Q5: Suppose $F(x) = F_l(z_l) + F_{\setminus l}(z_{\setminus l})$ where $x = z_l \cup z_{\setminus l}$,

$z_l \cap z_{\setminus l} = \emptyset$

(a) Consider the partial dependence of $F(x)$ on z_l

$$\bar{F}_l(z_l) = E_{z_{\setminus l}}[F(x)]$$

$$= E_{z_{\setminus l}}[F_l(z_l) + F_{\setminus l}(z_{\setminus l})]$$

$$= E_{z_{\setminus l}}[F_l(z_l)] + E_{z_{\setminus l}}[F_{\setminus l}(z_{\setminus l})]$$

$$= \int F_l(z_l)\, p_{\setminus l}(z_{\setminus l})\, dz_{\setminus l} + \int F_{\setminus l}(z_{\setminus l})\, p_{\setminus l}(z_{\setminus l})\, dz_{\setminus l}$$

$$= F_L(z_L) \int p_{\setminus l}(z_{\setminus l})\, dz_{\setminus l} + \quad ''$$, since $F_L(z_L)$ is constant wrt $z_{\setminus L}$

$$= F_L(z_L) + \quad ''$$, since $\int p_{\setminus L}(z_{\setminus L})\, dz_{\setminus L} = 1$

$$= F_L(z_L) + \underbrace{\int F_{\setminus L}(z_{\setminus L})\, p_{\setminus L}(z_{\setminus L})\, dz_{\setminus L}}_{= \frac{1}{N}\sum_{i=1}^{N} F_{\setminus L}(z_{i \setminus L})}.$$

$= \frac{1}{N}\sum_{i=1}^{N} F_{\setminus L}(z_{i \setminus L})$ = some constant c given the data $\bar{X}$, for all values of z_L.

$$= F_L(z_L) + c \quad \square$$

(b) Consider

$$E_{z_{\setminus L}}[F(x) \mid z_L]$$

$$= E_{z_{\setminus L}}[F_L(z_L) \mid z_L] + E[F_{\setminus L}(z_{\setminus L}) \mid z_L]$$

$$= F_L(z_L) + E[F_{\setminus L}(z_{\setminus L}) \mid z_L]$$

$$= F_L(z_L) + \underbrace{\int F_{\setminus L}(z_{\setminus L})\, p_{\setminus L}(z_{\setminus L} \mid z_L)\, dz_{\setminus L}}$$

This second quantity is not a constant – the distribution of $z_{\sim l}$ can change based on different values of z_l, so

$$E_{z_{\sim l}}[F(x) \mid z_l] \neq F_l(z_l) + c \text{ constant.}$$

This is only a constant when

$$p(z_{\sim l} \mid z_l) = p(z_{\sim l}),$$ or $z_{\sim l}$ and z_l are indepen-dent.