Introduction to Set Theory

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> *Every mathematician agrees that every mathematician must know some set theory; the disagreement begins in trying to decide how much is some - Paul Halmos*

What is a set

A set is a collection of "things".

What "things" you may ask? Almost anything you'd like ...

A collection of three colours, red, blue, and yellow is a set. So is the collection of the names of capital cities of Australia (or the world for that matter).

We write a set by enclosing the collection of "things" inside $\{$brackets$\}$ with commas separating the different objects.

e.g. $\{\text{North}, \text{East}, \text{South}, \text{West}\}$ is a set

Often we consider sets that contain numbers. Other times we consider sets that contain sets!

e.g. $\{17, 42, 108\}$ is also a set

Sets do not care about multiples of the same “thing”, nor do they care about order

e.g. $\{\text{cat}, \text{dog}\}$ is the same as $\{\text{cat}, \text{dog}, \text{cat}\}$

e.g. $\{1, 2, 3\}$ is the same as $\{3, 2, 1\}$

We use the symbol $\in$ to represent "membership" to a set.

We write $1 \in \{1, 2, 3\}$ to say, 1 is a "member of" (or "belongs to") the set $\{1, 2, 3\}$. $4 \notin \{1, 2, 3\}$.

We usually denote a set with a capital letter (usually starting at the start of the alphabet).

We generally denote members of the set with a lower case letter (usually from the end of the alphabet).

For example you will often see something like. Let x be a member of the set A.

If all the members in one set, say set A, also belong to another set, say set B, we say that A is a subset of B. We write this as $A \subseteq B$.

B may very well contain more "things" than A.

For example the set $\{1, 2, 3\} \subseteq \{\cdots, -2, -1, 0, 1, 2, \cdots\}$.

Some famous sets

- The counting numbers $\{0, 1, 2, 3, 4, 5, \cdots\}$, otherwise known as the Natural Numbers, denoted by $\mathbb{N}$
- The Integers $\{\cdots, -2, -1, 0, 1, 2, \cdots\}$ denoted by $\mathbb{Z}$
- The Rational Numbers, i.e. those numbers that can be expressed as a ratio of two integers, denoted by $\mathbb{Q}$
- The Real Numbers, denoted by $\mathbb{R}$
- The set with no members $\{\}$, otherwise known as the Empty Set, denoted by $\emptyset$

What can we do with sets

Create a new set by combining the members of other sets. We call this the Union of sets. We write the union of two sets A and B as $A \cup B$.

e.g. Let $A = \{a, b, c\}$ and let $B = \{x, y\}$, then
$A \cup B = \{a, b, c, x, y\}$

What can we do with sets

Create a new set by including only the common members of other sets. We call this the Intersection of sets. We write the intersection of two sets C and D as $C \cap D$.

e.g. Let $C = \{-2, 0, 2\}$ and let $D = \{-1, 0, 1\}$, then $C \cap D = \{0\}$

What can we do with sets

Count them ... well, kind of, ... We call this the Cardinality of a set. Formally it is not really a "count", but we will get to that another day! The number of elements (the Cardinality) of a set A is is denoted by $|A|$

e.g. Let $A = \{a, b, c\}$, then $|A| = 3$.

We often write a set explicitly, by listing out all of the elements in the set.

This doesn't work when we want to write out infinite sets.

In these cases we use a "generating rule" (or rules) to define how a set is generated.

For example, we could define the even integers the following way:
$\{2x \text{ such that } x \in \mathbb{Z}\}$

Big results in set theory

I'd like to end this session by briefly bringing up some of the most famous results of Set Theory. I want to give the beginning of an answer to the question "Why do we care about sets?"

I don't expect anyone to completely understand any of this, but I think it is important to at least name of some of the most significant results. So that anyone interested can look it up themselves (or talk to me!).

- Formal ways of building the real numbers (via Dedekind cuts for example)
- There are more real numbers than natural numbers (Cantor's Diagonalisation Argument)
- Mathematics doesn't play as nice as we would have liked (Godel's Incompleteness Theorems)

Further reading

I'd recommend the following two resources!

- Stanford Encyclopedia of Philosophy - Set Theory
- Naive Set Theory by Paul Halmos