Eigenvectors

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An eigenvector of a square matrix is a vector whose direction does not change when multiplied on the left by A

The column vector A multiplied by the vector r is equal to some scalar multiple of the vector r. Meaning the direction hasn't changed but the length may have. The scalar is called the eigenvalue.

$A\,\underline{r} = \lambda\underline{r}$ where $\lambda \in \mathbb{R}$ and is called an *eigenvalue.*

(the magnitude may change)

Equivalently, the matrix multiplied by the vector r is equal to some lambda multiplied by the vector r.

Hence we say that lambda is the eigenvalue corresponding to eigenvector r.

Note that we exclude the trivial case where vector r = 0.

Example
If we have the 2x2 matrix A;

$$A = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$$

has eigenvectors

$$\underline{r}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \underline{r}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

with associated eigenvalues $\lambda_1 = 8$ and $\lambda_2 = 2$, since

$$\begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 8 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix},$$

$$\text{i.e. } A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 8\begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad A\begin{bmatrix} 1 \\ -1 \end{bmatrix} = 2\begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Theorem

Lambda is an eigenvalue of the matrix A if and only if the determinant of the matrix A minus lambda * the identity matrix is equal to 0.

This equation is called the characteristic equation of A
It is a polynomial of degree n in lambda

Proof:

$$\begin{aligned} & \lambda \text{ is an eigenvalue of } A \\ \Leftrightarrow \quad & A\underline{v} = \lambda\,\underline{v} \text{ for some nonzero } \underline{v} \\ \Leftrightarrow \quad & (A - \lambda\,I)\underline{v} = \underline{0} \text{ for some nonzero } \underline{v} \\ \Leftrightarrow \quad & |\,A - \lambda\,I\,| = 0. \end{aligned}$$

The eigenvalues of a real matrix may be complex. For example, consider the rotation matrix corresponding to theta = 3pi / 2

Let A be a square matrix of order n. A number λ is called an *eigenvalue* of A if $A\underline{v} = \lambda\underline{v}$ for some non-zero column vector $\underline{v}$. When this is the case we call $\underline{v}$ an *eigenvector* of A corresponding to λ.

Problem. Find the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} -5 & 3 \\ 6 & -2 \end{bmatrix}$.

Solution. We have

$$A - \lambda I = \begin{bmatrix} -5 & 3 \\ 6 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -5-\lambda & 3 \\ 6 & -2-\lambda \end{bmatrix}$$

$$\begin{aligned} \text{so} \quad |A - \lambda I| = 0 &\iff (-5-\lambda)(-2-\lambda) - 18 = 0 \\ &\iff \lambda^2 + 7\lambda - 8 = 0 \\ &\iff (\lambda - 1)(\lambda + 8) = 0 \\ &\iff \lambda = 1 \text{ or } \lambda = -8. \end{aligned}$$

Hence the eigenvalues of A are 1 and -8. To find the eigenvectors we consider each eigenvalue in turn.

$$\begin{aligned} (x, y) \text{ is an eigenvector with eigenvalue } 1 &\iff \begin{bmatrix} -5 & 3 \\ 6 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 1 \begin{bmatrix} x \\ y \end{bmatrix} \\ &\iff \begin{cases} -6x + 3y = 0 \\ 6x - 3y = 0 \end{cases} \\ &\iff y = 2x. \end{aligned}$$

Hence any nonzero vector of the form $(x, 2x)$ or, equivalently, any nonzero multiple of $(1, 2)$ is an eigenvector corresponding to the eigenvalue 1.

$$\begin{aligned} (x, y) \text{ is an eigenvector with eigenvalue } -8 &\iff \begin{bmatrix} -5 & 3 \\ 6 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = -8 \begin{bmatrix} x \\ y \end{bmatrix} \\ &\iff \begin{cases} 3x + 3y = 0 \\ 6x + 6y = 0 \end{cases} \\ &\iff y = -x \end{aligned}$$

Hence any nonzero vector of the form $(x, -x)$ or equivalently any nonzero multiple of $(1, -1)$ is an eigenvector corresponding to the eigenvalue -8.

Diagonal Matrices
All non-zero entries lie on the diagonal.
Multiplying is quick and easy

Problem. Find a diagonal matrix D and an invertible matrix P such that $P^{-1}AP = D$, where A is the matrix:

$$\begin{bmatrix} 4 & 2 & 2 \\ -1 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix}.$$

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Solution. The characteristic equation, $|A - \lambda I| = 0$, is

$$\begin{vmatrix} 4-\lambda & 2 & 2 \\ -1 & 1-\lambda & -1 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0$$

or
$$(2-\lambda)((4-\lambda)(1-\lambda) + 2) = 0,$$

i.e.
$$0 = (2-\lambda)(\lambda^2 - 5\lambda + 6) = (2-\lambda)(\lambda - 2)(\lambda - 3),$$

so the eigenvalues of A are 2, 2 and 3.

Now (x, y, z) is an eigenvector corresponding to the eigenvalue 2 if and only if

$$\begin{aligned} \begin{bmatrix} 4 & 2 & 2 \\ -1 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 2 \begin{bmatrix} x \\ y \\ z \end{bmatrix} &\iff \begin{cases} 2x + 2y + 2z = 0 \\ -x - y - z = 0 \\ 0 = 0 \end{cases} \\ &\iff x + y + z = 0. \end{aligned}$$

So any vector of the form $(x, y, -x-y)$ or $x(1, 0, -1) + y(0, 1, -1)$ where x and y are not both zero is an eigenvector corresponding to the repeated eigenvalue 2. In this case, $(1, 0, -1)$ and $(0, 1, -1)$ may be taken as independent eigenvectors corresponding to the repeated eigenvalue 2.

Finally, (x, y, z) is an eigenvector corresponding to the eigenvalue 3 if and only if

$$\begin{aligned} \begin{bmatrix} 4 & 2 & 2 \\ -1 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 3 \begin{bmatrix} x \\ y \\ z \end{bmatrix} &\iff \begin{cases} x + 2y + 2z = 0 \\ -x - 2y - z = 0 \\ z = 0 \end{cases} \\ &\iff \begin{cases} z = 0 \\ x = -2y. \end{cases} \end{aligned}$$

Hence any vector of the form $(-2y, y, 0)$ or $y(-2, 1, 0)$, where $y \neq 0$, is an eigenvector corresponding to the eigenvalue 3.

Hence we can say that $P^{-1}AP = D$ where:

$$P = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ -1 & -1 & 0 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$