AN INVESTIGATION INTO PROBLEM SOLVING IN THE CALCULUS II CLASSROOM

by

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ABSTRACT

The importance of tertiary education has grown to new heights, especially in the United States. A critical component of successful modern professionals remains the ability to employ problem-solving strategies and techniques. This study seeks to investigate initial problem-solving strategies employed by post-secondary students enrolled in Calculus II when presented with problems common to integral calculus. In-person pair-wise interviews were conducted asking six participants to sort integrals into categories based on the technique they would use to solve it. Participant responses were analyzed using a concept image composed of general and topic-specific symbolic forms, related conceptual images and concept definitions, and associated cognitive resources. Results indicate participants successfully sort by technique initially, suggesting technique choice is not a significant cause of error. Though a single cause of error cannot be established from this investigation, remarks from participants allude to other potential sources, including algebraic and arithmetic operations.

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General Integration	
Symbolic Form	Conceptual Schema
integration operator	definite vs. indefinite
function	variable being integrated with respect to has differential being applied to it
differential	
$$\int f(x)\,\mathrm{d}x$$	

Table 1: A general concept image for all integration problems. All problems will contain an integration operator, a function, and a differential operator expressing the variable being integrated. All problems are classified as either “definite” or “indefinite”.

Reverse Power Rule	
Symbolic Form	Conceptual Schema
polynomial functions	conceptually inverse to the derivative power rule
	increase power first, then divide by the new power
$$\int a_n x^n + \cdots + a_0\,\mathrm{d}x$$	

Table 2: Concept image for reversing the power rule. This applies to integration of polynomial functions and is procedurally and conceptually inverse to the derivative power rule.

Reverse Chain Rule	
Symbolic Form	Conceptual Schema
three functions; two must relate through differentiation, two must be composed	occasionally a function can be separated across expression
	derivative of inside functions could be constant
	occasionally may need scalar multiplication to match inside function with its derivative
$\int f(\ g(x)\ )\ g'(x)\,\mathrm{d}x$	

Table 3: Concept image for reversing the chain rule. Three identifiable functions are being integrated, with one function being the derivative of another. Conceptually inverse to the chain rule for derivatives. Some deep structures include the derivative of a function being constant or the need to factor a function by a constant to establish it as the derivative of another.

U-Substitution	
Symbolic Form	Conceptual Schema
three distinct functions; two must relate through differentiation, two must be composed	occasionally a function can be separated across expression
	derivative of inside functions could be constant
	occasionally may need scalar multiplication to match inside function with its derivative
	occasional need for substitution for extra factor of integrated variable in terms of substituted variable
$\int f(\ g(x)\ )\ g'(x)\,\mathrm{d}x$	

Table 4: Concept image for using substitution. The symbolic form is shared with reversing the chain rule. There are exceptions to this overlap though, where the problem could have an extra factor of the variable, meaning the problem must use substitution and cannot reverse the chain rule. The conceptual schema highlights situations where the substituted expression occurs more than once throughout the problem, as well as all other situations shared with reversing the chain rule.

Integration by Parts	
Symbolic Form	Conceptual Schema
two functions, separable (1)	use product rule derivation combined with treating integral as a quantity to break an infinite loop in rare cases
generally, one should differentiate into a constant to avoid entering an infinite loop (2)	occasionally only one function is explicitly written, meaning consideration of the constant function "1" associated with the differential
1	2
$\int f(x)\, g(x)\, \mathrm{d}x$	$\int a_n x^n \sin(x)\, \mathrm{d}x$

Table 5: Concept image for integration by parts. Two functions must be chosen before using the rule, where one function eventually differentiates to zero to avoid infinite loops. However, there is a deep structure noted in the conceptual schema where the original integral appears after applying multiple rounds of integration by parts. This situation calls for treating the integral as nothing more than a quantity, which can be added or subtracted to the original left side of the equation, the original problem being solved.

Long Division	
Symbolic Form	Conceptual Schema
A rational function (1)	after performing division, understanding any remainder must become a rational function itself; remainder must be numerator of a fraction with the original divisor as denominator
degree of numerator must be greater than degree of denominator (2)	
1	2
$\int \frac{f(x)}{g(x)} \, \mathrm{d}x$	$\int \frac{a_n x^n + \cdots + a_0}{b_m x^m + \cdots + b_0} \, \mathrm{d}x$

Table 6: Concept image for simplification by long division. This is part of a collection of methods taught to simplify functions before integrating. The symbolic form is a rational function where the degree of the numerator must be greater than or equal to the degree of the denominator. The conceptual schema simply includes situations where a remainder exists, and how it must be treated as its own rational function with the original divisor as the denominator.

Partial Fractions	
Symbolic Form	Conceptual Schema
ultimately, a rational function (1)	denominator must be factorable, as well as fully factored, before beginning decomposition
degree of numerator must be less than degree of denominator	a function entirely raised to a power must be represented with the same degree of multiplicity in decomposition (2)
	a function whose degree is greater than one must have an associated numerator in decomposition with degree one less than that function's (2)
	solving for constants before beginning integration

	understanding solving systems of equations
1	2
$\int \frac{f(x)}{g(x)}\,\mathrm{d}x$	$\int \frac{a_n x^n + \cdots + a_0}{(c_1 + x)^p \cdots (c_m + x^q)}\,\mathrm{d}x$

Table 7: Concept image for partial fraction decomposition. The symbolic form includes a rational function where the degree of the numerator is strictly less than the degree of the denominator. The conceptual schema includes conditional information, including the rules before applying partial fractions. The denominator must be factorable and furthermore completely factored before beginning decomposition. While building the equation, if a factor contains a function whose degree is greater than one, the resulting polynomial in the numerator must contain a leading term with degree one less than the denominator. If a function is completely factored with degree greater than one, more than one fraction must be constructed to represent it, where the numerator of each fraction contains a constant and the denominator of each consecutive fraction contains one more factor of the function than the previous one.

Trigonometric Identities	
Symbolic Form	Conceptual Schema
there must be trigonometric function(s) present	an understanding of derivative relationships and identities between trigonometric functions
multiple factors (two or more) of a given function that can be broken down (1)	as an extension, strategizing by focusing problems into sin/cos and sec/tan (2) (3)
	ultimately, looking to simplify the problem using identities and derivative definitions until u-substitution or reverse chain rule can be used

1	2	3
$\int (f(x))^n \, (g(x))^m \, \mathrm{d}x$	$\int (\sin(x))^n \, (\cos(x))^m \, \mathrm{d}x$	$\int (\tan(x))^n \, (\sec(x))^m \, \mathrm{d}x$

Table 8: Concept image for using trigonometric identities. The symbolic form requires exclusively trigonometric functions present. Usually, there are multiple factors of each present function that can be broken down. Understanding the relationships between functions through taking derivatives is crucial, and furthermore understanding when to restructure the problem to contain only sine and cosine functions or tangent and secant functions. Ultimately, integration using substitution is the goal after all simplification.

Trigonometric Substitution	
Symbolic Form	Conceptual Schema
sum or difference of squares must be present (1)	must understand Pythagorean theorem in relation to sums of squares and trigonometric functions relating to sides of a triangle
common forms include square roots, either in numerators or denominators of fractions, attached to factors of the integrated variable (2)	understanding of Pythagorean theorem to correctly identify sides of the triangle related to the problem
	understanding fractional exponents and rewritten forms of them
	understanding trigonometric identities relating functions
1	2
$\int a^2 + (f(x))^2 \, \mathrm{d}x$	$\int \frac{1}{\sqrt{a^2 + (f(x))^2}} \, \mathrm{d}x$

Table 9: Concept image of substitution using trigonometric quantities and Pythagorean relationships. The symbolic form requires no trigonometric functions to be initially

present, but rather a sum or difference of squares, mimicking various forms of the Pythagorean formula. Extra factors of the variable are also possible and can be substituted out of the problem as well. Conceptually, there must be an understanding of Pythagorean relationships between the sides of a triangle and trigonometric functions. Algebraically, one may need to complete the square within the root. Ultimately, multiple substitutions should cause most factors within the problem to cancel and produce a simple function to antidifferentiate.

	Confident	Unconfident
Person A		
Person B		

Table 10: An organizational matrix to classify each person's "confidence" in each technique of integration. Though this question follows the initial sorting procedure, participants were asked to judge their confidence in general.

	Easy	Hard
Correct		
Incorrect		

Table 11: An organizational matrix to classify sorted problems as correct and incorrect related to participant indication of "easy" and "hard" problems. In each interview,

problems used as subjects to follow-up questions were chosen at random based on this list of “easy” and “hard” problems.

	Confident	Unconfident
Person A	U-sub, long division, algebraic simplification, by parts	Partial fractions, power rule, trig sub
Person B	U-sub, algebraic simplification, power rule	By parts; iffy on trig sub, trig identity

Table 12: This matrix shows the confidence ratings expressed by the participants in general integration techniques from Interview 1. Each provided individual answers, so some techniques are duplicated in the confidence rating columns.

	Easy	Hard
Correct	1, 2, 4, 7, 11, 19, 20	13, 14, 15, 17, 18
Incorrect	–	–

Table 13: This matrix shows problems categorized as “easy” and “hard” sorted against their sorting correctness from Interview 1.

	Confident	Unconfident
Person A	U-sub, partial fractions, long division, trig sub, trig identities	Trig identities

Person B	Algebraic simplification, u-sub, partial fractions, long division, trig sub	Trig identities

Table 14: This matrix shows the confidence ratings expressed by the participants in general integration techniques from Interview 2.

	Easy	Hard
Correct	2, 3, 14, 18, 19, 20	4
Incorrect	–	**11** (at the end, it was moved to a more correct spot)

Table 15: This matrix shows problems categorized as “easy” and “hard” sorted against their sorting correctness from Interview 2.

	Confident	Unconfident
Person A	The power rule, by parts	Trig sub, trig identity, long division
Person B	Maybe long division	Trig sub, trig identity, partial fractions, u-sub

Table 16: This matrix shows the confidence ratings expressed by the participants in general integration techniques from Interview 3.

	Easy	Hard
Correct	7, 13, 16	1, 2, 5
Incorrect	–	–

Table 17: This matrix shows problems categorized as “easy” and “hard” sorted against their sorting correctness from Interview 3.

INTRODUCTION

An important aspect of education is how the student interacts with new material. Modern pedagogical approaches shift focus from traditional memorization and recitation to active learning frameworks designed around student self-discovery and key experiences (Yair 2008). A current goal remains to produce “reflexive professionals, capable of problem-solving” (Santos et. al, 2019). Simultaneously, there has been a continuous effort to “strengthen the scientific workforce” in recent years, particularly in the United States (Lichtenberger & George-Jackson, 2013). Encouragement for undergraduate students to pursue degrees in Science, Technology, Engineering, and Mathematics (STEM) has dominated post-secondary education in the last 60 years (National Academy of Sciences, Engineering, and Medicine, 2018). Therefore, focusing on the robustness and quality of such undergraduate education must remain a high priority. This paper proposes an investigative approach using modern methods from education research adjoined with concepts from cognitive learning theory and aims to contribute to exposing sources of potential student misconceptions. The investigation is focused on student problem-solving procedures in a section of MAT127 (Calculus II) at the University of Maine. The concept of integration and various procedural techniques of integration are introduced. The vehicle for analyzing student problem-solving for this investigation will be a sorting task asking participants to choose the “correct” integration technique based on visual aspects and conceptual understanding. While the skill of manual integration may be considered obsolete, especially by students, the related problem-solving strategies can never be considered such.

BACKGROUND

To begin to understand research on problem-solving, it is important to first understand the advent of problem-based learning (PBL). It is widely agreed that the integration of PBL began at McMasters University in the 1970s in medical school pedagogy (Hung et. al, 2008). A gradual spread of PBL integration followed in various disciplines and generally resulted in a "robust positive effect…on the skills of students" (Dochy et. al., 2003). Disciplines with PBL applications in higher education include engineering design (Hasna, 2008), biomechanics (Mandeville & Stoner, 2015), business management (Scherpereel & Bowers, 2006), and ecological restoration (Schaeffer & Gonzalez, 2013).

After centering a learning model around problems, a natural next direction is to investigate how students are approaching these problems. In particular, "most mathematics educators agree that the development of students' problem-solving abilities is a primary objective" (Lester, 1994). Beginning with work by Polya in 1945 and amplified by introducing problem-based learning, problem-solving research was particularly active throughout the 1970s and early 1980s (Schoenfeld, 2007). While activity has declined, problem-solving research remains active in fields of higher education such as graduate physics research (Leak et. al., 2017), integrated systems biology (MacLeod & Nersessian, 2016), civil engineering (Akinci-Ceylan et. al., 2022), and business (Kemery & Stickney, 2014).

The concept of flexibility in problem-solving introduced by Star & Rittle-Johnson (2007) as "knowledge [of both] multiple strategies and the relative efficiencies of those strategies" is particularly relatable to the scope of this investigation. The primary purpose of choosing an integration technique is to optimize problem-solving efficiency. To determine from qualitative results from this investigation if this is happening, theoretical and conceptual frameworks can be employed.

Defined by Kivunja (2018), a theoretical framework "comprises the theories expressed by experts…for data analysis and interpretation of results". Specifically, theoretical frameworks "entered mathematics education at its very beginning as an academic field" (Reid, 2014). Some researchers have adopted predetermined frameworks from other researchers, such as Carlson & Bloom's (2005) Multidimensional Problem-Solving Framework (Dawkins & Epperson 2014, Savic 2015), while others construct custom frameworks tailored to their study.

This investigation is based on a theoretical framework constructed using three components: (1) the symbolic form by Sherin (2001), (2) the concept image and concept definition from Tall & Vinner (1981), and (3) cognitive resources from Hammer (2000) associated with each. The theoretical framework and its components are examined in closer detail in its self-titled chapter.

Established research specifically in undergraduate mathematics education has focused on areas including conceptual understanding (De Zeeu et al, 2015, Tapare, 2013, Spencer-Tyre, 2019, West, 2023), procedural understanding (Maciejowski & Star, 2016, Nadaei et. al., 2022, Oberg, 2000), and metacognition (Radmehr & Drake, 2020, Smith, 2013). This investigation aims to contribute primarily to research on procedural understanding with the possibility of results associated with conceptual understanding and metacognition.

THEORETICAL FRAMEWORK

This investigation is guided by the theoretical principles of the concept image and concept definition (Tall & Vinner, 1981), the symbolic form (Sherin, 2001), and cognitive resources (Jones, 2013). Each symbol, idea, and principle a person references when analyzing a problem is considered a cognitive resource. A person constructs a concept image by combining many individual cognitive resources into one collection. Each concept image is unique to each person, and they reference it when building understanding for a given principle.

Each concept image consists of a symbolic form and a conceptual schema. The symbolic form gathers all visual information the problem offers and distinguishes each separate item to build a pattern that a person could use when analyzing a problem. The conceptual schema of each concept image draws on its symbolic form as well as various cognitive resources to produce a list of principles relevant to appropriately applying that concept.

For this study, lead researchers constructed all concept images included in this study. Subsequently, each concept image is a reference standard created by the researchers themselves, built from their perception of each symbolic form and cognitive resources they would utilize. Participants will have their own unique concept images and may reference cognitive resources that the researchers did not include. Therefore, a concept image will not be used for determining the correctness of a student's own concept image. Rather, the cognitive resources composing this study's concept image will be compared to those used by the student while they analyze each problem (Jones, 2013).

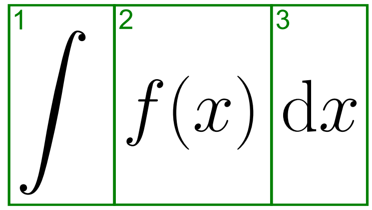

Figure 1: A visualization of the symbolic form for general integration. It is divided into three sections grouping together the symbols. Section 1 denotes the integral symbol, Section 2 denotes the function, and Section 3 denotes the differential operator.

As an example, the concept image of general integration will be explored in further detail. Figure 1 highlights this concept image's symbolic form. It consists of three symbols: (1) the integral symbol, (2) a function to be integrated, and (3) a differential operator. The conceptual schema highlights both definite and indefinite integrals, though all integrals used in this study are indefinite. It also includes the differential operator which signifies the variable being integrated. One can form this connection by observing the variable "x" within the parentheses of the function "f(x)" and "x" appearing next to the differential operator "d". This conclusion relies on connecting visual cues from the symbolic form to principles from the conceptual schema. The researchers are interested in understanding what relationships participants will establish between specific visuals in problems and their own cognitive resources. Visualizations for a symbolic form of each technique of integration, determined by the researchers, are included below in Table 1.

Reverse Chain Rule/U-Substitution $$\int f(\ g(x)\ )\ g'(x)\,\mathrm{d}x$$	One Step/Reverse Power Rule $$\int a_n x^n + \cdots + a_0\,\mathrm{d}x$$
Algebraic Simplification $$\int \frac{f(x)}{g(x)}\ \mathrm{d}x$$	By Parts $$\int f(x)\ g(x)\ \mathrm{d}x$$
Partial Fractions $$\int \frac{a_n x^n + \cdots + a_0}{(c_1 + x)^p \cdots (c_m + x^q)}\,\mathrm{d}x$$	Long Division $$\int \frac{a_n x^n + \cdots + a_0}{b_m x^m + \cdots + b_0}\,\mathrm{d}x$$
Trigonometric Identities $$\int (\tan(x))^n\ (\sec(x))^m\ \mathrm{d}x$$	Trigonometric Substitution $$\int \frac{1}{\sqrt{a^2 + (f(x))^2}}\,\mathrm{d}x$$

Table 1: Example visualizations of the symbolic form of each integration technique. Some techniques have more than one visualization. The visualizations chosen for this table represent symbolic forms of problems included in this investigation.

METHOD

Construction of Integration Concept Images

A theoretical framework was first constructed by the researchers before analyzing participant responses to particular problems (see Theoretical Framework). The researchers determined that problems in Integral Calculus, known as integrals, have a general symbolic form, regardless of the technique used to solve them.

As explained in the Theoretical Framework section, each integral is composed of (1) a large symbol indicating integration, (2) an integrand, and (3) a differential operator indicating the variable integrated with respect to. While determining their concept images, the researchers first considered each symbolic form. They were concerned with what symbols are *necessary* for each problem category as well as the pattern these symbols are *typically* arranged in. A list of symbols similar to the one outlined above was established and then visualized into the diagrams in the Appendix.

While the researchers were considering the relationships within each symbolic form, their cognitive resources were established and included as part of the conceptual schema. Each person, and therefore each researcher, utilizes different and often unique cognitive resources, so an individual analysis before a group determination was deemed important and necessary. The researchers recognize the determined concept images are their own interpretations. However, these concept images are used only as standardized reference points for comparison while analyzing participant responses.

Construction of the Problem Set

Researchers chose a list of twenty integrals to include in the total set of problems. They individually evaluated each problem themselves by indicating their choice of correct technique as well as a list of incorrect techniques that students might choose. They then compared analyses and agreed upon correct and potential incorrect choices associated with each problem.

While the researchers were choosing problems, the difficulty of sorting each problem was considered. Problems with visual similarities to a technique's "usual" symbolic form were favored. The goal of this investigation was not to evaluate the *strength* of students, but to analyze student problem-solving strategies in *usual* situations. Including problems with an unusual appearance or those that required advanced conceptual understanding would have strayed from the focus of this project.

Considerations of varying difficulty levels are addressed in Limitations and Future Work. A total of twenty problems were chosen for this investigation. See the Appendix to view the full list of problems and their associated correct and incorrect techniques, and see Table 1 in Results for the problems sorted in each interview.

Composition of the Interview and Sorting Process

For participant data collection, in-person, pair-wise interviews were conducted after the initial sorting process. The focus of the study requires qualitative data gathered from physically sorting problems into categories. Researchers determined interviews to be optimal for generating follow-up questions after sorting as opposed to passive options like surveys.

Participants for these interviews were students enrolled in a section of MAT127 (Calculus II) at the University of Maine. Pairs were determined based off of indicated availability. A total of six participants indicated interest in this study and all six individuals were interviewed. No incentive was offered for participation other than potential benefit from performing the task itself.

Problems were printed on pieces of paper and could be physically moved into categories. Therefore, an in-person setting was the best option rather than virtual. Based on the theoretical framework, a template of questions was constructed to ensure a level of standardization among each interview. The researchers included questions about the participants' confidence in their initial sort, which problems they found easy or hard, and if they would like to recategorize any problems after all other questions.

The setting of the interview was dependent on availability, but all interviews were conducted in the same physical orientation. That is, the interviewer sat across from both participants sitting side-by-side. The interviewer always faced the participants from the beginning to the end of the interview.

The interviewer first arranged the techniques into categories in a horizontal line across from the participants. The interviewer then chose eight problems at random from the total set of problems contained in a plastic bag. The problems are arranged into a horizontal line at random far beneath the categories to indicate each problem was not being placed into an initial category by the interviewer. This provided a brief period of time where the participants were able to analyze problems before the sorting process began. The interviewer would then start an audio recording, verbally indicate the number of the recording for reference purposes, and read the following instructions: “In front of you are a list of problems asking you to integrate a given function. Please sort each problem into the category of the technique you would use to solve it. You do not need to solve the integrals. Be sure to talk with each other when sorting.”

The participants were given as much time as needed to perform the sorting task. If participants finished the sorting process quickly, additional problems for sorting were offered. After completing the sorting process, participants were asked to indicate their confidence level in their initial sort. Matrices were used to classify “confident” and “unconfident” integration techniques for each participant as well as “easy” and “hard” problems to sort relative to the correctness of their sort. Follow-up questions were dependent on participant response and asked based on a problem indicated as “easy” or “hard” by the participants. This problem was chosen by the interviewer.

Participants were asked to point out specific visual aspects of the selected problem for analysis against the control symbolic form determined by the researchers. Participants were also encouraged to identify any conceptual aspects of a technique which they related with visual aspects from the problem itself for as much relation to the researchers’ concept image as possible. Of course, participant responses depended on their *own* concept image, and any relationships drawn between theirs and the researchers' concept image were also determined by the researchers.

Ultimately, participants were given the opportunity to sort any problem into another category after answering these questions. Relationships among responses from each interview were determined qualitatively using transcriptions of recorded audio with references to the researchers' theoretical framework of concept images.

RESULTS

Results indicate successful initial choice of integration technique. All but one problem was correctly sorted after one round of sorting across all interviews. Participants in Interview 1 correctly sorted the following problems: 1, 2, 4, 7, 11, 13, 14, 15, 17, 18, 19, 20. Participants in Interview 2 correctly sorted the following problems: 2, 3, 4, 11, 14, 18, 19, 20. The same participants incorrectly sorted problem 11 initially but later elected to correct themselves. Participants in Interview 3 correctly sorted the following problems: 1, 2, 5, 7, 9, 13, 16, 19. A table visualizing the problems sorted in each interview can be found below, and the total set of problems with their symbolic form and correct technique choice is available in the Appendix.

Total interview time ranged from approximately eleven minutes to seventeen minutes, and total initial sorting time ranged approximately between two minutes and six minutes. Participants in Interview 1 accepted an offer to sort four more problems, and hence sorted a total of twelve compared to the total of eight sorted in Interviews 2 and 3.

	Interview 1	Interview 2	Interview 3
1			
2			
3			
4			
5			
6			
7			
8			
9			
10			
11			
12			
13			

14			
15			
16			
17			
18			
19			
20			

Table 1: A visualization of the problems sorted in each interview. Green cells indicate correctly sorted problems, while orange cells indicate incorrect sorting with later correction. Note Interview 1's greater number of total problems sorted, as they chose to sort more after completing the sorting process very quickly. Additionally, note problems 2 and 19, which appeared across all three interviews. Each problem is discussed further in Analysis and Discussion.

	Interview 1		Interview 2		Interview 3	
Person	A	B	A	B	A	B
Reverse Chain Rule/U-Substitution						
One Step/Reverse Power Rule						
By Parts						
Algebraic Simplification						
Long Division						
Partial Fractions						
Trigonometric Identities						
Trigonometric Substitution						

Table 2: A visualization of the indicated confidence levels in each integration technique for each interview's participants. The second row denotes Person A and Person B under a column for each interview. Note the high confidence distribution for U-Substitution and low confidence distribution for both Trigonometric Identities and Substitution.

While sorting, all participants referenced visual cues related to the symbolic forms as well as cognitive resources related to conceptual schema. Visual cues included degrees of polynomial functions, squared quantities, relationships between expressions and their derivatives, and differences between the numerator and denominator of a fraction. All participants willingly answered follow-up questions asking for answers related to aspects of the concept image. Participants were discouraged from attempting to solve the problems while sorting, though six notable responses in Interview 2 were based on how they might completely solve the problem. Nevertheless, these decisions were made with many references to the symbolic form of the problem as well as the participants' own cognitive resources.

Confidence in each integration technique varied with each participant, though 5 out of 6 rated themselves "unconfident" in trigonometric identities and 4 out of 6 in trigonometric substitution. Additionally, 4 out of 6 participants indicated confidence in u-substitution, and one participant indicated confidence *only* in long division.

Participants in Interview 1 justified many decisions with references to the symbolic form. Confidence was indicated in algebra-dense topics, including Algebraic Simplification, U-Substitution, and Reverse Power Rule. This confidence became apparent through many responses with visual-to-conceptual justification: "The numerator being one means there's not a whole lot you can do with trying to manipulate any kind of algebra to cancel anything out", "It's a very standard form. If you ever see some constant plus or minus an x squared and it's in something where it's hard to mess with it like for example this is in a square root so then its trig sub pretty obviously", "Unless you have another factor that is also in a square root, then you are going to have a lot of difficulty trying to cancel anything out of that expression".

Participants in Interview 2 drew upon aspects of the greater integration process, especially the process of anti-differentiating the function, or completely solving the integral. Trying to solve the integral was discouraged in both the initial directions and additionally during the sorting process, though of course it cannot be completely removed from the participants' train of thought. They each seemed to use the greater integration process to generate visual-to-conceptual justifications: “You don’t need to break it up anymore so it can’t be partial fractions”, “It’s still weird to me that [11] becomes natural log with no relationship to the coefficient [of the x term]”. In particular, one participant indicated strength in U-Substitution: “There’s a component to a higher power and something difficult and something to a lower power, and that’s pretty much a classic recognition of u-sub”, “I think of u-sub almost instantly”. The other participant responded to this by stating, “U-sub is kind of the second or third thing I think about”, and further reinforced their differences: “More process of elimination [used by Person B] than immediate first step [used by Person A]?” “Yeah”.

Participants in Interview 3 used a visual-to-conceptual pipeline similar to the researchers: “When you take the derivative of one of these it equals the other”, “Expanding the denominator made it look more similar to [19] so I felt more comfortable putting it [in partial fractions]”, “There’s two squared [quantities]...that you could easily use trig sub for”. Additionally, several comments were made about the balance of strengths and weaknesses between the two participants: “Our strengths are literally opposites”, “We’d do really good on tests together”.

Problems were chosen at random and therefore varied with each interview, though Problems 2 and 19 appeared in all interviews. Problem 19 garnered comments from participants about its symbolic form and the cognitive resources required to correctly sort and solve it. Comments included “If it wasn’t factored, I think it would be harder” and “Condensing [19] into something that isn’t factored might give people more pause in trying to figure out what to do with it”. All confidence ratings, “easy” and “hard” problems, and sorting correctness ratings are available below.

	Confident	Unconfident
Person A	U-sub, long division, algebraic simplification, by parts	Partial fractions, power rule, trig sub
Person B	U-sub, algebraic simplification, power rule	By parts; iffy on trig sub, trig identity

Table 2: Technique confidence ratings indicated by participants in Interview 1.

	Confident	Unconfident
Person A	U-sub, partial fractions, long division, trig sub, trig identities	Trig identities
Person B	Algebraic simplification, u-sub, partial fractions, long division, trig sub	Trig identities

Table 3: Technique confidence ratings indicated by participants in Interview 2.

	Confident	Unconfident
Person A	The power rule, by parts	Trig sub, trig identity, long division
Person B	Maybe long division	Trig sub, trig identity, partial fractions, u-sub

Table 4: Technique confidence ratings indicated by participants in Interview 3.

	Easy	Hard
Correct	1, 2, 4, 7, 11, 19, 20	13, 14, 15, 17, 18
Incorrect	–	–

Table 5: “Easy” and “hard” problems indicated by participants in Interview 1 by correctness of sort.

	Easy	Hard
Correct	2, 3, 14, 18, 19, 20	4
Incorrect	–	**11** (at the end, it was moved to a more correct spot)

Table 6: “Easy” and “hard” problems indicated by participants in Interview 2, by correctness of sort.

	Easy	Hard
Correct	7, 13, 16	1, 2, 5
Incorrect	–	–

Table 7: “Easy” and “hard” problems indicated by participants in Interview 3, by correctness of sort.

ANALYSIS AND DISCUSSION

As indicated above, the results suggest participants can successfully choose integration techniques informed by both aspects of the researcher's theoretical framework: symbolic forms and conceptual schemas. Responses were primarily aligned with the following trends.

Providing aspects of a symbolic form as justification was the most common form of response. Examples include: "…algebraic simplification because you can just break this up into three [terms]…and divide"; "There's a component to a higher power and something difficult and something to a lower power, and that's pretty much a classic recognition of u-sub"; "There's two squared [quantities]...that you could easily use trig sub for"; "You can pull x cubed out of the top and cross that out and you just have x on the bottom". Notice the reference to symbols in the first portion of their responses followed by the conclusion they arrived at. This form of response mimics the visual-to-conceptual mentality of the researchers when the integrals for this study were being chosen. Therefore, the standard theoretical framework's symbolic form proved to provide a reliable medium for analyzing responses.

Participants also show a satisfactory understanding of conceptual aspects of integration techniques by arriving at conclusions from symbols related to their own cognitive resources: "There's three [rules] for trig sub: secant, sine and tangent, and the tangent is the g(x) squared plus a squared, this is pretty clear [here]"; "Even if it is natural log like we were saying, if I were to take what I think the answer is and take the derivative of that, I think we would end up with different integral. If I reverse engineer it, it's wrong". In particular, note the first previous response. The knowledge of trigonometric substitution rules cannot be derived from this investigation's problem set alone. This participant applied their previous experience and associated cognitive resources with the symbolic form of the problem. The trigonometric substitution rules are a particular form of the detail listed in the researchers' Trigonometric Substitution conceptual schema regarding an understanding of the Pythagorean theorem. However, this participant utilized a different cognitive resource derived from the *same* concept. This shows a particular difference between the concept images of the participant and the researchers while also highlighting how each connects back to the same foundational concept.

Some responses using conceptual justification relate to completely solving the problem, especially from Interview 2. Participants in Interview 2 requested paper, which was denied by the interviewer as it does not align with the task at hand. It is not unfounded nor concerning that participants would choose a trial-and-error approach using each technique until it works. This sorting procedure is different from material given to students in MAT127 because it relies *entirely* on visual information for decision-making. Material in MAT127 includes a visual-to-conceptual process used similar to the researchers' process in addition mathematical steps using algebra, calculus, or trigonometry. This visual-to-conceptual first step within the greater process of mathematical integration is the focus of this investigation. Granting participants the ability to focus on any other aspects of integration would have strayed from the investigation's intended purpose.

Even with prior indications that they were "unconfident" in a particular technique, all participants were able to correctly justify their sort using symbolic forms. In some cases, participants compared forms of problems: “The form of [trig substitution] questions [like 2 and 4 is] really consistent”; “Comparing that to some of the long division problems like 7 and 20, these are written in order and the numerator is larger than the denominator…you are able to go and simplify it out with long division”. Participants use understanding of the symbolic composition of problems commonly associated with a given technique. Importantly, when a participant was unable to sort the problem themselves, their partner would step in to provide their thoughts: “Are you sure that’s a trig identity?”, “I’m not so I’m thinking it through now”; “Isn’t there a square root on trig sub?”, “Yeah”; “Could you expand this and then do long division?”, “Wouldn’t it have to be higher on top? The larger power on top?”, “Yeah…”. There was a clear indication across all three interviews of a yin-yang combination of strengths and weaknesses. From the previous responses, it is conclusive that participants recognized this and relied on each other to increase the strength of their sorting decisions.

Collaboration between participants with an uneven distribution of confidence did result in the more confident participant initiating discussion about a particular problem. Most discussions in Interview 3 were initiated by the more confident participant, though that did not stop the other participant from contributing to the discussion:

> “I feel like this one could also go under long division if you factor it out”
>
> “I don’t know, because I thought we had to have the numerator, like the x cubed is higher than x, so I don’t know for sure if the x can do x squared”
>
> “Comparing problem 5 to problem 7, in the numerator we have a greater degree than in the denominator, and you’re saying that’s not the same thing that we have in problem 5?”

“Mmhmm”

This implies increased confidence and comfort when working in groups to solve problems rather than alone. Participants indicated explicitly as such: “We’d do really good on tests together”, “Our strengths are literally opposites”. A current learning model integrated into a section of the University of Maine Calculus II (MAT127) course recitation is a problem-based group work approach, encouraging combined efforts to problem solving with the support of graduate and undergraduate learning assistants. This investigation suggests the current model is, and can be, beneficial to students’ confidence while solving problems. For mathematics specifically, improving problem-solving confidence could mean reducing the loss of “joy and confidence in mathematics” and subsequent dropout rates for students in STEM disciplines (Bressoud et. al., 2012).

Participants referenced potential direct and indirect causes of arriving at incorrect solutions to problems. As mentioned, integration techniques related to trigonometry were commonly categorized as “unconfident”. Participants occasionally mentioned insufficient knowledge of trigonometry as well: “We did [trig identities] two years ago and I…just derived them last minute instead of learning them”; “Are you sure that’s a trig identity?” “I’m not so I’m thinking it through now”; “I don’t know trig”. Curiously, participants in Interview 2 categorized themselves individually as “confident” in trig substitution while “unconfident” in trig identities. This could be due to insufficient understanding of trigonometric identities: “We did [trig identities] two years ago and I…just derived them last minute instead of learning them”.

Additionally, participants exuded more confidence when justifying symbolic forms relating trigonometry to polynomial expressions rather than explicit trigonometric functions: “I think you could do this with trig identities. [Because] I don’t know how else you would do it either” shows little confidence and slight desperation compared to “There’s three [rules] for trig sub: secant, sine and tangent, and the tangent is the g of x squared plus ‘a’ squared, this is pretty clear [here]”. Nevertheless, the participants in Interview 2 remained determined to sort all problems correctly, even under “unconfident” topics.

Across all interviews, participants also referenced insufficient algebra skills: “Sometimes my algebra skills are a little rusty”. It is often the case that those who “[withdraw] from calculus often lack algebraic fluency” (Dawkins & Epperson, 2014). For example, optimization problems require both strong calculus and algebra skills. Roots of a function must be found, and without a strong understanding of how to factor or simplify a polynomial function to solve for a variable, the solution cannot be obtained. In integral calculus, these same manipulations must often be performed on integrals, especially when using Algebraic Simplification, Long Division, and Partial Fractions techniques. Participants were able to proficiently use algebra to justify decisions: “You can split up the fraction so you have 12 over 3x”, “Because if you expand this then it’s not helpful because you can’t cancel anything out”, “The numerator being one means there’s not a whole lot you can do with trying to manipulate any kind of algebra to cancel anything out”. This shows participants have a correct understanding of algebraic manipulations when approaching a problem without actually performing any algebra. Any algebraic errors are either physical miscalculations or incorrect applications of certain methods. Hence, this serves as further reinforcement that participants are successful in sorting problems initially even through self-reported difficulties, reducing the significance of this sorting as a cause of incorrectness.

LIMITATIONS AND FUTURE WORK

An exploratory study brings with it a level of crudeness meant to be refined through future work. While this investigation produced important qualitative data, more interviews would provide a greater range of participants each with unique concept images and understandings of integration techniques. While an increase in interviews means an increase in qualitative data interpretation and therefore more time, it introduces the possibility of measurable statistical correlation between responses from participants and problems chosen for interviews, similar to the responses related to problem 19. Providing an incentive for participants could generate more interest; there was no incentive offered for participation in this study.

Conducting the interviews using a medium other than problems printed on pieces of paper could affect the sorting process. Problems were displayed to students briefly before the interview began, allowing them some time for analysis before officially beginning the sort. Electronic media has the potential to reduce this, though simply restricting the view of the problems from the participants until officially beginning the sort would be an improvement.

A list of problems with a wider range of difficulty could also test participant conceptual understanding more rigorously. A participant stated that, “Condensing [19] into something that isn’t factored might give people more pause in trying to figure out what to do with it”. Both partial fractions problems in the problem set followed the “standard” form of the denominator with a factored polynomial. Some literature includes an “index of difficulty” and related scoring procedure (Angco, 2021) which could be used to further understand where the “limit” of participant understanding lies. Should a factor similar to a difficulty index be implemented into this investigation, the descriptor "difficulty" would refer to how "non-obvious" the correct technique associated with the problem is. In other words, the more difficult the problem, the less obvious its correct associated technique is to choose. This difficulty could refer to both visual cues provided by the problem's symbolic form as well as the conceptual cognitive resources needed for correct determination.

Additionally, a survey recording a brief history of each participant could supply reason for a participant's confidence or sorting ability. Participant history is common supplemental information to record in surveys and interviews, and hindsight indicates it could benefit this investigation. Such history could include rating of potential math anxiety levels, demographic information, and indication of being a "first-generation" undergraduate student. Asking if the participant has been exposed to material taught in MAT127 prior to taking the class is especially relevant, as "the conceptions that students bring from their previous mathematical experiences strongly influence how they make sense of the calculus concepts they encounter" (Ferrini-Mundy & Lauten, 1994).

This investigation can be applied to varying concepts across a range of courses, including series convergence and divergence tests also in MAT127. Future work could consist of implementing and refining this investigation's concept in other settings. For example, this investigation could be adapted as a form of assessment using pre-test and post-test evaluation (Bonate, 2000).

While the focus of this investigation is to determine if the stage of choosing integration technique is a cause of error using the sorting task, the sorting task *itself* could be assessed for potential improvement. As referenced above, students draw upon previous mathematical experiences when making sense of current experiences (Ferrini-Mundy & Lauten, 1994). The potential benefit of this sorting task could be measured using pre-test and post-test evaluation. Alternatively, the task itself could be used as a form of evaluation and could be integrated as a simple pre-test or post-test with another activity in between.

With additional interviews, analysis could be performed using natural language processing (NLP). NLP refers to "how computers can be used to understand…natural language text to do useful things" (Chowdhary & Chowdhary, 2020). Correlation between words, phrases, and associated quantitative data could be measured with statistical significance, raising the results of this investigation from suggestions to proven conclusions. Deriving new quantitative data *from* qualitative data could form paths to new conclusions as well as other possibilities for future work.

PERSONAL MEANING

While assisting with MAT127 at the University of Maine, I heard several comments made by students on their feelings about both Calculus II and the subject of calculus as a whole. Sentiments of these comments included frustration, confusion, hopelessness, and disinterest. Each of these feelings would always lead to someone asking, “Why do I need to take this class? Why do we need to learn this? I’ll never need to do this manually, there are calculators and A.I. for this”. My response had always grown, starting with “This is just a weed-out class, you can get through it!” and later adding “It’s important to know *how* your computers are giving you answers”. The most recent answer includes a source of inspiration for this investigation: “It’s not that you’ll need to do these problems specifically, but you’ll need to be quite a good problem-solver”. This is the primary reason why this project is not just a math education study, but a general problem-solving investigation. There are a lot of automatic tools to complete tasks, but these tools often only help with predictable situations. Professionals will encounter situations where they cannot rely on established machinery, but rather their own intuition, education, and problem-solving skills.

I must say I am surprised by the results. The participants completed the task with much more correctness than I personally anticipated. However, the interactions I have with students while evaluating their ability to choose a correct integration technique are when they are choosing *individually*. I believe this investigation’s choice to have participants sort in pairs greatly improved the correctness of every interview’s sort. I believe the students think this as well based on their responses. I find this a satisfying result since it gives the current function of the recitation extra relevance. Students are encouraged to complete worksheets in groups, with some of these groups being pairs of two. Based off of the responses recorded, I believe many correct justifications are being exchanged between them, preventing the spread of misunderstanding and misinformation.

Finally, I am glad to hear from participants how the sorting task itself was enjoyable and helpful. One student agreed to use the sorting task as a studying resource before their final exam in MAT127. It gives me hope that choosing to investigate the benefits of the sorting procedure will yield positive results. I think the physical visualization of a task usually performed mentally is a primary benefit, and am interested to see what other benefits can be uncovered. To be frank, it feels fantastic knowing something I created and implemented has helped others to learn. I would love to maximize this impact, even in subjects other than mathematics. The improvement of education is what I signed up for when I decided to work as a Maine Learning Assistant, and I am happy to have conducted research allowing me to answer my own questions while benefiting the education of others.

CONCLUSION

This investigation aimed to observe undergraduate student problem solving processes in integral calculus by attempting to conclude if initial integration technique choice is a significant source of error during solving. Results from three in-person, pairwise interviews with six participants total indicate the initial choice of integration technique is not a source of error in the procedure of mathematical integration for undergraduate students enrolled in a section of MAT127 (Calculus II) at the University of Maine. However, it is important not to generalize this conclusion and apply it to all MAT127 students due to the small sample size of participants. All problems except one were sorted correctly across all interviews.

Participants provided justifications for their sorts relating to visual and conceptual cognitive resources, including This implies at least foundational understanding of concepts introduced in MAT127. Participants embraced collaboration with some expressing a preference for collaboration. Qualitative analysis of audio transcription data supports these conclusions, and furthermore alludes to algebra and trigonometry insufficiencies as potential sources of error.

In summary, takeaways from this investigation are as follows: (1) even with a small sample size, results show initial sorting is not a significant cause of error among participants in collaborative environments like this investigation, (2) participant responses show positive reactions to completing the task in groups due to opposite strengths and weaknesses, and (3) alternative potential sources of error, including trigonometric understanding, are areas to investigate. As a small sample exploratory study, this investigation was a successful foundation for potential future work to be derived from.

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APPENDIX

1. This is the exact text referenced to maintain a standard flow in each interview.

Interview Process

Before Sorting Process*

→ In front of you are a list of problems asking you to integrate a given function. Please sort each problem into the category of the technique you would use to solve it. You do not need to solve the integrals. Be sure to talk with each other when sorting.

After Sorting Process

→ How confident do you feel regarding how you have sorted these problems?

→ Are there any techniques of integration you feel particularly confident in?

→ Are there any techniques of integration you feel particularly unconfident in?

→ How did you each work together to complete the task?

→ Which problems were easy to sort?

Pick easy problem*

→ Why did you choose this category?

→ Are there aspects of this problem that caused you to choose this category?

→ Which problems were hard to sort?

Pick hard problem*

→ Why did you choose this category?

→ Are there aspects of this problem that caused you to choose this category?

If there are unaddressed leftovers*

→ Why did you leave this problem unsorted?

→ Are there any categories you were thinking of sorting this problem into?

After Initial Questioning

→ Are there any problems you would move to a different category?

If Anything Was Moved

→ Why did you move this problem to this category?

2. This is the collection of integrals rendered using LaTeX, along with their determined correct choice of technique of integration as well as potential incorrect choices. These integrals are inserted as images.

1.

$$\int 5x(1-4x^2)^{20}\,\mathrm{d}x$$

Correct Choices: Reverse Chain Rule, U-Substitution

Potential Incorrect Choices: Reverse Power Rule, Trig Substitution, By Parts

2.

$$\int \frac{1}{\sqrt{x^2+4}}\,\mathrm{d}x$$

Correct Choices: Trig Substitution

Potential Incorrect Choices: U-Substitution, Algebraic Simplification

3.

$$\int x\,\cos(x^2)\,\mathrm{d}x$$

Correct Choices: U-Substitution

Potential Incorrect Choices: By Parts, Trig Identity

4.

$$\int \tan(x)\sec^3(x)\,\mathrm{d}x$$

Correct Choices: Trig Identity

Potential Incorrect Choices: By Parts, Reverse Chain Rule

5.

$$\int \frac{(5x+1)}{(x+3)^2}\,\mathrm{d}x$$

Correct Choices: Partial Fractions

Potential Incorrect Choices: Long Division, Algebraic Simplification

6.	$$\int e^x \sin(2x)\,\mathrm{d}x$$ **Correct Choices:** By Parts **Potential Incorrect Choices:** Trig Identity, Reverse Chain Rule
7.	$$\int \frac{x^3 - 4x^2 - 24}{x - 5}\,\mathrm{d}x$$ **Correct Choices:** Long Division **Potential Incorrect Choices:** Partial Fractions, Algebraic Simplification
8.	$$\int x^4 \ln(x)\,\mathrm{d}x$$ **Correct Choices:** By Parts **Potential Incorrect Choices:** U-Substitution, Algebraic Simplification, Derivative Definition for Natural Log
9.	$$\int \sin^3(x)\cos^3(x)\,\mathrm{d}x$$ **Correct Choices:** Trig Identity **Potential Incorrect Choices:** By Parts, Reverse Chain Rule, U-Substitution
10.	$$\int \ln(300x)\,\mathrm{d}x$$ **Correct Choices:** By Parts **Potential Incorrect Choices:** Reverse Chain Rule, U-Substitution, Algebraic Simplification
11.	$$\int \frac{1}{1 - 10x}\,\mathrm{d}x$$ **Correct Choices:** U-Substitution, Reverse Chain Rule **Potential Incorrect Choices:** Algebraic Simplification, Long Division

12.

$$\int \cos^3(4x)\sin(4x)\,\mathrm{d}x$$

Correct Choices: U-Substitution, Reverse Chain Rule, Trig Identity

Potential Incorrect Choices: By Parts

13.

$$\int x^e + e^x + e^e\,\mathrm{d}x$$

Correct Choices: Reverse Power Rule

Potential Incorrect Choices: Algebraic Simplification

14.

$$\int \frac{1}{3}x^3 - 9x + \pi\,\mathrm{d}x$$

Correct Choices: Reverse Power Rule, U-Substitution Term-by-Term

Potential Incorrect Choices: Algebraic Simplification

15.

$$\int xe^{3x}\,\mathrm{d}x$$

Correct Choices: By Parts

Potential Incorrect Choices: U-Substitution, Reverse Chain Rule

16.

$$\int \frac{1}{100 - 81x^2}\,\mathrm{d}x$$

Correct Choices: Trig Substitution

Potential Incorrect Choices: Algebraic Simplification, Derivative Definition of Natural Log

17.

$$\int \frac{10x^3 - x^7 - 5x^{11}}{5x^8}\,\mathrm{d}x$$

Correct Choices: Algebraic Simplification

Potential Incorrect Choices: Partial Fractions, Long Division

18.

$$\int \frac{12x^3 - 2x^4 - x^5}{3x^4}\,\mathrm{d}x$$

Correct Choices: Algebraic Simplification

Potential Incorrect Choices: Partial Fractions, Long Division

19.

$$\int \frac{1}{x(x+1)(x-3)}\,\mathrm{d}x$$

Correct Choices: Partial Fractions

Potential Incorrect Choices: Long Division, Algebraic Simplification, Trig Substitution

20.

$$\int \frac{6x^3 + 20x^2 - 21}{3x+7}\,\mathrm{d}x$$

Correct Choices: Long Division

Potential Incorrect Choices: Partial Fractions, Algebraic Simplification

AUTHOR'S BIOGRAPHY

Joseph (Joe) Godinez came to the University of Maine from Pittsburgh, Pennsylvania in August of 2020. He has grown to love the state of Maine in a way he never could have imagined. He has met many people who have helped shape and refine his perception of the college experience. Joe enjoys trying out new recipes in the kitchen, expanding his taste in music, going on spur-of-the-moment adventures with friends, and learning as much new stuff as he can along the way.

Along with a Bachelor of Science in Mathematics with Honors, Joe will graduate in May 2024 with a deep appreciation for undergraduate education as a whole. He plans to attend graduate school for Statistics to expand his knowledge of the applications of mathematics. He hopes to one day find his way back to the undergraduate classroom to help educate the world's next generation of professionals and keep evolving his own understanding of what education truly is.