Sequences

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We expect a sequence (an) to converge to a limit l if its terms eventually get close to l. To make this precise, a number e, say, is used as a measure of “closeness to l"

Definition of a convergent sequence. A sequence (a_n) of real numbers is said to *converge to a limit* $l \in \mathbb{R}$ if for every $\epsilon > 0$ there is an integer N (which depends on ϵ) with $|a_n - l| < \epsilon$ for all $n > N$. When (a_n) converges to l we write

$$\lim_{n\to\infty} a_n = l \qquad \text{or} \qquad a_n \to l.$$

Note that $|a_n - l|$ is the distance between the points a_n and l on the real line. The definition says that *no matter how small* a positive number ϵ we take, the distance between a_n and l will eventually be smaller than ϵ, i.e., the numbers a_n will eventually lie between $l - \epsilon$ and $l + \epsilon$.

A useful tool to help identify whether a sequence converges is to break it down into a series of simlper sequences and then use the rules below:

Combination Rules for Convergent Sequences

If $(a_n), (b_n), (c_n)$ are convergent sequences with $a_n \to \alpha, b_n \to \beta, c_n \to \gamma$, then:

sum *rule*	$a_n + b_n \to \alpha + \beta$		*[CR(1)]*
scalar multiple *rule*	$\lambda a_n \to \lambda\alpha$	(*for* $\lambda \in \mathbb{R}$)	*[CR(2)]*
product *rule*	$a_n b_n \to \alpha\beta$		*[CR(3)]*
reciprocal *rule*	$1/a_n \to 1/\alpha$	(*provided* $\alpha \neq 0$)	*[CR(4)]*
quotient *rule*	$b_n/a_n \to \beta/\alpha$	(*provided* $\alpha \neq 0$)	*[CR(5)]*
‘hybrid’ *rule*	$b_n c_n/a_n \to \beta\gamma/\alpha$	(*provided* $\alpha \neq 0$)	*[CR(6)]*

Problem. Show that the sequence (a_n) defined by

$$a_n = \frac{(n+2)(2n-1)}{3n^2+1}$$

converges and find its limit.

We cannot use the quotient rule directly since the sequences in the numerator and denominator do not individually converge. Examples like this can be tackled by dividing throughout by the term which increases fastest.

Solution. Dividing numerator and denominator by n^2 gives

$$a_n = \frac{(n+2)(2n-1)}{3n^2+1} = \frac{(1+2/n)(2-1/n)}{3+1/n^2}.$$

Now $(1/n)$ and $(1/n^2)$ converge to 0, so applying the combination rules CR(1,2,3 and 5) gives

$$a_n \to \frac{(1+0)(2-0)}{3+0} \qquad \text{i.e.,} \qquad a_n \to \frac{2}{3}.$$

Subsequences

A subsequence is a sequence with some terms deleted.

Bounds

A sequence is bounded above/below is there exists a U/L for which all a_n < U or > L.
A sequence is called bounded if it is both bounded below and above.

Increasing / Decreasing Sequences

Divergent Sequences

A sequence a_n is said to diverge to infinity if for every k in R there is an N with a_n > K whenever n > N
If a_N diveges t infinity we write a_n -> infinity

E.g. fibonacci sequence diverges to infinity
-n diverges to -infinity
(-1)^n oscillates

Basic Convergent Sequences

Basic properties of convergent sequences

(1) *A convergent sequence has a unique limit.*
(2) *If $a_n \to l$, then every subsequence of (a_n) also converges to l.*
(3) *If $a_n \to l$ then $|a_n| \to |l|$.*
(4) **The squeeze rule.** *If $a_n \to l$ and $b_n \to l$ and $a_n \leq c_n \leq b_n$ for all n, then*

$c_n \to l$.

(5) *A convergent sequence* (a_n) *is bounded, i.e., there is a* $B > 0$ *with* $-B \leq a_n \leq B$ *for all* n.

(6) *Any increasing sequence which is bounded above converges. Any decreasing sequence which is bounded below converges.*

e.g. to prove that –1^n does not converge, we observe that the even subsequence is 1,1,1... which converges to 1, and the odd subsequence is –1,-1,-1... which converges to –1. However, as per boxed rule 2, if a sequence converges then all of its subsequence's also converge to the same limit l. This is not the case and as such the sequence does not converge.

Basic Convergent Sequences

Basic Convergent Sequences

(1)	$\lim_{n\to\infty} \frac{1}{n^p} = 0$	for any $p > 0$
(2)	$\lim_{n\to\infty} c^n = 0$	for any c with $\lvert c\rvert < 1$
(3)	$\lim_{n\to\infty} c^{1/n} = 1$	for any $c > 0$
(4)	$\lim_{n\to\infty} n^p c^n = 0$	for $p > 0$ and $\lvert c\rvert < 1$
(5)	$\lim_{n\to\infty} \frac{c^n}{n!} = 0$	for any $c \in \mathbb{R}$
(6)	$\lim_{n\to\infty} \left(1 + \frac{c}{n}\right)^n = e^c$	for any $c \in \mathbb{R}$