Matrices and Linear Transformations

07 February 2020 13:35

A function $T : \mathbb{R}^m \to \mathbb{R}^n$ is called a **linear transformation** if, for all $\underline{u}, \underline{v} \in \mathbb{R}^m$ and all $\lambda \in \mathbb{R}$, we have :

$$T(\underline{u}+\underline{v}) = T(\underline{u}) + T(\underline{v}) \quad \text{and} \quad T(\lambda \underline{u}) = \lambda T(\underline{u}).$$
$$(preservation\ of\ addition) \qquad (preservation\ of\ scalar\ multiplication)$$

T(vector 0) is vector 0. Meaning the transformation of the zero vector of length m is the zero vector of length n.

Problem. Which of the following functions are linear transformations:
(1) $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (x + y, x - y)$;
(2) $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (x + 1, y - 1)$;
(3) $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (x^2, y^2)$.

Solution.
(1) Let $\underline{a} = (a_1, a_2)$ and $\underline{b} = (b_1, b_2)$ then $T(\underline{a}) = (a_1 + a_2, a_1 - a_2)$ and $T(\underline{b}) = (b_1 + b_2, b_1 - b_2)$ so

$$\begin{aligned} T(\underline{a}+\underline{b}) &= T(a_1 + b_1, a_2 + b_2) \\ &= (a_1 + b_1 + a_2 + b_2, a_1 + b_1 - a_2 - b_2) \\ &= (a_1 + a_2, a_1 - a_2) + (b_1 + b_2, b_1 - b_2) \\ &= T(\underline{a}) + T(\underline{b}) \end{aligned}$$

and

$$\begin{aligned} T(\lambda\underline{a}) &= T(\lambda a_1, \lambda a_2) \\ &= (\lambda a_1 + \lambda a_2, \lambda a_1 - \lambda a_2) \\ &= \lambda(a_1 + a_2, a_1 - a_2) \\ &= \lambda T(\underline{a}). \end{aligned}$$

So T is a linear transformation.

(2) Here we have $T(0, 0) = (1, -1)$, i.e. $T(\underline{0}) \neq \underline{0}$, so T cannot be linear.

(3) Here we have

$$\begin{aligned} T((1, 0) + (1, 0)) &= T((2, 0)) = (4, 0), \\ T(1, 0) + T(1, 0) &= (1, 0) + (1, 0) = (2, 0), \end{aligned}$$

so T is not a linear transformation.

1

Here we see how to identify whether or not a function is a linear transformat property above in which the transformation of two vectors T(u + v) is the sam the vectors individually, added together – T(u) + T(v)

In the first example, this is checked by creating two vectors and substituing tl transformation equation. We see that T(a + b) = T(a) + T(b) so the transforma transformation.

With the second one, we see that T(0) != 0 so it cannot be linear

The 3rd T(a + b) is not equal to T(a) + T(b) when we sub in two vectors, so it is transformation

Projection

Projection

We define the projection of $\underline{x} \in \mathbb{R}^2$ onto nonzero vector $\underline{u} \in \mathbb{R}^2$ to be the vector $P_{\underline{u}}(\underline{x})$ with the properties:

(1) $P_{\underline{u}}(\underline{x})$ is a multiple of u;
(2) $\underline{x} - P_{\underline{u}}(\underline{x})$ is perpendicular to $\underline{u}$.

The projection of vector x onto vector u can be regarded as a function P such that

$$P_{\underline{u}}(\underline{x}) = \left(\frac{\underline{x}.\underline{u}}{|\underline{u}|^2}\right)\underline{u}$$

Using the rules of transformations that are discussed earlier, we can now show that the projection function is a linear transformation.

It is now easy to verify using properties of the scalar product that $P_{\underline{u}}$ is a linear transformation.
For any $\underline{x}, \underline{y} \in \mathbb{R}^2$ and any $\lambda \in \mathbb{R}$:

$$P_{\underline{u}}(\underline{x}+\underline{y}) = \left(\frac{(\underline{x}+\underline{y}).\underline{u}}{|\underline{u}|^2}\right)\underline{u} = \left(\frac{\underline{x}.\underline{u}}{|\underline{u}|^2}\right)\underline{u} + \left(\frac{\underline{y}.\underline{u}}{|\underline{u}|^2}\right)\underline{u} = P_{\underline{u}}(\underline{x}) + P_{\underline{u}}(\underline{y}),$$
$$P_{\underline{u}}(\lambda\underline{x}) = \left(\frac{(\lambda\underline{x}).\underline{u}}{|\underline{u}|^2}\right)\underline{u} = \lambda\left(\frac{\underline{x}.\underline{u}}{|\underline{u}|^2}\right)\underline{u}.$$

Every matrix defines a linear transformation

Theorem: Every matrix defines a linear transformation.

Equivalently, given $n \times m$ matrix M, the function $T : \mathbb{R}^m \to \mathbb{R}^n$ defined by $T(\underline{x}) = M\underline{x}$ for all $\underline{x} \in \mathbb{R}^m$ is a linear transformation.

Proof:

This is trivially true since

$$M(\underline{u}+\underline{v}) = M\underline{u} + M\underline{v} \quad \text{and} \quad M(\lambda\underline{u}) = \lambda(M\underline{u})$$

for any $m \times 1$ column vectors $\underline{u}$ and $\underline{v}$ and any $\lambda \in \mathbb{R}$.