Lastly, to show if $z \in \mathcal{C}((I-P_W)X) \Rightarrow (P_X - P_W)z = z$

Suppose $z \in \mathcal{C}((I-P_W)X)$. Then for some vector v,
$(I-P_W)Xv = z$.

Now, $(P_X - P_W)z = (P_X - P_W)(I-P_W)Xv = (P_X - P_W)(Xv - P_W Xv)$

$= P_X Xv - P_X P_W Xv - P_W Xv + P_W P_W Xv$

$= Xv - P_W Xv - P_W Xv + P_W Xv$ [Since $P_X Xv = Xv$ and P_W idempotent and because $P_X P_W = P_W$ shown earlier.]

$= Xv - P_W Xv = (I-P_W)Xv = z$

$\Rightarrow (P_X - P_W)z = z$ for $z \in \mathcal{C}((I-P_W)X)$

Thus, it has been shown that $P_X - P_W$ is the symmetric orthogonal projection onto $\mathcal{C}((I-P_W)X)$. $\square$