Integers

25 January 2020 12:27

To convert from base 10 to base b where b > 1, repeatedly divide the number by b and store the remainder in the new base form. Digits are stored from right to left so the final remainder is on the left hand side of the new base b number.

Modulus

We say a and b are congruent modulo n when a – b is a multiple of n

Twos Complement

To convert from binary to the negative twos complement of a binary number, flip the bits and add one.

e.g. 0011 would change to 1100 + 1 = 1101 = -3

Real Numbers

25 January 2020 12:56

Real numbers can be thought of as points on the number line
Rational numbers are an important subset of the reals that can be represented as m/n where n and m != 0

Not all numbers are rational, e.g. root 2.

- Supposing root 2 was irrational, then root 2 = m/n
- Since m/n is the simplest possible form, the GCD is 1
- M squared divided by n squared is therefore 2
- So m squared is equal to 2 times n squared
- So m is even as n squared has a factor of 2
- We can therefore say that m is equal to some 2k
- 2k squared = 4 (k squared) so 4(k squared) = 2(n squared)
- Both sides have a factor of two and as such gave a GCD of two
- Contradition of the initial GCD of 1 so supposition of rational is false

Basic Properties of the Reals

(1) Commutativity: x + y = y + x and x.y = y.x.
(2) Associativity: x + (y + z) = (x + y) + z and x.(y.z) = (x.y).z.
(3) Distributivity of . over + : x.(y + z) = x.y + x.z.
(4) There is an additive identity: There exists 0 ∈ R such that x + 0 = x.
(5) There is a multiplicative identity: There exists 1 ∈ R such that x.1 = x.
(6) The multiplicative and additive identities are distinct: 1 6= 0.
(7) Every element has an additive inverse: There exists (−x) ∈ R such that x+ (−x) = 0.
(8) Every non-zero element has a multiplicative inverse: If x 6= 0 then there exists x −1 ∈ R such that x.x−1 = 1.
(9) Transitivity of ordering: If x < y and y < z then x < z.
(10) The trichotomy law: Exactly one of the following is true: x < y, y < x or x = y.
(11) Preservation of ordering under addition: If x < y then x + z < y + z.
(12) Preservation of ordering under multiplication: If 0 < z and x < y then x.z < y.z.
(13) Completeness: Every non-empty subset of R that is bounded above has a least upper bound.

Upper Bound, Lower Bound, Supremum and Infimum

- Let S be a set of real numbers
- A real number u is called an upper bound of S if x ≤ u for all x ∈ S
- A real number l is called a lower bound of S if l ≤ x for every x ∈ S
- A real number U is called the least upper bound (supremum) of S if U is an upper bound of S and U ≤ u for every upper bound u of S
- A real number L is called the greatest lower bound (infimum) of S if L is a lower bound of S and l ≤ L for every lower bound l of S.

Consider the set S= [1,2), a subset of R. 1 is clearly the infimum as every element in S is >=1, and that wouldn't be the case if the supremum was bigger than it.
For the supremum, 2 isn't in S, but you can't rigorously define a number which is in S such that there do not exist any numbers in S greater than it. So if you say 1.99, I could say 1.999, and so on. There is no "biggest number" in S, so the supremum can't be in S

The Archimedean Property of R

- If x is a real number with x > 0 then there is an integer n > 0 with nx > 1

- Between any two distinct real numbers there are both rational and irrational numbers
- Every real number can be represented by a (possibly infinite) decimal expansion

The Archimedean property can be proved by contradiction. If there is no n with the stated property then we must have $n\epsilon \leq 1$ for every n. Thus the set $\{n\epsilon \mid n \in \mathbb{N}\}$ is bounded above, and so by the Completeness axiom has a least upper bound l. But now, for every n,

$$n\epsilon = (n+1)\epsilon - \epsilon \leq l - \epsilon,$$

so $l - \epsilon$ is also an upper bound of the set. But $l - \epsilon$ is smaller than the least upper bound l, giving a contradiction.

Complex Numbers

25 January 2020 13:59

$$\begin{aligned} a + ib = c + id &\Leftrightarrow a = c \text{ and } b = d, \\ (a+ib)+(c+id) &= (a+c)+i(b+d), \\ (a+ib)(c+id) &= (ac-bd)+i(bc+ad). \end{aligned}$$

A complex number a + ib can be represented by an ordered pair (a, b) of real numbers. The number a ∈ R is called the real part and b ∈ R is called the imaginary part. The set of all complex numbers will be denoted by C.

Complex Conjugate

The complex conjugate of an imaginary number a + ib is a – ib (simply a reflection in the real axis)

Polar Coordinates

The use of a distance and direction as a means of describing position is far more natural than using two distances on a grid. This means of location is used in polar coordinates and bearings.

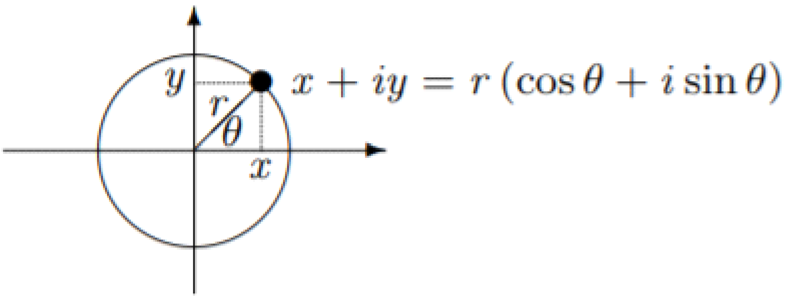
We say that (r,θ) are the polar coordinates of the point P, where r is the distance P is from the origin O and θ the angle between Ox and OP.

We can therefore express a complex number in polar coordinates

x = r cos θ, y = r sin θ

So x + iy = r(cosθ + isinθ)

r = sqrt(x^2 + y^2) , and θ satisfies tan θ = y/x



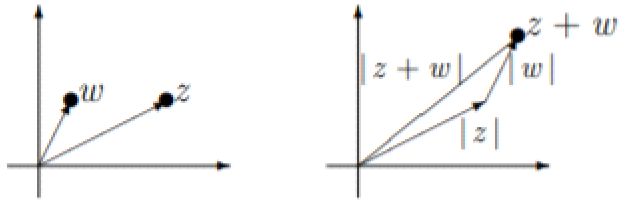
Modulus

The number $\sqrt{x^2+y^2}$ is called the *modulus* of $x+iy$ and denoted by $|x+iy|$. Geometrically it represents the distance between $x+iy$ and the origin of the complex plane.

Properties of the modulus. *For any* $z, w \in \mathbb{C}$*:*

(1) $|z| = |\overline{z}|$,
(2) $|z| = \sqrt{z\overline{z}}$,
(3) $z\overline{z} = |z|^2$,
(4) $|zw| = |z||w|$,
(5) $|z+w| \leq |z| + |w|$, *(the triangle inequality)*
(6) $||z| - |w|| \leq |z-w|$.

Triangle inequality arises from representing the complex numbers as points in the plane as in the diagram:



De Moivres Theorem

Gives a formula for computing powers of complex numbers

$$\big(r(\cos\theta + i\sin\theta)\big)^n = r^n\big(\cos(n\theta) + i\sin(n\theta)\big).$$

Fundamental Theorem of Algebra

Every polynomial equation of degree n with complex coefficients has exactly n (not necessarily distinct) solutions in C

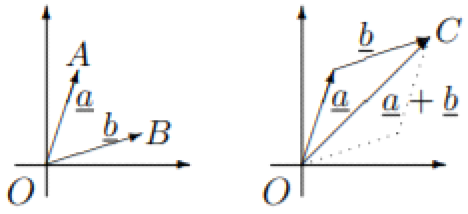
Vectors

25 January 2020 15:16

a + b = (a1 + b1, a2 + b2)
λa = (λa1, λa2)

a + b = (a1 + b1, a2 + b2, a3 + b3)
λa = (λa1, λa2, λa3)

Vector a + vector b = vector c where OABC is a parallelogram



Length and Distance

If $\underline{a} = (a_1, a_2) \in \mathbb{R}^2$ then we define the *length* $|\underline{a}|$ of $\underline{a}$ by

$$|\underline{a}| = \sqrt{a_1^2 + a_2^2}.$$

Similarly if $\underline{a} = (a_1, a_2, a_3) \in \mathbb{R}^3$, then we define the length of $\underline{a}$

$$|\underline{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}.$$

A vector is called a *unit vector* if its length is 1. The *distance* between $\underline{a}$ and $\underline{b}$ is defined to be $|\underline{b} - \underline{a}|$.

Normalisation

If we have a vector and need to find the unit vector then we carry out a process call normalisation
Take the modulus of the vector to get the length
Divide the vector by this value
e.g. |(2, -1)| = root 5
1/root5(2, -1) = 1
(2/root5, -1/root5) = 1

Scalar Product (Dot Product)

a.b = a1b1 + a2b2

Finding the Angle Between Vectors

$$\cos\theta = \cos(\beta - \alpha) = \cos\beta\cos\alpha + \sin\beta\sin\alpha = \frac{a_1b_1 + a_2b_2}{|\underline{a}|\,|\underline{b}|}.$$

Two vectors are orthogonal (perpendicular) if their dot product is 0

Linear Comb & Subspaces

25 January 2020 15:41

https://warwick.ac.uk/fac/sci/dcs/teaching/material/cs131/part2

In R^2 the vector (5, 3) can be written in the form (5, 3) = 5(1, 0) + 3(0, 1) and also in the form (5, 3) = 1(2, 0) + 3(1, 1). In each case we say that (5, 3) is a linear combination of the two vectors on the right hand side.

If we wanted to express (6,6) as a linear combination of two other vectors:
(6, 6) = α(0, 3) + β(2, 1)

We would create two simultaneous equations
(6, 6) = (2β, 3α + β)
So we have (6, 6) = 1(0, 3) + 3(2, 1)

Span

Span. If $U = \{\underline{u}_1, \underline{u}_2, \ldots, \underline{u}_m\}$ is a finite set of vectors in $\mathbb{R}^n$, then the *span* of U is the set of all linear combinations of $\underline{u}_1, \underline{u}_2, \ldots, \underline{u}_m$, and is denoted by span U. Hence

$$\text{span } U = \{\alpha_1\underline{u}_1 + \alpha_2\underline{u}_2 + \cdots + \alpha_m\underline{u}_m | \alpha_1, \alpha_2, \ldots, \alpha_m \in \mathbb{R}\}$$

- (1) If U = {u} contains just a single vector, then span {u} = {αu | α ∈ R} is the set of all multiples of u.
- (2) In R^2 if U = {(1, 0),(0, 1)} then the span of U is R 2 . To see this note that we can write an arbitrary vector (x, y) in R 2 as a linear combination of (1, 0) and (0, 1) as follows: (x, y) = x(1, 0) + y(0, 1).
- (3) In R^3 the span of the set {(1, 0, 0),(0, 1, 0),(0, 0, 1)} is R^3
- (4) In R^3 let u = (1, 0, 1) and v = (2, 0, 3). Then αu + βv = α(1, 0, 1) + β(2, 0, 3) = (α + 2β, 0, α + 3β), so any linear combination of u and v has 0 for its middle component. In fact any vector with middle component 0 is a linear combination of u and v
- (5) In general, if u and v are not parallel, then the span of {u, v} is the plane determined by the three points u, v and 0.

Subspace

A Subspace is a Vector Space included in another larger Vector Space. Therefore, all properties of a Vector Space, such as being closed under addition and scalar multiplication still hold true when applied to the Subspace.

S subspace of R^n is a nonempty subset S of R^n with the following properties:
(1) u, v ∈ S ⇒ u + v ∈ S
(2) u ∈ S, λ ∈ R ⇒ λu ∈ S.

S = (x, y, 0) is a subspace of R^3 as any vector of S will also be in a the vector space R^3

To check if a set S is a subspace of some other vector space, simply check that the set is closed under addition and scalar multiplication

The set {0} consisting of just the zero vector is a subspace of R^n
The set R n itself is a subspace of R^n

Linear Independence

02 February 2020 14:38

A set of vectors is linearly dependent if there exists a set of number $x_1 – x_n$ such that the sum of the product of $x_1 u_1$ all the way to $x_n u_n$ is 0.

A set of vectors is called linearly independent if it is not linearly dependent, meaning all of the coefficients must be 0 for the sum to be 0 (there exists no set such that the sum of 0 holds)

If vector u != 0, then the set {u} is linearly independent. For if $\alpha u = 0$ then since vector u != 0 we must have $\alpha = 0$

Any set containing the zero vector is linearly dependent (sum can equate to 0 without having all coefficients 0)

A set containing only two vectors is linearly dependent if and only if one is a multiple of the other

If there exists an alpha and beta such that $\alpha u + \beta v = 0$ then the vector set must be linearly dependent

Theorem

A set $\{u_1 , u_2 , . . . , u_m\}$ of nonzero vectors is linearly dependent if and only if some u_r is a linear combination of its predecessors $u_1 , . . . , u_{r-1}$.

The proof is here: https://warwick.ac.uk/fac/sci/dcs/teaching/material/cs131/part2/note6.pdf

Basis

02 February 2020 14:58

Basis

Let S be a subspace of Rn . A set of vectors is called a basis of S if it is a linearly independent set which spans S.

Example
The set {e1 , e2 , e3} where e1 = (1, 0, 0), e2 = (0, 1, 0) and e3 = (0, 0, 1) is a basis for R3
To verify this we have two things to show:
First, the vectors are linearly independent since: $\alpha(1, 0, 0) + \beta(0, 1, 0) + \gamma(0, 0, 1) = 0 \Longrightarrow (\alpha, \beta, \gamma) = (0, 0, 0) \Longrightarrow \alpha = \beta = \gamma = 0$
Second, the vectors are a spanning set since we can write any vector (x, y, z) in R3 as a linear combination thus: (x, y, z) = x(1, 0, 0) + y(0, 1, 0) + z(0, 0, 1)

Thus the set of vectors {e1, e2, e3} is a basis for the subspace R3

To prove that a set of vectors is a basis, proof linear independence and the fact it spans the subspace (recall linear independence means all coefficients must be 0 for the sum to be 0 and spanning means any vector in the subspace can be formed as a product of some combination of the vectors)

Standard Basis

In Rn, the standard basis is the set {e1, e2,..., en} where er is the vector with r th component 1 and all other components 0.
For example, the standard basis for R5 is {e1, e2, e3, e4, e5} where
e1 = (1, 0, 0, 0, 0)
e2 = (0, 1, 0, 0, 0)
e3 = (0, 0, 1, 0, 0)
e4 = (0, 0, 0, 1, 0)
e5 = (0, 0, 0, 0, 1)

Theorem

Let S be a subspace of Rn . If the set {v1 , v2 , . . . , vm} spans S then any linearly independent subset of S contains at most m vectors.

Dimension

02 February 2020 15:37

The dimension of a subspace of Rn is the number of vectors in a basis for the subspace.

Since the standard basis for R n contains n vectors it follows that R n has dimension n

Show that the set S = {(x, y, z)|x + 2y − z = 0} is a subspace of R3. Find a basis for, and the dimension of, S

We can write: S = {(x, y, x + 2y) | x, y ∈ R} = {x(1, 0, 1) + y(0, 1, 2) | x, y ∈ R} = span {(1, 0, 1),(0, 1, 2)}
This shows that S is a subspace, **since the span of any nonempty finite subset of Rn is a subspace**
The spanning set {(1, 0, 1),(0, 1, 2)} is **linearly independent since neither vector is a multiple of the other**, and hence is a basis. Thus the dimension of S is 2

Theorem

Let {v1 , v2 , . . . , vm} be a set of nonzero vectors that spans a subspace S of Rn . Then removing each vi which is a linear combination of its predecessors will leave a basis for S

To see why this works, note that each vector removed is a linear combination of the remaining ones, so the span is not altered by the removal. Also the remaining vectors are linearly independent, since none is a linear combination of its predecessors.

Problem

Find a basis for and the dimension of the subspace S of R 4 spanned by the set {(2, 1, 0, −3),(−1, 0, −1, 2),(1, 2, −3, 0),(0, 0, 0, 1),(0, 1, −2, 0)}.

Solution

To find a basis we remove from the spanning set any vector which is a linear combination of its predecessors.

The remaining set of vectors: {(2, 1, 0, −3),(−1, 0, −1, 2),(0, 0, 0, 1)} is a basis for S, and hence the dimension of S is 3

Theorem

When the dimension of a subspace is known, then the task of deciding whether a given set is a basis can be simplified by using the following result. Instead of checking if the vectors form a spanning set we need only count them.

Let S be an m-dimensional subspace of Rn , then:
(1) any subset of S containing more than m vectors is linearly dependent
(2) a subset of S is a basis if and only if it is a linearly independent set containing exactly m vectors

When S = Rn

Any subset of Rn containing more than n vectors is linearly dependent. A subset of R n is a basis if and only if it is a linearly independent set containing exactly n vectors.

Subspaces of R2

(1) There is one 0-dimensional subspace {0}.
(2) A one-dimensional subspace is spanned by a single non-zero vector. Hence the one dimensional

subspaces correspond to the straight lines through the origin. (3) There is only one two-dimensional subspace — R^2 itself.

Subspaces of R^3

(1) There is one 0-dimensional subspace {0}
(2) A one-dimensional subspace is spanned by a single non-zero vector. Hence the one dimensional subspaces correspond to the straight lines through the origin
(3) A two-dimensional subspace is spanned by two linearly independent vectors. Hence the two-dimensional subspaces correspond to planes which contain the origin.
(4) There is only one three-dimensional subspace — R^3 itself.

Matrix Algebra

02 February 2020 16:32

$$
\begin{aligned}
&(1) \quad A+(B+C)=(A+B)+C \\
&(2) \quad A+O=A=O+A \\
&(3) \quad A+(-A)=O=(-A)+A \\
&(4) \quad A+B=B+A \\
&(5) \quad (\lambda+\mu)A=\lambda A+\mu A \\
&(6) \quad \lambda(A+B)=\lambda A+\lambda B \\
&(7) \quad \lambda(\mu A)=(\lambda\mu)A.
\end{aligned}
$$

Identity matrix

The identity matrix of order n is the n × n diagonal matrix whose diagonal elements are all 1. It is denoted by I or In.

A * A^-1 = I
AI = A

The identity matrix is like the number 1 in normal numbers.

Transpose.

The transpose AT of a matrix A is obtained by interchanging the rows and columns. Thus if A = [aij] m×n then AT = [a 0 ij]n×m where a 0 ij = aji. For example, if

$$
\begin{array}{lll}
(1) & (A^T)^T = A & \\
(2) & (A+B)^T = A^T + B^T & \text{when } A+B \text{ exists} \\
(3) & (\lambda A)^T = \lambda A^T & \text{for any } \lambda \in \mathbb{R} \\
(4) & (AB)^T = B^T A^T & \text{when } AB \text{ exists.}
\end{array}
$$

Matrix Inverse

If A and B are square matrices of the same order, then B is called the inverse of A if AB = I = BA
It can be shown that if A has an inverse, then that inverse is unique. It will be denoted by A^−1
The determinant of a 2 × 2 matrix A = |a b, c d |is defined to be the number ad − bc and is denoted by det(A), |A|

Now if a 2 × 2 matrix A has an inverse, then det(A) det(A−1) = det(AA−1) = det(I) = 1 which means that det(A) 6= 0. Conversely if det(A) 6= 0 then it is easy to verify that A has an inverse

A 2×2 matrix A is *invertible* if and only if its determinant is nonzero. If $\det(A)\neq 0$ and $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then

$$A^{-1}=\frac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}=\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Linear Equations

02 February 2020 17:09

A system of two linear equations in unknowns x_1 and x_2:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned}$$

can be written in the following equivalent matrix form:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

To solve a system of equations in this form we can use a series of elementary row operations

Elementary row operations on matrices
(1) interchange two rows;
(2) multiply a row by a nonzero number;
(3) add a multiple of one row to another.

Each of these can also be regarded as an operation carried out on the rows of the matrices A and b. One advantage of this matrix approach is that we do not need to copy out the unknowns over and over again. The matrices A and b are usually combined to give the **augmented matrix** of the system.

System of equations	Augmented matrix
$$\begin{aligned} x_1 - x_2 + x_3 &= 1 \\ x_1 + x_2 + 2x_3 &= 0 \\ 2x_1 - x_2 + 3x_3 &= 2 \end{aligned}$$	$$\left[\begin{array}{ccc\|c} 1 & -1 & 1 & 1 \\ 1 & 1 & 2 & 0 \\ 2 & -1 & 3 & 2 \end{array}\right]$$

1

Row Equivalence

We say that two matrices A and B are row equivalent and we write A ~ B if A can be transformed to B using a finite (possibly 0) number of elementary row operations.

Row Echelon Form

A matrix is said to be in row echelon form if the first nonzero entry in each row is further to the right than the first nonzero entry in the previous row. A system of linear equations can be solved by using elementary row operations to reduce the augmented matrix to row echelon form.

Elementary matrices

Elementary row operations can be performed by multiplying a matrix on the left by a suitable "elementary matrix".

Calculation of Inverses

An important consequence of this is that we can use elementary row operations to calculate inverse matrices. If a sequence of elementary row operations transforms a square matrix A into I, then A is invertible and the same sequence transforms I into A−1

Solution. To perform elementary row operations simultaneously on the given matrix and I, we write them both as a single 3×6 matrix.

$$
\left[\begin{array}{rrr|rrr} 2 & -1 & 4 & 1 & 0 & 0 \\ 4 & 0 & 2 & 0 & 1 & 0 \\ 3 & -2 & 7 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{rrr|rrr} 1 & -\frac{1}{2} & 2 & \frac{1}{2} & 0 & 0 \\ 4 & 0 & 2 & 0 & 1 & 0 \\ 3 & -2 & 7 & 0 & 0 & 1 \end{array}\right] \qquad \tfrac{1}{2}\,\text{row1}
$$

$$
\sim \left[\begin{array}{rrr|rrr} 1 & -\frac{1}{2} & 2 & \frac{1}{2} & 0 & 0 \\ 0 & 2 & -6 & -2 & 1 & 0 \\ 0 & -\frac{1}{2} & 1 & -\frac{3}{2} & 0 & 1 \end{array}\right] \qquad \begin{array}{r} \text{row2} - 4\text{row1} \\ \text{row3} - 3\text{row1} \end{array}
$$

$$
\sim \left[\begin{array}{rrr|rrr} 1 & -\frac{1}{2} & 2 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & -3 & -1 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 1 & -\frac{3}{2} & 0 & 1 \end{array}\right] \qquad (\text{row2})/2
$$

$$
\sim \left[\begin{array}{rrr|rrr} 1 & -\frac{1}{2} & 2 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & -3 & -1 & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} & -2 & \frac{1}{4} & 1 \end{array}\right] \qquad \text{row3} + \tfrac{1}{2}\text{row2}
$$

$$
\sim \left[\begin{array}{rrr|rrr} 1 & -\frac{1}{2} & 2 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & -3 & -1 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 4 & -\frac{1}{2} & -2 \end{array}\right] \qquad -2\text{row3}
$$

$$
\sim \left[\begin{array}{rrr|rrr} 1 & -\frac{1}{2} & 0 & -\frac{15}{2} & 1 & 4 \\ 0 & 1 & 0 & 11 & -1 & -6 \\ 0 & 0 & 1 & 4 & -\frac{1}{2} & -2 \end{array}\right] \qquad \begin{array}{r} \text{row1} - 2\text{row3} \\ \text{row2} + 3\text{row3} \end{array}
$$

$$
\sim \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & -2 & \frac{1}{2} & 1 \\ 0 & 1 & 0 & 11 & -1 & -6 \\ 0 & 0 & 1 & 4 & -\frac{1}{2} & -2 \end{array}\right] \qquad \text{row1} + \tfrac{1}{2}\text{row2}.
$$

Hence

$$
\begin{bmatrix} 2 & -1 & 4 \\ 4 & 0 & 2 \\ 3 & -2 & 7 \end{bmatrix}^{-1} = \begin{bmatrix} -2 & \frac{1}{2} & 1 \\ 11 & -1 & -6 \\ 4 & -\frac{1}{2} & -2 \end{bmatrix}.
$$

Matrix Inverse

02 February 2020 17:21

Determinants

The determinant of a 3×3 matrix $A = [a_{ij}]$ is denoted by

$$\det(A), \quad |A|, \quad \text{or} \quad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix},$$

and is defined to be the number

$$a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

Recall that the transpose of A is the matrix A^T obtained by interchanging the rows and columns of A. The determinant of A^T is therefore given by

$$|A^T| = \begin{vmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{vmatrix}$$

Note that the determinant of A transposed is the same as the determinant of A

Problem. Compute the determinant of $A = \begin{bmatrix} 5 & 2 & 4 \\ 3 & 0 & 1 \\ -3 & -1 & -2 \end{bmatrix}$.

Solution. Expanding along the second row gives

$$|A| = -3 \begin{vmatrix} 2 & 4 \\ -1 & -2 \end{vmatrix} + 0 - 1 \begin{vmatrix} 5 & 2 \\ -3 & -1 \end{vmatrix} = -3(-4+4) - (-5+6) = -1.$$

The sign of the coefficient $(-1)^{i+j} = \pm 1$ is the ijth element of the following alternating pattern:

$$\begin{bmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}.$$

If A is an n x n matrix, then the matrix of cofactors is the matrix obtained by replacing each element of A by its corresponding cofactor.

The adjoint adj(A) of A is the transposition of the matrix of co factors

The matrix of cofactors is the matrix of 2x2 determinants as in the below diagram

The adjoint is given by

$$\operatorname{adj}(A) = \begin{bmatrix} \begin{vmatrix} 0 & 2 \\ 3 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 2 & 3 \end{vmatrix} \\ -\begin{vmatrix} 1 & 4 \\ 3 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 4 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} \\ \begin{vmatrix} 1 & 4 \\ 0 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 4 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} \end{bmatrix}^T$$

Matrix Inverse

Matrix inverse. A square matrix A is invertible if and only if its determinant is non-zero. If A is invertible then

$$A^{-1} = \frac{1}{|A|}\operatorname{adj}(A).$$

The matrix inverse of A is found by taking 1 over the determinant of A multiplied by the adjoint of A.

The matrix inverse can only be found for square matrices. It is possible for A.B to equal I even when the matrices are not square, however this does not prove the fact that B is the inverse of A, as it is possible to arrive at this result with non-square matrices.

Problem. Find the inverse of the matrix $A = \begin{bmatrix} 2 & 1 & 4 \\ 1 & 0 & 2 \\ 2 & 3 & 1 \end{bmatrix}$ and hence solve the system of equations

$$\begin{array}{rrrcr} 2x & +y & +4z & = & 2 \\ x & & +2z & = & 3 \\ 2x & +3y & +z & = & -6. \end{array}$$

Solution. The determinant of A is most easily calculated from the expansion by the second row:

$$|A| = (-1)\begin{vmatrix} 1 & 4 \\ 3 & 1 \end{vmatrix} + 0 - 2\begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} = (-1)(1-12) - 2(6-2) = 3.$$

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The adjoint is given by

$$\begin{aligned} \operatorname{adj}(A) &= \begin{bmatrix} \begin{vmatrix} 0 & 2 \\ 3 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 2 & 3 \end{vmatrix} \\ -\begin{vmatrix} 1 & 4 \\ 3 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 4 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} \\ \begin{vmatrix} 1 & 4 \\ 0 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 4 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} \end{bmatrix}^T \\ &= \begin{bmatrix} -6 & 3 & 3 \\ 11 & -6 & -4 \\ 2 & 0 & -1 \end{bmatrix}^T \\ &= \begin{bmatrix} -6 & 11 & 2 \\ 3 & -6 & 0 \\ 3 & -4 & -1 \end{bmatrix} \end{aligned}$$

Hence

$$A^{-1} = \frac{1}{|A|}\operatorname{adj}(A) = \frac{1}{3}\begin{bmatrix} -6 & 11 & 2 \\ 3 & -6 & 0 \\ 3 & -4 & -1 \end{bmatrix}.$$

$$A^{-1} = \frac{1}{|A|}\text{adj}(A) = \frac{1}{3}\begin{bmatrix} -6 & 11 & 2 \\ 3 & -6 & 0 \\ 3 & -4 & -1 \end{bmatrix}.$$

The system of equations can be written as

$$A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -6 \end{bmatrix},$$

and multiplying on the left of each side by A^{-1} gives

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1}\begin{bmatrix} 2 \\ 3 \\ -6 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -6 & 11 & 2 \\ 3 & -6 & 0 \\ 3 & -4 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ -6 \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \\ 0 \end{bmatrix}.$$

Hence the solution is $x = 3$, $y = -4$ and $z = 0$.

Matrices and Linear Independence

07 February 2020 13:20

Determinant of the matrix A is the same as the determinant of the transpose of A

Performing elementary row operations on the transpose is equivalent to performing the same operations on the columns of A.

Elementary column operations and determinants.
If B is the matrix obtained from A by
1. *multiplying a column of A by a number λ, then $|B| = \lambda|A|$;*
2. *interchanging two columns of A, then $|B| = -|A|$;*
3. *adding a multiple of one column of A to another, then $|B| = |A|$.*

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}.$$

The rows (a11 – a1n etc) are called row vectors
The columns (a11 – am1 etc) are called column vectors

A subset of Rn is a basis if and only if it is a linearly independent set containing n vectors.
We can check for this linear independence by computing a determinant.

Linear independence via determinant evaluation. A set of n vectors in $\mathbb{R}^n$ is linearly independent (and therefore a basis) if and only if it is the set of column vectors of a matrix with nonzero determinant.

Let U be the $n \times n$ matrix $[u_{ij}]$. If $|U| \neq 0$ then U is invertible and multiplying both sides of the above matrix equation on the left by U^{-1} gives $\alpha_1 = 0$, $\alpha_2 = 0, \dots, \alpha_n = 0$, i.e. the set $\{\underline{u}_1, \underline{u}_2, \dots, \underline{u}_n\}$ is linearly independent.

If $|U| = 0$ then $|U^T| = 0$ (since $|U| = |U^T|$), so the transpose U^T is not invertible. Hence U^T cannot be reduced to I by elementary row operations and so must be reducible to a matrix with a row of zeros. Therefore elementary column operations can be applied to U to produce a column of zeros. Hence some non-trivial linear combination of $\underline{u}_1, \underline{u}_2, \dots, \underline{u}_n$ is $\underline{0}$, so these vectors are linearly dependent.

Problem. Which of the following sets are a basis for $\mathbb{R}^3$?
(1) $\{(1, -1, 2), (0, 2, 3), (3, -5, 3)\}$
(2) $\{(-2, 3, 4), (2, 1, 3), (1, -2, -3)\}$

Check the determinant of the matrix - 0 means linearly dependent and as such a basis, !0 means linearly independent and so not a basis.

Matrices and Linear Transformations

07 February 2020 13:35

A function $T : \mathbb{R}^m \to \mathbb{R}^n$ is called a **linear transformation** if, for all $\underline{u}, \underline{v} \in \mathbb{R}^m$ and all $\lambda \in \mathbb{R}$, we have :

$$T(\underline{u} + \underline{v}) = T(\underline{u}) + T(\underline{v}) \quad \text{and} \quad T(\lambda \underline{u}) = \lambda T(\underline{u}).$$

$$(preservation\ of\ addition) \qquad (preservation\ of\ scalar\ multiplication)$$

T(vector 0) is vector 0. Meaning the transformation of the zero vector of length m is the zero vector of length n.

Problem. Which of the following functions are linear transformations:
(1) $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (x + y, x - y)$;
(2) $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (x + 1, y - 1)$;
(3) $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (x^2, y^2)$.

Solution.
(1) Let $\underline{a} = (a_1, a_2)$ and $\underline{b} = (b_1, b_2)$ then $T(\underline{a}) = (a_1 + a_2, a_1 - a_2)$ and $T(\underline{b}) = (b_1 + b_2, b_1 - b_2)$ so

$$\begin{aligned} T(\underline{a} + \underline{b}) &= T(a_1 + b_1, a_2 + b_2) \\ &= (a_1 + b_1 + a_2 + b_2, a_1 + b_1 - a_2 - b_2) \\ &= (a_1 + a_2, a_1 - a_2) + (b_1 + b_2, b_1 - b_2) \\ &= T(\underline{a}) + T(\underline{b}) \end{aligned}$$

and

$$\begin{aligned} T(\lambda \underline{a}) &= T(\lambda a_1, \lambda a_2) \\ &= (\lambda a_1 + \lambda a_2, \lambda a_1 - \lambda a_2) \\ &= \lambda(a_1 + a_2, a_1 - a_2) \\ &= \lambda T(\underline{a}). \end{aligned}$$

So T is a linear transformation.

(2) Here we have $T(0, 0) = (1, -1)$, i.e. $T(\underline{0}) \neq \underline{0}$, so T cannot be linear.

(3) Here we have

$$\begin{aligned} &T((1, 0) + (1, 0)) = T((2, 0)) = (4, 0), \\ &T(1, 0) + T(1, 0) = (1, 0) + (1, 0) = (2, 0), \end{aligned}$$

so T is not a linear transformation.

1

Here we see how to identify whether or not a function is a linear transformation. We're checking the property above in which the transformation of two vectors T(u + v) is the same as the transformation of the vectors individually, added together – T(u) + T(v)

In the first example, this is checked by creating two vectors and substituing their values into the transformation equation. We see that T(a + b) = T(a) + T(b) so the transformation is a linear transformation.

With the second one, we see that T(0) != 0 so it cannot be linear

The 3rd T(a + b) is not equal to T(a) + T(b) when we sub in two vectors, so it is not a linear transformation

Projection

Projection

We define the projection of $\underline{x} \in \mathbb{R}^2$ onto nonzero vector $\underline{u} \in \mathbb{R}^2$ to be the vector $P_{\underline{u}}(\underline{x})$ with the properties:

(1) $P_{\underline{u}}(\underline{x})$ is a multiple of u;

(2) $\underline{x} - P_{\underline{u}}(\underline{x})$ is perpendicular to $\underline{u}$.

The projection of vector x onto vector u can be regarded as a function P such that

$$P_{\underline{u}}(\underline{x}) = \left(\frac{\underline{x}.\underline{u}}{|\underline{u}|^2}\right)\underline{u}$$

Using the rules of transformations that are discussed earlier, we can now show that the projection function is a linear transformation.

It is now easy to verify using properties of the scalar product that $P_{\underline{u}}$ is a linear transformation. For any $\underline{x}, \underline{y} \in \mathbb{R}^2$ and any $\lambda \in \mathbb{R}$:

$$P_{\underline{u}}(\underline{x}+\underline{y}) = \left(\frac{(\underline{x}+\underline{y}).\underline{u}}{|\underline{u}|^2}\right)\underline{u} = \left(\frac{\underline{x}.\underline{u}}{|\underline{u}|^2}\right)\underline{u} + \left(\frac{\underline{y}.\underline{u}}{|\underline{u}|^2}\right)\underline{u} = P_{\underline{u}}(\underline{x}) + P_{\underline{u}}(\underline{y}),$$

$$P_{\underline{u}}(\lambda\underline{x}) = \left(\frac{(\lambda\underline{x}).\underline{u}}{|\underline{u}|^2}\right)\underline{u} = \lambda\left(\frac{\underline{x}.\underline{u}}{|\underline{u}|^2}\right)\underline{u}.$$

Every matrix defines a linear transformation

Theorem: Every matrix defines a linear transformation.

Equivalently, given $n \times m$ matrix M, the function $T : \mathbb{R}^m \to \mathbb{R}^n$ defined by $T(\underline{x}) = M\underline{x}$ for all $\underline{x} \in \mathbb{R}^m$ is a linear transformation.

Proof:

This is trivially true since

$$M(\underline{u}+\underline{v}) = M\underline{u} + M\underline{v} \qquad \text{and} \qquad M(\lambda\,\underline{u}) = \lambda(M\underline{u})$$

for any $m \times 1$ column vectors $\underline{u}$ and $\underline{v}$ and any $\lambda \in \mathbb{R}$.

Coordinates and Basis Change

07 February 2020 14:14

Let $V = \{\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_n\}$ be a basis for $\mathbb{R}^n$. If $\underline{x} \in \mathbb{R}^n$ then $\underline{x}$ has a unique expansion as a linear combination

$$\underline{x} = \alpha_1 \underline{v}_1 + \alpha_2 \underline{v}_2 + \cdots + \alpha_n \underline{v}_n$$

of these basis vectors. The coefficients $\alpha_1, \alpha_2, \ldots, \alpha_n$ are called the *coordinates* of $\underline{x}$ with respect to the basis V.

When you have a basis of a set of vectors V, then any vector X in the same subspace has a unique expansion as a linear combination of the vectors in V. The coefficients of a1, a2 .. an of the vectors in V are called the coordinates of x with respect to V.

Problem. Let $E = \{(1,0),(0,1)\}$ be the standard basis for $\mathbb{R}^2$ and let V be the basis $\{(1,-1),(2,3)\}$. Find the coordinates of the vector $(1,2)$ with respect to E and with respect to V.

Solution. We have

$$(1,2) = 1(1,0) + 2(0,1)$$

so the coordinates of $(1,2)$ with respect to E are $[1,2]$. Also

$$(1,2) = \alpha(1,-1) + \beta(2,3) \iff \left\{ \begin{array}{rcl} \alpha + 2\beta &=& 1 \\ -\alpha + 3\beta &=& 2 \end{array} \right. \iff \left\{ \begin{array}{rcl} \beta &=& 3/5 \\ \alpha &=& -1/5 \end{array} \right.$$

so the coordinates of $(1,2)$ with respect to V are $[-1/5, 3/5]$.

The Matrix of a Linear Transformation

Problem. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation given by

$$T(x,y) = (y, x+y, x) \qquad (x,y) \in \mathbb{R}^2.$$

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Find the matrix of T with respect to the basis $V = \{(1,1),(1,-1)\}$ of $\mathbb{R}^2$ and the basis $W = \{(1,2,0),(2,1,0),(0,0,1)\}$ of $\mathbb{R}^3$. If a vector $\underline{u}$ has coordinates $[2,3]$ with respect to V, then what are the coordinates of $T(\underline{u})$ with respect to W?

Solution. We have

$$T(1,1) = (1,2,1) = (1,2,0) + 0(2,1,0) + (0,0,1)$$

and

$$T(1,-1) = (-1,0,1) = \alpha(1,2,0) + \beta(2,1,0) + \gamma(0,0,1)$$

where

$$\begin{array}{rcl} \alpha + 2\beta &=& -1 \\ 2\alpha + \beta &=& 0 \\ \gamma &=& 1 \end{array} \qquad \text{or} \qquad \begin{array}{rcl} \alpha &=& 1/3 \\ \beta &=& -2/3 \\ \gamma &=& 1\,. \end{array}$$

We have now expressed the images of the vectors in V as linear combinations of the vectors in W:

$$\begin{array}{rcl} T(1,1) &=& 1(1,2,0) + 0(2,1,0) + 1(0,0,1) \\ T(1,-1) &=& \frac{1}{3}(1,2,0) - \frac{2}{3}(2,1,0) + 1(0,0,1) \end{array}$$

so the matrix of T with respect to V and W is

$$\begin{bmatrix} 1 & 1/3 \\ 0 & -2/3 \\ 1 & 1 \end{bmatrix}.$$

If $\underline{u}$ has coordinates $[2,3]$ with respect to V then

$$\begin{bmatrix} 1 & 1/3 \\ 0 & -2/3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix}$$

so the coordinates of $T(\underline{u})$ with respect to W are $[3,-2,5]$.

Finding the matrix of a function T with respect to a basis V in R2 and basis W in R3

A function that can be expressed as a matrix is by definition a linear transformation. Hence we know that the vectors in V can be expressed in some linear combination of the vectors in W.

Subbing the values of v1 into the function we see T(1,1) = (1,2,1) which can be expressed as some linear combination of the vectors in W. We can solve simultaneously to find alpha, beta and gamma.

Once we have found the values of alpha, beta and gamma for both vectors in V, we can write them as column vectors to form the matrix T.

Change of Basis

If we have two different bases in R n then a given vector will have different coordinates with respect to each basis. The change in coordinates can be described by a transition matrix

The coordinates of a vector are the coefficients of the linear combination of the base vectors required to make the vector.

If two bases exist for a subspace then two different sets of co-ordinates exist to make a vector in the subspace. Change of basis handles the conversion between these two sets of co-ordinates.

The set of co-ordinates A can be converted to the set B by multiplying A by a transition matrix M

$$\begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{bmatrix} = M \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$$

We express each basis vector as a linear combination of the vectors in the other basis. Once the co-efficients for every vector in basis A have been found, these co-efficients can be expressed as a transformation matrix.

Problem. Find the transition matrix from the basis $\{(1,-1),(1,1)\}$ to the basis $\{(1,2),(2,3)\}$ of $\mathbb{R}^2$. If a vector has coordinates $[2,-1]$ with respect to the first basis, what are its coordinates with respect to the second?

Solution. We have

$$(1,-1) = \alpha(1,2) + \beta(2,3) \iff \begin{cases} \alpha + 2\beta &= 1 \\ 2\alpha + 3\beta &= -1 \end{cases} \iff \begin{cases} \alpha &= -5 \\ \beta &= 3 \end{cases}$$

$$(1,1) = \alpha(1,2) + \beta(2,3) \iff \begin{cases} \alpha + 2\beta &= 1 \\ 2\alpha + 3\beta &= 1 \end{cases} \iff \begin{cases} \alpha &= -1 \\ \beta &= 1 \end{cases}$$

Thus the old basis vectors can be written as linear combinations of the new ones as:

$$\begin{aligned} (1,-1 &= -5(1,2) + 3(2,3) \\ (1,1) &= -1(1,2) + 1(2,3) \end{aligned}$$

and so the transition matrix is $\begin{bmatrix} -5 & -1 \\ 3 & 1 \end{bmatrix}$.

If a vector has coordinates $[2,-1]$ with respect to the first basis then since

$$\begin{bmatrix} -5 & -1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -9 \\ 5 \end{bmatrix},$$

the coordinates with respect to the second basis are $[-9,5]$.

Eigenvectors

06 February 2020 13:08

An eigenvector of a square matrix is a vector whose direction does not change when multiplied on the left by A

The column vector A multiplied by the vector r is equal to some scalar multiple of the vector r. Meaning the direction hasn't changed but the length may have. The scalar is called the eigenvalue.

$A\,\underline{r} = \lambda \underline{r}$ where $\lambda \in \mathbb{R}$ and is called an *eigenvalue.*

(the magnitude may change)

Equivalently, the matrix multiplied by the vector r is equal to some lambda multiplied by the vector r.

Hence we say that lambda is the eigenvalue corresponding to eigenvector r.

Note that we exclude the trivial case where vector r = 0.

Example
If we have the 2x2 matrix A;

$$A = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$$

has eigenvectors

$$\underline{r}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \underline{r}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

with associated eigenvalues $\lambda_1 = 8$ and $\lambda_2 = 2$, since

$$\begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 8 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix},$$

$$\text{i.e. } A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 8\begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \text{and} \qquad A\begin{bmatrix} 1 \\ -1 \end{bmatrix} = 2\begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Theorem

Lambda is an eigenvalue of the matrix A if and only if the determinant of the matrix A minus lambda * the identity matrix is equal to 0.

This equation is called the characteristic equation of A
It is a polynomial of degree n in lambda

Proof:

$$\begin{aligned} & \lambda \text{ is an eigenvalue of } A \\ \Leftrightarrow \quad & A\underline{v} = \lambda\, \underline{v} \text{ for some nonzero } \underline{v} \\ \Leftrightarrow \quad & (A - \lambda I)\underline{v} = \underline{0} \text{ for some nonzero } \underline{v} \\ \Leftrightarrow \quad & | A - \lambda I | = 0. \end{aligned}$$

The eigenvalues of a real matrix may be complex. For example, consider the rotation matrix corresponding to theta = 3pi / 2

> Let A be a square matrix of order n. A number λ is called an *eigenvalue* of A if $A\underline{v} = \lambda\underline{v}$ for some non-zero column vector $\underline{v}$. When this is the case we call $\underline{v}$ an *eigenvector* of A corresponding to λ.

Problem. Find the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} -5 & 3 \\ 6 & -2 \end{bmatrix}$.

Solution. We have

$$A - \lambda I = \begin{bmatrix} -5 & 3 \\ 6 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -5-\lambda & 3 \\ 6 & -2-\lambda \end{bmatrix}$$

$$\begin{aligned} \text{so} \quad |A - \lambda I| = 0 \quad & \Longleftrightarrow \quad (-5-\lambda)(-2-\lambda) - 18 = 0 \\ & \Longleftrightarrow \quad \lambda^2 + 7\lambda - 8 = 0 \\ & \Longleftrightarrow \quad (\lambda - 1)(\lambda + 8) = 0 \\ & \Longleftrightarrow \quad \lambda = 1 \text{ or } \lambda = -8. \end{aligned}$$

Hence the eigenvalues of A are 1 and -8. To find the eigenvectors we consider each eigenvalue in turn.

$$\begin{aligned} (x, y) \text{ is an eigenvector with eigenvalue } 1 \iff & \begin{bmatrix} -5 & 3 \\ 6 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 1 \begin{bmatrix} x \\ y \end{bmatrix} \\ \iff & \begin{cases} -6x + 3y = 0 \\ 6x - 3y = 0 \end{cases} \\ \iff & y = 2x. \end{aligned}$$

Hence any nonzero vector of the form $(x, 2x)$ or, equivalently, any nonzero multiple of $(1, 2)$ is an eigenvector corresponding to the eigenvalue 1.

$$\begin{aligned} (x, y) \text{ is an eigenvector with eigenvalue } -8 \iff & \begin{bmatrix} -5 & 3 \\ 6 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = -8 \begin{bmatrix} x \\ y \end{bmatrix} \\ \iff & \begin{cases} 3x + 3y = 0 \\ 6x + 6y = 0 \end{cases} \\ \iff & y = -x \end{aligned}$$

Hence any nonzero vector of the form $(x, -x)$ or equivalently any nonzero multiple of $(1, -1)$ is an eigenvector corresponding to the eigenvalue -8.

Diagonal Matrices
All non-zero entries lie on the diagonal.
Multiplying is quick and easy

Problem. Find a diagonal matrix D and an invertible matrix P such that $P^{-1}AP = D$, where A is the matrix:

$$\begin{bmatrix} 4 & 2 & 2 \\ -1 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix}.$$

3

Solution. The characteristic equation, $|A - \lambda I| = 0$, is

$$\begin{vmatrix} 4-\lambda & 2 & 2 \\ -1 & 1-\lambda & -1 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0$$

or

$$(2-\lambda)((4-\lambda)(1-\lambda)+2) = 0,$$

i.e.

$$0 = (2-\lambda)(\lambda^2 - 5\lambda + 6) = (2-\lambda)(\lambda-2)(\lambda-3),$$

so the eigenvalues of A are 2, 2 and 3.

Now (x, y, z) is an eigenvector corresponding to the eigenvalue 2 if and only if

$$\begin{bmatrix} 4 & 2 & 2 \\ -1 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 2 \begin{bmatrix} x \\ y \\ z \end{bmatrix} \iff \left\{ \begin{array}{rcl} 2x + 2y + 2z & = & 0 \\ -x - y - z & = & 0 \\ 0 & = & 0 \end{array} \right.$$

$$\iff x + y + z = 0.$$

So any vector of the form $(x, y, -x-y)$ or $x(1, 0, -1) + y(0, 1, -1)$ where x and y are not both zero is an eigenvector corresponding to the repeated eigenvalue 2. In this case, $(1, 0, -1)$ and $(0, 1, -1)$ may be taken as independent eigenvectors corresponding to the repeated eigenvalue 2.

Finally, (x, y, z) is an eigenvector corresponding to the eigenvalue 3 if and only if

$$\begin{bmatrix} 4 & 2 & 2 \\ -1 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 3 \begin{bmatrix} x \\ y \\ z \end{bmatrix} \iff \left\{ \begin{array}{rcl} x + 2y + 2z & = & 0 \\ -x - 2y - z & = & 0 \\ z & = & 0 \end{array} \right.$$

$$\iff \left\{ \begin{array}{rcl} z & = & 0 \\ x & = & -2y. \end{array} \right.$$

Hence any vector of the form $(-2y, y, 0)$ or $y(-2, 1, 0)$, where $y \neq 0$, is an eigenvector corresponding to the eigenvalue 3.

Hence we can say that $P^{-1}AP = D$ where:

$$P = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ -1 & -1 & 0 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

Sequences

12 February 2020 10:12

We expect a sequence (an) to converge to a limit l if its terms eventually get close to l. To make this precise, a number e, say, is used as a measure of "closeness to l"

Definition of a convergent sequence. A sequence (a_n) of real numbers is said to *converge to a limit* $l \in \mathbb{R}$ if for every $\epsilon > 0$ there is an integer N (which depends on ϵ) with $|a_n - l| < \epsilon$ for all $n > N$. When (a_n) converges to l we write

$$\lim_{n\to\infty} a_n = l \qquad \text{or} \qquad a_n \to l.$$

Note that $|a_n - l|$ is the distance between the points a_n and l on the real line. The definition says that *no matter how small* a positive number ϵ we take, the distance between a_n and l will eventually be smaller than ϵ, i.e., the numbers a_n will eventually lie between $l - \epsilon$ and $l + \epsilon$.

A useful tool to help identify whether a sequence converges is to break it down into a series of simlper sequences and then use the rules below:

Combination Rules for Convergent Sequences

If $(a_n), (b_n), (c_n)$ *are convergent sequences with* $a_n \to \alpha, b_n \to \beta, c_n \to \gamma$, *then:*

sum *rule*	$a_n + b_n \to \alpha + \beta$	*[CR(1)]*
scalar multiple *rule*	$\lambda a_n \to \lambda\alpha$ (*for* $\lambda \in \mathbb{R}$)	*[CR(2)]*
product *rule*	$a_n b_n \to \alpha\beta$	*[CR(3)]*
reciprocal *rule*	$1/a_n \to 1/\alpha$ (*provided* $\alpha \neq 0$)	*[CR(4)]*
quotient *rule*	$b_n/a_n \to \beta/\alpha$ (*provided* $\alpha \neq 0$)	*[CR(5)]*
'hybrid' *rule*	$b_n c_n/a_n \to \beta\gamma/\alpha$ (*provided* $\alpha \neq 0$)	*[CR(6)]*

Problem. Show that the sequence (a_n) defined by

$$a_n = \frac{(n+2)(2n-1)}{3n^2+1}$$

converges and find its limit.

We cannot use the quotient rule directly since the sequences in the numerator and denominator do not individually converge. Examples like this can be tackled by dividing throughout by the term which increases fastest.

Solution. Dividing numerator and denominator by n^2 gives

$$a_n = \frac{(n+2)(2n-1)}{3n^2+1} = \frac{(1+2/n)(2-1/n)}{3+1/n^2}.$$

Now $(1/n)$ and $(1/n^2)$ converge to 0, so applying the combination rules CR(1,2,3 and 5) gives

$$a_n \to \frac{(1+0)(2-0)}{3+0} \qquad \text{i.e.,} \qquad a_n \to \frac{2}{3}.$$

Subsequences

A subsequence is a sequence with some terms deleted.

Bounds

A sequence is bounded above/below is there exists a U/L for which all a_n < U or > L.
A sequence is called bounded if it is both bounded below and above.

Increasing / Decreasing Sequences

Divergent Sequences

A sequence a_n is said to diverge to infinity if for every k in R there is an N with a_n > K whenever

n > N
If a_N diveges t infinity we write a_n -> infinity

E.g. fibonacci sequence diverges to infinity
-n diverges to -infinity
(-1)^n oscillates

Basic Convergent Sequences

Basic properties of convergent sequences

(1) *A convergent sequence has a unique limit.*
(2) *If $a_n \to l$, then every subsequence of (a_n) also converges to l.*
(3) *If $a_n \to l$ then $|a_n| \to |l|$.*
(4) **The squeeze rule.** *If $a_n \to l$ and $b_n \to l$ and $a_n \leq c_n \leq b_n$ for all n, then $c_n \to l$.*
(5) *A convergent sequence (a_n) is bounded, i.e., there is a $B > 0$ with $-B \leq a_n \leq B$ for all n.*
(6) *Any increasing sequence which is bounded above converges. Any decreasing sequence which is bounded below converges.*

e.g. to prove that –1^n does not converge, we observe that the even subsequence is 1,1,1... which converges to 1, and the odd subsequence is –1,-1,-1... which converges to –1. However, as per boxed rule 2, if a sequence converges then all of its subsequence's also converge to the same limit l. This is not the case and as such the sequence does not converge.

Basic Convergent Sequences

Basic Convergent Sequences

(1)	$\lim_{n\to\infty} \frac{1}{n^p} = 0$	for any $p > 0$
(2)	$\lim_{n\to\infty} c^n = 0$	for any c with $|c| < 1$
(3)	$\lim_{n\to\infty} c^{1/n} = 1$	for any $c > 0$
(4)	$\lim_{n\to\infty} n^p c^n = 0$	for $p > 0$ and $|c| < 1$
(5)	$\lim_{n\to\infty} \frac{c^n}{n!} = 0$	for any $c \in \mathbb{R}$
(6)	$\lim_{n\to\infty} \left(1 + \frac{c}{n}\right)^n = e^c$	for any $c \in \mathbb{R}$

Recurrence

13 February 2020 13:33

A sequence defined by its predecessors
A rule which defines each term using the preceding terms

Linear Recurrence

We are looking to solve linear recurrences with constant coefficients.

Auxiliary equation. The equation $\lambda^2 + a\lambda + b = 0$ is called the *auxiliary equation* of the recurrence $x_n + ax_{n-1} + bx_{n-2} = 0$.

If the auxiliary equation has two distinct solutions λ_1 and λ_2, then it is easy to verify that $x_n = A\lambda_1^n + B\lambda_2^n$ is a solution of the recurrence for any constants A and B. If the first two terms of the sequence (x_n) are given, then they can be used to find the values of A and B.

If the auxiliary equation has only a single solution for λ then $\sqrt{a^2 - 4b} = 0$, so $b = a^2/4$ and $\lambda = -a/2$. In this case $x_n = A\lambda^n$ is *a* solution of the recurrence for any A, but is not the *general* solution.

Non-homogenous Recurrences

Solution of the recurrence $x_n + ax_{n-1} + bx_{n-2} = f(n)$.

(1) Find the general solution $x_n = h_n$ of the homogeneous recurrence:

$$x_n + ax_{n-1} + bx_{n-2} = 0$$

(solution will contain two arbitrary constants).

(2) Find *any* particular solution $x_n = p_n$ of the original recurrence:

$$x_n + ax_{n-1} + bx_{n-2} = f(n).$$

(3) The general solution of the original recurrence is then given by $x_n = h_n + p_n$.

Finding a solution for f(n) = k is not generally easy.

- If f is constant, say $f(n) = k$ for all n, then it is easy to find a constant particular solution (provided $1 + a + b \neq 0$). For if $x_n = c$ for all n, then substituting into the recurrence $x_n + ax_{n-1} + bx_{n-2} = k$ gives $c + ac + bc = k$ or $c = k/(1 + a + b)$.
- For a more complicated (polynomial) $f(n)$, try to find a particular solution which is also a polynomial in n, e.g., try $x_n = k$ or $x_n = k_1 n + k_2$ or $x_n = k_1 n^2 + k_2 n + k_3$, ...

Example. Find the general solution of the recurrence

$$x_n - 10.1x_{n-1} + x_{n-2} = -2.7n.$$

Solution. The homogeneous recurrence $x_n - 10.1x_{n-1} + x_{n-2} = 0$ has auxiliary equation

$$\lambda^2 - 10.1\lambda + 1 = 0 \qquad \text{or} \qquad (\lambda - 10)(\lambda - 1/10) = 0,$$

and so has general solution $x_n = A(10^n) + B/10^n$.

To find a particular solution of

$$x_n - 10.1x_{n-1} + x_{n-2} = -2.7n,$$

we try $x_n = Cn + D$. This gives

$$Cn + D - 10.1(C(n-1) + D) + C(n-2) + D = -8.1Cn + 8.1D - 8.1C = -2.7n.$$

Since this holds *for all* n, we have

$$-8.1C = -2.7,\ 8.1D - 8.1C = 0 \qquad \text{so} \qquad C = D = 1/3.$$

Hence the general solution of the recurrence is

$$x_n = 10^n A + \frac{B}{10^n} + \frac{n+1}{3}.$$

Series

19 February 2020 15:06

(1) a sequence (an) called the sequence of terms,
(2) a sequence (sn) called the sequence of partial sums

Sum of a Series. If the sequence (s_n) of partial sums converges to s then we say that the series $\sum a_n$ *converges to the sum* s and write

$$\sum_{n=0}^{\infty} a_n = s.$$

Otherwise we say that the series *diverges*.

Example. The ***geometric series*** $\sum r^n$ converges to $1/(1-r)$ provided that $|r| < 1$. To see this, we calculate the nth partial sum:

$$s_n = 1 + r + r^2 + \cdots + r^n.$$

Multiplying both sides by r gives

$$rs_n = r + r^2 + \cdots + r^n + r^{n+1}.$$

Subtracting these equations gives:

$$s_n - rs_n = 1 - r^{n+1} \qquad \text{or} \qquad s_n = \frac{1-r^{n+1}}{1-r}.$$

If $|r| < 1$ then $r^{n+1} \to 0$ so $s_n \to 1/(1-r)$. Hence

$$\boxed{\sum_{n=0}^{\infty} r^n = \frac{1}{1-r} \qquad \text{for } |r| < 1.}$$

To find the value that the sequence coverges to w can use the rul that r^n converges to 1/(1-r) when |r| < 1

Problem. Show that the series $\sum_{n=1}^{\infty} 1/(4n^2-1)$ converges and find its sum.

Solution. Taking partial fractions we have

$$\frac{1}{4n^2-1} = \frac{1}{(2n-1)(2n+1)} = \frac{1/2}{2n-1} - \frac{1/2}{2n+1}.$$

1

Hence the n^{th} partial sum is

$$\begin{aligned} s_n &= \sum_{r=1}^{n} \tfrac{1}{4r^2-1} = \tfrac{1}{2}\left(\sum_{r=1}^{n} \tfrac{1}{2r-1} - \sum_{r=1}^{n} \tfrac{1}{2r+1}\right) \\ &= \tfrac{1}{2}\left(1 + \tfrac{1}{3} + \tfrac{1}{5} + \cdots + \tfrac{1}{2n-3} + \tfrac{1}{2n-1} - \tfrac{1}{3} - \tfrac{1}{5} - \cdots - \tfrac{1}{2n-3} - \tfrac{1}{2n-1} - \tfrac{1}{2n+1}\right) \\ &= \frac{1}{2}\left(1 - \frac{1}{2n+1}\right). \end{aligned}$$

Hence $s_n \to 1/2$, so the series converges and

$$\sum_{n=1}^{\infty} \frac{1}{4n^2-1} = \frac{1}{2}.$$

Basic Properties of Convergent Series

(1) **Sum Rule:** *if $\sum a_n$ converges to s and $\sum b_n$ converges to t, then $\sum(a_n + b_n)$ converges to $s + t$.*
(2) **Multiple rule:** *if $\sum a_n$ converges to s and $\lambda \in \mathbb{R}$, then $\sum \lambda a_n$ converges to λs.*
(3) *If the series $\sum a_n$ converges, then the sequence (a_n) converges to 0.*
(4) *If the series $\sum |a_n|$ converges, then the series $\sum a_n$ also converges.*

The Comparison Test

Often it is easier to determine whether or not a series converges rather than its actual sum.

The Comparison Test

Suppose that $0 \leq a_n \leq b_n$ *for every* n, *then*

(1) *if* $\sum b_n$ *converges then so does* $\sum a_n$;
(2) *if* $\sum a_n$ *diverges then so does* $\sum b_n$.

Problem. Show that the series $\sum_{n=1}^{\infty} 1/n^k$ diverges for $k \leq 1$.

Solution. It was shown earlier that $\sum 1/n$ diverges. For $k \leq 1$, we have $0 \leq 1/n \leq 1/n^k$, so by the second part of the Comparison Test, the series $\sum 1/n^k$ diverges since $\sum 1/n$ diverges.

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Problem. Determine whether the following series converge or diverge.

$$\text{(i)} \sum \frac{n+2}{n^3 - n^2 + 1}, \qquad \text{(ii)} \sum \frac{n^2+4}{2n^3 - n + 1}.$$

Solution. For each $n \geq 2$ we have

$$n + 2 \leq n + 2n = 3n,$$
$$n^3 - n^2 + 1 \geq n^3 - n^2 = n^2(n-1) \geq n^2(n/2) = n^3/2,$$

since $n - 1 \geq n/2$ for $n \geq 2$.

So $$0 \leq \frac{n+2}{n^3 - n^2 + 1} \leq \frac{3n}{n^3/2} = \frac{6}{n^2}.$$

$\sum 1/n^2$ converges, so $\sum 6/n^2$ converges by the Multiple rule, and now the series (i) converges by the Comparison Test.

We can get the equation into inequality form and then using the comparison rule deduce that the series converges. In this instance, the limits of the equation are reduced to 0 as the lower bound and 6/n^2 on the upper bound. Since we know 1/n^2 converges, by the comparison rule the original equation also converges.

The Ratio Test

The Ratio Test.

If $|a_{n+1}/a_n| \to L$ *then*

(1) *if* $0 \leq L < 1$ *then the series* $\sum a_n$ *converges,*
(2) *if* $L > 1$ *(or* L *is* ∞*) then the series* $\sum a_n$ *diverges,*
(3) *if* $L = 1$ *then the test is inconclusive and the series may or may not converge.*

Problem. Does the series $\sum a_n$ converge or diverge when $a_n = n/2^n$?

Solution. Trying the Ratio Test we get

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{n+1}{2^{n+1}} \times \frac{2^n}{n}\right| = \frac{n+1}{2n} = \frac{1+1/n}{2} \to \frac{1}{2}.$$

Since this limit is less than 1 we conclude that the series $\sum n/2^n$ converges.

Problem. Show that the series $\sum n^k r^n$ converges where $k > 0$ and $0 \leq r < 1$.

Solution. We have

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)^k r^{n+1}}{n^k r^n}\right| = r\left(1 + \frac{1}{n}\right)^k,$$

which converges to r (< 1). So the series $\sum n^k r^n$ converges by the Ratio Test.

Summary of the ratio test:

- To prove whether a series converges or diverges, especially useful for those series containing factorials:
- Take the absolute value of term a_n+1 divided by term a_n
- Simplify and observe the value of the result as n increases.
- If the result is < 1 then the series converges. If the result is > 1 then the series diverges. If the result is exactly 21 then the result is inconclusive

Basic Series

Basic Convergent Series

(1) $\sum_{n=0}^{\infty} r^n = 1/(1-r)$ for any r with $|r| < 1$.
(2) the series $\sum 1/n^k$ converges for any $k > 1$.
(3) the series $\sum n^k r^n$ converges for $k > 0$ and $|r| < 1$.
(4) $\sum_{n=0}^{\infty} c^n/n! = e^c$ for any $c \in \mathbb{R}$.

Basic Divergent Series

- the series $\sum 1/n^k$ diverges for any $k \leq 1$

Power Series. A *power series* is a series of the form $\sum a_n x^n$ where it is usual to start the index at $n = 0$. So the first term is a_0, the second $a_1 x$, and so on.

Lemma. *If $\sum a_n R^n$ converges for some $R \geq 0$, then $\sum a_n x^n$ converges for every x with $|x| < R$.*

Proof. If the series $\sum a_n R^n$ converges then the sequence $(a_n R^n)$ converges to 0 (and so is bounded). Hence there is a B with $|a_n R^n| < B$ for every n. Now

$$|a_n x^n| \leq |a_n R^n||x/R|^n \leq B|x/R|^n \qquad \text{for all } n.$$

If $|x| < R$ then $\sum |x/R|^n$ is a convergent geometric series, and therefore $\sum |a_n x^n|$ converges by the Comparison Test. It follows that $\sum a_n x^n$ converges. □

Radius of Convergence

Radius of Convergence. We say that $R \geq 0$ is the *radius of convergence* of a power series $\sum a_n x^n$ if the series converges whenever $|x| < R$ and diverges whenever $|x| > R$. If the series converges for *all* x we say the radius of convergence is ∞. If the power series $\sum a_n x^n$ has radius of convergence R, then it defines a function $f : (-R, R) \to \mathbb{R}$ given by

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \qquad \text{for all } x \in (-R, R).$$

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Basic Properties of Power Series

Let

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \qquad x \in (-R_1, R_1),$$

$$g(x) = \sum_{n=0}^{\infty} b_n x^n \qquad x \in (-R_2, R_2),$$

where $R_1, R_2 > 0$, and let R be the minimum of R_1 and R_2.
Then:

(1) if $f(x) = g(x)$ for all $x \in (-R, R)$, then $a_n = b_n$ for each n.

Also for any $x \in (-R, R)$:

(2) **sum rule**:

$$f(x) + g(x) = \sum_{n=0}^{\infty} (a_n + b_n) x^n;$$

(3) **multiple rule**:

$$\lambda f(x) = \sum_{n=0}^{\infty} \lambda a_n x^n \text{ for any } \lambda \in \mathbb{R};$$

(4) **product rule**:

$$f(x)g(x) = \sum_{n=0}^{\infty} (a_0 b_n + a_1 b_{n-1} + \cdots + a_{n-1} b_1 + a_n b_0) x^n.$$

Decimal Representation of Real Numbers

22 February 2020 17:01

Recall that if $|r| < 1$ then the *geometric series* (Note 17) $\sum_{n\geq 0} r^n$ has sum $1/(1-r)$, i.e.,

$$1 + r + r^2 + r^3 + \cdots = \frac{1}{1-r}.$$

If $a_1, a_2, \cdots$ is a sequence of decimal digits, so that each a_i belongs to $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, then $.a_1a_2a_3\cdots$ denotes the real number

$$\frac{a_1}{10} + \frac{a_2}{10^2} + \frac{a_3}{10^3} + \cdots .$$

Representing Repeating Decimals as Quotient

Repeating decimal as rational number. A repeating decimal represents a rational number and can be expressed as a quotient of two integers by summing an appropriate geometric series as the following problem illustrates.

Problem. Express the repeating decimal $0.59\dot{1}0\dot{2}$ as the quotient of two integers.

Solution. We have

$$\begin{aligned} 0.59\dot{1}0\dot{2} &= 0.59102102\cdots = \tfrac{59}{100} + \tfrac{102}{10^5} + \tfrac{102}{10^8} + \tfrac{102}{10^{11}} + \cdots \\ &= \tfrac{59}{100} + \tfrac{102}{10^5}\left(1 + \tfrac{1}{10^3} + \tfrac{1}{10^6} + \cdots\right) = \tfrac{59}{100} + \tfrac{102}{10^5}\left(\tfrac{1}{1-1/10^3}\right) \\ &= \tfrac{59}{100} + \tfrac{102}{100}\left(\tfrac{1}{10^3-1}\right) = \tfrac{59}{100} + \tfrac{102}{100}\,\tfrac{1}{999} = \tfrac{59\times 999+102}{99900} \\ &= 59043/99900. \end{aligned}$$

Note that termianting decimal can also be represented as a non-terminating decimal e.g. 1 = 0.99999..

Decimal Expansion of a Rational Number

Problem

Find the decimal expansion of $5/14$.

Solution

$$\begin{aligned} 50 &= 14 \cdot 3 + 8 \\ 80 &= 14 \cdot 5 + 10 \\ 100 &= 14 \cdot 7 + 2 \\ 20 &= 14 \cdot 1 + 6 \\ 60 &= 14 \cdot 4 + 4 \\ 40 &= 14 \cdot 2 + 12 \\ 120 &= 14 \cdot 8 + 8 \end{aligned}$$

Therefore $5/14 = 0.3\dot{5}7142\dot{8}$