COMS 4771 Machine Learning 2020
Problem Set #1

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Problem 1: Statistical Estimators

(i) We are given that $x_1, \ldots, x_n$ are drawn independently from

$p(x|\theta = (a,b)) \propto \begin{cases} 1 & \text{if } a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$.

That is, $x_i \overset{\text{iid}}{\sim} \text{unif}[a,b]$, $\forall i \in \{1, \ldots, n\}$. Then for each i, the pdf of x_i is

$p(x_i|\theta) = \begin{cases} \frac{1}{b-a} & \text{if } a \leq x_i \leq b \\ 0 & \text{otherwise} \end{cases}$. Therefore, the likelihood function is

$$\mathcal{L}(\theta|X) = \prod_{i=1}^{n} p(x_i|\theta) = \prod_{i=1}^{n} \frac{1}{b-a} = \frac{1}{(b-a)^n}, \quad \text{for} \quad a \leq x_i \leq b \ , \ \forall i \in \{1, \ldots, n\}$$

The constraints on θ can be written equivalently as $a \leq \min_i\{x_i\}$ and $b \geq \max_i\{x_i\}$.

The values of a and b that maximize $\frac{1}{(b-a)^n}$ are equivalent to the values of a and b that minimize $b-a$. Subject to the constraints, $b-a$ is minimized when $a = \min_i\{x_i\}$ and $b = \max_i\{x_i\}$ (both of which are feasible). The MLE estimate of $\theta = (a,b)$, denoted by θ_{ML}, is then

$$\theta_{ML} = \arg\max_{\theta} \mathcal{L}(\theta|X) = \arg\max_{a \leq x_i \leq b} \frac{1}{(b-a)^n} = \arg\min_{a \leq x_i \leq b}(b-a) = (\min_i\{x_i\} \ , \ \max_i\{x_i\})$$

Therefore,

$$\theta_{ML} = (\min\{x_1, \ldots, x_n\}, \ \max\{x_1, \ldots, x_n\})$$

(ii) *Proof.* For an arbitrary, differentiable function g, let Γ be such that $g : \Omega \longrightarrow \Gamma$, where Ω is the parameter space. That is, $\Gamma := \{\tau : g(\theta) = \tau\}$. For each $\tau \in \Gamma$, define $\Theta_\tau := \{\theta : g(\theta) = \tau\}$, and note that $\Theta_\tau \subseteq \Omega$. Finally, let $\hat{\tau}$ be the MLE of $g(\theta)$. That is,

$$\hat{\tau} = \arg\max_{\tau \in \Gamma} \left(\max_{\theta \in \Theta_\tau} \log \mathcal{L}(\theta|\mathbf{x}) \right)$$

Since $\Theta_\tau \subseteq \Omega$, $\max_{\theta \in \Theta_\tau} \log \mathcal{L}(\theta|\mathbf{x}) \leq \max_{\theta \in \Omega} \log \mathcal{L}(\theta|\mathbf{x}) = \log \mathcal{L}(\theta_{ML}|\mathbf{x})$, for all $\tau \in \Gamma$.

That is, since $\log \mathcal{L}(\theta|\mathbf{x})$ is maximized by θ_{ML} over all $\theta \in \Omega$, then it also maximizes $\log \mathcal{L}(\theta|\mathbf{x})$ over $\Theta_\tau \subseteq \Omega$, for all $\tau \in \Gamma$. More specifically,

$$\max_{\tau \in \Gamma} \left(\max_{\theta \in \Theta_\tau} \log \mathcal{L}(\theta|\mathbf{x}) \right) = \max_{\theta \in \Omega} \log \mathcal{L}(\theta|\mathbf{x}) = \log \mathcal{L}(\theta_{ML}|\mathbf{x})$$

Then it must be the case that $\theta_{ML} \in \Theta_{\hat{\tau}} = \{\theta : g(\theta) = \hat{\tau}\}$

$\implies g(\theta_{ML}) = \hat{\tau}$. Hence, $g(\theta_{ML})$ is the MLE of $g(\theta)$.

□

(iii)

- *Consistent and unbiased*: (i) $\hat{\mu} = \frac{1}{N} \sum_{i=1}^{N} X_i$ and (ii) a linear combination of the data with unequal weights on each data point:

  $\hat{\mu} = \sum_{i=1}^{N} \gamma_i X_i$ where $\gamma_i \neq \gamma_j$ and $\sum_{i=1}^{N} \gamma_i = 1$

  For each estimator, we will show why each estimate is consistent and unbiased.

  (i) $\boxed{\hat{\mu} = \frac{1}{N} \sum_{i=1}^{N} X_i}$

  Unbiased: $\mathbb{E}[\hat{\mu}] = \mathbb{E}\left[\frac{1}{N} \sum_{i=1}^{N} X_i\right] = \frac{1}{N} \sum_{i=1}^{N} \mathbb{E}[X_i] = \frac{1}{N} \sum_{i=1}^{N} \mu = \mu$

  Consistent:

  $$\lim_{N \to \infty} \mathbb{E}[\hat{\mu}] = \lim_{N \to \infty} \mathbb{E}\left[\frac{1}{N} \sum_{i=1}^{N} X_i\right] = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \mu = \lim_{N \to \infty} \mu = \mu$$

  and

  $$\lim_{N \to \infty} \mathrm{Var}[\hat{\mu}] = \lim_{N \to \infty} \mathrm{Var}\left[\frac{1}{N} \sum_{i=1}^{N} X_i\right] = \lim_{N \to \infty} \frac{1}{N^2} \sum_{i=1}^{N} \mathrm{Var}[X_i] = \lim_{N \to \infty} \frac{N\sigma^2}{N^2} = \lim_{N \to \infty} \frac{\sigma^2}{N} = 0$$

  (ii) $\boxed{\hat{\mu} = \mathbb{E}\left[\sum_{i=1}^{N} \gamma_i X_i\right] \text{ where } \gamma_i \neq \gamma_j \text{ and } \sum_{i=1}^{N} \gamma_i = 1}$

  Unbiased:

  $$\mathbb{E}[\hat{\mu}] = \mathbb{E}\left[\sum_{i=1}^{N} \gamma_i X_i\right] = \sum_{i=1}^{N} \gamma_i \mathbb{E}[X_i] = \sum_{i=1}^{N} \gamma_i \mu = \mu \sum_{i=1}^{N} \gamma_i = \mu \times 1 = \mu$$

Consistent:

$$\lim_{N\to\infty} \mathbb{E}[\hat{\mu}] = \lim_{N\to\infty} \mathbb{E}\left[\sum_{i=1}^{N} \gamma_i X_i\right] = \lim_{N\to\infty} \sum_{i=1}^{N} \gamma_i \mu = \lim_{N\to\infty} \mu = \mu$$

- *Consistent, but not unbiased*: (i) $\hat{\mu} = \frac{1}{N-1}\sum_{i=1}^{N} X_i$ and (ii) $\hat{\mu} = \frac{1}{N}\sum_{i=1}^{N} X_i + \frac{1}{N}$

  For each estimator, we will show why each estimate is consistent and biased.

(i) $\boxed{\hat{\mu} = \frac{1}{N-1}\sum_{i=1}^{N} X_i}$

Biased: $\mathbb{E}[\hat{\mu}] = \mathbb{E}\left[\frac{1}{N-1}\sum_{i=1}^{N} X_i\right] = \frac{1}{N-1}\sum_{i=1}^{N} \mathbb{E}[X_i] = \frac{1}{N}\sum_{i=1}^{N} \mu = \frac{N\mu}{N-1} \neq \mu$

Consistent:

$$\lim_{N\to\infty} \mathbb{E}[\hat{\mu}] = \lim_{N\to\infty} \mathbb{E}\left[\frac{1}{N-1}\sum_{i=1}^{N} X_i\right] = \lim_{N\to\infty} \frac{1}{N-1}\sum_{i=1}^{N} \mu = \lim_{N\to\infty} \frac{N\mu}{N-1} = \mu$$

and

$$\begin{aligned}\lim_{N\to\infty} \mathrm{Var}[\hat{\mu}] &= \lim_{N\to\infty} \mathrm{Var}\left[\frac{1}{N-1}\sum_{i=1}^{N} X_i\right] \\ &= \lim_{N\to\infty} \left(\frac{1}{N-1}\right)^2 \sum_{i=1}^{N} \sigma^2 = \lim_{N\to\infty} \frac{N\sigma^2}{(N-1)^2} = \lim_{N\to\infty} \frac{\sigma^2}{N-2+\frac{1}{N}} = 0\end{aligned}$$

(ii) $\boxed{\hat{\mu} = \frac{1}{N}\sum_{i=1}^{N} X_i + \frac{1}{N}}$

Biased:

$$\begin{aligned}\mathbb{E}[\hat{\mu}] = \mathbb{E}\left[\frac{1}{N}\sum_{i=1}^{N} X_i + \frac{1}{N}\right] &= \mathbb{E}\left[\frac{1}{N}\sum_{i=1}^{N} X_i\right] + \mathbb{E}\left[\frac{1}{N}\right] = \frac{1}{N}\sum_{i=1}^{N} \mathbb{E}[X_i] + \frac{1}{N} \\ &= \frac{1}{N}\sum_{i=1}^{N} \mu + \frac{1}{N} = \mu + \frac{1}{N} \neq \mu\end{aligned}$$

Consistent:

$$\lim_{N\to\infty} \mathbb{E}[\hat{\mu}] = \lim_{N\to\infty} \mathbb{E}\left[\frac{1}{N}\sum_{i=1}^{N} X_i + \frac{1}{N}\right] = \lim_{N\to\infty} \left(\frac{1}{N}\sum_{i=1}^{N} \mu \; + \; \frac{1}{N}\right) = \lim_{N\to\infty} \left(\mu + \frac{1}{N}\right) = \mu$$

$$\lim_{N\to\infty} \mathrm{Var}[\hat{\mu}] = \lim_{N\to\infty} \mathrm{Var}\left[\frac{1}{N}\sum_{i=1}^{N} X_i + \frac{1}{N}\right] = \lim_{N\to\infty} \frac{1}{N^2}\sum_{i=1}^{N} \mathrm{Var}[X_i] = \lim_{N\to\infty} \frac{N\sigma^2}{N^2}$$
$$= \lim_{N\to\infty} \frac{\sigma^2}{N} = 0$$

- *Not consistent, but unbiased*: (i) $\hat{\mu} = X_k \in \{X_1, \ldots, X_N\}$ and (ii) $\hat{\mu} = \frac{X_1+X_2}{2}$
  For each estimator, we will show why each estimate is not consistent, but unbiased.

  (i) $\boxed{\hat{\mu} = X_k}$
  Unbiased: $\mathbb{E}[\hat{\mu}] = \mathbb{E}[X_k] = \mu$
  Not consistent: Inconsistent since X_k is fixed and will not change as $N \longrightarrow \infty$. That is,

  $$\lim_{N\to\infty} \mathrm{Var}[\hat{\mu}] = \lim_{N\to\infty} \mathrm{Var}[X_k] = \lim_{N\to\infty} \sigma^2 = \sigma^2 \neq 0$$

  (ii) $\boxed{\hat{\mu} = \frac{X_1+X_2}{2}}$
  Unbiased: $\mathbb{E}[\hat{\mu}] = \mathbb{E}\left[\frac{X_1+X_2}{2}\right] = \frac{1}{2}\mathbb{E}[X_1] + \frac{1}{2}\mathbb{E}[X_2] = \frac{1}{2}\mu + \frac{1}{2}\mu = \mu$
  Not consistent:

  $$\lim_{N\to\infty} \mathrm{Var}[\hat{\mu}] = \lim_{N\to\infty} \mathrm{Var}\left[\frac{X_1+X_2}{2}\right] = \lim_{N\to\infty} \frac{1}{2}\sigma^2 = \frac{\sigma^2}{2} \neq 0$$

- *Neither consistent, nor unbiased*: (i) $X_k + \alpha$ and (ii) αX_k, for some $X_k \in \{X_1, \ldots, X_N\}$ and a fixed constant $\alpha > 1$
  For each estimator, we will show why each estimate is neither consistent, nor unbiased.

  (i) $\boxed{\hat{\mu} = X_k + \alpha}$
  Biased: $\mathbb{E}[\hat{\mu}] = \mathbb{E}[X_k + \alpha] = \mu + \alpha \neq \mu$
  Not consistent:

  $$\lim_{N\to\infty} \mathbb{E}[\hat{\mu}] = \lim_{N\to\infty} \mathbb{E}[X_k + \alpha] = \lim_{N\to\infty} \mu + \alpha = \mu + \alpha \neq \mu$$

  and

  $$\lim_{N\to\infty} \mathrm{Var}[\hat{\mu}] = \lim_{N\to\infty} \mathrm{Var}[X_k + \alpha] = \lim_{N\to\infty} \mathrm{Var}[X_k] = \lim_{N\to\infty} \sigma^2 = \sigma^2 > 0$$

  (ii) $\boxed{\hat{\mu} = \alpha X_k}$
  Biased: $\mathbb{E}[\hat{\mu}] = \mathbb{E}[\alpha X_k] = \alpha\mu \neq \mu$

Not consistent:

$$\lim_{N\to\infty} \mathbb{E}[\hat{\mu}] = \lim_{N\to\infty} \mathbb{E}[\alpha X_k] = \lim_{N\to\infty} \alpha\mu = \alpha\mu \neq \mu$$

and

$$\lim_{N\to\infty} \mathrm{Var}[\hat{\mu}] = \lim_{N\to\infty} \mathrm{Var}[\alpha X_k] = \lim_{N\to\infty} \alpha^2\sigma^2 = \alpha^2\sigma^2 > 0$$

Problem 2: On Forecasting Product Demand

1. Expected profit π:

$$\pi = \begin{cases} (P-C)Q & D \geq Q \\ (P-C)D - C(Q-D) & D < Q \end{cases}$$

First, we will define f as the PDF of D and F as the CD of D.
If the retailer buys Q items:

$$\begin{aligned}
\mathbb{E}_\pi(D) &= \int_Q^\infty (P-C)Qf(D)dD + \int_0^Q [(P-C)Df(D) - C(Q-D)f(D)]dD \\
&= \int_Q^\infty (P-C)Qf(D)dD + \int_0^Q (P-C)Df(D)dD - \int_0^Q C(Q-D)f(D)dD \\
&= (P-C)Q\int_Q^\infty f(D)dD + (P-C)\int_0^Q Df(D)dD - CQ\int_0^Q f(D)dD \\
&+ C\int_0^Q Df(D)dD
\end{aligned}$$

Since

$$F(Q) = \int_0^Q f(D)dD \implies \int_Q^\infty f(D)dD = 1 - F(Q)$$

we have:

$$\begin{aligned}
\mathbb{E}_\pi(D) &= (P-C)Q(1-F(Q)) + P\int_0^Q Df(D)dD - \cancel{C\int_0^Q Df(D)dD} \\
&- CQF(Q) + \cancel{C\int_0^Q Df(D)dD}
\end{aligned}$$

Simplifying further:

$$\begin{aligned}
\mathbb{E}_\pi(D) &= (P-C)Q - PQF(Q) + CQF(Q) - CQF(Q) + P\int_0^Q Df(D)dD \\
&= \boxed{(P-C)Q - PQF(Q) + P\int_0^Q Df(D)dD}
\end{aligned}$$

2. Derivative with respect to Q (using equation from above):

$$\begin{aligned}
\frac{d\mathbb{E}_\pi(D)}{dQ} &= (P-C) - PF(Q) - PQf(Q) + P\left(\frac{d}{dQ}\int_0^Q Df(D)dD\right) \\
&= (P-C) - PF(Q) - PQf(Q) + PQF(Q) \\
&= \boxed{(P-C) - PF(Q)}
\end{aligned}$$

To find critical points:

$$\begin{aligned}\frac{d\,\mathbb{E}_\pi(D)}{dQ} = P - C - PF(Q) &= 0\\ PF(Q) &= P - C\\ F(Q) &= 1 - \frac{C}{P}\end{aligned}$$

Thus, the optimal quality Q^* is when $F(Q^*) = 1 - \frac{C}{P}$.

Problem 3: Evaluating Classifiers

(i) We get an error when $x_i > t, y_i = y_2$ or when $x_i \leq t, y_i = y_1$

$$\begin{aligned} P[f_t(X) \neq y] &= \int_x P[f_t(\vec{x}) \neq y | X = \vec{x}] P[X] dx \\ &= \int_x \left[P[f_t(\vec{x}) = y_1, Y = y_2 | X = \vec{x}] + P[f_t(\vec{x}) = y_2, Y = y_1 | X = \vec{x}] \right] dx \\ &= \int_t^{\infty} P[f_t(\vec{x}) = y_1, Y = y_2 | X = \vec{x}] dx + \int_{-\infty}^{t} P[f_t(\vec{x}) = y_2, Y = y_1 | X = \vec{x}] dx \end{aligned}$$

(ii) We will find the optimal threshold via critical points:

$$\begin{aligned} \frac{d}{dx} \left(\int_t^{\infty} P[f_t(\vec{x}) = y_1, Y = y_2 | X = \vec{x}] dx + \int_{-\infty}^{t} P[f_t(\vec{x}) = y_2, Y = y_1 | X = \vec{x}] dx \right) &= 0 \\ P[f_t(\vec{x}) = y_1, Y = y_2 | X = t] dx - P[f_t(\vec{x}) = y_2, Y = y_1 | X = t] dx &= 0 \\ P[f_t(\vec{x}) = t, Y = y_2] &= P[f_t(\vec{x}) = t, Y = y_1] \\ P(X = t | Y = y_1) P(Y = y_1) &= P(X = t | Y = y_2) P(Y = y_2) \end{aligned}$$

(iii) Assuming that the underlying population distribution has equal class priors ($P[Y = y_1] = P[Y = y_2]$) and the individual class conditionals ($P[X|Y = y_1]$ and $P[X|Y = y_2]$) are distributed as Gaussians.

(a) An example of class conditionals such that for some threshold value t, the rule f_t achieves the Bayes error rate.
Since they are distributed as Gaussians, the thresholding will achieve the same error rate as Bayes when t is the intersection of the gaussians (the midpoint between the means if the variance is the same). The Bayes error rate, which is achieved when effectively selecting the label with the higher probability, is exactly matched when the two class conditionals are separated (different μ), and have the same variance, and the threshold is simply the intersection of the class conditionals.

(b) An example setting of class conditionals such that for no threshold value t, the rule f_t achieves the Bayes error rate.
Class conditionals with the same mean and different variance will mean that any value of t will yield a higher error rate than the Bayes error rate. In this case, the Bayes error rate is achieved by predicting: class one $\rightarrow$ class two $\rightarrow$ class one. Since thresholding can only provide one transition, for any value of t, it will have a higher error rate.

Problem 4: Analyzing iterative optimization

(i) *Proof.* We first show that M is *symmetric*.
Recall that a (square) matrix M is symmetric $\iff M = M^\top$
Clearly, $M = A^\top A$ is a square matrix ($A^\top A$ a $d \times d$ matrix). Consider the following

$$M^\top = (A^\top A)^\top = A^\top (A^\top)^\top = A^\top A = M \implies M^\top = M$$

Thus, M is symmetric.

To prove that M is also *positive semi-definite*, it suffices to show that for any $x \in \mathbb{R}^d,\ x^\top M x \geq 0$. As such, consider an arbitrary vector $\mathbf{x} \in \mathbb{R}^d$ and let $\mathbf{w} = A\mathbf{x}$, where $\mathbf{w} \in \mathbb{R}^n$ and $\mathbf{w} = [w_1,\ w_2,\ \ldots, w_n]^\top$. We then have that,

$$\mathbf{x}^\top M \mathbf{x} = \mathbf{x}^\top A^\top A \mathbf{x} = (A\mathbf{x})^\top A\mathbf{x} = \|A\mathbf{x}\|_2^2 = \|\mathbf{w}\|_2^2 = \sum_{i=1}^{n} w_i^2 \geq 0$$

$\implies M$ is positive semi-definite. □

(ii) *Proof.* Proof by induction on N.

Base case: $(N = 1, 2)$

For $N = 1$ we have

$$\begin{aligned}
\beta^{(1)} &= \beta^{(0)} + \eta A^\top (b - A\beta^{(0)}) && \text{(by definition of the Richardson iteration)} \\
&= \eta A^\top b = \eta v && \text{(since } \beta^{(0)} \text{ is the zero vector and } v = A^\top b) \\
&= \eta I v = \eta \underbrace{(I - \eta M)^0}_{=I} v && \text{(Note: } (I - \eta M) \text{ is a square matrix since } I \text{ and } M \text{ are square)} \\
&= \eta \sum_{k=0}^{0} (I - \eta M)^k v && \text{Thus, it holds for } N = 1.
\end{aligned}$$

For $N = 2$ we have

$$\begin{aligned}
\beta^{(2)} &= \beta^{(1)} + \eta A^\top (b - A\beta^{(1)}) = \eta v + \eta (A^\top b - A^\top A \eta v) && \text{(since } \beta^{(1)} = \eta v \text{ from the above)} \\
&= \eta v + \eta (v - M \eta v) && \text{(since } M = A^\top A \text{ and } v = A^\top b) \\
&= \eta \underbrace{(I - \eta M)^0}_{=I} v + \eta (I - \eta M) v && (\eta \text{ a real number}) \\
&= \eta [(I - \eta M)^0 v + (I - \eta M)^1 v] \\
&= \eta \sum_{k=0}^{1} (I - \eta M)^k v && \text{Thus, the result holds for } N = 2.
\end{aligned}$$

(Inductive hypothesis) Now assume the following holds for $k = 1, 2, \ldots, N-1$
That is, assume the following holds:

$$\beta^{(N-1)} = \eta \sum_{k=0}^{N-2} (I - \eta M)^k v$$

From the definition of the Richardson iteration, the N^{th} iterate is

$$\begin{aligned}
\beta^{(N)} &= \beta^{(N-1)} + \eta A^\top (b - A\beta^{(N-1)}) = \beta^{(N-1)} + \eta(v - M\beta^{(N-1)}) \\
&= \eta \sum_{k=0}^{N-2} (I - \eta M)^k v + \eta \left[v - M \left(\eta \sum_{k=0}^{N-2} (I - \eta M)^k v \right) \right] && \text{(from the induction hypothesis)} \\
&= \eta \sum_{k=0}^{N-2} (I - \eta M)^k v - \eta M \left(\eta \sum_{k=0}^{N-2} (I - \eta M)^k v \right) + \eta v \\
&= (I - \eta M) \left(\eta \sum_{k=0}^{N-2} (I - \eta M)^k v \right) + \eta v \\
&= \eta \sum_{k=0}^{N-2} (I - \eta M)^{k+1} v + \eta v \\
&= \eta \sum_{k=0}^{N-1} (I - \eta M)^k v + \eta v && \text{(rearrange indices)} \\
&= \eta \sum_{k=1}^{N-1} (I - \eta M)^k v + \eta \underbrace{(I - \eta M)^0}_{=I} v && \text{(adding } k = 0 \text{ summand)} \\
&= \eta \sum_{k=0}^{N-1} (I - \eta M)^k v
\end{aligned}$$

Hence, $\beta^{(N)} = \eta \sum_{k=0}^{N-1} (I - \eta M)^k v$ □

(iii) We are given that the eigenvalues of M are $\lambda_1, \lambda_2, \ldots, \lambda_d$. Then, the eigenvalues of $I - \eta M$ are $1 - \eta\lambda_i$, for all $i = 1, \ldots, d$. Indeed, without loss of generality, let $\mathbf{x}$ be the eigenvector associated with λ_i, then $M\mathbf{x} = \lambda_i \mathbf{x} \implies (\eta M)\mathbf{x} = (\eta\lambda_i)\mathbf{x} \implies I\mathbf{x} - (\eta M)\mathbf{x} = I\mathbf{x} - \eta\lambda_i\mathbf{x} = 1\mathbf{x} - \eta\lambda_i\mathbf{x} \implies (I - \eta M)\mathbf{x} = (1 - \eta\lambda_i)\mathbf{x}$.

We also claim that since $((1 - \eta\lambda_i),\ \mathbf{x})$ is the eigenvalue–eigenvector pair for $(I - \eta M)$, then $((1 - \eta\lambda_i)^k,\ \mathbf{x})$ is the eigenvalue–eigenvector pair for $(I - \eta M)^k$, $k \in \mathbb{N} \cup \{0\}$.

Proof of claim: For any $i \in \{1, 2, \ldots, d\}$ we have

$$\begin{aligned}
(I - \eta M)\mathbf{x} = (1 - \eta\lambda_i)\mathbf{x} &\implies (I - \eta M)^2 \mathbf{x} = (1 - \eta\lambda_i)(I - \eta M)\mathbf{x} = (1 - \eta\lambda_i)^2 \mathbf{x} \\
&\implies (I - \eta M)^3 \mathbf{x} = (1 - \eta\lambda_i)^2 (I - \eta M)\mathbf{x} = (1 - \eta\lambda_i)^3 \mathbf{x} \\
&\vdots \\
&\implies (I - \eta M)^k \mathbf{x} = (1 - \eta\lambda_i)^{k-1} (I - \eta M)\mathbf{x} = (1 - \eta\lambda_i)^k \mathbf{x}
\end{aligned}$$

□

Using the above results, we have the following

$$\begin{aligned}
&\eta I\mathbf{x} + \eta(I - \eta M)\mathbf{x} + \eta(I - \eta M)^2 \mathbf{x} + \cdots + \eta(I - \eta M)^{N-1} \mathbf{x} = \\
&\qquad \eta\mathbf{x} + \eta(1 - \eta\lambda_i)\mathbf{x} + \eta(1 - \eta\lambda_i)^2 \mathbf{x} + \cdots + \eta(1 - \eta\lambda_i)^{N-1} \mathbf{x}
\end{aligned}$$

$$\implies \eta(I + (I - \eta M) + (I - \eta M)^2 + \cdots + (I - \eta M)^{N-1})\mathbf{x} = \eta(1 + (1 - \eta\lambda_i) + (1 - \eta\lambda_i)^2 + \cdots + (1 - \eta\lambda_i)^{N-1})\mathbf{x}$$

$$\implies \left(\eta \sum_{k=0}^{N-1} (I - \eta M)^k\right)\mathbf{x} = \left(\eta \sum_{k=0}^{N-1} (1 - \eta\lambda_i)^k\right)\mathbf{x} = \left(\frac{1 - (1 - \eta\lambda_i)^N}{\lambda_i}\right)\mathbf{x}\,,\ \forall i = 1, 2, \ldots, d$$

Thus, the eigenvalues of $\eta \sum_{k=0}^{N-1} (I - \eta M)^k$ are

$$\frac{1 - (1 - \eta\lambda_1)^N}{\lambda_1}\ ,\ \frac{1 - (1 - \eta\lambda_2)^N}{\lambda_2}\ ,\ \cdots\ ,\ \frac{1 - (1 - \eta\lambda_d)^N}{\lambda_d}$$

.

(iv) *Proof.*
Note that

$$\begin{aligned}
\hat{\beta} - \beta^{(N)} &= \left(\hat{\beta} + \eta A^\top (b - A\hat{\beta})\right) - \left(\beta^{(N-1)} + \eta A^\top (b - A\beta^{(N-1)})\right) \\
&= \left(\hat{\beta} + \eta v - \eta M\hat{\beta}\right) - \left(\beta^{(N-1)} + \eta v - \eta M\beta^{(N-1)}\right) \\
&= \left((I - \eta M)\hat{\beta} + \cancel{\eta v}\right) - \left((I - \eta M)\beta^{(N-1)} + \cancel{\eta v}\right) \\
&= (I - \eta M)(\hat{\beta} - \beta^{(N-1)}) \\
&= (I - \eta M)^2((\hat{\beta} - \beta^{(N-2)}) \\
&= (I - \eta M)^3(\hat{\beta} - \beta^{(N-3)}) \\
&\vdots \\
&= (I - \eta M)^N(\hat{\beta} - \beta^{(0)}) \\
&= (I - \eta M)^N\hat{\beta}
\end{aligned}$$

Now consider,

$$\begin{aligned}
\|\beta^{(N)} - \hat{\beta}\|_2^2 = \|\hat{\beta} - \beta^{(N)}\|_2^2 &= \|(I - \eta M)^N\hat{\beta}\|_2^2 \\
&\leq \|(I - \eta M)^N\|_2^2\|\hat{\beta}\|_2^2 \\
&\leq (\|(I - \eta M)\|_2^2)^N\|\hat{\beta}\|_2^2 \\
&= \|I - \eta M\|_2^{2N}\|\hat{\beta}\|_2^2 \\
&\leq (1 - 2\eta\lambda_{min})^N\|\hat{\beta}\|_2^2 \\
&\leq e^{-2\eta\lambda_{min}N}\|\hat{\beta}\|_2^2
\end{aligned}$$

□

Problem 5: Designing socially aware classifiers

(i) It is not enough just to remove the sensitive attribute A from the dataset because it is possible that other attributes in the feature vector are correlated with this attribute. For example, let's say we are hiring for a Computer Science job, and we want to look at the sensitive attribute *gender*. We can simply remove the Male/Female flag feature from the dataset. However, suppose that another feature in the dataset is where the applicant got their education. What if the process of college admissions is biased in terms of gender? Then this schooling feature will be highly correlated with gender. Thus, it is not always sufficient to remove ONLY the sensitive feature in question.

(ii) Notation: $\Sigma[\hat{Y} = y]$ will represent the count of $\hat{Y} = y$ for the total population, where similarly $\Sigma_a[\hat{Y} = y]$ will be the count of $\hat{Y} = y$ for the population of sensitive attribute a. Also, for convenience, we will use N to represent the number of instances in the whole population and N_a to represent the number of instances in the population of sensitive attribute a.

$$\mathbb{P}_0[\hat{Y} = 1] = \mathbb{P}_1[\hat{Y} = 1]$$

$$\frac{\Sigma_0[\hat{Y} = 1]}{\Sigma_0[\hat{Y} = 1] + \Sigma_0[\hat{Y} = 0]} = \frac{\Sigma_1[\hat{Y} = 1]}{\Sigma_1[\hat{Y} = 1] + \Sigma_1[\hat{Y} = 0]}$$

$$\frac{\Sigma_0[\hat{Y} = 1]}{N_0} = \frac{\Sigma_1[\hat{Y} = 1]}{N - N_0}$$

$$\frac{\Sigma_0[\hat{Y} = 1]}{N_0} = \frac{\Sigma[\hat{Y} = 1] - \Sigma_0[\hat{Y} = 1]}{N - N_0}$$

$$(N - N_0)\Sigma_0[\hat{Y} = 1] = N_0\Sigma[\hat{Y} = 1] - N_0\Sigma_0[\hat{Y} = 1]$$

$$(N-\Sigma_0[\hat{Y}=1]-\Sigma_0[\hat{Y}=0])\Sigma_0[\hat{Y}=1] = (\Sigma_0[\hat{Y}=1]+\Sigma_0[\hat{Y}=0])\Sigma[\hat{Y}=1]-(\Sigma_0[\hat{Y}=1]+\Sigma_0[\hat{Y}=0])\Sigma_0[\hat{Y}=1]$$

$$N\Sigma_0[\hat{Y}=1]-(\Sigma_0[\hat{Y}=1])^2-\Sigma_0[\hat{Y}=0]\Sigma_0[\hat{Y}=1] = \Sigma_0[\hat{Y}=1]\Sigma[\hat{Y}=1]+\Sigma_0[\hat{Y}=0]\Sigma[\hat{Y}=1]-(\Sigma_0[\hat{Y}=1])^2-\Sigma_0[\hat{Y}=0]\Sigma_0[\hat{Y}=1]$$

$$N\Sigma_0[\hat{Y} = 1] = \Sigma[\hat{Y} = 1](\Sigma_0[\hat{Y} = 1] + \Sigma_0[\hat{Y} = 0])$$

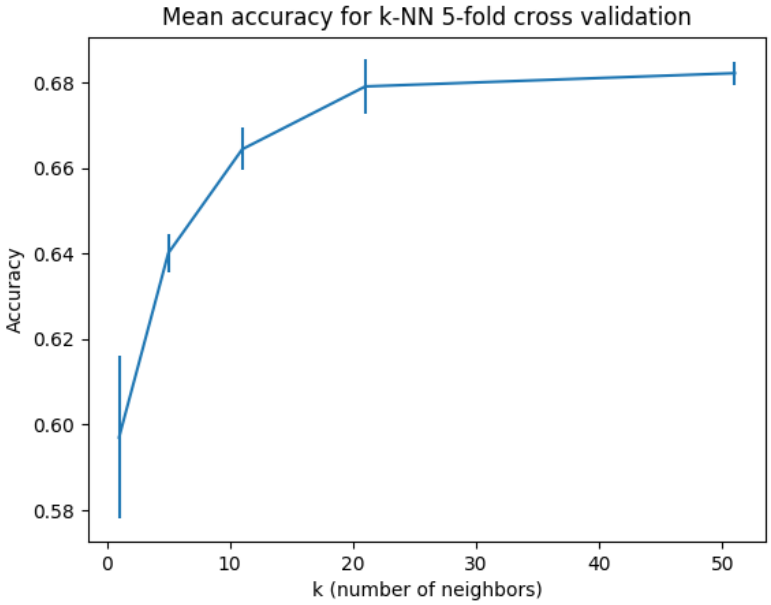
$$\frac{\Sigma_0[\hat{Y} = 1]}{\Sigma_0[\hat{Y} = 1] + \Sigma_0[\hat{Y} = 0]} = \frac{\Sigma[\hat{Y} = 1]}{N}$$

$$\mathbb{P}_a[\hat{Y} = 1] = \mathbb{P}[\hat{Y} = 1]$$

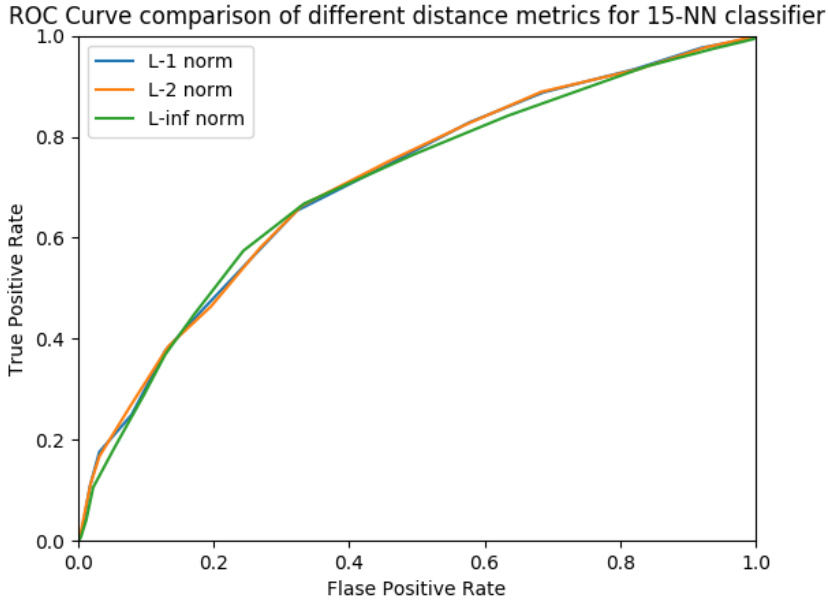
(iii) $\mathbb{P}[\hat{Y} = 1] = \mathbb{P}_a[\hat{Y} = 1]$ is equivalent to $\mathbb{E}[\hat{Y} = y] = \mathbb{E}[\hat{Y}] \forall a, y \in \{0, 1\}$.

In general, $\mathbb{P}[\hat{Y} = y|A = a] = \mathbb{P}[\hat{Y} = y] \forall a \in \mathbb{N}, \forall y \in \mathbb{R}$ [1]

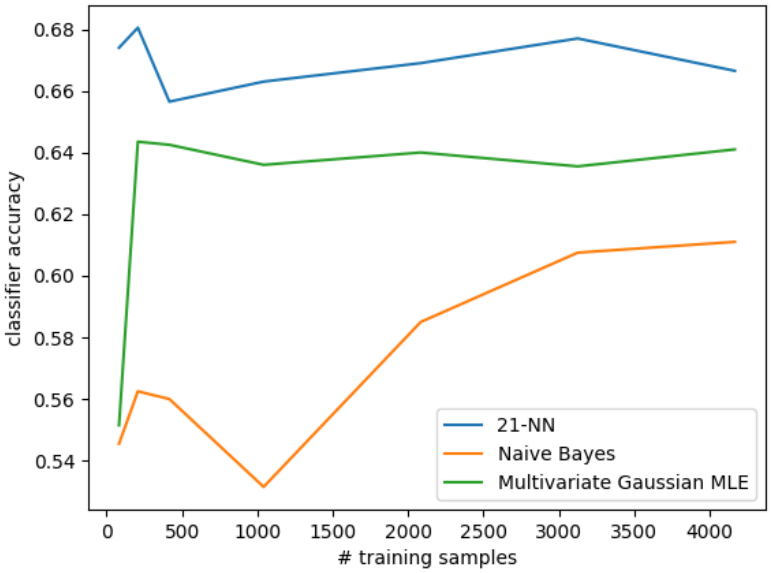
(iv) We consider different values of k for our k-NN classifier:



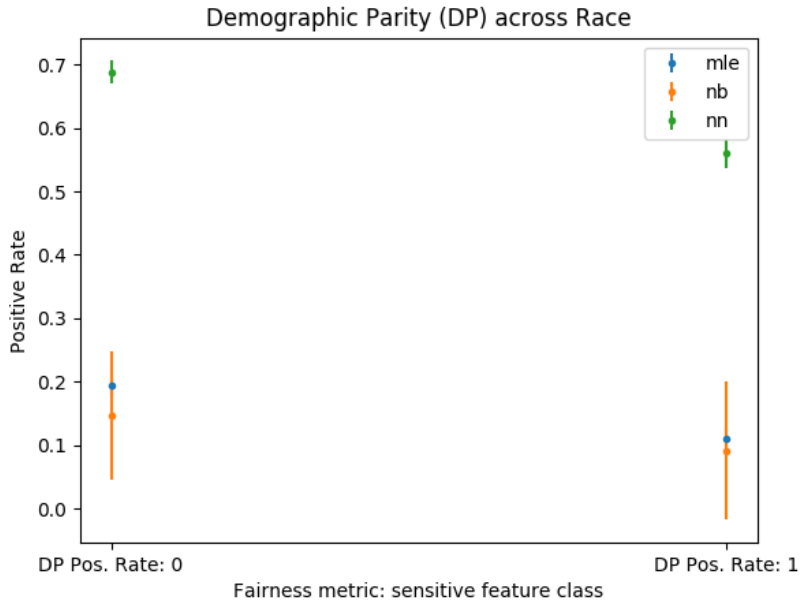
We also consider different distance metrics (L_1, L_2, L_∞):



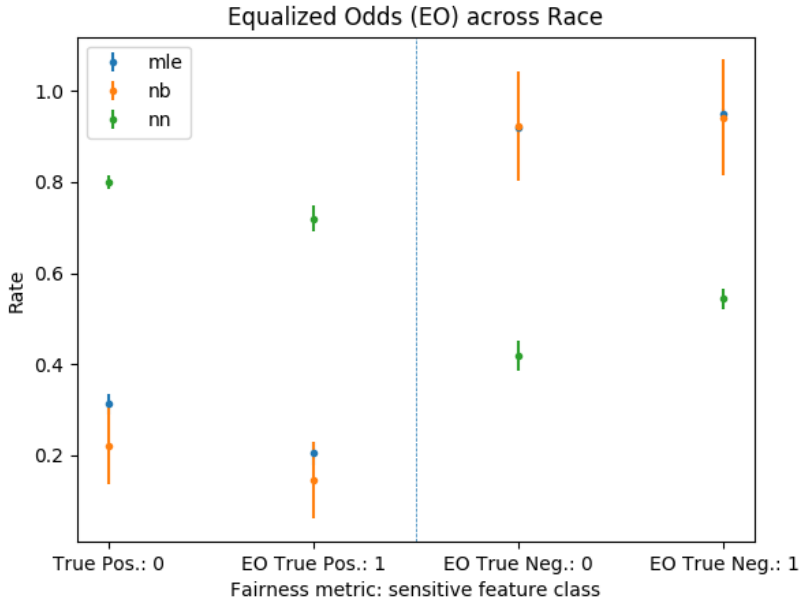
(v) The nearest neighbors classifier works best for this task:



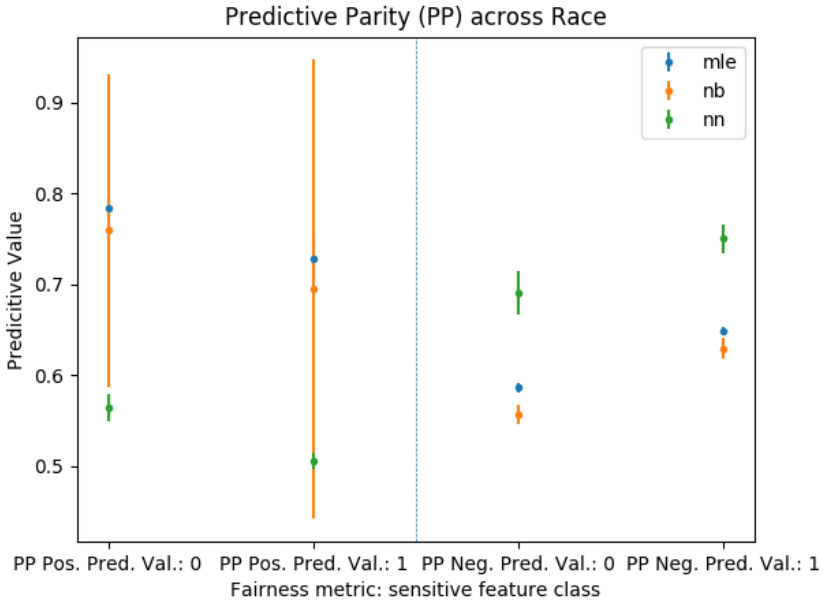
(vi) First, we consider Demographic Parity. The Naive Bayes classifier is the fairest, because the other two have statistically significant differences (outside 2 standard deviations) between the Positive rate for the two different races:



Next, we have Equalized Odds. Once again, Naive Bayes methods do not show a statistically significant difference across the sensitive feature for either the true positive or true negative rates:



Finally, we test for Predictive Parity. In the case of Positive predictive value, Naive Bayes is most fair. In contrast, Nearest Neighbors seems to be most fair for Negative Predictive Value, though all 3 methods struggle with fairness for this metric:



(vii) Once again, we can consider the situation of hiring fairness. Equalized Odds (EO) is an appropriate fairness metric for hiring. It is superior to Demographic Parity (DP) because DP can still be achieved with "laziness." Laziness could come into play if we consider that only a subset of the applicants are qualified for the job. It would be possible to achieve DP across the two gender groups by hiring only the qualified candidates from one group, while randomly hiring any candidates from the second. This does not equate to outcome equality, given that the random candidates will be

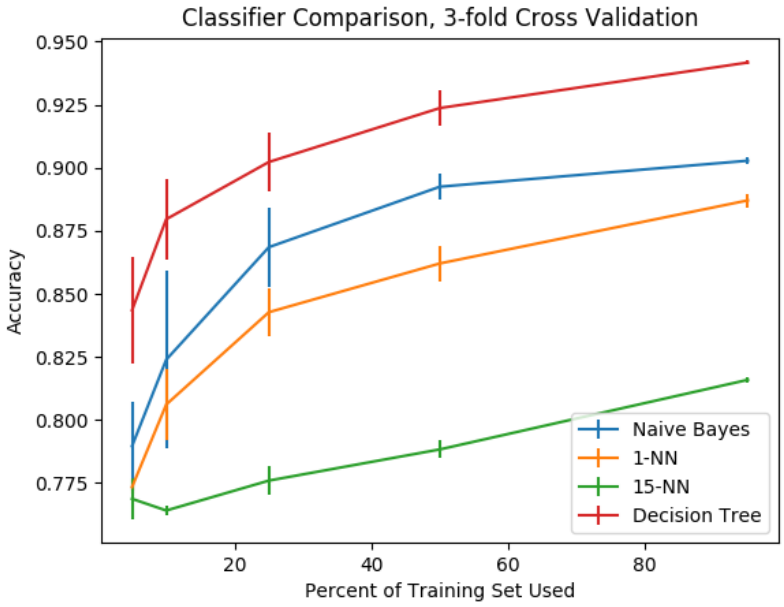
less likely to succeed in the long run. EO avoids this issue by looking at the true positive and true negative rate, as opposed to just the positive rate. However, EO still does have flaws. If once again we consider that only some candidates are qualified from each demographic, we could achieve EO by hiring the same proportion of candidates from each of the two qualified pools (one for each gender). However, if one of the two pools has significantly more qualified candidates than the other, the outcome could still be imbalanced in terms of overall hiring counts across gender. [2]

Problem 6: Email spam classification case study

(i) We have reduced the feature space by stemming and removing stop words and punctuation. We reduced the space further by only considering words that appeared at least 10 times in the entire dataset. This left us with approximately 4,000 features.

(ii) We implemented nearest neighbors and 2 versions of Naive Bayes: one with binary features (i.e. just presence/absence of a word) and one with word counts distributed as Gaussians. Please see our code for further details. Also note that our Decision Tree code did not come together in time, so for the purpose of analysis, we used a prepackaged decision tree classifier.

(iii) Decision tree is the best classifier for this task, followed by naive bayes.



References

[1] Alekh Agarwal, Alina Beygelzimer, Miroslav Dudik, John Langford, and Hanna Wallach. *A Reductions Approach to Fair Classification*. ICML, 2018.

[2] Solon Barocas and Moritz Hardt. *Fairness in Machine Learning*. NIPS, 2017.