Stochastic Calculus I Notes

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Chapter 1 - Binomial No-Arbitrage Pricing

1.1 - The One Period Model

No Arbitrage: Refers to the idea that there is no 'free-money' in the market (i.e. stock prices don't go negative).

A stock price at time in is $S_n(H/T)$ and depends only on S_{n-1} and the coin toss H or T. The probability of H is p and T is $q = 1 - p$. We'll also call proportion that the stock increases in outcome H as $u = \frac{S_1(H)}{S_0}$ and decrease as $d = \frac{S_1(T)}{S_0}$. Often $d = 1/u$

The **interest rate** r is defined by the yield of $1 + r$ at time one given an investment of 1 at time 0.

No-Arbitrage in the math (1.1.1) is the assumption that

$$0 < d < 1 + r < u$$

European Call Option: Owner may buy one share of stock at time 1 for **strike price** K. Assume $S_1(T) < K < S_1(H)$. If H occurs, the option can be **exercised** yielding profit $S_1(H) - K$. So, the option at time 1 is worth $(S_1 - K)^+ = \max\{S_1(H) - K, 0\}$.

Example 1.1.1
Let $S(0) = 4, u = 2, d = 1/2, r = 1/4$. Then $S_1(H) = 8$, and $S_1(T)2$. Suppose $K = 5$, the initial account is $X_0 = 1.2$ and we buy $\Delta_0 = 1/2$ shares at time 0. So, we must borrow .8 to buy, giving us *cash position* $X_0 - \Delta_0 S_0 = -.8$. At time one this becomes $(1 + r)(X_0 - \Delta_0 S_0) = -1$ or a debt of 1. Also at time one our assets will become $(1/2)S_1(H) = 4$ or $(1/2)S_1(T) = 1$. Thus we have either:

$$\begin{cases} X_1(H) = .5S_1(H) + (1 + r)(X_0 - \Delta_0 S_0) = 3 \\ X_1(T) = .5S_1(T) + (1 + r)(X_0 - \Delta_0 S_0) = 0 \end{cases}$$

Since the value of the portfolio X agree with the value of the option, we have *replicated* the option by trading in money markets and stocks. So, $X_0 = 1.2$ is the **no-arbitrage price of the option at time 0**. If a seller sold the option at 1.2, and someone bought it, they could pay out the option using 1.2 and put the remaining .01 into the money market. If it was sold at 1.19 then the buyer could set up the opposite of the option, buy the option, and place the remaining .01 in an independent money market account. This is an arbitrage.

Assumption made in Ex 1.1.1:

1. Stocks can be subdivided
2. Investing interest rate is the same as borrowing
3. Purchase price of stock is same as selling (zero *bid-ask spread*)
4. A stock can only take on two possible positions at end of time 1.

The first three bullets are the **Black-Scholes-Merton option-pricing formula**.

A **derivative security** is a security that pays $V_1(H)$ at time 1 if H occurs and $V_1(T)$ if tails. A European call option is a kind of derivative security. The **European put option** is another derivative which pays off $(K - S_1)^+$ at time 1 where K is constant (note this is inverse of the call). Another is **forward contract** with value $S_1 - K$ at time 1.

Derivative price V_0 for a derivative paying $V_1(H), V_1(T)$, we replicate the option as in Ex 1.1.1. The value of the position at time 1 is

$$X_1 = \Delta_0 S_1 + (1+r)(X_0 - \Delta_0 S_0) = (1+r)X_0 + \Delta_0(S_1 - (1+r)S_0)$$

To balance this so that $X_1(H) = V_1(H)$ and $X_1(T) = V_1(T)$ (our given options), we choose initial X_0 and Δ_0. That is we solve:

$$X_0 + \Delta_0 \left(\frac{1}{1+r} S_1(H/T) - S_0 \right) = \frac{1}{1+r} V_1(H/T)$$

Two equations, we two unknowns X_0, Δ_0. The solution is, if $\tilde{p} = \frac{1+r-d}{u-d}$ and $\tilde{q} = \frac{u-1-r}{u-d}$ (p,q called the *risk neutral probabilities*):

$$X_0 = \frac{1}{1+r} \left(\tilde{p} V_1(H) + \tilde{q} V_1(T) \right)$$

Note this gives a way for pricing securities: $X_0 = V_0$ where V_0 should be the price of a security that pays out $V_1(H), V_1(T)$ at time 1 (this is the risk-neutral pricing formula for the one-period binomial model). This also creates the:

Delta Hedging Formula:

$$\Delta_0 = \frac{V_1(H) - V_1(T)}{S_1(H) - S_1(T)}$$

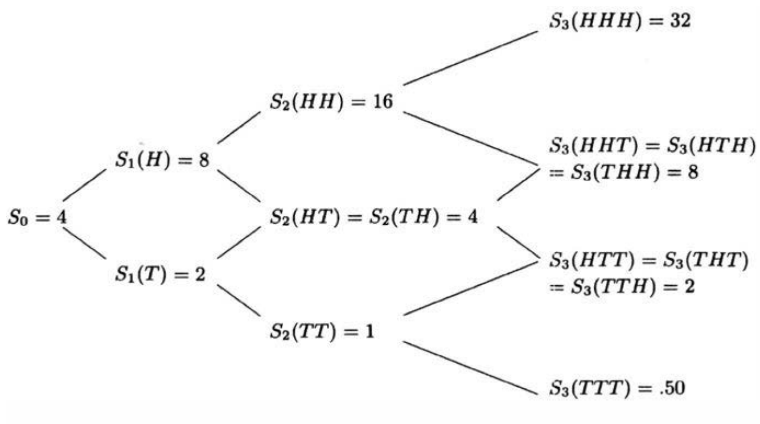

Fig. 1.2.2. A particular three-period model.

Figure 1: Fig 1.2.2 ($u = d^{-1} = 2, S_0 = 4, r = 1/4$)

2.2 - Multiperiod Binomial Model

For a two period model, we extend the reasoning from the one period model, this time doing the analysis twice on the 3 possible outcomes of time 2 (TT, HT = TH, and HH). We can generalize the findings for time n:

The Wealth Equation: $X_{n+1} = \Delta_n S_{n+1} + (1+r)(X_n - \Delta_n S_n)$
This equation depends on the implicit random variables (the arguments of X_i, S_i so on).

Note: The contract of a derivative security is contingent and depends on the outcome of the coin toss. Additionally, we denote V_n as the no arbitrage price of the derivative security at time n.

Theorem 1 *(1.2.2)*
Using the no-Arbitrage assumptions and the formulas for $\tilde{p}$ and $\tilde{q}$, let V_N be the random variable depending on $\omega_1, \omega_2, ..., \omega_n$ coin tosses. Define recursively backwards the random variables $V_{n-1}, ..., V_0$ by

$$V_n(\omega_1, ..., \omega_n) = \frac{1}{1+r}\left(\tilde{p}V_{n+1}(\omega_1, ..., \omega_n, H) + \tilde{q}V_{n+1}(\omega_1, ..., \omega_n, T)\right)$$

and define

$$\Delta_n(\omega_1, ..., \omega_n) = \frac{V_{n+1}(\omega_1, ..., \omega_n, H) - V_{n+1}(\omega_1, ..., \omega_n, T)}{S_{n+1}(\omega_1, ..., \omega_n, H) - S_{n+1}(\omega_1, ..., \omega_n, T)}$$

Then the options $V_0, ..., V_n$ are priced with no arbitrage, that is if $X_0 = V_0$ define forward the portfolio values $X_1, X_2, ..., X_n$ then

$$X_N(\omega_1, ..., \omega_n) = V_N(\omega_1, ..., \omega_n) \text{ for all } \omega_1, ..., \omega_n$$

The multi-period binomial model is *complete*, i.e. every derivative can be recreated by investing in the stock and money markets.

Example 1.2.4: Look-back option (refer to book, pg 14).

1.3 - Computational Considerations

The multi-period binomial model is impractical computationally as it grows exponentially.

Example 1.3.1 Consider the paths in Fig 1.2.2, which lead to 4 different outcomes. The simplification is that $HHT = THH = HTH$ and $HTT = TTH = THT$ for the purposes of the three period model.