SYK Two-Point Explorations

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1. Construct a Majorana operator following the Jordan-Wigner Transformation:

$$\chi_i \rightarrow \begin{cases} \left(\prod_{j<k} \sigma_j^z\right) \sigma_k^x, \ i \text{ even} \\ \left(\prod_{j<k} \sigma_j^z\right) \sigma_k^y, \ i \text{ odd} \end{cases}$$

Easy does it. Now use these operators to construct the SYK Hamiltonian

$$H = (i^2) \sum_{i<j<k<l} \frac{J_{ijkl}}{4!} \chi_i \chi_j \chi_k \chi_l$$

Where

$$E[J_{ijkl}] = 0, \ E[J_{ijkl}^2] = \frac{6J^2}{N^3}$$

The equation for our two-point function:

$$G_R(\tau) = \frac{\text{Tr}[e^{(-\beta+\tau)H} \chi_i(\tau) e^{-\tau H} \chi_i(0)]}{\text{Tr}[e^{(-\beta+\tau)H} e^{-\tau H}]}$$

Notice we're working with χ_i. We make $\beta = 1$ for simplicity. Let us see what this looks like for N = 12 Majorana fermions.

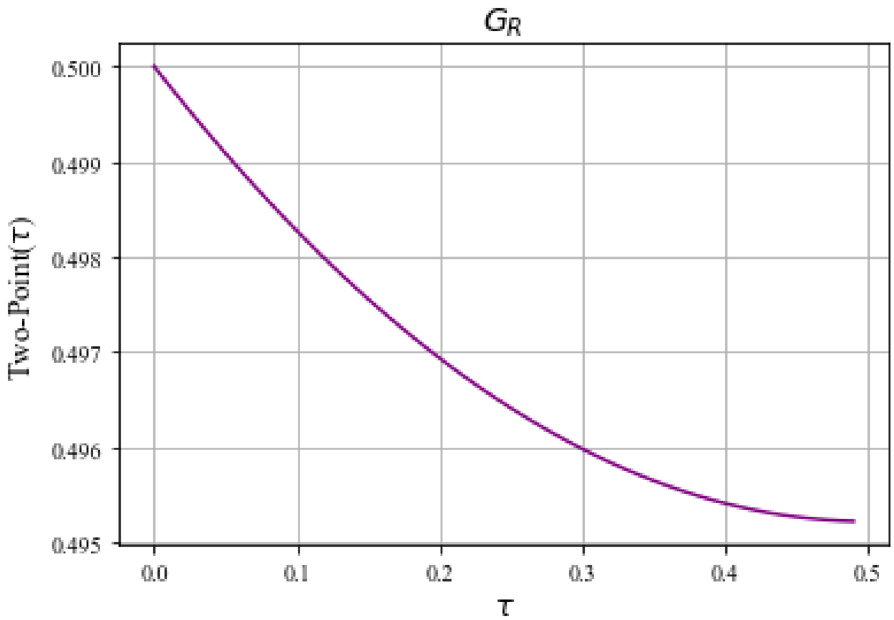

Figure 1: Normalized Two-Point Result. 12 Majorana operators with 4-interaction SYK Hamiltonian. The number of Majoranas will be increased. This is promising testament to a working script for small N. $\beta = 1$.

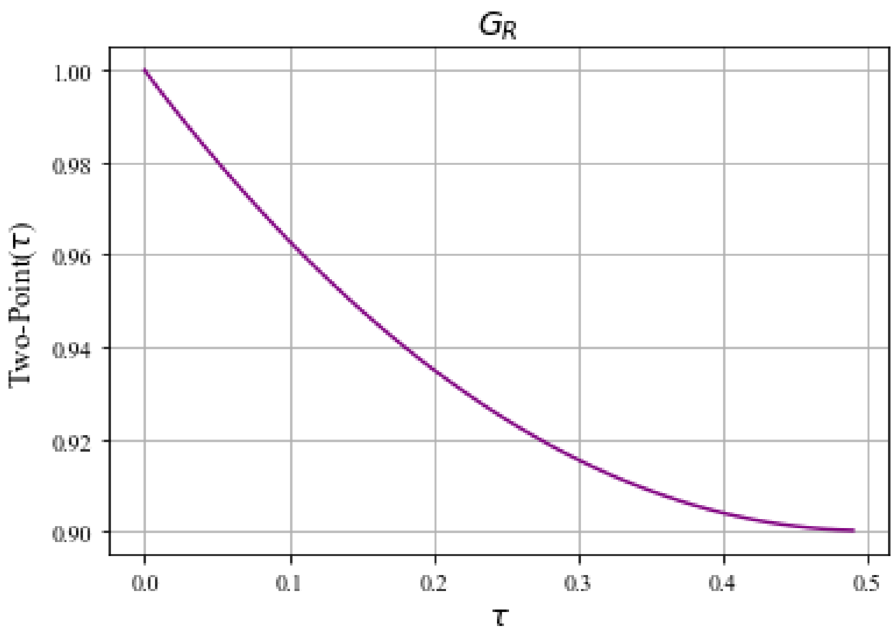

Figure 2: Normalized Two-Point Result. 12 Majorana operators with 4-interaction SYK Hamiltonian. The number of Majoranas will be increased. Notice here that a scaling factor $\frac{1}{\sqrt{2}}$ wasn't applied to the Majorana operators, making the Two-point's initial value 1 instead of .5. A nearly identical form is achieved. $\beta = 1$.

We're now considering $N = 20$ as sufficiently large for revealing dynamics in the large N limit. It's interesting to see whether this Majorana fermion system's two-point dynamics will converge when using a coupling distribution that isn't the classical J_{ijkl} used in the SYK model. Let's introduce some arbitrary bimodal distribution straddling zero.

Peak 1:

$$E[\gamma_{ijkl}] = \frac{-2.5 * 6\ J^2}{N^3},\ E[\gamma_{ijkl}^2] = \frac{6J^2}{N^3}$$

and Peak 2:

$$E[\omega_{ijkl}] = \frac{2.5 * 6J^2}{N^3},\ E[\omega_{ijkl}^2] = \frac{6J^2}{N^3}$$

Very straightforward. The numerical procedure is the same as the conventional SYK simulation with $N = 20$. Let's compare these results to see if the two-point dynamics are different.

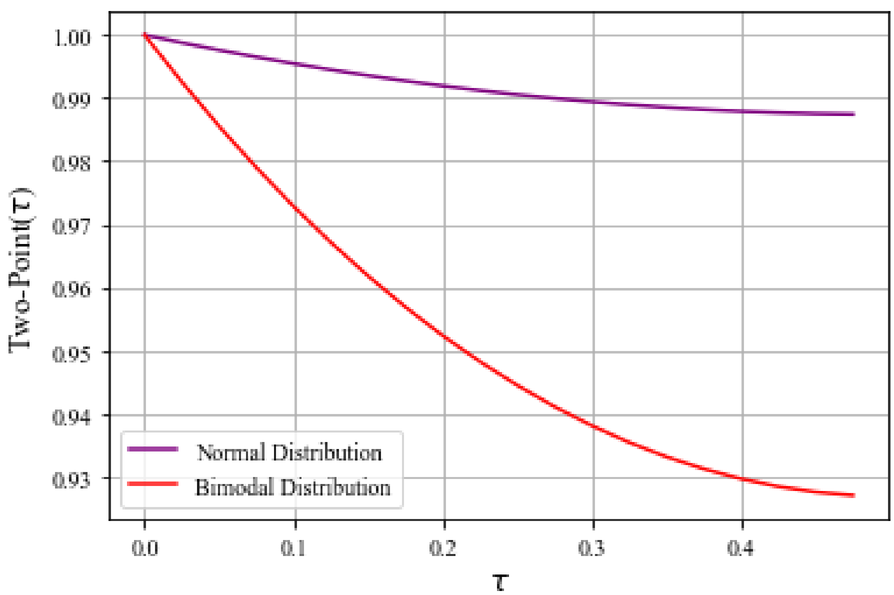

Figure 3: $N = 20$ two-point function comparison between classical normal distribution and the bimodal distribution constructed as above. There exists a discrepancy between these two functions in the limit of expected convergence across coupling distributions. $\beta = 1$.

Another thing to test is a non-normal coupling distribution with the same variance. The most simple is a uniform distribution with characteristics as follows:

$$E[\kappa_{ijkl}] = 0,\ E[\kappa_{ijkl}^2] = \frac{6J^2}{N^3}$$

Now we have a coupling distribution that has symmetry around 0 and the same variance as the classical SYK model.

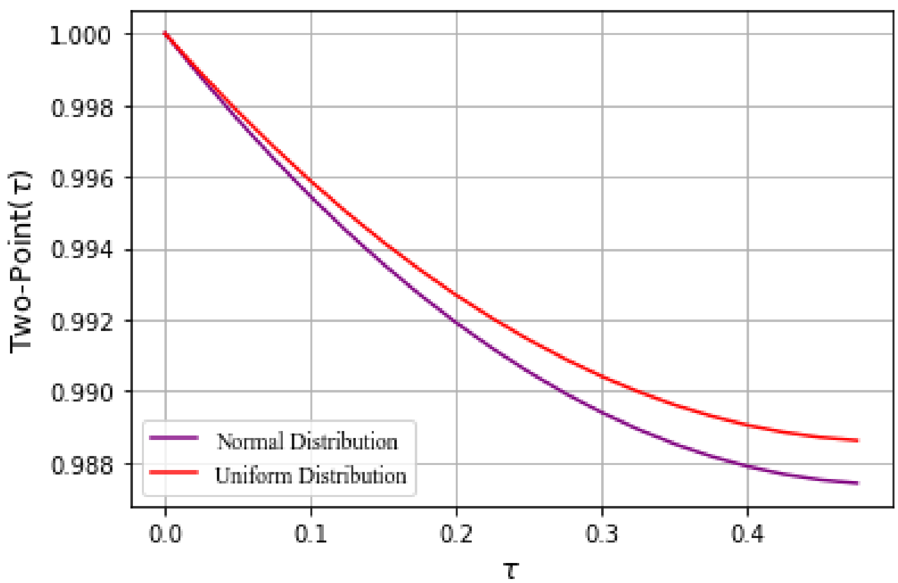

Figure 4: $N = 20$ two-point function comparison between classical normal coupling distribution and a uniform coupling distribution with the same expected value and variance. The two-point function in the figure is averaged over all $N = 20$ Majorana fermions. There exists a slight discrepancy between these two functions in the limit of expected convergence across coupling distributions. $\beta = 1$.