exponents. In $(3^2)^3$, the first exponent is 2 and the second exponent is

3. The power rule tells us that we can just multiply those exponents and get $2 \cdot 3 = 6$, which means that

$$(3^2)^3 = 3^6$$

In general, we can write the **power rule** as

$$(a^m)^n = a^{mn}$$

Let's do some examples with the power rule.

Example

Use the power rule for exponents to simplify the expression.

$$(2^2)^4$$

To use the power rule, we just multiply the exponents.

$$2^{2 \cdot 4}$$

$$2^8$$

$$256$$

We'll try one more example.

Example

Use the power rule for exponents to simplify the expression.

$$(3^2)^2$$

We can apply the power rule and multiply the exponents.

$$3^{2(2)}$$

$$3^4$$

$$81$$

Negative and other exponent rules

Now we'll look at a few more exponent rules, starting with the negative exponent rule.

Negative exponent rule

The **negative exponent rule** tells us that, if the exponent is negative, it can be change into a positive exponent by taking the reciprocal.

$$a^{-m} = \frac{1}{a^m} \text{ and } a^m = \frac{1}{a^{-m}}$$

For example,

$$2^{-5} = \frac{1}{2^5} = \frac{1}{32}$$

And we can still use all the rules we learned earlier for addition and subtraction, and multiplication and division with exponents, even when we have negative. In fact, let's do one more example where we apply the quotient rule, even with a negative exponent.

Example

Simplify the rational expression.

$$\frac{x^3}{x^{-2}}$$

It doesn't matter that the exponent in the denominator is negative. We can still use the quotient rule since the bases are the same, and subtract the exponent in the denominator from the exponent in the numerator.

$$\frac{x^3}{x^{-2}} = x^{3-(-2)} = x^{3+2} = x^5$$

We also know that, when we have the quotient of two power terms, and the exponents are equal, that we can rewrite the expression as

$$\frac{a^n}{b^n} = \left(\frac{a}{b}\right)^n$$

as long as b is non-zero. Similarly,

$$\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$$

Let's do an example with these rules.

Example

Rewrite the expression.

$$\frac{x^3}{y^3}$$

The base of the expression in the numerator and the base of the expression in the denominator are not the same, so we can't use the quotient rule for exponents. But the exponents are the same, so

$$\frac{x^3}{y^3} = \left(\frac{x}{y}\right)^3$$

Let's do an example with negative exponents.

Example

Rewrite the expression.

$$\frac{x^{-5}}{y^{-5}}$$

The base of the expression in the numerator and the base of the expression in the denominator aren't the same, so we can't use the quotient rule for exponents. But the exponents are the same, so

$$\frac{x^{-5}}{y^{-5}} = \left(\frac{x}{y}\right)^{-5}$$

Then to make the exponent positive, we can take the reciprocal of the base.

$$\left(\frac{x}{y}\right)^{-5} = \left(\frac{y}{x}\right)^5$$

Zero exponent rule

The **zero exponent rule** tells us that, if a real number has an exponent equal to 0, then its value is equal to 1.

$$a^0 = 1$$

For example,

$$2^0 = 1$$

Radicals

We can think about radicals (also called "roots") as the opposite of exponents.

We already know that the expression x^2 with the exponent of 2 means "multiply x by itself two times". The opposite operation would be "what do we have to multiply by itself two times in order to get x?" That's where radicals come in. If we see

$$\sqrt{x}$$

it means "the number we have to multiply by itself to get x." The symbol that contains the x is the **radical sign**, and the expression inside the symbol - in this case x - is the **radicand**.

Instead of saying that an expression is *inside* the radical sign, however, we usually say that it's *under* the radical sign. When we see $\sqrt{x}$, we can call it "the square root of x." But there are other kinds of roots of x too (which are indicated by little numbers tucked into the left side of the radical sign):

$\sqrt[3]{x}$ "the cube root of x"

$\sqrt[4]{x}$ "the fourth root of x"

$\sqrt[5]{x}$ "the fifth root of x"

"The cube root of x" means the number that's multiplied by itself three times in order to get x; "the fourth root of x" means the number that's multiplied by itself four times in order to get x, and so on.

Since roots are the opposite operation of exponents, we can convert between roots and exponents. For example, taking the square root of x is the same as raising x to the $1/2$ power. To see this, apply the power rule for exponents:

$$(x^{\frac{1}{2}})^2 = x^{(\frac{1}{2}\cdot 2)} = x^1 = x$$

Here's how to convert between roots and exponents.

$\sqrt{x}$	is the same as	$x^{\frac{1}{2}}$
$\sqrt[3]{x}$	is the same as	$x^{\frac{1}{3}}$
$\sqrt[4]{x}$	is the same as	$x^{\frac{1}{4}}$
$\sqrt[5]{x}$	is the same as	$x^{\frac{1}{5}}$

Now let's do an example where we simplify a radical.

Example

Simplify the radical expression.

$$\sqrt{9}$$

We're taking the square root of 9, which means we need to figure out what number we have to multiply by itself in order to get 9.

If we multiply 3 by itself, we get 9, which means that the square root of 9 is 3. So we can say

$$\sqrt{9} = 3$$

If we're given a number that x stands for, we want $\sqrt{x}$ to represent only one number - and a number that everyone will agree on. Notice, however, that we can get 9 not only by multiplying 3 by itself, but also by multiplying -3 by itself:

$$(+3)(+3) = 9 = (-3)(-3)$$

The way we get around this is that everyone agrees that both $\sqrt{9}$ and $9^{\frac{1}{2}}$ mean the positive number that we can multiply by itself in order to get 9:

$$\sqrt{9} = 3 = 9^{\frac{1}{2}}$$

Also

$$\sqrt{0} = 0 = 0^{\frac{1}{2}}$$

because 0 is the only number that when multiplied by itself gives 0. Now notice that there is no negative number that when multiplied by itself gives a negative number. (A positive number multiplied by itself is positive, 0 multiplied by itself is 0, and a negative number multiplied by itself is positive.) So $\sqrt{x}$ and $x^{\frac{1}{2}}$ are undefined if x is negative.

If x stands for any positive number, there's one and only one positive number that when multiplied by itself gives x. So $\sqrt{x}$ and $x^{\frac{1}{2}}$ are defined, and they represent that "one and only one positive number."

Sometimes the argument of a radical is not a perfect square, although it may contain a perfect square within its factors. When we're trying to simplify roots, we need to factor the radicand and take out any factor that's a perfect square. The rule we apply is

$$\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$$

We'll factor the radicand, then separate the perfect square factor into its own radical, and then simplify that radical individually.

Let's do an example.

Example

Simplify the radical.

$$\sqrt{40}$$

The radicand 40 is not itself a perfect square, but we can factor 40 as $4 \cdot 10$.

$$\sqrt{4 \cdot 10}$$

Now we can separate each factor into its own radical,

$$\sqrt{4}\sqrt{10}$$

and then $\sqrt{4} = 2$, so $\sqrt{40}$ simplifies to

$$2\sqrt{10}$$

We know the simplification is done, because the remaining radicand 10 is not a perfect square, nor does it contain any factors that are perfect squares.

In later sections, we'll look at the specific rules we use to handle radical expressions.

Adding and subtracting radicals

In this section we'll talk about how to add and subtract terms containing radicals.

When we have two terms that contain the same type of root (the radical in both terms is a square root, the radical in both terms is a cube root, etc.) and identical radicands (the expressions under the radical signs in the two terms are the same), they are like terms, and adding and subtracting is really simple.

Example

Simplify the expression.

$$\sqrt{3}+4\sqrt{3}$$

Here, the two terms contain the same type of root (a square root), and the radicands are the same, which means they're like terms, and we can just do the addition. The coefficient of $\sqrt{3}$ in the first term is understood to be 1, so we can rewrite the expression as

$$1\sqrt{3}+4\sqrt{3}$$

Now we can think of this as follows: "If we have one square root of 3, and we add to that four square roots of 3, then how many square roots of 3 (total) do we have?" Well, one of them plus four of them is five of them, total. So we get

$$5\sqrt{3}$$

If the radicals aren't the same, then they aren't like terms, and we can't combine them. So if we have $\sqrt{2}$ and $\sqrt{3}$, we can't add them or subtract one of them from the other. The radicands are different (one is 2 and the other is 3), so they aren't like terms and we can't combine them.

Example

Find the difference.

$$\sqrt{5}-\sqrt{3}$$

Because the radicands are different, $\sqrt{5}$ and $\sqrt{3}$ aren't like terms, so we can't add them or subtract one of them from the other. Therefore, we can't simplify this expression at all.

It isn't always true that terms with the same type of root but different radicands can't be added or subtracted. Sometimes we can simplify a radical within itself, and end up with like terms.

Example

Find the sum.

$$\sqrt{2} + \sqrt{8}$$

At first it looks as if we can't combine these terms, since the radicands are different, and therefore they're not like terms. But $\sqrt{8}$ can be simplified.

$$\sqrt{8}$$

$$\sqrt{4 \cdot 2}$$

When a radicand can be factored, the radical can be expressed as a product of radicals with the individual factors as the radicands, so here we get

$$\sqrt{4} \cdot \sqrt{2}$$

$$2\sqrt{2}$$

Which means that the sum $\sqrt{2} + \sqrt{8}$ can be rewritten as

$$\sqrt{2} + 2\sqrt{2}$$

And now we have

$$1\sqrt{2} + 2\sqrt{2}$$

$$3\sqrt{2}$$

So when it comes to adding and subtracting radicals, we want to remember that only like terms can be combined, which means that those terms have to contain the same type of root and their radicands have to be the same.

But even when the radicands are different, sometimes one or both of the radicals can be rewritten in a way that will actually make the radicands the same. So watch out for opportunities to rewrite the radicals in one or both of the terms before concluding that they're not like terms and can't be combined.

Multiplying radicals

When we multiply two radicals with the same type of root (both square roots, both cube roots, and so on), we simply multiply the **radicands**, which are the expressions under the radical signs, and put the product under a radical sign.

Let's do an example, first with two different radicands.

Example

Find the product.

$$\sqrt{3}\sqrt{2}$$

When we see two radicals next to each other like this, it means we're supposed to multiply them.

To multiply two square roots, we just multiply the radicands and put the product under a radical sign. That is, the product of two square roots is equal to the square root of the product of the radicands.

$$\sqrt{3 \cdot 2}$$

$$\sqrt{6}$$

It's helpful to remember that we can use this rule for multiplication of radicals to go in the opposite direction as well. In other words, if we're given $\sqrt{6}$, we can factor the 6 as $3 \cdot 2$, then rewrite $\sqrt{6}$ as $\sqrt{3 \cdot 2}$, and finally rewrite the square root of the product (of 3 and 2) as the product of their square roots.

$$\sqrt{6}$$

$$\sqrt{3 \cdot 2}$$

$$\sqrt{3}\sqrt{2}$$

Sometimes rewriting a radical as a product of radicals can help us solve a problem we're working on, so it's helpful to remember that we can go both ways with this rule for multiplication of radicals.

The product of square roots theorem tells us that, if m and/or n are nonnegative real numbers, then

$$\sqrt{m}\sqrt{n} = \sqrt{mn} \qquad \text{and} \qquad \sqrt{mn} = \sqrt{m}\sqrt{n}$$

Let's do another example where we multiply two square roots, this time with equal radicands.

Example

Find the product.

$$\sqrt{5}\sqrt{5}$$

Let's follow the same steps we did before, where we rewrite the product of the square roots as the square root of the product of the radicands.

$$\sqrt{5 \cdot 5}$$

$$\sqrt{25}$$

But now we need to realize that $\sqrt{25}$ is just 5, since 5 multiplied by itself is equal to 25. So we can write $\sqrt{25}$ as just 5.

Which brings us to the point that when we multiply two identical square roots, the result is just the same as the radicand. So

$\sqrt{5}\sqrt{5}$	is the same as	5
$\sqrt{7}\sqrt{7}$	is the same as	7
$\sqrt{12}\sqrt{12}$	is the same as	12

Similarly, when we have a product of three identical cube roots, we get the number that's equal to the radicand,

$$\sqrt[3]{-21}\sqrt[3]{-21}\sqrt[3]{-21} = -21$$

and when we have four identical fourth roots, we get the number that's equal to the radicand.

$$\sqrt[4]{13}\sqrt[4]{13}\sqrt[4]{13}\sqrt[4]{13} = 13$$

And so on for higher-numbered roots.

Let's do one more example of multiplication of square roots - this time where one of them has a coefficient other than 1.

Example

Find the product.

$$(4\sqrt{2}) \cdot \sqrt{3}$$

The fact that we have a coefficient other than 1 doesn't change anything. We can leave the coefficient in front and multiply just the square roots.

$$(4\sqrt{2}) \cdot \sqrt{3}$$

$$4(\sqrt{2} \cdot \sqrt{3})$$

$$4\sqrt{2 \cdot 3}$$

$$4\sqrt{6}$$

Dividing radicals

Dividing radicals is really similar to multiplying radicals. Remember that when we multiply radicals with the same type of root, we just multiply the radicands and put the product under a radical sign. So

$$\sqrt{3}\sqrt{2}$$

$$\sqrt{3 \cdot 2}$$

$$\sqrt{6}$$

We'll do the same when we divide radicals, because when we divide one radical by another with the same type of root, we just divide the radicands and put the quotient under a radical sign.

Example

Find the quotient.

$$\frac{\sqrt{6}}{\sqrt{3}}$$

Since we're dividing one square root by another, we can simply divide the radicands and put the quotient under a radical sign. That is, the quotient of square roots is equal to the square root of the quotient of the radicands.

$$\sqrt{\frac{6}{3}}$$

$$\sqrt{2}$$

Just as with multiplication of radicals, we can reverse this process and go the other way. So, if we wanted to, we could recognize that 2 is the same as $6/3$, and we could rewrite $\sqrt{2}$ as the square root of $6/3$, and finally rewrite that as the quotient of the square roots of 6 and 3.

$$\sqrt{2}$$

$$\sqrt{\frac{6}{3}}$$

$$\frac{\sqrt{6}}{\sqrt{3}}$$

Let's do an example where the radicands are the same.

Example

Simplify the radical expression.

$$\frac{\sqrt{5}}{\sqrt{5}}$$

We could follow the steps we did in the previous example.

$$\sqrt{\frac{5}{5}}$$

$$\sqrt{1}$$

$$1$$

This should remind us that when the roots are of the same type and the radicands are equal, the result will always be 1, because anything divided by itself is 1 (except, of course, that 0 divided by itself is undefined!).

Mathematicians don't like to end up with a radical in the denominator of a fraction. When there's a square root in the denominator, we can turn it into a rational number by multiplying the numerator and denominator of the fraction by that square root and then simplifying. That process is known as **rationalizing the denominator**, because the result has a rational number in the denominator.

Example

Rationalize the denominator.

$$\frac{7}{\sqrt{5}}$$

To get rid of the radical in the denominator, we'll multiply the numerator and denominator by $\sqrt{5}$.

$$\frac{7\sqrt{5}}{\sqrt{5}\sqrt{5}}$$

Now we have $\sqrt{5}\sqrt{5}$ in the denominator, which is equal to 5, so we get

$$\frac{7\sqrt{5}}{5}$$

Radical expressions

When we work with radicals, we'll run into all different kinds of radical expressions, and we'll want to use the rules we've learned for working with radicals in order to simplify them. This could include any combination of addition, subtraction, multiplication, and division of radicals.

Just so we remember, here are some rules for radicals that we'll use:

$$\sqrt{ab} = \sqrt{a}\sqrt{b}$$

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$$

$$\sqrt{a}\sqrt{a} = a$$

Let's do an example where we have to simplify a radical expression.

Example

Simplify the radical expression.

$$3\sqrt{2} + 6\sqrt{8} - \sqrt{18}$$

In order to add or subtract terms containing square roots, the radicands have to be the same. Otherwise, those terms aren't like terms, and we can't simplify the sum or difference.

Even though the radicands in the square roots in this expression aren't the same, we may be able to simplify some of them in order to get identical radicands. Since 8 and 18 can be factored as $4 \cdot 2$ and $9 \cdot 2$, respectively, we could rewrite the expression as

$$3\sqrt{2} + 6\sqrt{4 \cdot 2} - \sqrt{9 \cdot 2}$$

We know that the square root of a product is equal to the product of the square roots with the individual factors as the radicands. So we can rewrite $\sqrt{4 \cdot 2}$ and $\sqrt{9 \cdot 2}$ as $\sqrt{4}\sqrt{2}$ and $\sqrt{9}\sqrt{2}$, respectively. Also, 4 and 9 are perfect squares, so we can take their square roots.

$$3\sqrt{2} + 6\sqrt{4}\sqrt{2} - \sqrt{9}\sqrt{2}$$

$$3\sqrt{2} + 6(2)\sqrt{2} - 3\sqrt{2}$$

$$3\sqrt{2} + 12\sqrt{2} - 3\sqrt{2}$$

Now the radicands in all three terms are the same, so all three terms are like terms, and can be combined.

$$(3 + 12 - 3)\sqrt{2}$$

$$12\sqrt{2}$$

Powers of 10

We want to start getting comfortable with powers of 10, since we'll be using them all the time for scientific notation. When we talk about powers of 10, we mean the result of raising 10 to some power:

$$10^0 = 1$$

$$10^1 = 10$$

$$10^2 = 100$$

$$10^3 = 1{,}000$$

$$10^4 = 10{,}000$$

etc.

Notice how the power (exponent) on the 10 is the same as the number of 0's in the power of 10 (in the number to the right of the equals sign). For example, the exponent in 10^4 is 4, and there are four 0's in $10{,}000$.

The key thing to remember then is that, when we multiply a number by a power of 10, all we do is count the 0's in the power of 10, and then move the decimal point that many places to the right in the other number, and that gives us the product.

Example

Find the product.

$$67 \times 1{,}000$$

Since we're multiplying by a power of 10, we need to count the 0's in the power of 10. There are three 0's in $1{,}000$, so we need to move the decimal point in 67 three places to the right. Since 67 has no decimal point (and so it looks as if there's no way to move a decimal point to the right), we have to first put a decimal point to the right of the 7 (which gives $67.$), and then (so that we'll be able to move the decimal point three places to the right) we put three 0's to the right of the decimal point.

At that stage in the process, we have 67.000, so when we move the decimal point three places to the right, we get

$$67{,}000$$

We can divide by powers of 10 just as easily. When we multiply by a power of 10, we move the decimal point to the right, but when we divide by a power of 10 (which is equivalent to multiplying by the number we would get if we raised 10 to the corresponding negative power), we move the decimal point to the left.

Notice the pattern in powers of 10:

$$10^0 = 1$$

$$10^{-1} = 0.1$$

$$10^{-2} = 0.01$$

$$10^{-3} = 0.001$$

etc.

Let's do an example.

Example

Do the division.

$$4.3 \div 100$$

There are two zeros in 100, and since we're dividing, we need to move the decimal point two places to the left. Since 4.3 has only one digit to the left of the decimal point (and so it looks as if there's no way to move the decimal point two places to the left), we have to first put a 0 to the left of the 4 (to give us a total of two digits to the left of the decimal point).

At that stage, we have 04.3, so when we move the decimal point two places to the left, we get $.043$. In any decimal number, however, we always want to have at least one digit to the left of the decimal point. Since there is no such digit in $.043$, the part that's to the left of the decimal point is understood to be 0, so we put a 0 to the left of the decimal point and get

$$0.043$$

Let's do another example.

Example

Simplify the expression.

$$510.75 \times 10^{-2}$$

When the power of 10 is negative, we move the decimal point to the left by the number of places indicated by the exponent. Since the exponent is negative 2, we move the decimal point to the left two places, and we get

$$510.75 \times 10^{-2}$$

$$5.1075$$

So, regardless of the number we start with, if we multiply that number by a power of 10, then we move the decimal point to the right (by the number of places equal to the number of 0's in that power of 10). If we divide a number by a power of 10, then we move the decimal point to the left (by the number of places equal to the number of 0's in that power of 10).

Scientific notation

Scientific notation is a tool we use to write numbers that would otherwise be really tedious to express. Kind of like the way exponents allow us to express

$$x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x$$

as

$$x^9$$

which is a lot easier, scientific notation allows us to express

$$964{,}000{,}000{,}000{,}000{,}000{,}000{,}000$$

as

$$9.64 \times 10^{23}$$

which is also a lot easier. Notice how the number we started with was a huge, huge number. It's actually hard for us to know what that number is, because we have to spend time counting the 0's just to figure out its actual value.

But with scientific notation, we're able to express that number without all the 0's and write a simpler number that quickly tells us how big the number is.

So scientific notation can be used to express really, really big numbers like that one, but it can also be used to express really, really small numbers like

$$0.000000000000000000000000782$$

A number in scientific notation always has two parts: a decimal number, which comes first, and a power of 10 written in exponential form, which comes second. There are some requirements that these two parts must satisfy.

The decimal number can be positive or negative, and it has to have exactly one digit to the left of the decimal point. Also, in proper scientific notation, the digit to the left of the decimal point cannot be 0.

The exponent (in the power of 10 written in exponential form) can be positive, negative, or 0, but the base has to be 10.

Here are some examples of numbers that are in proper scientific notation:

$$7.1805 \times 10^6$$

$$4.32 \times 10^{-8}$$

$$-6.9 \times 10^7$$

$$-5.777 \times 10^{-9}$$

And here are some examples of numbers that aren't in proper scientific notation:

21.4×10^7 (more than one digit to the left of the decimal point)

0.382×10^{-4} (the digit to the left of the decimal point is 0)

2.698×8^3 (the base in the exponential expression is 8, not 10)

Let’s use an example to talk about how to express a really big number or really small number in scientific notation.

Example

Express the number in scientific notation.

$$0.0000000000000000000000782$$

As written, this number is really difficult to understand, because it’s hard to know how many 0’s are included after the decimal point without spending time to carefully count them. So this is a great number to express in scientific notation.

When we take a number in decimal form and express it in proper scientific notation, the digits we include in the decimal number part are called the “**significant figures**.”

The significant figures in the number in this example are the 7, the 8, and the 2. We say that 7 is the first significant figure, 8 is the second significant figure, and 2 is the third significant figure. We want to move the decimal point until it’s just to the right of the first significant figure. Which means we want it to be just to the right of the 7.

Now the question is, “How many places do we need to move the decimal point in order to get it just to the right of the 7?” If we count all the 0’s to the right of the decimal point, and the 7, in

$$0.0000000000000000000000782$$

we can see that we need to move the decimal point 25 places to the right (one for each of the twenty-four 0's to the right of the decimal point, and one more for the 7). This means that in scientific notation the decimal number part of this number will be 7.82, but then because we moved the decimal point 25 places to the right, we have to multiply the 7.82 by 10^{-25}, so we get

$$7.82 \times 10^{-25}$$

Now we might wonder how we know that the exponent has to be negative (that it's -25, and not 25). Well, here's the general rule for the sign of the exponent (in the power of 10 written in exponential form) when we take a number in decimal form and express it in scientific notation:

> If we have to move the decimal point to the right (to get exactly one nonzero digit to the left of the decimal point), the exponent is negative.
>
> If we have to move the decimal point to the left, the exponent is positive.
>
> If we don't have to move the decimal point at all (if there is already exactly one digit to the left of the decimal place, and that digit isn't 0), the exponent is 0.

Since we moved the decimal point 25 places to the right in this example, the exponent has to be -25.

$$7.82 \times 10^{-25}$$

Multiplying scientific notation

When we multiply two numbers in scientific notation, we want to follow the same set of steps each time.

1. Multiply their decimal numbers.

2. Multiply their powers of 10. By the rules of exponents, we add the exponents when we do this.

3. Express the results together in proper scientific notation.

Let's do an example where we work through these steps.

Example

Find the product.

$$(3.4 \times 10^{-6})(2.14 \times 10^{13})$$

Let's follow the steps we outlined above. First, we'll multiply the decimal numbers.

$$3.4 \times 2.14 = 7.276$$

Now we'll multiply the powers of 10, adding the exponents.

$$10^{-6} \times 10^{13} = 10^{-6+13} = 10^{7}$$

Next, we'll multiply these two values together.

$$7.276 \times 10^7$$

This is already in proper scientific notation so we can leave it as is. But if there were two or more digits to the left of the decimal point, or if we had a 0 in the ones (units) place, the decimal point, then we'd have to express it in proper scientific notation.

Dividing scientific notation

Dividing one number by another in scientific notation is really similar to multiplying two numbers in scientific notation, because we're basically following the same steps.

1. Divide their decimal numbers.

2. Divide their powers of 10. By the rules of exponents, we subtract the exponents when we do this.

3. Express the results together in proper scientific notation.

Let's do an example.

Example

Find the quotient.

$$(3.4 \times 10^{-6}) \div (2.14 \times 10^{13})$$

Let's follow the steps we outlined above. First, we'll divide the decimal numbers.

$$3.4 \div 2.14 = 1.588785$$

Now we'll divide the powers of 10, subtracting the exponents.

$$10^{-6} \div 10^{13} = 10^{-6-13} = 10^{-19}$$

Next, we'll write down the multiplication problem in which the numbers to be multiplied are the results of those two divisions.

$$1.588785 \times 10^{-19}$$

This is already in proper scientific notation, because there's just one nonzero digit to the left of the decimal point, so we can leave it as is. But if there were two or more digits to the left of the decimal point, or if we had a 0 in the ones (units) place, then we'd have to express it in proper scientific notation.

Multiplying and dividing scientific notation

Sometimes we'll have problems where we're asked to multiply and divide scientific notation at the same time. It doesn't matter if we do the multiplication first and then the division, or the division first and then the multiplication.

Either way, our process for multiplying scientific notation doesn't change, and our process for dividing scientific notation doesn't change.

Remember, when we multiply scientific notation we multiply the decimal numbers, then multiply the powers of 10, adding the exponents. And when we divide scientific notation we divide the decimal numbers, then divide the powers of 10, subtracting the exponents. And in both cases, we need to make sure to express the answer in proper scientific notation.

Let's do an example.

Example

Simplify the expression.

$$\frac{(2.3 \times 10^{-4})(6.4 \times 10^{12})}{4.2 \times 10^{10}}$$

There are a few ways that we could simplify this expression, but we'll choose to do the multiplication in the numerator first, and then once we have the numerator simplified we'll divide by the denominator.

Starting in the numerator, we'll multiply the decimal numbers.

$$2.3 \times 6.4 = 14.72$$

Then we'll multiply the powers of 10 from the numerator.

$$10^{-4} \times 10^{12} = 10^{-4+12} = 10^{8}$$

Which means that the numerator simplifies to

$$14.72 \times 10^{8}$$

and the full expression is now

$$\frac{14.72 \times 10^{8}}{4.2 \times 10^{10}}$$

Technically, the number in the numerator isn't in proper scientific notation, since in 14.72 there are two digits to the left of the decimal point, but we don't have to deal with that yet. We can divide by the denominator first, and then deal with it at the end if we need to.

Let's divide the decimal numbers.

$$14.72 \div 4.2 \approx 3.5$$

Remember that the squiggly lines $\approx$ mean that 3.5 is an approximate result. If we were to give that result to 5 decimal places, it would be 3.50476, so by rounding it to the nearest tenth, we get 3.5.

Now we'll divide the powers of 10.

$$10^{8} \div 10^{10} = 10^{8-10} = 10^{-2}$$

Multiplying our results, we get

$$3.5 \times 10^{-2}$$

Since this is already in proper scientific notation, there's nothing else we need to do.

Estimating scientific notation

Most of the time we'll be finding exact values with scientific notation, but sometimes it's nice just to get a quick estimate of the value of an expression.

This is a simple process where we round the decimal numbers, and then do our multiplication and/or division. Obviously we don't get a result that's quite as accurate, but the process is a little quicker, so it's a trade-off.

Example

Estimate the value of the expression.

$$\frac{(2.3 \times 10^{-4})(6.4 \times 10^{12})}{4.2 \times 10^{10}}$$

First, we'll round each of the decimal numbers to the nearest whole number.

$$\frac{(2 \times 10^{-4})(6 \times 10^{12})}{4 \times 10^{10}}$$

Then we'll express the fraction as the product of two fractions, one for the whole numbers and the other for the powers of 10.

$$\frac{2 \times 6}{4} \cdot \frac{10^{-4} \times 10^{12}}{10^{10}}$$

Now we'll simplify.

$$\frac{12}{4} \cdot \frac{10^{-4+12}}{10^{10}}$$

$$3 \cdot \frac{10^{8}}{10^{10}}$$

$$3 \cdot 10^{8-10}$$

$$3 \times 10^{-2}$$

If we were to compute the exact value of this expression with a calculator and then round the answer to the nearest tenth (to one decimal place), we would get 3.5×10^{-2}. Not only was our estimate not too far off, but we were able to arrive at the estimate faster than we could have if we'd computed the exact value.

We can also use scientific notation to estimate a product of several numbers. Let's look at an example of that.

Example

Find the product.

$$(13)(476)(52{,}450)(975)(143)$$

Instead of computing the exact value, we can use scientific notation to get an estimate of it. We'll do that by first expressing all the numbers in scientific notation.

$$(1.3 \times 10^1)(4.76 \times 10^2)(5.245 \times 10^4)(9.75 \times 10^2)(1.43 \times 10^2)$$

Then we'll round each of the decimal numbers to the nearest whole number.

$$(1 \times 10^1)(5 \times 10^2)(5 \times 10^4)(10 \times 10^2)(1 \times 10^2)$$

We'll express the product as the product of two products, one for the whole numbers and the other for the powers of 10, and then we'll simplify.

$$(1 \times 5 \times 5 \times 10 \times 1)(10^1 \times 10^2 \times 10^4 \times 10^2 \times 10^2)$$

$$250 \times 10^{1+2+4+2+2}$$

$$250 \times 10^{11}$$

Write this estimate in scientific notation.

$$2.5 \times 10^2 \times 10^{11}$$

$$2.5 \times 10^{2+11}$$

$$2.5 \times 10^{13}$$

This isn't an exact value for the product we were asked to find, but it's a pretty quick way to get a decent estimate.

W W W . K R I S T A K I N G M A T H . C O M