Matrices and Linear Independence

07 February 2020 13:20

Determinant of the matrix A is the same as the determinant of the transpose of A

Performing elementary row operations on the transpose is equivalent to performing the same operations on the columns of A.

Elementary column operations and determinants.
If B is the matrix obtained from A by
1. *multiplying a column of A by a number λ, then $|B| = \lambda|A|$;*
2. *interchanging two columns of A, then $|B| = -|A|$;*
3. *adding a multiple of one column of A to another, then $|B| = |A|$.*

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}.$$

The rows (a11 – a1n etc) are called row vectors
The columns (a11 – am1 etc) are called column vectors

A subset of Rn is a basis if and only if it is a linearly independent set containing n vectors.
We can check for this linear independence by computing a determinant.

Linear independence via determinant evaluation. A set of n vectors in $\mathbb{R}^n$ is linearly independent (and therefore a basis) if and only if it is the set of column vectors of a matrix with nonzero determinant.

Let U be the $n \times n$ matrix $[u_{ij}]$. If $|U| \neq 0$ then U is invertible and multiplying both sides of the above matrix equation on the left by U^{-1} gives $\alpha_1 = 0,\ \alpha_2 = 0, \dots, \alpha_n = 0$, i.e. the set $\{\underline{u}_1, \underline{u}_2, \dots, \underline{u}_n\}$ is linearly independent.

If $|U| = 0$ then $|U^T| = 0$ (since $|U| = |U^T|$), so the transpose U^T is not invertible. Hence U^T cannot be reduced to I by elementary row operations and so must be reducible to a matrix with a row of zeros. Therefore elementary column operations can be applied to U to produce a column of zeros. Hence some non-trivial linear combination of $\underline{u}_1, \underline{u}_2, \dots, \underline{u}_n$ is $\underline{0}$, so these vectors are linearly dependent.

Problem. Which of the following sets are a basis for $\mathbb{R}^3$?
(1) $\{(1,-1,2), (0,2,3), (3,-5,3)\}$
(2) $\{(-2,3,4), (2,1,3), (1,-2,-3)\}$

Check the determinant of the matrix - 0 means linearly dependent and as such a basis, !0 means linearly independent and so not a basis.