Coordinates and Basis Change

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Let $V = \{\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_n\}$ be a basis for $\mathbb{R}^n$. If $\underline{x} \in \mathbb{R}^n$ then $\underline{x}$ has a unique expansion as a linear combination

$$\underline{x} = \alpha_1\underline{v}_1 + \alpha_2\underline{v}_2 + \cdots + \alpha_n\underline{v}_n$$

of these basis vectors. The coefficients $\alpha_1, \alpha_2, \ldots, \alpha_n$ are called the *coordinates* of $\underline{x}$ with respect to the basis V.

When you have a basis of a set of vectors V, then any vector X in the same subspace has a unique expansion as a linear combination of the vectors in V. The coefficients of a1, a2 .. an of the vectors in V are called the coordinates of x with respect to V.

Problem. Let $E = \{(1,0),(0,1)\}$ be the standard basis for $\mathbb{R}^2$ and let V be the basis $\{(1,-1),(2,3)\}$. Find the coordinates of the vector $(1,2)$ with respect to E and with respect to V.

Solution. We have

$$(1,2) = 1(1,0) + 2(0,1)$$

so the coordinates of $(1,2)$ with respect to E are $[1,2]$. Also

$$(1,2) = \alpha(1,-1) + \beta(2,3) \iff \left\{ \begin{array}{rcl} \alpha + 2\beta &=& 1 \\ -\alpha + 3\beta &=& 2 \end{array} \right. \iff \left\{ \begin{array}{rcl} \beta &=& 3/5 \\ \alpha &=& -1/5 \end{array} \right.$$

so the coordinates of $(1,2)$ with respect to V are $[-1/5, 3/5]$.

The Matrix of a Linear Transformation

Problem. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation given by

$$T(x,y) = (y, x+y, x) \qquad (x,y) \in \mathbb{R}^2.$$

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Find the matrix of T with respect to the basis $V = \{(1,1),(1,-1)\}$ of $\mathbb{R}^2$ and the basis $W = \{(1,2,0),(2,1,0),(0,0,1)\}$ of $\mathbb{R}^3$. If a vector $\underline{u}$ has coordinates $[2,3]$ with respect to V, then what are the coordinates of $T(\underline{u})$ with respect to W?

Solution. We have

$$T(1,1) = (1,2,1) = (1,2,0) + 0(2,1,0) + (0,0,1)$$

and

$$T(1,-1) = (-1,0,1) = \alpha(1,2,0) + \beta(2,1,0) + \gamma(0,0,1)$$

where

$$\begin{array}{rcl} \alpha + 2\beta &=& -1 \\ 2\alpha + \beta &=& 0 \\ \gamma &=& 1 \end{array} \qquad \text{or} \qquad \begin{array}{rcl} \alpha &=& 1/3 \\ \beta &=& -2/3 \\ \gamma &=& 1\,. \end{array}$$

We have now expressed the images of the vectors in V as linear combinations of the vectors in W:

$$\begin{array}{rcl} T(1,1) &=& 1(1,2,0) + 0(2,1,0) + 1(0,0,1) \\ T(1,-1) &=& \frac{1}{3}(1,2,0) - \frac{2}{3}(2,1,0) + 1(0,0,1) \end{array}$$

so the matrix of T with respect to V and W is

$$\begin{bmatrix} 1 & 1/3 \\ 0 & -2/3 \\ 1 & 1 \end{bmatrix}.$$

If $\underline{u}$ has coordinates $[2,3]$ with respect to V then

$$\begin{bmatrix} 1 & 1/3 \\ 0 & -2/3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix}$$

so the coordinates of $T(\underline{u})$ with respect to W are $[3,-2,5]$.

Finding the matrix of a function T with respect to a basis V in R2 and basis W in F

A function that can be expressed as a matrix is by definition a linear transformat that the vectors in V can be expressed in some linear combination of the vectors

Subbing the values of v1 into the function we see T(1,1) = (1,2,1) which can be e: combination of the vectors in W. We can solve simultaneously to find alpha, bet

Once we have found the values of alpha, beta and gamma for both vectors in V, column vectors to form the matrix T.

Change of Basis

If we have two different bases in R n then a given vector will have different coordinates with respect to each basis. The change in coordinates can be described by a transition matrix

The coordinates of a vector are the coefficients of the linear combination of the base vectors required to make the vector.

If two bases exist for a subspace then two different sets of co-ordinates exist to make a vector in the subspace. Change of basis handles the conversion between these two sets of co-ordinates.

The set of co-ordinates A can be converted to the set B by multiplying A by a transition matrix M

$$\begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{bmatrix} = M \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$$

Problem. Find the transition matrix from the basis $\{(1,-1),(1,1)\}$ to the basis $\{(1,2),(2,3)\}$ of $\mathbb{R}^2$. If a vector has coordinates $[2,-1]$ with respect to the first basis, what are its coordinates

We express each basis vector as a linear with respect to the second?

Solution. We have

$$(1,-1)=\alpha(1,2)+\beta(2,3) \iff \left\{\begin{array}{rcl}\alpha+2\beta &=& 1\\ 2\alpha+3\beta &=& -1\end{array}\right. \iff \left\{\begin{array}{rcl}\alpha &=& -5\\ \beta &=& 3\end{array}\right.$$

$$(1,1)=\alpha(1,2)+\beta(2,3) \iff \left\{\begin{array}{rcl}\alpha+2\beta &=& 1\\ 2\alpha+3\beta &=& 1\end{array}\right. \iff \left\{\begin{array}{rcl}\alpha &=& -1\\ \beta &=& 1\end{array}\right.$$

Thus the old basis vectors can be written as linear combinations of the new ones as:

$$\begin{array}{rcl}(1,-1 &=& -5(1,2)+3(2,3)\\ (1,1) &=& -1(1,2)+1(2,3)\end{array}$$

and so the transition matrix is $\begin{bmatrix}-5 & -1\\ 3 & 1\end{bmatrix}$.

If a vector has coordinates $[2,-1]$ with respect to the first basis then since

$$\begin{bmatrix}-5 & -1\\ 3 & 1\end{bmatrix}\begin{bmatrix}2\\ -1\end{bmatrix}=\begin{bmatrix}-9\\ 5\end{bmatrix},$$

the coordinates with respect to the second basis are $[-9,5]$.

combination of the vectors in the other basis. Once the co-efficients for every vector in basis A have been found, these co-efficients can be expressed as a transformation matrix.