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$$\Rightarrow \begin{bmatrix} I_q \beta_1 \\ I_{(p-q)}\beta_2 \end{bmatrix} = \begin{bmatrix} (W^TW)^{-1}W^Ty \\ (Z^TZ)^{-1}Z^Ty \end{bmatrix}$$ β_1 of size $q \times 1$
β_2 of size $(p-q)\times 1$
so product on LHS well-defined.

$$\Rightarrow \begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{bmatrix} = \begin{bmatrix} (W^TW)^{-1}W^Ty \\ (Z^TZ)^{-1}Z^Ty \end{bmatrix}$$

Hence, $\hat{\beta}_1 = (W^TW)^{-1}W^Ty$
and $\hat{\beta}_2 = (Z^TZ)^{-1}Z^Ty$ // $\square$

(b.) Show that $\hat{\beta}_1$ and $\hat{\beta}_2$ are independent.

The relationship between $\hat{\beta}_1$ and $\hat{\beta}_2$ is usually only considered jointly. That is, it is measured on a joint distribution. Therefore if it can be shown that $Cov(\hat{\beta}_1, \hat{\beta}_2) = 0$ (and hence their correlation, $\rho = 0$) then this will imply that $\hat{\beta}_1$ and $\hat{\beta}_2$ are independent.

$$Cov(\hat{\beta}_1, \hat{\beta}_2) = Cov\left[(W^TW)^{-1}W^Ty,\ (Z^TZ)^{-1}Z^Ty\right] \quad (\text{by part (a)})$$

$$= (W^TW)^{-1}W^T\, Cov[y,y] \cdot \left[(Z^TZ)^{-1}Z^T\right]^T$$
$$= (W^TW)^{-1}W^T\, Var[y] \cdot Z\left[(Z^TZ)^{-1}\right]^T$$

since $Var[y] = Var[\varepsilon]$ for our model, we have

$$= (W^TW)^{-1}W^T\, Var[\varepsilon]\, Z\left[(Z^TZ)^T\right]^{-1}$$

Since $Var[\varepsilon]$ is a scalar matrix with elements σ^2 on its diagonal we have

$$= (W^TW)^{-1}W^T \sigma^2 I_n Z (Z^TZ)^{-1}$$

Since σ^2 a scalar,

$$= \sigma^2 (W^TW)^{-1}W^T I_n Z (Z^TZ)^{-1}$$
$$= \sigma^2 (W^TW)^{-1}W^TZ(Z^TZ)^{-1} = \sigma^2 (W^TW)^{-1} 0 \cdot (Z^TZ)^{-1}$$

since $W^TZ = 0$, $= 0$

Hence, $Cov(\hat{\beta}_1, \hat{\beta}_2) = 0$ $\Longrightarrow$