Real Numbers

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Real numbers can be thought of as points on the number line
Rational numbers are an important subset of the reals that can be represented as m/n where n and m != 0

Not all numbers are rational, e.g. root 2.

- Supposing root 2 was irrational, then root 2 = m/n
- Since m/n is the simplest possible form, the GCD is 1
- M squared divided by n squared is therefore 2
- So m squared is equal to 2 times n squared
- So m is even as n squared has a factor of 2
- We can therefore say that m is equal to some 2k
- 2k squared = 4 (k squared) so 4(k squared) = 2(n squared)
- Both sides have a factor of two and as such gave a GCD of two
- Contradition of the initial GCD of 1 so supposition of rational is false

Basic Properties of the Reals

(1) Commutativity: x + y = y + x and x.y = y.x.
(2) Associativity: x + (y + z) = (x + y) + z and x.(y.z) = (x.y).z.
(3) Distributivity of . over + : x.(y + z) = x.y + x.z.
(4) There is an additive identity: There exists 0 ∈ R such that x + 0 = x.
(5) There is a multiplicative identity: There exists 1 ∈ R such that x.1 = x.
(6) The multiplicative and additive identities are distinct: 1 6= 0.
(7) Every element has an additive inverse: There exists (−x) ∈ R such that x+ (−x) = 0.
(8) Every non-zero element has a multiplicative inverse: If x 6= 0 then there exists x −1 ∈ R such that x.x−1 = 1.
(9) Transitivity of ordering: If x < y and y < z then x < z.
(10) The trichotomy law: Exactly one of the following is true: x < y, y < x or x = y.
(11) Preservation of ordering under addition: If x < y then x + z < y + z.
(12) Preservation of ordering under multiplication: If 0 < z and x < y then x.z < y.z.
(13) Completeness: Every non-empty subset of R that is bounded above has a least upper bound.

Upper Bound, Lower Bound, Supremum and Infimum

- Let S be a set of real numbers
- A real number u is called an upper bound of S if x ≤ u for all x ∈ S
- A real number l is called a lower bound of S if l ≤ x for every x ∈ S
- A real number U is called the least upper bound (supremum) of S if U is an upper bound of S and U ≤ u for every upper bound u of S
- A real number L is called the greatest lower bound (infimum) of S if L is a lower bound of S and l ≤ L for every lower bound l of S.

Consider the set S= [1,2), a subset of R. 1 is clearly the infimum as every element in S is >=1, and that wouldn't be the case if the supremum was bigger than it.
For the supremum, 2 isn't in S, but you can't rigorously define a number which is in S such that there do not exist any numbers in S greater than it. So if you say 1.99, I could say 1.999, and so on. There is no "biggest number" in S, so the supremum can't be in S

The Archimedean Property of R

- If x is a real number with x > 0 then there is an integer n > 0 with nx > 1
- Between any two distinct real numbers there are both rational and irrational numbers
- Every real number can be represented by a (possibly infinite) decimal expansion

The Archimedean property can be proved by contradiction. If there is no n with the stated property then we must have $n\epsilon \leq 1$ for every n. Thus the set $\{n\epsilon \mid n \in \mathbb{N}\}$ is bounded above, and so by the Completeness axiom has a least upper bound l. But now, for every n,

$$n\epsilon = (n+1)\epsilon - \epsilon \leq l - \epsilon,$$

so $l - \epsilon$ is also an upper bound of the set. But $l - \epsilon$ is smaller than the least upper bound l, giving a contradiction.