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Problem 2:

To show that $P_X - P_W$ is the symmetric orthogonal projection onto $\mathcal{C}((I-P_W)X)$ it suffices to show that

1) $P_X - P_W$ is idempotent
2) For any $z \in \mathbb{R}^n$ $(P_X - P_W)z \in \mathcal{C}((I-P_W)X)$
3) If $z \in \mathcal{C}((I-P_W)X) \Rightarrow (P_X - P_W)z = z$

1) Px - Pw Idempotent:

$$(P_X - P_W)(P_X - P_W) = P_X P_X - P_X P_W - P_W P_X + P_W P_W$$
$$= P_X - P_X P_W - P_W P_X + P_W \quad (*)$$
(Since P_X and P_W idempotent)

Now for any $z \in \mathbb{R}^n$, we have that $P_W z \in \mathcal{C}(W)$ since P_W is the projection onto $\mathcal{C}(W)$.
Then, since $\mathcal{C}(W) \subseteq \mathcal{C}(X)$ $P_W z \in \mathcal{C}(X)$
Since P_X projection onto $\mathcal{C}(X)$, then for any $v \in \mathcal{C}(X)$, $P_X v = v$. Thus, $P_X P_W z = P_X(P_W z) = P_W z$
$\Rightarrow P_X P_W = P_W$
Taking the transpose of both sides: $(P_X P_W)^T = P_W^T$
$\Rightarrow P_W^T P_X^T = P_W^T$ where P_W is symmetric
$\Rightarrow P_W^T P_X^T = P_W^T = P_W = P_X P_W$
and $P_W^T P_X^T = P_W P_X$ since $P_W^T P_X^T = P_W^T$, a symmetric matrix $\Rightarrow P_W P_X = P_X P_W$.

Then, by $(*)$

$$(P_X - P_W)(P_X - P_W) = P_X - P_X P_W - P_W P_X + P_W$$
$$= P_X - 2P_X P_W + P_W \quad (P_X P_W = P_W P_X)$$
$$= P_X - 2P_W + P_W \quad (P_X P_W = P_W)$$
$$= P_X - P_W.$$

Thus, $P_X - P_W$ idempotent.

$\Longrightarrow$