- For any $z \in \mathbb{R}^n$, $(P_X - P_W)z \in \mathcal{C}((I-P_W)X)$:

Since $\mathcal{C}(X)$ is the orthogonal complement of $\mathcal{N}(X^T)$ then any $z \in \mathbb{R}^n$ can be written as $z = u + w$ where $u \in \mathcal{C}(X)$ and $w \in \mathcal{N}(X^T)$.
and since $u \in \mathcal{C}(X)$ then for some vector v $Xv = u$.
$\Rightarrow z = Xv + w$

If it is the case that $(P_X - P_W)z \in \mathcal{C}((I-P_W)X)$ then for some v, $(P_X - P_W)z = (I - P_W)Xv$.
It suffices to show that the above equality holds.

Since P_X is the projection onto $\mathcal{C}(X)$ then $I - P_X$ is the projection onto $\mathcal{N}(X^T)$ since $\mathcal{N}(X^T)$ is the orthogonal complement of $\mathcal{C}(X)$.
Moreover, since $\mathcal{C}(W) \subseteq \mathcal{C}(X)$ then
$\mathcal{N}(X^T) \subseteq \mathcal{N}(W^T)$.
Thus, for any $w \in \mathcal{N}(X^T) \Rightarrow w \in \mathcal{N}(W^T)$
Since $I - P_X$ projection onto $\mathcal{N}(X^T)$ then
$(I - P_X)w = w$
Since $I - P_W$ projection onto $\mathcal{N}(W^T)$ then
$(I - P_W)w = w$.

- Hence, $(I - P_X)w = (I - P_W)w$
$\Rightarrow w - P_X w = w - P_W w \Rightarrow P_X w = P_W w$
(This can also be argued to be true since $I - P_X$ is the unique projection onto $\mathcal{N}(X^T)$ thus $I - P_W = I - P_X$)

Finally, $(P_X - P_W)z = (P_X - P_W)(Xv + w) =$
$= P_X Xv + P_X w - P_W Xv - P_W w$
$= Xv - P_W Xv + P_X w - P_W w$
$= (I - P_W)Xv + 0$
(since P_X proj. onto $\mathcal{C}(X)$ and $Xv \in \mathcal{C}(X)$ $P_X Xv = Xv$)
(from above: since $P_X w = P_W w$)

$\Rightarrow (P_X - P_W)z = (I - P_W)Xv$
Therefore $(P_X - P_W)z \in \mathcal{C}((I - P_W)X)$. //