Series

19 February 2020 15:06

(1) a sequence (an) called the sequence of terms,
(2) a sequence (sn) called the sequence of partial sums

Sum of a Series. If the sequence (s_n) of partial sums converges to s then we say that the series $\sum a_n$ *converges to the sum s* and write

$$\sum_{n=0}^{\infty} a_n = s.$$

Otherwise we say that the series *diverges.*

Example. The ***geometric series*** $\sum r^n$ converges to $1/(1-r)$ provided that $|r| < 1$. To see this, we calculate the nth partial sum:

$$s_n = 1 + r + r^2 + \cdots + r^n.$$

Multiplying both sides by r gives

$$rs_n = r + r^2 + \cdots + r^n + r^{n+1}.$$

Subtracting these equations gives:

$$s_n - rs_n = 1 - r^{n+1} \qquad \text{or} \qquad s_n = \frac{1-r^{n+1}}{1-r}.$$

If $|r| < 1$ then $r^{n+1} \to 0$ so $s_n \to 1/(1-r)$. Hence

$$\boxed{\sum_{n=0}^{\infty} r^n = \frac{1}{1-r} \qquad \text{for } |r| < 1.}$$

To find the value that the sequence coverges to w can use the rul that r^n converges to 1/(1-r) when |r| < 1

Problem. Show that the series $\sum_{n=1}^{\infty} 1/(4n^2 - 1)$ converges and find its sum.

Solution. Taking partial fractions we have

$$\frac{1}{4n^2-1} = \frac{1}{(2n-1)(2n+1)} = \frac{1/2}{2n-1} - \frac{1/2}{2n+1}.$$

1

Hence the n^{th} partial sum is

$$\begin{aligned} s_n &= \textstyle\sum_{r=1}^{n} \frac{1}{4r^2-1} = \frac{1}{2}\left(\sum_{r=1}^{n} \frac{1}{2r-1} - \sum_{r=1}^{n} \frac{1}{2r+1}\right) \\ &= \textstyle\frac{1}{2}\left(1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-3} + \frac{1}{2n-1} - \frac{1}{3} - \frac{1}{5} - \cdots - \frac{1}{2n-3} - \frac{1}{2n-1} - \frac{1}{2n+1}\right) \\ &= \frac{1}{2}\left(1 - \frac{1}{2n+1}\right). \end{aligned}$$

Hence $s_n \to 1/2$, so the series converges and

$$\sum_{n=1}^{\infty} \frac{1}{4n^2-1} = \frac{1}{2}.$$

Basic Properties of Convergent Series

(1) **Sum Rule:** *if $\sum a_n$ converges to s and $\sum b_n$ converges to t, then $\sum(a_n + b_n)$ converges to $s + t$.*
(2) **Multiple rule:** *if $\sum a_n$ converges to s and $\lambda \in \mathbb{R}$, then $\sum \lambda a_n$ converges to λs.*
(3) *If the series $\sum a_n$ converges, then the sequence (a_n) converges to 0.*
(4) *If the series $\sum |a_n|$ converges, then the series $\sum a_n$ also converges.*

The Comparison Test

Often it is easier to determine whether or not a series converges rather than its actual sum.

The Comparison Test

Suppose that $0 \le a_n \le b_n$ for every n, then
(1) *if $\sum b_n$ converges then so does $\sum a_n$;*
(2) *if $\sum a_n$ diverges then so does $\sum b_n$.*

Problem. Show that the series $\sum_{n=1}^{\infty} 1/n^k$ diverges for $k \le 1$.

Solution. It was shown earlier that $\sum 1/n$ diverges. For $k \le 1$, we have $0 \le 1/n \le 1/n^k$, so by the second part of the Comparison Test, the series $\sum 1/n^k$ diverges since $\sum 1/n$ diverges.

3

We can get the equation into

Problem. Determine whether the following series converge or diverge.

$$\text{(i)} \sum \frac{n+2}{n^3-n^2+1}, \qquad \text{(ii)} \sum \frac{n^2+4}{2n^3-n+1}.$$

Solution. For each $n \geq 2$ we have

$$n+2 \leq n+2n = 3n,$$
$$n^3-n^2+1 \geq n^3-n^2 = n^2(n-1) \geq n^2(n/2) = n^3/2,$$

since $n-1 \geq n/2$ for $n \geq 2$.

So
$$0 \leq \frac{n+2}{n^3-n^2+1} \leq \frac{3n}{n^3/2} = \frac{6}{n^2}.$$

$\sum 1/n^2$ converges, so $\sum 6/n^2$ converges by the Multiple rule, and now the series (i) converges by the Comparison Test.

inequality form and then using the comparison rule deduce that the series converges. In this instance, the limits of the equation are reduced to 0 as the lower bound and 6/n^2 on the upper bound. Since we know 1/n^2 converges, by the comparison rule the original equation also converges.

The Ratio Test

The Ratio Test.
If $|a_{n+1}/a_n| \to L$ then
(1) *if $0 \leq L < 1$ then the series $\sum a_n$ converges,*
(2) *if $L > 1$ (or L is ∞) then the series $\sum a_n$ diverges,*
(3) *if $L = 1$ then the test is inconclusive and the series may or may not converge.*

Problem. Does the series $\sum a_n$ converge or diverge when $a_n = n/2^n$?

Solution. Trying the Ratio Test we get

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{n+1}{2^{n+1}} \times \frac{2^n}{n}\right| = \frac{n+1}{2n} = \frac{1+1/n}{2} \to \frac{1}{2}.$$

Since this limit is less than 1 we conclude that the series $\sum n/2^n$ converges.

Problem. Show that the series $\sum n^k r^n$ converges where $k > 0$ and $0 \leq r < 1$.

Solution. We have

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)^k r^{n+1}}{n^k r^n}\right| = r\left(1+\frac{1}{n}\right)^k,$$

which converges to r (< 1). So the series $\sum n^k r^n$ converges by the Ratio Test.

Summary of the ratio test:

- To prove whether a series converges or diverges, especially useful for those series containing factorials:
- Take the absolute value of term a_n+1 divided by term a_n
- Simplify and observe the value of the result as n increases.
- If the result is < 1 then the series converges. If the result is > 1 then the series diverges. If the result is exactly 21 then the result is inconclusive

Basic Series

Basic Convergent Series
(1) $\sum_{n=0}^{\infty} r^n = 1/(1-r)$ for any r with $|r| < 1$.
(2) the series $\sum 1/n^k$ converges for any $k > 1$.
(3) the series $\sum n^k r^n$ converges for $k > 0$ and $|r| < 1$.
(4) $\sum_{n=0}^{\infty} c^n/n! = e^c$ for any $c \in \mathbb{R}$.

Basic Divergent Series
- the series $\sum 1/n^k$ diverges for any $k \leq 1$

Power Series. A *power series* is a series of the form $\sum a_n x^n$ where it is usual to start the index at $n = 0$. So the first term is a_0, the second $a_1 x$, and so on.

Lemma. *If $\sum a_n R^n$ converges for some $R \geq 0$, then $\sum a_n x^n$ converges for every x with $|x| < R$.*

Proof. If the series $\sum a_n R^n$ converges then the sequence $(a_n R^n)$ converges to 0 (and so is bounded). Hence there is a B with $|a_n R^n| < B$ for every n. Now

$$|a_n x^n| \leq |a_n R^n||x/R|^n \leq B|x/R|^n \qquad \text{for all } n.$$

If $|x| < R$ then $\sum |x/R|^n$ is a convergent geometric series, and therefore $\sum |a_n x^n|$ converges by the Comparison Test. It follows that $\sum a_n x^n$ converges. □

Radius of Convergence

Radius of Convergence. We say that $R \geq 0$ is the *radius of convergence* of a power series $\sum a_n x^n$ if the series converges whenever $|x| < R$ and diverges whenever $|x| > R$. If the series converges for *all* x we say the radius of convergence is ∞. If the power series $\sum a_n x^n$ has radius of convergence R, then it defines a function $f : (-R, R) \to \mathbb{R}$ given by

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \qquad \text{for all } x \in (-R, R).$$

5

Basic Properties of Power Series

Let

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} a_n x^n \qquad x \in (-R_1, R_1), \\ g(x) &= \sum_{n=0}^{\infty} b_n x^n \qquad x \in (-R_2, R_2), \end{aligned}$$

where $R_1, R_2 > 0$, and let R be the minimum of R_1 and R_2.
Then:

(1) if $f(x) = g(x)$ for all $x \in (-R, R)$, then $a_n = b_n$ for each n.

Also for any $x \in (-R, R)$:

(2) **sum rule**:

$$f(x) + g(x) = \sum_{n=0}^{\infty} (a_n + b_n) x^n;$$

(3) **multiple rule**:

$$\lambda f(x) = \sum_{n=0}^{\infty} \lambda a_n x^n \text{ for any } \lambda \in \mathbb{R};$$

(4) **product rule**:

$$f(x)g(x) = \sum_{n=0}^{\infty} (a_0 b_n + a_1 b_{n-1} + \cdots + a_{n-1} b_1 + a_n b_0) x^n.$$