An Investigation into Problem Solving in Integral Calculus

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Brief Abstract

- **Motivation**: importance of understanding and improving tertiary education and problem-solving skills have grown to new heights
- **Aim**: investigate initial problem-solving strategies employed by integral calculus students
- **Method**: in-person, pair-wise interviews where participants sort integrals by technique used to solve it
- **Results**: suggest successful technique choice is not a cause of error; point to other sources of error in the overall problem-solving process

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Introduction

Modern Approach to Education

- Shift from traditional memorization and recitation to **student self-discovery** [Yair, 2008]
- Aims to help produce professionals that are "**reflexive**" and **strong problem-solvers** [Santos et al., 2019], particularly in **STEM disciplines**
- There must be continuous **effort to keep improving** education

Motivation for this Investigation

- Mathematics courses are **prerequisites** for many STEM disciplines
- Also an **excellent setting** for observing problem-solving
- Inspiration from **personal experience** with students struggling to successfully and correctly solve calculus problems
- Combine all of these aspects by observing student approaches to **solving integral calculus problems**

Background

Problem-Based Learning

- First implemented in medical education at McMasters University in 1970s [Hung et al., 2008], has spread to various other areas
- Leads to learning centered around problem-solving; **important to study** this new learning model
- In math education specifically, ”the development of students’ problem solving abilities is a **primary objective**” [Lester, 1994]

Established Problem-Solving Research

- Particularly active from 1970s to early 1980s
- Remains active in higher education research specifically
- Concept of **flexibility in problem-solving** [Star and Rittle-Johnson, 2008] is particularly relevant
- Defined as “knowledge [of both] multiple strategies and the relative efficiencies of those strategies”
- Flexibility is **important to choosing techniques** for integral calculus problems

Tools for Studying Problem-Solving

- Many studies produce qualitative results; need a reliable **way to analyze** them
- Use implementation of both **theoretical and conceptual frameworks**
- In particular, theoretical frameworks are built by ”theories expressed by experts” [Kivunja, 2018]
- Theoretical frameworks “entered mathematics education **at its very beginning** as an academic field” [Reid, 2014]

Our Tools for Studying Problem-Solving

- Concerned with problem-solving in **integral calculus** specifically
- Use problems common to this subject known as **integrals**
- Observe participants analyzing integrals through **pair-wise interviews**
- Analyze problem-solving strategies by referencing predetermined **theoretical framework**

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Theoretical Framework

Foundational Components

1. **Conceptual Image** [Tall and Vinner, 1981], [Jones, 2013]
    - Contains **concept definition** and **conceptual schema**
    - The **definition** is a "form of words used to specify that concept"; the **schema** lists details important for understanding the concept
2. **Symbolic Form** [Sherin, 2001]
    - "Associates a simple conceptual schema with an **arrangement of symbols**" (p. 482)
    - Great for responses referencing **visual aspects** of a problem
3. **Cognitive Resources** [Hammer, 2000]
    - Defined as "**fine-grained elements** of knowledge in a person's cognition"
    - Many small details could influence problem-solving approaches; this is a general category for them

How the Framework Will Be Used

- Need **standards** for general integration and each technique; the framework provides a standard
- Symbolic forms for what problems **typically** look like for the technique used to integrate it
- Concept images for each technique including associated cognitive resources
- Same process for **all** integration problems in general

Method

Construction of the "Standards"

- All theoretical frameworks for general integration and each technique were **determined by researchers** piloting this investigation
- Symbolic forms consider **all visual parts** of a problem needed to correctly choose a technique
- Conceptual schema consider **concepts** and **cognitive resources** used to justify using each technique
- Tables detailing symbolic forms and conceptual schema are listed under "List of Tables" in the paper

Example: Framework for General Integration

$$\int f(x)\,\mathrm{d}x$$

Here is the visual **symbolic form** for any general integration problem.

Example: Framework for General Integration

$$\int f(x)\,\mathrm{d}x$$

It consists of an integral symbol, a function, and a differential operator.

Example: Framework for General Integration

$$\int f(x)\,\mathrm{d}x$$

It consists of an integral symbol, a function, and a differential operator.

Example: Framework for General Integration

$$\int f(x)\,\mathrm{dx}$$

It consists of an integral symbol, a function, and a differential operator.

Example: Framework for General Integration

$$\int f(x)\,\mathrm{d}x$$

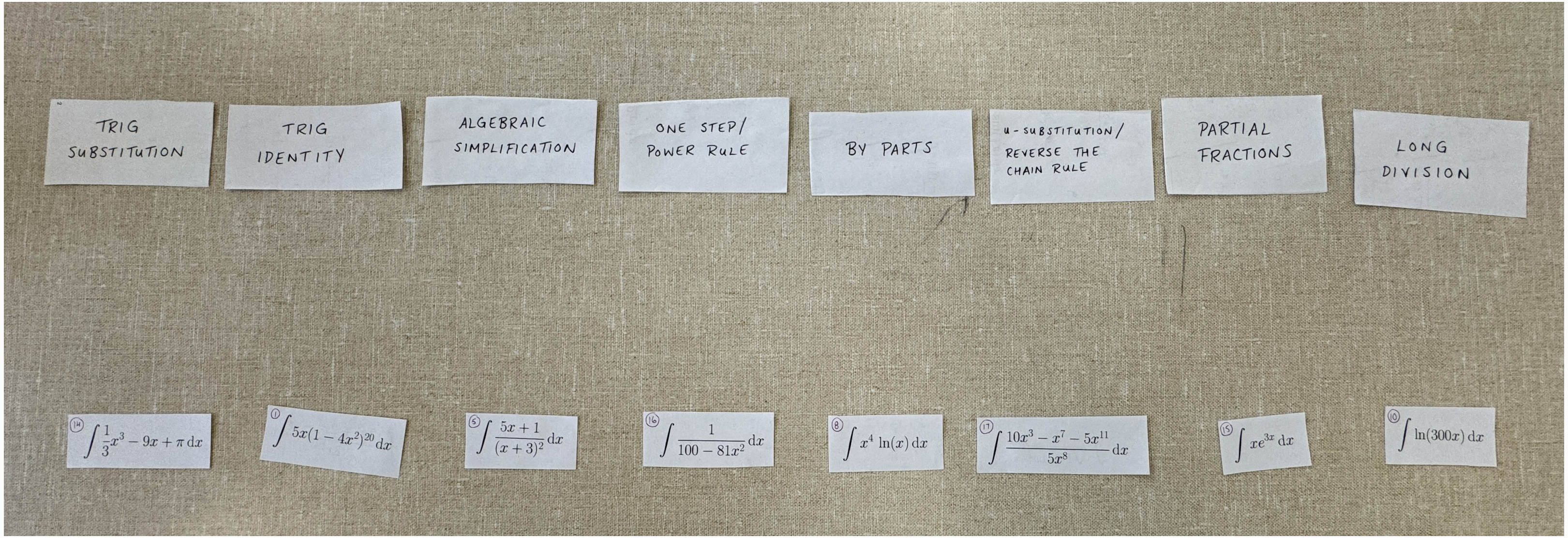
Symbolic Form	**Conceptual Schema**
integration operator	definite vs. indefinite
function	variable being integrated has differential applied to it
differential operator	

Table 1: Concept image for general integration problems.

Construction of the Interview

- Interviews were conducted **in-person**
- **Pairs of participants** were encouraged to perform the task together
- Researchers constructed a **template** for a standardized flow for questioning
- Participants performed sorting task then answered questions
- Primary areas of questioning include:
  - Confidence in initial sort
  - "Confident" and "unconfident" techniques
  - "Easy" and "hard" problems to sort
  - Why given "easy" and "hard" problems were such
  - After questioning, if they would like to re-sort any problems

Setup for Sorting Process



Additional Pieces of the Interview

- Sorting procedure used problems **printed on pieces of paper**; participants **physically moved** problems into categories
- Follow-up questions **depended** on participant response
- Audio from each interview was recorded and transcribed for analysis

Results

Overall, What Happened?

- Total of **three interviews** conducted
- **All but one problem was sorted correctly, across all interviews**
- All participants were **happy to answer** any follow-up questions
- Participants referenced **visual and conceptual aspects** of the problems and techniques when justifying
- Confidence in each integration technique **varied** between participants
- **5/6** said they were "unconfident" in **trigonometric identities**
- **4/6** said they were "unconfident" in **trigonometric substitution**
- One participant indicated "confident" **only for long division**

Problems Sorted in Each Interview

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1																				
2																				
3																				

Table 2: Visualization of specific problems sorted in each interview. Green indicates correctly sorted, yellow indicates incorrect initial sort with later correct sort, and white indicates the problem was not present for the sort. Note columns 2 and 19, indicating problems 2 and 19 appeared across all three interviews.

Participant Confidence Ratings

	1	2	3	4	5	6
Reverse Chain Rule/U-Substition						
Reverse Power Rule						
By Parts						
Algebraic Simplification						
Long Division						
Partial Fractions						
Trig Identities						
Trig Substitution						

Table 3: Visualization of participant confidence self-rating for integration techniques. Green indicates ”confident”, red indicates ”unconfident”, and white indicates no mention.

Analysis

Cool Things from Interview 1

- "This one I think is algebraic simplification because you can **break this up** into three and just divide – cancel powers."
- "You can **split up the fraction** so you have 12 over 3x"
- "There's **three [rules] for trig sub**: secant, sine and tangent,, and the tangent is the g(x) squared plus a squared, this is pretty clear [here]"
- "Something like algebraic simplification, you can cancel some terms, you can **make the whole equation a lot simpler**. Or long division where you have a...linear or exponential function in the bottom and another function on top you can go about separating out these terms and dividing them"

More Cool Things from Interview 1

- “If you’re looking at a problem, usually the higher power is **listed first**”
- “e is a constant...you see a letter and you think ‘oh this is **going to be complicated** in some way’ but because e is a constant the problem ends up being **a lot easier to solve than it looks**”
- “I probably would **not really know** how to solve [13] because the e does make it so complicated for me to understand”
- “We have **different** strengths and weaknesses”

Cool Things from Interview **2**

- “You **don’t need to break it up** anymore so it can’t be partial fractions”
- “I think you could do this with trig identities. [Because] I don’t know **how else you would do it** either”
- “Can we have paper?” ”No”
- “Sometimes my algebra skills are a little **rusty**”
- “I don’t know trig”
- “I’m looking at it and sometimes I feel like I’m just **jumping too quickly**...to natural log”

More Cool Things from Interview **2**

- "I think of u-sub **almost instantly**" - Person A
- "U-sub is kind of the **second or third thing** I think about" - Person B

- "What's going on with 11? **Can you break that up?**" "What kind of cancellation are you looking for?" "The coefficient of the variable happened to come out when the variable was to the first [power]. It essentially became **coefficient over coefficient** and canceled out to 1. **I don't feel super confident** in that [mindset] anymore"

Cool Things from Interview **3**

- "When you take **the derivative of one** of these it **equals the other**"
- "I think you could factor it but **I don't think it would help**"
- "Isn't there a square root on trig sub?" "Yeah"
- "Because if you expand this then **it's not helpful** because you can't cancel anything out. So you could but **it doesn't really do anything**"
- "[I am unconfident in] long division because **I was never taught it**"

More Cool Things from Interview **3**

- "We'd do really good on tests together" "Our **strengths are literally opposites**"
- "There's two squared [quantities]...that you could **easily use trig sub** for"
- "The numerator [of 5] is so long, I feel like it's **longer than we usually see** [in] partial fractions"
- "If you first see [that problem] you might **want to do u-sub**...because sin and cosine are different like derivatives of each other"
- "Could you expand this and then do long division?" "Wouldn't it **have to be higher on top**? The larger power on top?" "Yeah..."

Symbolic Form of **Problem 19**

$$\int \frac{1}{x(x+1)(x-3)}\,\mathrm{d}x$$

Comments About **Problem 19**

- "If it wasn't factored, I think it would be **harder**"
- "Condensing [19] into something that **isn't factored** might give people more pause in trying to figure out what to do with it"
- "Does problem 19 look like what you would call a **'typical' partial fractions problem**?" "Yeah"
- "Expanding the denominator made it look more similar to [19] so I felt more comfortable putting it [in partial fractions]"

What the Results Suggest

- Choosing a technique is **not a source of error** in overall integration, regardless of technique
- Participants indicate **low confidence in trigonometric topics**
- Participants heavily reference **visual details** while choosing a technique
- Preference completing the task with **at least another person**
- For Problem 19, justification came primarily from visual aspect of factored form

Limitations and Future Work

Limitations

- Only three interviews, six participants; **need more trials** for robust statistical significance
- **No incentive** for participants, would entice more people to participate
- Participants had a brief period of time to **process questions before sorting**
- Problems were **chosen for being straightforward**; wider range of difficulty would affect results
- Random choice was from the interviewer, could have skewed the problem set a bit

Where to Go From Here

- Many directions to explore!
- Sorting **individually**, groups larger than two, etc.
- Using different mediums; **electronic** interface for sorting, problem generation
- **Natural Language Processing** (NLP) is applicable; used for correlation and similarity, classification, automated discourse analysis, etc.
- Setting could use **other topics** in calculus and math education; all about problem solving, no matter the problems

Where Else to Go From Here

- Check if the task increases sorting accuracy with **pre-test post-test evaluation**
- **Multiple** sorting tasks separated by assessments to measure potential benefit
- Sorting procedure **itself** could be an assessment tool

Conclusion

What Was Gained

- **Greater understanding** of student problem-solving when approaching this portion of integration
- Results suggest many things: success in choosing techniques, low confidence in trig, like working in pairs
- Learned what to keep and what to change in the investigation
- Personally, how I can adjust my focus as an educator

What Can Be Gained

- Statistical significance from more samples
- More information regarding the benefit of the sorting task on accuracy of choosing correct integration techniques
- Understanding in various other areas using this task

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