A Theory of Collusion with Partial Mutual Understanding: Supplemental Appendices

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1 Appendix to Section 4.2

In deriving sufficient conditions for $\left(S^L, S^F\right)$ to be a subgame perfect equilibrium, let us first consider S^L and have ρ denote the lagged maximum price. If $\rho = p^N$ then $S^L\left(p^N\right) = p'$ which is optimal iff p' is at least as profitable as p^N,

$$\begin{aligned} & \pi\left(p', p^N\right) + \delta\pi\left(p^M, p'\right) + \left(\frac{\delta^2}{1-\delta}\right)\pi\left(p^M, p^M\right) \\ \geq\ & \pi\left(p^N, p^N\right) + \delta\pi\left(p', p^N\right) + \delta^2\pi\left(p^M, p'\right) + \left(\frac{\delta^3}{1-\delta}\right)\pi\left(p^M, p^M\right) \end{aligned} \tag{1}$$

and at least as profitable as p^M,

$$\begin{aligned} & \pi\left(p', p^N\right) + \delta\pi\left(p^M, p'\right) + \left(\frac{\delta^2}{1-\delta}\right)\pi\left(p^M, p^M\right) \\ \geq\ & \pi\left(p^M, p^N\right) + \left(\frac{\delta}{1-\delta}\right)\pi\left(p^M, p^M\right). \end{aligned} \tag{2}$$

(1) and (2) can be simplified to:

$$\pi\left(p', p^N\right) + \delta\pi\left(p^M, p'\right) + \delta^2\pi\left(p^M, p^M\right) \geq \pi\left(p^N, p^N\right) + \delta\pi\left(p', p^N\right) + \delta^2\pi\left(p^M, p'\right) \tag{3}$$

$$\pi\left(p', p^N\right) + \delta\pi\left(p^M, p'\right) \geq \pi\left(p^M, p^N\right) + \delta\pi\left(p^M, p^M\right) \tag{4}$$

If $\delta \simeq 1$ then (3) is true, and (4) is true when:

$$\pi\left(p', p^N\right) + \pi\left(p^M, p'\right) > \pi\left(p^M, p^N\right) + \pi\left(p^M, p^M\right) \tag{5}$$

Now suppose $\rho = p'$. $S^L(p') = p^M$ is optimal iff p^M is at least as profitable as p^N,

$$\pi\left(p^M, p'\right) + \left(\frac{\delta}{1-\delta}\right)\pi\left(p^M, p^M\right) \geq \pi\left(p^N, p'\right) + \left(\frac{\delta}{1-\delta}\right)\pi\left(p^N, p^N\right), \tag{6}$$

and at least as profitable as p',

$$\begin{aligned} \pi\left(p^M, p'\right) + \left(\frac{\delta}{1-\delta}\right)\pi\left(p^M, p^M\right) \quad \geq \quad & \pi\left(p', p'\right) + \delta\pi\left(p^M, p'\right) \\ & + \left(\frac{\delta^2}{1-\delta}\right)\pi\left(p^M, p^M\right). \end{aligned} \tag{7}$$

If $\delta \simeq 1$ then (6) and (7) hold. Finally, if $\rho = p^M$ then $S^L\left(p^M\right) = p^M$ is optimal iff:

$$\left(\frac{1}{1-\delta}\right)\pi\left(p^M, p^M\right) \geq \max\left\{\pi\left(p^N, p^M\right), \pi\left(p', p^M\right)\right\} + \left(\frac{\delta}{1-\delta}\right)\pi\left(p^N, p^N\right), \tag{8}$$

which holds if $\delta \simeq 1$. In sum, S^L is subgame perfect if $\delta \simeq 1$ and (5) holds.

Next, let us turn to S^F. If $\rho = p^N$ then $S^F\left(p^N\right) = p^N$ is optimal iff p^N is at least as profitable as p',

$$\pi\left(p^N, p'\right) + \delta\pi\left(p', p^M\right) + \left(\frac{\delta^2}{1-\delta}\right)\pi\left(p^M, p^M\right) \geq \pi\left(p', p'\right) + \delta\pi\left(p', p^M\right) + \left(\frac{\delta^2}{1-\delta}\right)\pi\left(p^M, p^M\right), \tag{9}$$

and is at least as profitable as p^M,

$$\pi\left(p^N, p'\right) + \delta\pi\left(p', p^M\right) + \left(\frac{\delta^2}{1-\delta}\right)\pi\left(p^M, p^M\right) \geq \pi\left(p^M, p'\right) + \left(\frac{\delta}{1-\delta}\right)\pi\left(p^M, p^M\right). \tag{10}$$

Both conditions hold for all δ.[1] If $\rho = p'$ then $S^F(p') = p'$ is optimal iff p' is at least as profitable as p^N,

$$\pi\left(p', p^M\right) + \left(\frac{\delta}{1-\delta}\right)\pi\left(p^M, p^M\right) \geq \pi\left(p^N, p'\right) + \left(\frac{\delta}{1-\delta}\right)\pi\left(p^N, p^N\right), \tag{11}$$

and is at least as profitable as p^M,

$$\pi\left(p', p^M\right) + \left(\frac{\delta}{1-\delta}\right)\pi\left(p^M, p^M\right) \geq \left(\frac{1}{1-\delta}\right)\pi\left(p^M, p^M\right). \tag{12}$$

(11) holds for $\delta \simeq 1$, and (12) holds for all δ. Finally, if $\rho = p^M$ then $S^F\left(p^M\right) = p^M$ is optimal iff (8) is true. In sum, S^F is subgame perfect if $\delta \simeq 1$.

[1] Note that $\pi\left(p^N, p'\right) > \pi\left(p', p'\right)$ for if that was not the case then p' would be a static Nash equilibrium and thereby violation the assumption that p^N is the unique Nash equilibrium. Similarly, it must be true that $\pi\left(p', p^M\right) > \pi\left(p^M, p^M\right)$.

To evaluate when (5) holds, consider:

$$\begin{aligned}
\pi\left(p', p^N\right) + \pi\left(p^M, p'\right) &> \pi\left(p^M, p^N\right) + \pi\left(p^M, p^M\right) \Leftrightarrow \\
\pi\left(\frac{p^M + p^N}{2}, p^N\right) - \pi\left(p^M, p^N\right) &> \pi\left(p^M, p^M\right) - \pi\left(p^M, \frac{p^M + p^N}{2}\right) \Leftrightarrow
\end{aligned}$$

$$-\int_{\frac{p^M+p^N}{2}}^{p^M} \left(\frac{\partial \pi\left(p, p^N\right)}{\partial p_1}\right) dp_1 > \int_{\frac{p^M+p^N}{2}}^{p^M} \left(\frac{\partial \pi\left(p^M, p\right)}{\partial p_2}\right) dp_2. \tag{13}$$

Assuming linear demand and constant marginal cost,

$$\pi\left(p_i, \mathbf{p}_{-i}\right) = \left(a - bp_i + d\left(\frac{1}{n-1}\right)\sum_{j \neq i} p_j\right)\left(p_i - c\right), \text{where } a > bc > 0, b > d > 0,$$

(13) is

$$\begin{aligned}
-\int_{\frac{p^M+p^N}{2}}^{p^M} \left(a + bc - 2bp_1 + dp^N\right) dp_1 &> \int_{\frac{p^M+p^N}{2}}^{p^M} d\left(p^M - c\right) dp_2 \Leftrightarrow \\
-\left(a + bc + dp^N\right)\left(\frac{p^M - p^N}{2}\right) + b\left[\left(p^M\right)^2 - \left(\frac{p^M + p^N}{2}\right)^2\right] &> d\left(p^M - c\right)\left(\frac{p^M - p^N}{2}\right)
\end{aligned}$$

which, after some manipulations, is equivalent to

$$3bp^M + bp^N > 2a + 2bc + 2dp^N + 2dp^M - 2dc. \tag{14}$$

Substituting

$$p^N = \frac{a + bc}{2b - d},\ p^M = \frac{a + (b-d)c}{2(b-d)}$$

and again performing some manipulations, (14) is equivalent to

$$[a + (b-d)c]\left[(6b - 4d)(b-d) + d^2\right] + 2(b - 2d)(b-d)dc > 0. \tag{15}$$

The first term is positive because $b > d$, while the second term is non-negative when $b \geq 2d$. Hence, if products are sufficiently differentiated then (15) is true. When instead $b < 2d$ then (15) holds when $c \simeq 0$. Hence, if cost is sufficiently small then (15) is true

2 Appendix to Section 5.2

The objective is to show that there exists beliefs such that each strategy $T \in \{1, 2, ...\}$ is sequentially rational, and that this set of strategies satisfies A1-A4. The prior beliefs of firm i are assumed to have full support on $T_j \in \{1, 2, ...\}$ but, in order to simplify the analysis, zero probability will be assigned to $T_j = \infty$. (It is explained later that the ensuing analysis is robust to allowing for a small positive probability attached to $T_j = \infty$.) For each strategy in $\{1, 2, ...\}$, prior beliefs on $\{1, 2, ...\}$ are found such that the strategy is sequentially rational.

Given that all elements of $\{1, 2, ...\}$ then satisfy $\mathcal{A}^0$ using beliefs with support $\{1, 2, ...\}$, all of those strategies satisfy $\mathcal{A}^1$ as well and so forth; hence, $\{1, 2, ...\}$ satisfies A1. As these beliefs will be constructed to comply with Bayes Rule, A4 is also satisfied. Finally, A3 is satisfied given each firm uses a strategy from $\{1, 2, ...\}$, prior beliefs have full support on $\{1, 2, ...\}$, and posterior beliefs satisfy Bayes Rule.

Consider the following prior beliefs of firm i on firm j's strategy:

T_j	Prior Probability
1	1/2
2	$(1/2)^2$
$\vdots$	$\vdots$
$t'_j - 1$	$(1/2)^{t'_j - 1}$
t'_j	$(1/2)^{t'_j} (1/x)$
$\vdots$	$\vdots$
$t'_j + x - 1$	$(1/2)^{t'_j} (1/x)$
$t'_j + x$	$(1/2)^{t'_j + 1} (1/x)$
$\vdots$	$\vdots$
$t'_j + 2x - 1$	$(1/2)^{t'_j + 1} (1/x)$
$\vdots$	$\vdots$

(16)

where $t'_j \in \{1, 2, ...\}$. For $T_j \in \left\{1, ..., t'_j - 1\right\}$, the probability assigned to the rival firm using a strategy that has it lead in period T_j is $(1/2)^{T_j}$ and thus is exponentially declining. Starting with period t'_j, the probability assigned over every x periods exponentially decays and that probability mass is uniformly distributed within a window of x periods.

Let T_1 represent the strategy of firm 1 and assume firm 1's prior beliefs on firm 2's strategy are as specified in (16). The proof will first derive sufficient conditions for it to be sequentially rational for firm 1 to wait in period t (assuming neither firm has yet raised price) when $t < t'_2$; in other words, sequential rationality requires $T_1 \geq t'_2$. Next, sufficient conditions are derived for it to be sequentially rational for firm 1 to raise price when $t = t'_2$. Finally, it is shown that if $t > t'_2$ then it is sequentially rational for firm 1 to raise price. Thus, sequential rationality implies $T_1 = t'_2$. As this will be shown for an arbitrary t'_2 then every strategy in $\{1, 2, ...\}$ is sequentially rational for some prior beliefs.

Suppose firm 1's prior beliefs on firm 2's strategy are as specified in (16) and $t'_2 > 1$. (The case of $t'_2 = 1$ is covered when I examine $t = t'_2$.) Consider a strategy for firm 1 with $T_1 > 1$ so that firm 1 does not raise price in period 1. According to (16) with $t'_2 > 1$, firm 1 assigns probability 1/2 to $T_2 = 1$ and, in that event, firm 2 raises price in period 1 so firm 1's payoff is $\pi^F + \left(\frac{\delta}{1-\delta}\right) \pi^M$. Also with probability 1/2, firm 1 believes $T_2 > 1$ in which case firm 1's period 1 profit from $T_1 > 1$ is π^N, while a lower bound on its expected future payoff is $\pi^N / (1 - \delta)$ (which firm 1 can achieve by pricing at p^N in all ensuing periods). Thus, a lower bound on firm 1's expected payoff from $T_1 > 1$ is

$$\left(\frac{1}{2}\right) \left[\pi^F + \left(\frac{\delta}{1-\delta}\right) \pi^M\right] + \left(\frac{1}{2}\right) \left(\frac{1}{1-\delta}\right) \pi^N. \quad (17)$$

In comparison, firm 1's expected payoff from $T_1 = 1$ is

$$\left[\left(\frac{1}{2}\right)\pi^L + \left(\frac{1}{2}\right)\pi^M\right] + \left(\frac{\delta}{1-\delta}\right)\pi^M. \tag{18}$$

Hence, if $t_2' > 1$ then it is sequentially rational for firm 1 not to raise price in period 1 when (17) exceeds (18):

$$\left(\frac{1}{2}\right)\left[\pi^F + \left(\frac{\delta}{1-\delta}\right)\pi^M\right] + \left(\frac{1}{2}\right)\left(\frac{1}{1-\delta}\right)\pi^N > \left(\frac{1}{2}\right)\pi^L + \left(\frac{1}{2}\right)\pi^M + \left(\frac{\delta}{1-\delta}\right)\pi^M \Rightarrow$$

$$1 - \left(\frac{\pi^M - \pi^N}{\pi^F - \pi^L}\right) > \delta. \tag{19}$$

Given the beliefs in (16) with $t_2' > 1$, if (19) holds then $T_1 = 1$ is not sequentially rational for firm 1.

Now suppose it is period 2 and neither firm raised price in period 1. Firm 1 then infers that $T_2 \geq 2$. If prior beliefs are (16) with $t_2' > 2$, Bayes Rule implies firm 1's posterior beliefs are those in (16) divided by the probability that $T_2 \geq 2$ (which is 1/2). By the same analysis which showed that it is not sequentially rational for firm 1 to raise price in period 1 when $t_2' > 1$, it is not sequentially rational for firm 1 to raise price in period 2 when $t_2' > 2$. That this property also holds for period 2 is because the posterior probability that firm 2 raises price in period 2, given it did not raise price in period 1, equals the prior probability that firm 2 raises price in period 1. In fact, this property holds for all $t < t_2'$ so that, if neither firm has raised price come period t, it is not sequentially rational for firm 1 to raise price in period t. In sum, if (19) holds then it follow from prior beliefs (16), posterior beliefs satisfying Bayes Rule, and sequential rationality that $T_1 \geq t_2'$.

Now suppose it is period $t = t_2'$ and neither firm raised price over periods $1, ..., t_2' - 1$. The posterior beliefs of firm 1 on firm 2's strategy are

T_j	Posterior Probability as of Period t_j'
t_j'	$(1/2)(1/x)$
$t_j' + 1$	$(1/2)(1/x)$
$\vdots$	$\vdots$
$t_j' + x - 1$	$(1/2)(1/x)$
$t_j' + x$	$(1/2)^2(1/x)$
$\vdots$	$\vdots$
$t_j' + 2x - 1$	$(1/2)^2(1/x)$
$t_j' + 2x$	$(1/2)^3(1/x)$
$\vdots$	$\vdots$

(20)

As a first step, let us show that $T_1 = T'$ is preferable to $T_1 = T' + 1$ for all $T' \in \{t_2', ..., t_2' + x - 2\}$; that is, it is better to lead in period t than wait and lead in period $t + 1$ for all $t \in \{t_2', ..., t_2' + x - 2\}$. In comparing T' and $T' + 1$, first note that they yield the same profit sequence if $T_2 < T'$ as then firm 2 raises price first. Hence, we can focus on the payoffs associated with when $T_2 \geq T'$. Next note that both strategies always yield the same profits prior to T' and the same profit of π^M starting with period $T'+2$, so we need only consider how expected profits differ in periods T' and $T'+1$. With probability $(1/2)(1/x)$, firm 2's strategy is T' so it raises price in period T' in which case the period T' profit to firm 1 from strategy T' is π^M and from strategy $T'+1$ is π^F; both strategies yield the same profit starting in period $T'+1$. With probability $(1/2)(1/x)$, firm 2's strategy is $T'+1$ in which case the period T' profit to firm 1 from strategy T' is π^L and from strategy $T'+1$ is π^M; both strategies yield the same profit starting in period $T'+1$. And with probability

$$\left(\frac{1}{2}\right)\left(\frac{x-1-(T'+1-t_2')}{x}\right)+\sum_{y=2}^{\infty}\left(\frac{1}{2}\right)^y$$

firm 2's strategy exceeds $T'+1$ in which case firm 1's payoff over periods T' and $T'+1$ from strategy T' is $\pi^L+\delta\pi^M$ and from strategy $T'+1$ is $\pi^N+\delta\pi^L$; and both strategies yield the same profit starting in period $T'+2$. Thus, the difference between the expected payoff from strategy T' and strategy $T'+1$ is

$$\begin{aligned}&\left(\frac{1}{2}\right)\left(\frac{1}{x}\right)\left[\left(\pi^M+\delta\pi^M\right)-\left(\pi^F+\delta\pi^M\right)\right] \\ &+\left(\frac{1}{2}\right)\left(\frac{1}{x}\right)\left[\left(\pi^L+\delta\pi^M\right)-\left(\pi^N+\delta\pi^M\right)\right] \\ &+\left[\left(\frac{1}{2}\right)\left(\frac{x-1-(T'+1-t_2')}{x}\right)+\sum_{y=2}^{\infty}\left(\frac{1}{2}\right)^y\right]\left[\left(\pi^L+\delta\pi^M\right)-\left(\pi^N+\delta\pi^L\right)\right]\end{aligned} \tag{21}$$

For x sufficiently large, the sign of this expression is the same as the sign of the third term. Hence, (21) is positive if x is sufficiently large and

$$\pi^L+\delta\pi^M > \pi^N+\delta\pi^L. \tag{22}$$

(22) is equivalent to

$$\delta > \frac{\pi^N-\pi^L}{\pi^M-\pi^L}. \tag{23}$$

In sum, if x is sufficiently large and (23) holds then firm 1 prefers strategy T' to strategy $T'+1$ for all $T' \in \{t_2', ..., t_2'+x-2\}$.

Thus far, conditions have been derived whereby if $t=t_2'$ then firm 1 prefers to lead than to wait and lead in any period in $\{t_2'+1, ..., t_2'+x-2\}$. The next step is to show that firm 1 prefers to lead in period t_2' ($T_1=t_2'$) than to wait and lead in period $t_2'+x$ ($T_1=t_2'+x$). Note that, as of period t_2', the posterior probability assigned by firm 1 to $T_2=t_2'$ is $(1/2)(1/x)$ and to $T_2=t_2'+x$ is $(1/2)^2(1/x)$. The expected payoff from $T_1=t_2'$ is

$$\left(\frac{1}{2}\right)\left(\frac{1}{x}\right)\left(\frac{\pi^M}{1-\delta}\right)+\left[1-\left(\frac{1}{2}\right)\left(\frac{1}{x}\right)\right]\left(\pi^L+\frac{\delta\pi^M}{1-\delta}\right). \tag{24}$$

With probability $(1/2)(1/x)$, firm 2 also raises price in period t_2' so firm 1's current and future profit is π^M. With probability $1-(1/2)(1/x)$, firm 2 does not raise price in period t_2'

so firm 1's current profit is π^L and its future profit stream is π^M. The expected payoff from $T_1 = t_2' + x$ is

$$\begin{aligned}
&\left(\frac{1}{2}\right)\left(\frac{1}{x}\right)\left(\pi^F + \frac{\delta\pi^M}{1-\delta}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{x}\right)\left(\pi^N + \delta\pi^F + \frac{\delta^2\pi^M}{1-\delta}\right) + \\
&\cdots + \left(\frac{1}{2}\right)\left(\frac{1}{x}\right)\left(\pi^N + \delta\pi^N + \cdots + \delta^{x-2}\pi^N + \delta^{x-1}\pi^F + \frac{\delta^x\pi^M}{1-\delta}\right) \\
&+ \left(\frac{1}{2}\right)^2\left(\frac{1}{x}\right)\left(\pi^N + \delta\pi^N + \cdots + \delta^{x-1}\pi^N + \frac{\delta^x\pi^M}{1-\delta}\right) \\
&+ \left[1 - \left(\frac{1}{2}\right) - \left(\frac{1}{2}\right)^2\left(\frac{1}{x}\right)\right]\left(\pi^N + \delta\pi^N + \cdots + \delta^{x-1}\pi^N + \delta^x\pi^L + \frac{\delta^{x+1}\pi^M}{1-\delta}\right).
\end{aligned}$$

The first term is the probability that $T_2 = t_2'$ multiplied by the payoff in that event, the second term is the probability that $T_2 = t_2' + 1$ multiplied by the payoff in that event, and so forth; the penultimate term is the probability that $T_2 = t_2' + x$ multiplied by the payoff in that event, and the final term is the probability that $T_2 > t_2' + x$ multiplied by the payoff in that event. Collecting common profit terms,

$$\begin{aligned}
&\left(\frac{1}{2}\right)\left(\frac{1}{x}\right)\left(1 + \delta + \cdots + \delta^{x-1}\right)\pi^F \\
&+ \left(\frac{1}{2}\right)\left(\frac{1}{x}\right)\left(1 + \delta + \cdots + \delta^{x-1}\right)\frac{\delta\pi^M}{1-\delta} \\
&+ \left(\frac{1}{2}\right)\left(\frac{1}{x}\right)\left[1 + (1+\delta) + \cdots + \left(1 + \delta + \cdots + \delta^{x-2}\right)\right]\pi^N \\
&+ \left(\frac{1}{4x}\right)\left(\pi^N + \delta\pi^N + \cdots + \delta^{x-1}\pi^N + \frac{\delta^x\pi^M}{1-\delta}\right) \\
&+ \left(\frac{2x-1}{4x}\right)\left(\pi^N + \delta\pi^N + \cdots + \delta^{x-1}\pi^N + \delta^x\pi^L + \frac{\delta^{x+1}\pi^M}{1-\delta}\right)
\end{aligned} \tag{25}$$

In evaluating the third term, note that

$$\begin{aligned}
&1 + (1+\delta) + \cdots + \left(1 + \delta + \cdots + \delta^{x-2}\right) \\
=\ &\sum_{y=1}^{x-1}\left(\frac{1-\delta^y}{1-\delta}\right) = \left(\frac{1}{1-\delta}\right)\sum_{y=1}^{x-1}(1-\delta^y) \\
=\ &\left(\frac{1}{1-\delta}\right)\left((x-1) - \sum_{y=1}^{x-1}\delta^y\right) = \left(\frac{1}{1-\delta}\right)\left((x-1) - \delta\left(\frac{1-\delta^{x-1}}{1-\delta}\right)\right)
\end{aligned} \tag{26}$$

Substituting (26) and simplifying, (25) becomes

$$
\begin{aligned}
&\left(\frac{1}{2}\right)\left(\frac{1}{x}\right)\left(\frac{1-\delta^{x}}{1-\delta}\right)\pi^{F}+\left(\frac{1}{2}\right)\left(\frac{1}{x}\right)\left(\frac{1-\delta^{x}}{1-\delta}\right)\frac{\delta\pi^{M}}{1-\delta} \\
&+\left(\frac{1}{2}\right)\left(\frac{1}{x}\right)\left(\frac{1}{1-\delta}\right)\left((x-1)-\delta\left(\frac{1-\delta^{x-1}}{1-\delta}\right)\right)\pi^{N} \\
&+\left(\frac{1}{4x}\right)\left(\left(\frac{1-\delta^{x}}{1-\delta}\right)\pi^{N}+\frac{\delta^{x}\pi^{M}}{1-\delta}\right)+\left(\frac{2x-1}{4x}\right)\left(\left(\frac{1-\delta^{x}}{1-\delta}\right)\pi^{N}+\delta^{x}\pi^{L}+\frac{\delta^{x+1}\pi^{M}}{1-\delta}\right)
\end{aligned}
\tag{27}
$$

Using (24) and (27), the expected payoff from $T_1 = t_2'$ exceeds that from $T_1 = t_2' + x$ when

$$
\begin{aligned}
&\left(\frac{1}{2}\right)\left(\frac{1}{x}\right)\left(\frac{\pi^{M}}{1-\delta}\right)+\left[1-\left(\frac{1}{2}\right)\left(\frac{1}{x}\right)\right]\left(\pi^{L}+\frac{\delta\pi^{M}}{1-\delta}\right) \\
> &\left(\frac{1}{2}\right)\left(\frac{1}{x}\right)\left(\frac{1-\delta^{x}}{1-\delta}\right)\pi^{F}+\left(\frac{1}{2}\right)\left(\frac{1}{x}\right)\left(\frac{1-\delta^{x}}{1-\delta}\right)\frac{\delta\pi^{M}}{1-\delta} \\
&+\left(\frac{1}{2}\right)\left(\frac{1}{x}\right)\left(\frac{1}{1-\delta}\right)\left((x-1)-\delta\left(\frac{1-\delta^{x-1}}{1-\delta}\right)\right)\pi^{N} \\
&+\left(\frac{1}{4x}\right)\left(\left(\frac{1-\delta^{x}}{1-\delta}\right)\pi^{N}+\frac{\delta^{x}\pi^{M}}{1-\delta}\right)+\left(\frac{2x-1}{4x}\right)\left(\left(\frac{1-\delta^{x}}{1-\delta}\right)\pi^{N}+\delta^{x}\pi^{L}+\frac{\delta^{x+1}\pi^{M}}{1-\delta}\right)
\end{aligned}
\tag{28}
$$

Letting $x \to \infty$, (28) is

$$\pi^{L}+\frac{\delta\pi^{M}}{1-\delta}>\left(\frac{1}{1-\delta}\right)\pi^{N},$$

which is equivalent to (23).

In summing up the previous two steps, if (23) holds then, for x sufficiently large, at period $t = t_2'$: 1) firm 1 prefers $T_1 = t_2'$ to $T_1 = t_2' + 1$, prefers $T_1 = t_2' + 1$ to $T_1 = t_2' + 2$, ..., prefers $T_1 = t_2' + x - 2$ to $T_1 = t_2' + x - 1$; and 2) firm 1 prefers $T_1 = t_2'$ to $T_1 = t_2' + x$. By the structure of prior (and posterior beliefs), the analysis is the same starting from period $t_2' + x$ so that: 3) firm 1 prefers $T_1 = t_2' + x$ to $T_1 = t_2' + x + 1$, prefers $T_1 = t_2' + x + 1$ to $T_1 = t_2' + x + 2$, ..., prefers $T_1 = t_2' + 2x - 2$ to $T_1 = t_2' + 2x - 1$. By transitivity and (1)-(3), firm 1 prefers $T_1 = t_2'$ to T_1 for all $T_1 \in \{t_2' + 1, ..., t_2' + 2x - 1\}$. Iterating, firm 1 prefers $T_1 = t_2'$ to T_1 for all $T_1 > t_2'$.

In sum, if (19) and (23) hold then, for x sufficiently large, the sequentially rational strategy for firm 1 is $T_1 = t_2'$. Given that this argument works for all $t_2' \in \{1, 2, ...\}$, every $T_1 \in \{1, 2, ...\}$ is sequentially rational. Combining (19) and (23), it is required that

$$\delta \in \left(\frac{\pi^{N}-\pi^{L}}{\pi^{M}-\pi^{L}},1-\left(\frac{\pi^{M}-\pi^{N}}{\pi^{F}-\pi^{L}}\right)\right). \tag{29}$$

Note that there exist values for the discount factor whereby this condition holds because

$$
\begin{aligned}
1-\left(\frac{\pi^{M}-\pi^{N}}{\pi^{F}-\pi^{L}}\right) &> \frac{\pi^{N}-\pi^{L}}{\pi^{M}-\pi^{L}} \Leftrightarrow \\
\left(\pi^{M}-\pi^{L}\right)\left(\pi^{F}-\pi^{L}\right)-\left(\pi^{M}-\pi^{L}\right)\left(\pi^{M}-\pi^{N}\right) &> \left(\pi^{N}-\pi^{L}\right)\left(\pi^{F}-\pi^{L}\right) \Leftrightarrow \\
\left(\pi^{M}-\pi^{N}\right)\left(\pi^{F}-\pi^{L}\right) &> \left(\pi^{M}-\pi^{N}\right)\left(\pi^{M}-\pi^{L}\right) \Leftrightarrow \pi^{F}>\pi^{M},
\end{aligned}
$$

which is true because $\pi^M > \pi^N$ and $\pi^F > \pi^M$.

In concluding, let us explain why the analysis is robust to allowing firm i's prior beliefs assign a small positive probability to $T_j = \infty$. Modify the prior beliefs in (16) so that probability $\kappa \in (0, 1)$ is assigned to $T_j = \infty$ and the probabilities for all other T_j are scaled by $1 - \kappa$. Prior beliefs are now:

T_j	Prior Probability
∞	κ
1	$(1/2)(1-\kappa)$
2	$(1/2)^2(1-\kappa)$
$\vdots$	$\vdots$
$t'_j - 1$	$(1/2)^{t'_j-1}(1-\kappa)$
t'_j	$(1/2)^{t'_j}(1/x)(1-\kappa)$
$\vdots$	$\vdots$
$t'_j + x - 1$	$(1/2)^{t'_j}(1/x)(1-\kappa)$
$t'_j + x$	$(1/2)^{t'_j+1}(1/x)(1-\kappa)$
$\vdots$	$\vdots$
$t'_j + 2x - 1$	$(1/2)^{t'_j+1}(1/x)(1-\kappa)$
$\vdots$	$\vdots$

Now that there is some prior probability that firm j will never lead ($T_j = \infty$), firm i will have a stronger incentive to lead rather than wait. However, as long as κ is small relative to $1/t'_j$ - so that the posterior probability that $T_j = \infty$ is sufficiently small for $t < t'_j$ - then firm i will continue to prefer to wait for all $t < t'_j$. Thus, the sequential rationality of not leading before t'_j is robust to $\kappa > 0$ and small. Turning to the analysis that proves it is sequentially rational for firm i to lead for $t \geq t'_j$, it is reinforced when $\kappa > 0$. A firm will be more inclined to lead when it assigns positive probability to the rival firm never leading. While the associated analysis required that x is sufficiently large, note that x does not need to be sufficiently large relative to $1/\kappa$. For the proof to go through, we just need that κ is small relative to $1/t'_j$.