Analysis of Algorithms, I

CSOR W4231

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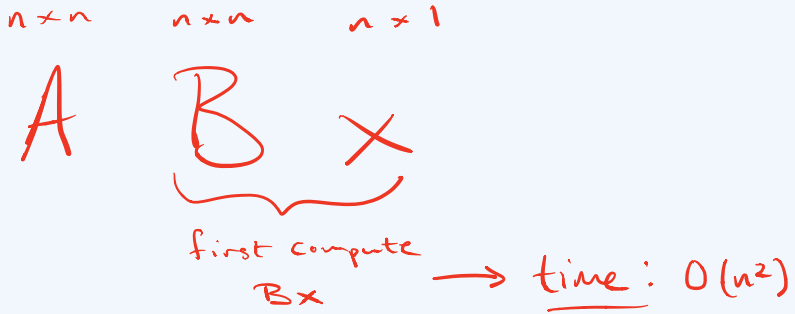
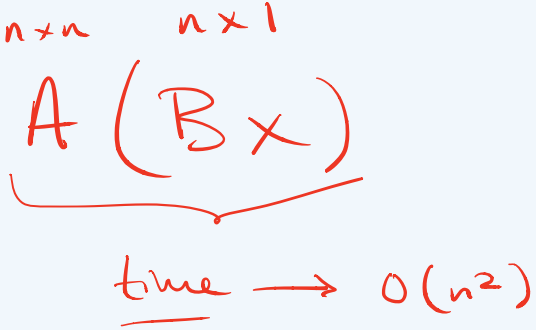
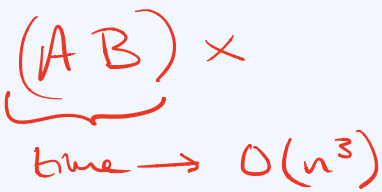
More dynamic programming: matrix chain multiplication

Outline

Today







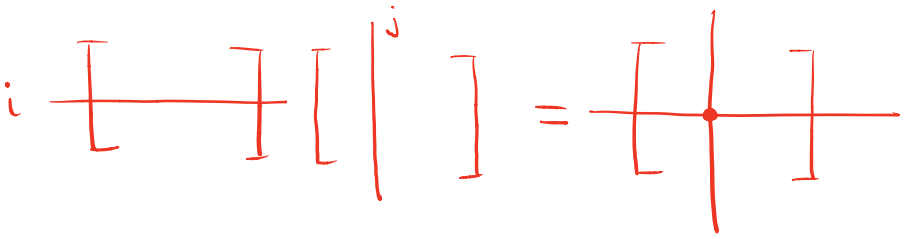
Usual way to multiply two matrices:

$$A \cdot B = C$$

$(m \times n)$ $(n \times p)$ $(m \times p)$

How many arithmetic operations are performed to compute C?

$$C_{ij} = \sum_{k=1}^{n} A_{ik} \cdot B_{kj}$$



arithmetic operations for C_{ij}: n multiplications + $n-1$ scalar additions

$= 2n-1$ operations

arithmetic operations for $C = mp \cdot n$ multiplications + $mp(n-1)$ additions

$\approx 2mp \cdot n$ operations

which is roughly twice the # of scalar multiplications.

$\Rightarrow$ Instead of computing the # of arithmetic operations to fill in the entire matrix, we are going to compute the # of scalar multiplications required to compute the entire matrix since we know that the total number of arithmetic operations is approx. twice the number of scalar multiplications.

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sc. mult. for $C = m \cdot n \cdot p$

Matrix chain multiplication example

Example 1.

Input: matrices A_1, A_2, A_3 of dimensions 6×1, 1×5, 5×2

Output:

- a way to compute the product $A_1 A_2 A_3$ so that the number of arithmetic operations performed is minimized;
- the minimum number of arithmetic operations required.

Useful observations

Remark 1.

- *We do not want to compute the actual product.*
- *Matrix multiplication is associative but not commutative (in general). Hence a solution to our problem corresponds to a parenthesization of the product.*
- *We want the optimal parenthesization and its cost, that is, the parenthesization that minimizes the number of arithmetic operations, as well as that number.*

Estimating #arithmetic operations

- Let A, B be matrices of dimensions $m \times n$, $n \times p$.
- Let $C = AB$. Then C is an $m \times p$ matrix such that

$$c_{ij} = \sum_{k=1}^{n} a_{ik} \cdot b_{kj}.$$

$\Rightarrow$ c_{ij} requires n scalar multiplications, $n - 1$ additions

$\Rightarrow$ #arithmetic operations to compute c_{ij} is dominated by #scalar multiplications

- Total #scalar multiplications to fill in C is mnp

Minimizing #scalar multiplications for $A_1A_2A_3$

Input: A_1, A_2, A_3 of dimensions 6×1, 1×5, 5×2 respectively

Given a parenthesization of the input matrices, its cost is the total # scalar multiplications to compute the product.

Two ways of computing $A_1A_2A_3$:

1. $(A_1A_2)A_3$: first compute A_1A_2, then multiply it by A_3
   - $6 \cdot 1 \cdot 5$ scalar multiplications for A_1A_2
   - $6 \cdot 5 \cdot 2$ scalar multiplications for $(A_1A_2)A_3$

   $\Rightarrow$ 90 scalar multiplications in total

2. $A_1(A_2A_3)$: first compute A_2A_3, then multiply A_1 by A_2A_3
   - $1 \cdot 5 \cdot 2$ scalar multiplications for A_2A_3
   - $6 \cdot 1 \cdot 2$ scalar multiplications for $A_1(A_2A_3)$

   $\Rightarrow$ 22 scalar multiplications in total

Remark 2.

Parenthesization $A_1(A_2A_3)$ improves over $(A_1A_2)A_3$ by over 75%.

(Fully) Parenthesized products of matrices

A_1A_2 : NOT parenthesized (A_1A_2) : fully parenthesized

Definition 2.

A product of matrices is fully parenthesized if it is

1. a single matrix; or
2. the product of two fully parenthesized matrices, surrounded by parentheses.

Examples: $((A_1A_2)A_3)$ and $(A_1(A_2A_3))$ are fully parenthesized.

Remark: we will henceforth refer to a *full parenthesization* simply as a *parenthesization*.

$A_1(A_2A_3)$: NOT parenthesized ; $(A_1(A_2A_3))$: Parenthesized

Matrix chain multiplication

A_1 A_2 A_3 A_n

$p_0 \times p_1$, $p_1 \times p_2$, $p_2 \times p_3$, $\cdots$, $p_{n-1} \times p_n$

Input: n matrices $A_1, A_2, \ldots, A_n$, with dimensions $p_{i-1} \times p_i$, for $1 \leq i \leq n$.

Output:

1. an **optimal** parenthesization of the input
   (i.e., a way to compute $A_1 \cdots A_n$ incurring minimum cost)

2. its **cost**
   (i.e., total # scalar multiplications to compute $A_1 \cdots A_n$)

Example: the optimal parenthesization for Example 1 is $(A_1(A_2A_3))$ and its cost is 22.

Want to compute/find the optimal parenthesization.

Optimal parenthesization for $A_1 A_2$? $(A_1 A_2)$

$(m \times n)$ $(n \times p)$

Brute-Force Algorithm

- Given $A_1, A_2, \ldots, A_n$
- Want to parenthesize this product such that you incur the minimum # of scalar multiplications to compute it.
- Brute-force: List every possible parenthesization and output the one that minimizes the cost.

Input	# possible parenthesizations
A_1	1
$A_1 A_2$	1
$(A_1 A_2) A_3$	2

Let $P(n)$ = # parenthesizations of n matrices

A product of matrices is fully parenthesized if it is

1. a single matrix; or
2. the product of two fully parenthesized matrices, surrounded by parentheses.

By this definition,

Any parenthesization will look like this:

$$((A_1 A_2 \cdots A_k)(A_{k+1} A_{k+2} \cdots A_n))$$

$$P(n) = \sum_{k=1}^{n-1} P(k) \cdot P(n-k)$$

# of parenth. for k matrices

# of parenth. for $n-k$ matrices.

For example, for $n=4$, there are 5 ways

$A_1 A_2 A_3 A_4$

1.) $(A_1 (A_2 (A_3 A_4)))$

2.) $(A_1 ((A_2 A_3) A_4))$

3.) $((A_1 A_2)(A_3 A_4))$

4.) $(((A_1 (A_2 A_3)) A_4)$

5.) $(((A_1 A_2) A_3) A_4)$

Parenthesize the 1st k matrices $A_1 \cdots A_k$.
For each of the $P(k)$ parenthesizations of $A_1 \cdots A_k$, you can parenthesize $A_{k+1} \cdots A_n$ in $P(n-k)$ ways. Therefore, there are $P(k) \cdot P(n-k)$ ways to parenthesize $A_1 \cdots A_n$, for a fixed k.

$$P(n) = \sum_{k=1}^{n-1} P(k) \cdot P(n-k)$$

$$= P(1)\,P(n-1) + P(2)\,P(n-2) + \cdots$$

$$\geq P(n-1) + P(n-2)$$

$$\geq F_n \geq 2^{n/2}$$

Notice:

$$F_n = F_{n-1} + F_{n-2}$$
$$\geq 2F_{n-2}$$
$$\geq 2 \cdot 2 F_{n-4}$$
$$\geq 2^3 F_{n-6}$$
$$\geq 2^{n/2} \cdot F_0 = 2^{n/2}$$

$$F_n = F_{n-1} + F_{n-2}$$

Note that $F_{n-1} \geq F_{n-2}$ since

$F_{n-1} = F_{n-2} + F_{n-3}$ and $F_{n-i} \geq 0$ $\forall i \in [1, n]$

Therefore, $F_n = F_{n-1} + F_{n-2}$

$$\geq F_{n-2} + F_{n-2}$$
$$= 2F_{n-2}$$
$$\geq 2 \cdot (2F_{n-4}) = 2^2 F_{n-4}$$
$$\geq 2^3 F_{n-6}$$
$$\vdots$$
$$\geq 2^{n/2} \cdot F_{n-n} = 2^{n/2} \cdot F_0 = 2^{n/2}$$

$F_{n-2} = F_{n-3} + F_{n-4}$

$F_n = F_{n-1} + F_{n-2}$

$= F_{n-2} + F_{n-3} + F_{n-2}$

$= 2F_{n-2} + F_{n-3}$

$\Rightarrow F_n \geq 2^{n/2}$

Let A^* be an opt. parenthesization of $A_1, \ldots, A_n$.

Let $OPT(n)$ = min cost to compute $A_1 \cdots A_n$

We know that $A^* = \left(\left(A_1 \cdots A_k^* \right) \left(A_{k+1}^* \cdots A_n \right) \right)$ ← footnote

Suppose that we know k^* (i.e., where the first subproduct ends)

Then given the form of A^*, we can compute the cost recursively as: (NOT YET CORRECT!!)

$$OPT(n) = OPT(k^*) + OPT(n-k^*) + P_0 P_{k^*} P_n$$

Not yet correct

NOT RIGHT!!

cost to compute $A_1 \cdots A_{n-k^*}$, but we want $A_{k^*} \cdots A_n$

Do we need to optimally compute the product from A_1 to A_{k^*}?

Yes; if you don't compute it optimally, then replace the suboptimal cost with the optimal cost which gives you a smaller cost contradicting optimality of $OPT(n)$.

footnote: $(A_1 \cdots A_{k^*})$ is a $P_0 \times P_{k^*}$ matrix and $(A_{k^*+1} \cdots A_n)$ is a $P_{k^*} \times P_n$ matrix

The subproblems we have defined are NOT expressive enough to give us a solution, so we need to introduce 2-dimensional subproblems:

$$OPT(1,n) = \min_{1 \le k \le n-1} \left\{ \begin{array}{l} OPT(1,k^*) + OPT(k^*+1,n) \\ + \; P_0 P_{k^*} P_n \end{array} \right\}$$

- Boundary Conditions:
  - $OPT(i,i) = 0$ (multiplying 1 matrix)
  - $OPT(i,i+1) = P_{i-1} \cdot P_i \cdot P_{i+1}$

$P_{i-1} \times P_i$ $\quad$ $P_i \times P_{i+1}$

$A_i \cdot A_{i+1}$

Possible subproblems: $OPT(1,1), OPT(1,2), \ldots, OPT(1,n)$

$OPT(2,2), \ldots, OPT(2,n)$

$\vdots$

$OPT(n,n)$

Smallest subproblems are of the form $OPT(i,i)$

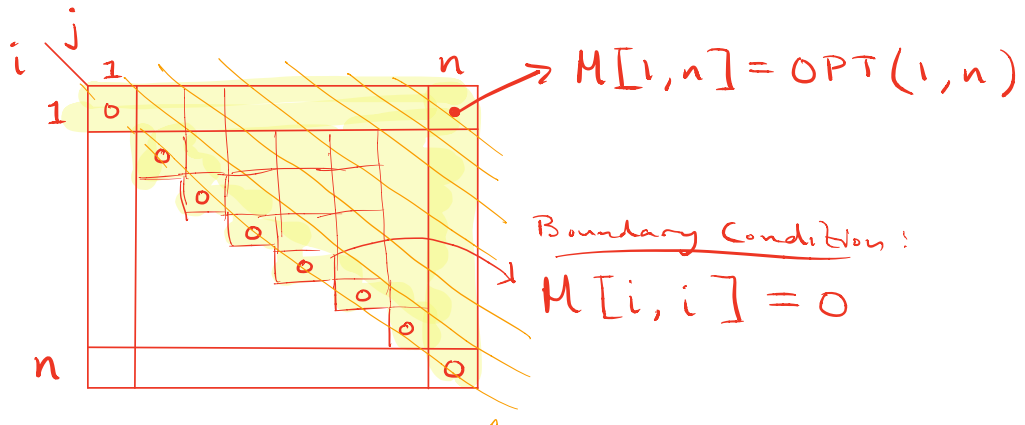
- Space: $\Theta(n^2)$

  since need to store $\binom{n}{2} + n$ subproblems.

- Time per subproblem (i, j): $\Theta(j-i) = O(n)$

- Total time: $O(n^3)$ (a tight bound)

$$OPT(i,j) = \min_{i \le k \le j-1} \left\{ \begin{array}{l} OPT(i,k) + OPT(k+1, j) \\ + P_{i-1} P_k P_j \end{array} \right\}$$



order to fill in the matrix: Fill in main diag. first, then slightly larger subproblems that appear above the diagonal and so on.

Practice and write out this DP solution and how the matrix is filled out. Particularly the pseudocode. Many DP problems have solutions of this kind, so practice this!!

Today

Brute-force approach

- $A_1, \ldots, A_n$ are matrices of dimensions $p_{i-1} \times p_i$ for $1 \leq i \leq n$.
- Consider the product $A_1 \cdots A_n$.
- Let $P(n) = \#$parenthesizations of the product $A_1 \cdots A_n$.
- Then $P(0) = 0$, $P(1) = 1$, $P(2) = 1$
- By Definition 2, for $n > 2$, every possible parenthesization of $A_1 \cdots A_n$ can be decomposed into the product of two parenthesized subproducts for some $1 \leq k \leq n - 1$:

$$((A_1 A_2 \cdots A_k)(A_{k+1} \cdots A_n))$$

Computing #possible parenthsizations

- Given k, the number of parenthesizations for the product

$$((A_1 A_2 \cdots A_k)(A_{k+1} \cdots A_n))$$

can be computed recursively:

$$P(k) \cdot P(n-k)$$

- There are $n-1$ possible values for k. Hence

$$P(n) = \sum_{k=1}^{n-1} P(k) \cdot P(n-k), \text{ for } n > 1$$

Bounding $P(n)$

- We may obtain a crude yet sufficient for our purposes **lower bound** for $P(n)$ as follows

$$\begin{aligned} P(n) &\geq P(1) \cdot P(n-1) + P(2) \cdot P(n-2) \\ &\geq P(n-1) + P(n-2) \end{aligned} \tag{1}$$

- By strong induction on n, we can show that $P(n) \geq F_n$, the n-th Fibonacci number.

- Hence $P(n) = \Omega(2^{n/2})$.
  - In fact, $P(n) = \Omega(2^{2n}/n^{3/2})$ (e.g., see your textbook).

$\Rightarrow$ Brute force requires exponential time.

Today

A second attempt: divide and conquer

Notation: $A_{1,n}$ is the optimal parenthesization of the product $A_1 \cdots A_n$, that is, the optimal way to compute this product.

By Definition 2, there exists $1 \leq k^* \leq n-1$ such that $A_{1,n}$ may be decomposed as the product of two fully parenthesized subproducts:

$$A_{1,n} = ((A_1 \cdots A_{k^*})(A_{k^*+1} \cdots A_n))$$

Optimal substructure

Notation: $A_{i,j}$ is the optimal parenthesization of the product $A_i \cdots A_j$.

Fact 3.

There exists $1 \leq k^* \leq n-1$ *such that*

$$A_{1,n} = (A_{1,k^*} \cdot A_{k^*+1,n}).$$

That is, the optimal parenthesization of the input can be decomposed into the optimal parenthesizations of two subproblems.

The cost of multiplying two matrices

- Recall that matrix A_i has dimensions $p_{i-1} \times p_i$.
- Then $(A_1 \cdots A_k)$ is a $p_0 \times p_k$ matrix;
- $(A_{k+1} \cdots A_n)$ is a $p_k \times p_n$ matrix.

$\Rightarrow$ The total #scalar multiplications required for multiplying matrix $(A_1 \cdots A_k)$ by matrix $(A_{k+1} \cdots A_n)$ is

$$p_0 p_k p_n.$$

Proof of Fact 3

Notation:

- $A_{i,j}$ is the optimal parenthesization of $A_i \cdots A_j$
- $OPT(i,j)$ = cost of $A_{i,j}$ = optimal cost to compute $A_i \cdots A_j$

Since $A_{1,n} = ((A_1 \cdots A_{k^*})(A_{k^*+1} \cdots A_n))$, its cost is given by

$$OPT(1,n) = OPT(1,k^*) + OPT(k^*+1,n) + p_0 p_{k^*} p_n,$$

where

- $OPT(1,k^*), OPT(k^*+1,n)$ are the costs for optimally *(why?)* computing $A_1 \cdots A_{k^*}$, $A_{k^*+1} \cdots A_n$ respectively
- $p_0 p_{k^*} p_n$ is the fixed cost for multiplying $(A_1 \cdots A_{k^*})$ by $(A_{k^*+1} \cdots A_n)$.

Recursive computation of $OPT(1, n)$

Notation: $OPT(1, n) =$ optimal cost for computing $A_1 \cdots A_n$

- **Issue:** we do not know k^*!
- **Solution:** consider every possible value of k.

$$OPT(1,n) = \begin{cases} 0 & \text{, if } n = 1 \\ \min\limits_{1 \le k < n} \left\{ OPT(1,k) + OPT(k+1,n) + p_0 p_k p_n \right\} & \text{, o.w.} \end{cases}$$

Remark 3.

This recurrence gives rise to an exponential recursive algorithm. However we can use dynamic programming to obtain an efficient solution.

Introducing subproblems

Notation: $OPT(i,j) =$ optimal cost for computing $A_i \cdots A_j$

$$OPT(i,j) = \begin{cases} 0 & \text{, if } i = j \\ \min\limits_{i \le k < j} \left\{ OPT(i,k) + OPT(k+1,j) + p_{i-1} p_k p_j \right\} & \text{, if } i < j \end{cases}$$

Remark 4.

- *Only* $\Theta(n^2)$ *subproblems.*
- *If subproblems are computed from smaller to larger, then only* $\Theta(j-i) = O(n)$ *work per subproblem: each term inside the* min *computation requires time* $O(1)$ *(why?).*

Today

1. Matrix chain multiplication
2. A first attempt: brute-force
3. A second attempt: divide and conquer
4. Organizing DP computations

Bottom-up computation of subproblems

Define matrix $M[1:n, 1:n]$, $S[1:n-1, 2:n]$ such that

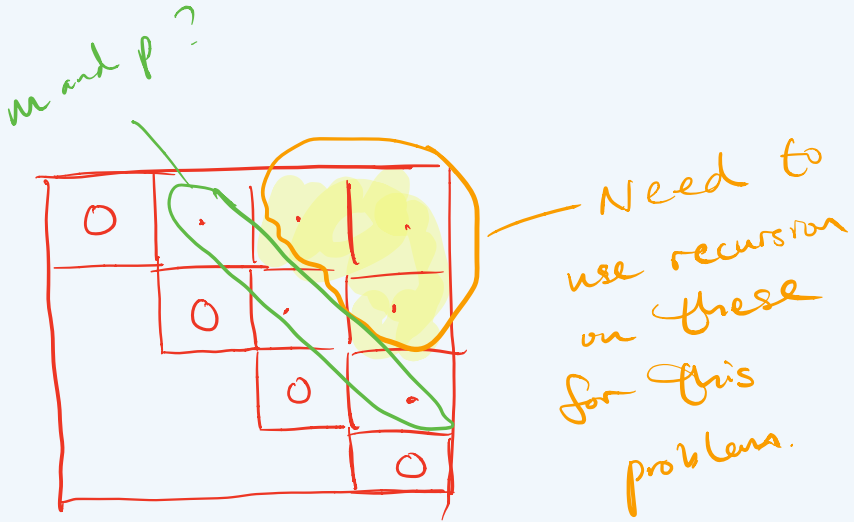
$$\begin{aligned} M[i,j] &= OPT(i,j), && \text{for } 1 \le i \le j \le n \\ S[i,j] &= k, \text{ if } A_{i,j} = A_{i,k}A_{k+1,j}, && \text{for } 1 \le i < j \le n \end{aligned}$$

- Only need fill in the upper triangle of M, where $i \le j$
- Start from the main diagonal, proceed diagonal by diagonal
- Last entry to fill in: $M[1,n]$, the cost of the optimal parenthesization of the entire product $A_1 \cdots A_n$
- Running time: $O(n^3)$
  - $\Theta(n^2)$ entries to fill in
  - each entry requires $\Theta(j-i) = O(n)$ work
- Space: $\Theta(n^2)$

Example

Input

- 6×1 matrix A_1
- 1×5 matrix A_2
- 5×2 matrix A_3
- 2×3 matrix A_4



Output

- the cost of the optimal parenthesization of $A_1A_2A_3A_4$ (by filling in the dynamic programming table M)

Computing the cost of the optimal parenthesization in $O(n^3)$ (from CLRS)

```
MATRIX-CHAIN-ORDER(p)
 1  n = p.length − 1
 2  let m[1..n, 1..n] and s[1..n − 1, 2..n] be new tables
 3  for i = 1 to n
 4      m[i, i] = 0
 5  for l = 2 to n                    // l is the chain length
 6      for i = 1 to n − l + 1
 7          j = i + l − 1
 8          m[i, j] = ∞
 9          for k = i to j − 1
10              q = m[i, k] + m[k + 1, j] + p_{i−1} p_k p_j
11              if q < m[i, j]
12                  m[i, j] = q
13                  s[i, j] = k
14  return m and s
```

Reconstructing the optimal parenthesization (from CLRS)

Time to reconstruct optimal parenthesization? $O(n)$

```
PRINT-OPTIMAL-PARENS(s, i, j)
1  if i == j
2       print "A"_i
3  else print "("
4       PRINT-OPTIMAL-PARENS(s, i, s[i, j])
5       PRINT-OPTIMAL-PARENS(s, s[i, j] + 1, j)
6       print ")"
```

Memoized recursion

Use the original recursive algorithm together with M:

- initialize M to ∞ above the main diagonal and to 0 on the main diagonal.
- to solve a subproblem, look up its value in M
  - if it is ∞, solve the subproblem **and** store its cost in M;
  - else, directly use its value from M.

Remark 5.

- *The memoized recursive algorithm solves every subproblem once, thus overcoming the main source of inefficiency of the original recursive algorithm.*
- *Running time:* $O(n^3)$.

Memoized recursion pseudocode (from CLRS)

```
MEMOIZED-MATRIX-CHAIN(p)
1  n = p.length − 1
2  let m[1..n, 1..n] be a new table
3  for i = 1 to n
4      for j = i to n
5          m[i, j] = ∞
6  return LOOKUP-CHAIN(m, p, 1, n)

LOOKUP-CHAIN(m, p, i, j)
1  if m[i, j] < ∞
2      return m[i, j]
3  if i == j
4      m[i, j] = 0
5  else for k = i to j − 1
6          q = LOOKUP-CHAIN(m, p, i, k)
                + LOOKUP-CHAIN(m, p, k + 1, j) + p_{i−1} p_k p_j
7          if q < m[i, j]
8              m[i, j] = q
9  return m[i, j]
```

Dynamic programming vs Divide & Conquer

- They both combine solutions to subproblems to generate a solution to the whole problem.
- However, divide and conquer starts with a large problem and divides it into small pieces.
- While dynamic programming works from the bottom up, solving the smallest subproblems first and building optimal solutions to steadily larger problems.