Linear Equations

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A system of two linear equations in unknowns x_1 and x_2:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned}$$

can be written in the following equivalent matrix form:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

To solve a system of equations in this form we can use a series of elementary row operations

Elementary row operations on matrices

(1) interchange two rows;
(2) multiply a row by a nonzero number;
(3) add a multiple of one row to another.

Each of these can also be regarded as an operation carried out on the rows of the matrices A and b. One advantage of this matrix approach is that we do not need to copy out the unknowns over and over again. The matrices A and b are usually combined to give the **augmented matrix** of the system.

System of equations	Augmented matrix
$\begin{aligned} x_1 - x_2 + x_3 &= 1 \\ x_1 + x_2 + 2x_3 &= 0 \\ 2x_1 - x_2 + 3x_3 &= 2 \end{aligned}$	$\left[\begin{array}{rrr\|r} 1 & -1 & 1 & 1 \\ 1 & 1 & 2 & 0 \\ 2 & -1 & 3 & 2 \end{array}\right]$

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Row Equivalence

We say that two matrices A and B are row equivalent and we write A ~ B if A can be transformed to B using a finite (possibly 0) number of elementary row operations.

Row Echelon Form

A matrix is said to be in row echelon form if the first nonzero entry in each row is further to the right than the first nonzero entry in the previous row. A system of linear equations can be solved by using elementary row operations to reduce the augmented matrix to row echelon form.

Elementary matrices

Elementary row operations can be performed by multiplying a matrix on the left by a suitable "elementary matrix".

Calculation of Inverses

An important consequence of this is that we can use elementary row operations to calculate inverse matrices. If a sequence of elementary row operations transforms a square matrix A into I, then A is invertible and the same sequence transforms I into A−1

Solution. To perform elementary row operations simultaneously on the given matrix and I, we write them both as a single 3×6 matrix.

$$\left[\begin{array}{rrr|rrr} 2 & -1 & 4 & 1 & 0 & 0 \\ 4 & 0 & 2 & 0 & 1 & 0 \\ 3 & -2 & 7 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{rrr|rrr} 1 & -\frac{1}{2} & 2 & \frac{1}{2} & 0 & 0 \\ 4 & 0 & 2 & 0 & 1 & 0 \\ 3 & -2 & 7 & 0 & 0 & 1 \end{array}\right] \qquad \tfrac{1}{2}\,\text{row1}$$

$$\sim \left[\begin{array}{rrr|rrr} 1 & -\frac{1}{2} & 2 & \frac{1}{2} & 0 & 0 \\ 0 & 2 & -6 & -2 & 1 & 0 \\ 0 & -\frac{1}{2} & 1 & -\frac{3}{2} & 0 & 1 \end{array}\right] \qquad \begin{array}{l} \text{row2} - 4\text{row1} \\ \text{row3} - 3\text{row1} \end{array}$$

$$\sim \left[\begin{array}{rrr|rrr} 1 & -\frac{1}{2} & 2 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & -3 & -1 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 1 & -\frac{3}{2} & 0 & 1 \end{array}\right] \qquad (\text{row2})/2$$

$$\left[\begin{array}{rrr|rrr} 1 & -\frac{1}{2} & 2 & \frac{1}{2} & 0 & 0 \end{array}\right.$$

$$\begin{aligned}
&\sim \left[\begin{array}{rrr|rrr} 0 & 1 & -3 & -1 & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} & -2 & \frac{1}{4} & 1 \end{array}\right] && \text{row3} + \tfrac{1}{2}\text{row2} \\
&\sim \left[\begin{array}{rrr|rrr} 1 & -\frac{1}{2} & 2 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & -3 & -1 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 4 & -\frac{1}{2} & -2 \end{array}\right] && -2\text{row3} \\
&\sim \left[\begin{array}{rrr|rrr} 1 & -\frac{1}{2} & 0 & -\frac{15}{2} & 1 & 4 \\ 0 & 1 & 0 & 11 & -1 & -6 \\ 0 & 0 & 1 & 4 & -\frac{1}{2} & -2 \end{array}\right] && \begin{array}{r}\text{row1} - 2\text{row3} \\ \text{row2} + 3\text{row3}\end{array} \\
&\sim \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & -2 & \frac{1}{2} & 1 \\ 0 & 1 & 0 & 11 & -1 & -6 \\ 0 & 0 & 1 & 4 & -\frac{1}{2} & -2 \end{array}\right] && \text{row1} + \tfrac{1}{2}\text{row2}.
\end{aligned}$$

Hence

$$\begin{bmatrix} 2 & -1 & 4 \\ 4 & 0 & 2 \\ 3 & -2 & 7 \end{bmatrix}^{-1} = \begin{bmatrix} -2 & \frac{1}{2} & 1 \\ 11 & -1 & -6 \\ 4 & -\frac{1}{2} & -2 \end{bmatrix}.$$