Matrix Inverse

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Determinants

The determinant of a 3×3 matrix $A = [a_{ij}]$ is denoted by

$$\det(A), \quad |A|, \quad \text{or} \quad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix},$$

and is defined to be the number

$$a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

Recall that the transpose of A is the matrix A^T obtained by interchanging the rows and columns of A. The determinant of A^T is therefore given by

$$|A^T| = \begin{vmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{vmatrix}$$

Note that the determinant of A transposed is the same as the determinant of A

Problem. Compute the determinant of $A = \begin{bmatrix} 5 & 2 & 4 \\ 3 & 0 & 1 \\ -3 & -1 & -2 \end{bmatrix}$.

Solution. Expanding along the second row gives

$$|A| = -3\begin{vmatrix} 2 & 4 \\ -1 & -2 \end{vmatrix} + 0 - 1\begin{vmatrix} 5 & 2 \\ -3 & -1 \end{vmatrix} = -3(-4+4) - (-5+6) = -1.$$

The sign of the coefficient $(-1)^{i+j} = \pm 1$ is the ijth element of the following alternating pattern:

$$\begin{bmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}.$$

If A is an n x n matrix, then the matrix of cofactors is the matrix obtained by replacing each element of A by its corresponding cofactor.

The adjoint adj(A) of A is the transposition of the matrix of co factors

The matrix of cofactors is the matrix of 2x2 determinants as in the below diagram

The adjoint is given by

$$\operatorname{adj}(A) = \begin{bmatrix} \begin{vmatrix} 0 & 2 \\ 3 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 2 & 3 \end{vmatrix} \\ -\begin{vmatrix} 1 & 4 \\ 3 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 4 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} \\ \begin{vmatrix} 1 & 4 \\ 0 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 4 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} \end{bmatrix}^T$$

Matrix Inverse

Matrix inverse. A square matrix A is invertible if and only if its determinant is non-zero. If A is invertible then

$$A^{-1} = \frac{1}{|A|}\operatorname{adj}(A).$$

The matrix inverse of A is found by taking 1 over the determinant of A multiplied by the adjoint of A.

The matrix inverse can only be found for square matrices. It is possible for A.B to equal I even when the matrices are not square, however this does not prove the fact that B is the inverse of A, as it is possible to arrive at this result with non-square matrices.

Problem. Find the inverse of the matrix $A = \begin{bmatrix} 2 & 1 & 4 \\ 1 & 0 & 2 \\ 2 & 3 & 1 \end{bmatrix}$ and hence solve the system of equations

$$\begin{array}{rrrcl} 2x & +y & +4z & = & 2 \\ x & & +2z & = & 3 \\ 2x & +3y & +z & = & -6. \end{array}$$

Solution. The determinant of A is most easily calculated from the expansion by the second row:

$$|A| = (-1)\begin{vmatrix} 1 & 4 \\ 3 & 1 \end{vmatrix} + 0 - 2\begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} = (-1)(1-12) - 2(6-2) = 3.$$

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The adjoint is given by

$$\begin{aligned} \operatorname{adj}(A) &= \begin{bmatrix} \begin{vmatrix} 0 & 2 \\ 3 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 2 & 3 \end{vmatrix} \\ -\begin{vmatrix} 1 & 4 \\ 3 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 4 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} \\ \begin{vmatrix} 1 & 4 \\ 0 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 4 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} \end{bmatrix}^T . \\ &= \begin{bmatrix} -6 & 3 & 3 \\ 11 & -6 & -4 \\ 2 & 0 & -1 \end{bmatrix}^T \\ &= \begin{bmatrix} -6 & 11 & 2 \\ 3 & -6 & 0 \\ 3 & -4 & -1 \end{bmatrix} \end{aligned}$$

Hence

$$A^{-1} = \frac{1}{|A|}\operatorname{adj}(A) = \frac{1}{3}\begin{bmatrix} -6 & 11 & 2 \\ 3 & -6 & 0 \\ 3 & -4 & -1 \end{bmatrix}.$$

The system of equations can be written as

$$A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -6 \end{bmatrix},$$

and multiplying on the left of each side by A^{-1} gives

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1}\begin{bmatrix} 2 \\ 3 \\ -6 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -6 & 11 & 2 \\ 3 & -6 & 0 \\ 3 & -4 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ -6 \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \\ 0 \end{bmatrix}.$$

Hence the solution is $x = 3$, $y = -4$ and $z = 0$.