Recurrence

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A sequence defined by its predecessors
A rule which defines each term using the preceding terms

Linear Recurrence

We are looking to solve linear recurrences with constant coefficients.

Auxiliary equation. The equation $\lambda^2 + a\lambda + b = 0$ is called the *auxiliary equation* of the recurrence $x_n + ax_{n-1} + bx_{n-2} = 0$.

If the auxiliary equation has two distinct solutions λ_1 and λ_2, then it is easy to verify that $x_n = A\lambda_1^n + B\lambda_2^n$ is a solution of the recurrence for any constants A and B. If the first two terms of the sequence (x_n) are given, then they can be used to find the values of A and B.

If the auxiliary equation has only a single solution for λ then $\sqrt{a^2 - 4b} = 0$, so $b = a^2/4$ and $\lambda = -a/2$. In this case $x_n = A\lambda^n$ is *a* solution of the recurrence for any A, but is not the *general* solution.

Non-homogenous Recurrences

Solution of the recurrence $x_n + ax_{n-1} + bx_{n-2} = f(n)$.

(1) Find the general solution $x_n = h_n$ of the homogeneous recurrence:

$$x_n + ax_{n-1} + bx_{n-2} = 0$$

(solution will contain two arbitrary constants).

(2) Find *any* particular solution $x_n = p_n$ of the original recurrence:

$$x_n + ax_{n-1} + bx_{n-2} = f(n).$$

(3) The general solution of the original recurrence is then given by $x_n = h_n + p_n$.

Finding a solution for f(n) = k is not generally easy.

- If f is constant, say $f(n) = k$ for all n, then it is easy to find a constant particular solution (provided $1 + a + b \neq 0$). For if $x_n = c$ for all n, then substituting into the recurrence $x_n + ax_{n-1} + bx_{n-2} = k$ gives $c + ac + bc = k$ or $c = k/(1 + a + b)$.
- For a more complicated (polynomial) $f(n)$, try to find a particular solution which is also a polynomial in n, e.g., try $x_n = k$ or $x_n = k_1 n + k_2$ or $x_n = k_1 n^2 + k_2 n + k_3, \ldots$.

Example. Find the general solution of the recurrence

$$x_n - 10.1x_{n-1} + x_{n-2} = -2.7n.$$

Solution. The homogeneous recurrence $x_n - 10.1x_{n-1} + x_{n-2} = 0$ has auxiliary equation

$$\lambda^2 - 10.1\lambda + 1 = 0 \qquad \text{or} \qquad (\lambda - 10)(\lambda - 1/10) = 0,$$

and so has general solution $x_n = A(10^n) + B/10^n$.

To find a particular solution of

$$x_n - 10.1x_{n-1} + x_{n-2} = -2.7n,$$

we try $x_n = Cn + D$. This gives

$$Cn + D - 10.1(C(n-1) + D) + C(n-2) + D = -8.1Cn + 8.1D - 8.1C = -2.7n.$$

Since this holds *for all* n, we have

$$-8.1C = -2.7,\ 8.1D - 8.1C = 0 \qquad \text{so} \qquad C = D = 1/3.$$

Hence the general solution of the recurrence is

$$x_n = 10^n A + \frac{B}{10^n} + \frac{n+1}{3}.$$