CSOR W4231 Analysis of Algorithms - Spring 2021
Homework #5

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Problem 1

i. From the conditions provided in the problem, each node $v \in V$ must satisfy the following demand constraint:

$$f^{\text{in}}(v) - f^{\text{out}}(v) = d(v)$$

By the flow conservation condition for feasible flows: $\displaystyle\sum_{v\in V} f^{\text{in}}(v) = \sum_{v\in V} f^{\text{out}}(v)$

$$\implies \sum_{v\in V} f^{\text{in}}(v) - \sum_{v\in V} f^{\text{out}}(v) = 0$$

We then have that

$$\sum_{v\in V} d(v) \;=\; \sum_{v\in V} f^{\text{in}}(v) - f^{\text{out}}(v) \;=\; \sum_{v\in V} f^{\text{in}}(v) - \sum_{v\in V} f^{\text{out}}(v) \;=\; 0$$

$$\implies \sum_{v\in V} d(v) = 0$$

$$\implies \sum_{v\in V} d(v) = \underbrace{\sum_{d(v)>0} d(v)}_{=\sum_{v\in T} d(v)} + \underbrace{\sum_{d(v)<0} d(v)}_{=\sum_{v\in S} d(v)} + \underbrace{\sum_{d(v)=0} d(v)}_{=0\ (\forall v\in V\setminus S\cup T)} = 0$$

$$\implies \sum_{v\in T} d(v) + \sum_{v\in S} d(v) + \cancelto{0}{\sum_{v\in V\setminus S\cup T} d(v)} = 0$$

$$\implies \sum_{v\in T} d(v) + \sum_{v\in S} d(v) = 0$$

$$\implies \sum_{v\in T} d(v) \;=\; -\sum_{v\in S} d(v)$$

Hence, a necessary condition for a feasible circulation with demands to exist is

$$\sum_{v \in T} d(v) \;=\; -\sum_{v \in S} d(v) \qquad \text{and/or} \qquad \sum_{v \in V} d(v) = 0$$

Furthermore, the demand at nodes in S or T must not exceed the capacity of the edges leaving or entering the nodes in S or T, respectively. However, this condition is captured by the combination of the capacity constraints and the demand constraints.

ii. Let $G = (V, E)$ be the flow network given in the problem. From G, construct a new network $G' = (V', E')$ where $V' = V \cup \{s, t\}$ and $E' = E \cup E_N$. The node s is a new, additional source node in G' and t is a new, additional sink in G'. The new set of edges E_N is a set of directed edges from s to all nodes in the set S and directed edges from all nodes in T to t. That is, $E_N = \{(s, v), (u, t) : v \in S \text{ and } u \in T\}$. For each $(u, t) \in E_N$, with $u \in T$, assign a capacity $c_{(u,t)} = d(u)$. For each $(s, v) \in E_N$, with $v \in S$, assign a capacity $c_{(s,v)} = -d(v)$, since $d(v) < 0$ for all $v \in S$ and capacities must be positive.

For any feasible circulation in G, the maximum flow can be achieved in G' by assigning a flow value $f(u, t) = c_{(u,t)} = d(u)$ for all $(u, t) \in E_N$ and $u \in T$, and a flow value $f(s, v) = c_{(s,v)} = -d(v)$ for all $(s, v) \in E_N$ and $v \in S$. That is, assign flow values equal to the capacity of the corresponding edge. Then clearly, the max flow value will be $\sum_{v \in T} d(v)$. On the other hand, if the max flow is $\sum_{v \in T} d(v)$ in G', then this implies that every $e \in E_N$ is fully saturated at capacity (i.e., $f(e) = c_e$), which gives a feasible solution to the max flow problem on the original network G, and thus a feasible circulation in G.

Problem 2

Problem 3

Problem 4

Problem 5