Decimal Representation of Real Numbers

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Recall that if $|r| < 1$ then the *geometric series* (Note 17) $\sum_{n\geq 0} r^n$ has sum $1/(1-r)$, i.e.,

$$1 + r + r^2 + r^3 + \cdots = \frac{1}{1-r}.$$

If $a_1, a_2, \cdots$ is a sequence of decimal digits, so that each a_i belongs to $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, then $.a_1a_2a_3\cdots$ denotes the real number

$$\frac{a_1}{10} + \frac{a_2}{10^2} + \frac{a_3}{10^3} + \cdots .$$

Representing Repeating Decimals as Quotient

Repeating decimal as rational number. A repeating decimal represents a rational number and can be expressed as a quotient of two integers by summing an appropriate geometric series as the following problem illustrates.

Problem. Express the repeating decimal $0.59\dot{1}0\dot{2}$ as the quotient of two integers.

Solution. We have

$$\begin{aligned} 0.59\dot{1}0\dot{2} &= 0.59102102\cdots = \tfrac{59}{100} + \tfrac{102}{10^5} + \tfrac{102}{10^8} + \tfrac{102}{10^{11}} + \cdots \\ &= \tfrac{59}{100} + \tfrac{102}{10^5}\left(1 + \tfrac{1}{10^3} + \tfrac{1}{10^6} + \cdots\right) = \tfrac{59}{100} + \tfrac{102}{10^5}\left(\tfrac{1}{1-1/10^3}\right) \\ &= \tfrac{59}{100} + \tfrac{102}{100}\left(\tfrac{1}{10^3-1}\right) = \tfrac{59}{100} + \tfrac{102}{100}\,\tfrac{1}{999} = \tfrac{59\times 999+102}{99900} \\ &= 59043/99900. \end{aligned}$$

Note that termianting decimal can also be represented as a non-terminating decimal e.g. 1 = 0.99999..

Decimal Expansion of a Rational Number

Problem

Find the decimal expansion of $5/14$.

Solution

$$\begin{aligned} 50 &= 14 \cdot 3 + 8 \\ 80 &= 14 \cdot 5 + 10 \\ 100 &= 14 \cdot 7 + 2 \\ 20 &= 14 \cdot 1 + 6 \\ 60 &= 14 \cdot 4 + 4 \\ 40 &= 14 \cdot 2 + 12 \\ 120 &= 14 \cdot 8 + 8 \end{aligned}$$

Therefore $5/14 = 0.3\dot{5}7142\dot{8}$