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EN.553.633 Homework #5

Problem 2.21) Let the random variable X have pdf

$$f(x) = \begin{cases} \frac{1}{4} & , 0 < x < 1 \\ x - \frac{3}{4} & , 1 \le x \le 2 \end{cases}.$$

Generate a random variable from $f(x)$, using

(a) the inverse-transform method.

Solution: To generate a random variable from $f(x)$, we first need the cdf of X and subsequently need the inverse cdf of X. Then with $f(x)$ defined as above, we have:

For $0 < x < 1$:

$$F(x) = \int_0^x \frac{1}{4}\,dt = \frac{1}{4}t\Big|_0^x = \frac{1}{4}x$$

For $1 \le x \le 2$:

$$F(x) = \int_0^x f(t)\,dt = \int_0^1 \frac{1}{4}\,dt + \int_1^x t - \frac{3}{4}\,dt$$

$$= \frac{t}{4}\Big|_0^1 + \frac{t^2}{2} - \frac{3}{4}t\Big|_1^x = \frac{1}{4} + \left(\frac{x^2}{2} - \frac{3}{4}x\right) - \left(\frac{1}{2} - \frac{3}{4}\right)$$

$$= \frac{x^2}{2} - \frac{3}{4}x + \frac{1}{2}$$

For $x > 2$: $F(x) = 1$, For $x < 0$: $F(x) = 0$

Then, the cdf of X is

$$F(x) = \begin{cases} 0 & , x < 0 \\ \frac{1}{4}x & , 0 < x < 1 \\ \frac{1}{2}x^2 - \frac{3}{4}x + \frac{1}{2} & , 1 \le x \le 2 \\ 1 & , x > 2 \end{cases} \Longrightarrow$$

The inverse of $F(x)$ is then,

For $x \in (0,1)$: $u = F(x) = \frac{x}{4} \Rightarrow 4u = x = F^{-1}(u)$

For $x \in [1,2]$: $u = F(x) = \frac{1}{2}x^2 - \frac{3}{4}x + \frac{1}{2}$

$$= \frac{1}{2}\left(x^2 - \frac{3}{2}x + 1\right) = \frac{1}{2}\left(x^2 - \frac{3}{2}x + \left(\frac{3}{4}\right)^2 + 1 - \left(\frac{3}{4}\right)^2\right)$$

(Completing the Square) $$= \frac{1}{2}\left(x - \frac{3}{4}\right)^2 + \frac{7}{32}$$

$$\Rightarrow u = F(x) = \frac{1}{2}\left(x - \frac{3}{4}\right)^2 + \frac{7}{32}$$

$$\Rightarrow 2u - \frac{7}{16} = \left(x - \frac{3}{4}\right)^2 \Rightarrow \sqrt{2u - \frac{7}{16}} = x - \frac{3}{4}$$

(since $x \in [1,2] \Rightarrow x - \frac{3}{4} > 0$
$\Rightarrow$ we only take positive square root)

$$\Rightarrow x = \sqrt{2u - \frac{7}{16}} + \frac{3}{4} = F^{-1}(u)$$

which can be simplified: $x = \frac{1}{4}\left(4\sqrt{2u - \frac{7}{16}} + 3\right)$
$= \frac{1}{4}\left(\sqrt{32u - 7} + 3\right)$

and $x \in (0,1) \Rightarrow 0 < u < \frac{1}{4}$ and $x \in [1,2] \Rightarrow \frac{1}{4} \le u \le 1$

$$\Rightarrow F^{-1}(u) = \begin{cases} 4u & , 0 < u < 1/4 \\ \frac{1}{4}\left(\sqrt{32u-7} + 3\right) & , \frac{1}{4} \le u \le 1 \end{cases}$$

Now we have the inverse cdf of X to input into our algorithm to generate r.v. from $f(x)$. The method/algorithm then goes
1.) We first generate a U from $U(0,1)$.
2.) If $U \in (0, 1/4)$, let $X = 4U$ and Return X
3.) If $U \in [1/4, 1]$, let $X = \frac{1}{4}(\sqrt{32U - 7} + 3)$ and Return X.

part (b) on next page $\longrightarrow$

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Problem A: Consider a Markov chain with state space $\{1, 2, 3\}$ and with a stationary probability distribution of $\pi = [\frac{3}{11}, \frac{3}{11}, \frac{5}{11}]$.

(a) Determine the values of the entries in the transition matrix P under the constraint that the values in the upper left 2×2 block are equal to each other (i.e. $p_{11} = p_{12} = p_{21} = p_{22}$) and that the sum of the entries in the first column is $13/15$.

Solution: Suppose that for some Markov chain $X = \{X_t : t \in \mathbb{N}\}$ with state space $\mathcal{E} = \{1, 2, 3\}$ and transition matrix P, that the stationary probability distribution, π, is such that $\pi = [\frac{3}{11}, \frac{3}{11}, \frac{5}{11}]$. Then by Markov Convergence Theorem, we have $\pi = \pi P$

$$(*) \quad \Rightarrow [\tfrac{3}{11}, \tfrac{3}{11}, \tfrac{5}{11}] \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix} = [\tfrac{3}{11}, \tfrac{3}{11}, \tfrac{5}{11}]$$

For convenience and accuracy of notation, relabel the entries of P as $a, b, c, \ldots i$:

$$P = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

We are given the constraints $a = b = d = e$ and $a + d + g = 13/15$

$\Rightarrow 2a + g = 13/15$, by the first constraint.

Then we have,

$$[\tfrac{3}{11}, \tfrac{3}{11}, \tfrac{5}{11}] \begin{bmatrix} a & a & c \\ a & a & f \\ g & h & i \end{bmatrix} = [\tfrac{6}{11}a + \tfrac{5}{11}g, \tfrac{6}{11}a + \tfrac{5}{11}h, \tfrac{3}{11}c + \tfrac{3}{11}f + \tfrac{5}{11}i]$$

$$= [\tfrac{3}{11}, \tfrac{3}{11}, \tfrac{5}{11}]$$

We also know, by properties of transition matrices, that the sum of the components in each row is 1

i.e.
$$\begin{aligned} 2a + c &= 1 \\ 2a + f &= 1 \\ g + f + i &= 1 \end{aligned}$$

$\Longrightarrow$

continued...

We then have the system of equations:

$$\left.\begin{aligned} 2a + c &= 1 \\ 2a + f &= 1 \\ g + h + i &= 1 \end{aligned}\right\} \text{By transition matrix property}$$

$$\left.\begin{aligned} \tfrac{6}{11}a + \tfrac{5}{11}g &= \tfrac{3}{11} \\ \tfrac{6}{11}a + \tfrac{5}{11}h &= \tfrac{3}{11} \\ \tfrac{3}{11}c + \tfrac{3}{11}f + \tfrac{5}{11}i &= \tfrac{5}{11} \end{aligned}\right\} \text{By Markov Convergence Theorem}$$

$$\left.2a + g = \tfrac{13}{15}\right\} \text{Constraint}$$

* The first 2 equations imply that $c = f$
* The 4^{th} and 5^{th} equations imply that $g = h$

We then have:

(Eq. 1) $2a + c = 1$

(Eq. 2) $2g + i = 1$

(Eq. 3) $\frac{6}{11}a + \frac{5}{11}g = \frac{3}{11}$ $\Rightarrow$

(Eq. 4) $\frac{6}{11}c + \frac{5}{11}i = \frac{5}{11}$

(Eq. 5) $2a + g = \frac{13}{15}$

$\frac{5}{11} \times (\text{Eq. 5}) - (\text{Eq. 3})$:

$$\frac{5}{11}\left(2a + g = \frac{13}{15}\right)$$
$$-\left(\frac{6}{11}a + \frac{5}{11}g = \frac{3}{11}\right)$$
$$= \frac{4}{11}a = \frac{4}{33} \Rightarrow \boxed{a = \frac{1}{3}}$$

* Plugging a into (Eq. 1): $c = 1 - 2\left(\frac{1}{3}\right) = 1/3 \Rightarrow \boxed{c = 1/3}$

$\Rightarrow$ From Eq. 4: $\frac{6}{11}\left(\frac{1}{3}\right) + \frac{5}{11}i = \frac{5}{11} \Rightarrow i = \frac{11}{5}\left(\frac{5}{11} - \frac{2}{11}\right)$
$= 3/5 \Rightarrow \boxed{i = 3/5}$

and lastly from (Eq. 2) $g = \frac{1}{2}\left(1 - \frac{3}{5}\right) = 1/5 \Rightarrow \boxed{g = 1/5}$

$$\Rightarrow P = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/5 & 1/5 & 3/5 \end{bmatrix}$$

To write algorithm a bit clearer (more legible)

1.) Generate $X \sim U[0,2]$ (i.e. from $g(x) = 1/2$ on $x \in [0,2]$)

2.) Generate $Y \sim U[0, 5/2]$ (i.e. from $U[0, c\,g(x)]$)

3.) { If $Y \leq f(X)$
Then Accept $\leftarrow$ True;
Return X;
If not, return to (1) }

end.

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$$\Rightarrow \quad P = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/5 & 1/5 & 3/5 \end{bmatrix}$$

(b) Using the P in part (a), carry out a simulation of 5000 (see Code)

The code generated 0.2973 for $X_2 = 1$
0.2588 for $X_2 = 2$
and 0.4071 for $X_3 = 3$.

(c) Code generated : 0.2728 for $X_{20} = 1$
0.2789 for $X_{20} = 2$
0.4583 for $X_{20} = 3$

We can see that now that the # of steps has increased to 20, the values approach the probabilities in the lim. dist. $\pi = [3/11, 3/11, 5/11]$

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Problem B.) (a.) If $X \sim N(\mu, \sigma^2)$, derive $E[e^X]$.

Solution: Since $X \sim N(\mu, \sigma^2)$, then

$$E[e^X] = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^x e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx =$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}[(x-\mu)^2 - 2\sigma^2 x]} dx$$

* Focusing only on the exponent of $e^{-\frac{1}{2\sigma^2}[(x-\mu)^2 - 2\sigma^2 x]}$:

$$-\frac{1}{2\sigma^2}[(x-\mu)^2 - 2\sigma^2 x] = \frac{-1}{2\sigma^2}[x^2 - 2x\mu + \mu^2 - 2\sigma^2 x] =$$

$$= -\frac{1}{2\sigma^2}[x^2 - 2(\mu+\sigma^2)x + \mu^2] =$$

$$= -\frac{1}{2\sigma^2}[x^2 - 2(\mu+\sigma^2)x + \mu^2 + (2\mu\sigma^2 + \sigma^4) - (2\mu\sigma^2 + \sigma^4)]$$

$$= -\frac{1}{2\sigma^2}[x^2 - 2(\mu+\sigma^2)x + (\mu^2 + 2\mu\sigma^2 + \sigma^4)] + \frac{1}{2\sigma^2}(2\mu\sigma^2 + \sigma^4)$$

$$= -\frac{1}{2\sigma^2}[x^2 - 2(\mu+\sigma^2)x + (\mu+\sigma^2)^2] + \mu + \frac{\sigma^2}{2}$$

$$= -\frac{1}{2\sigma^2}[x - (\mu+\sigma^2)]^2 + \mu + \frac{\sigma^2}{2}$$

Now, putting this back into the exponent above:

$$\frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^x e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}[x-(\mu+\sigma^2)]^2 + \mu + \frac{\sigma^2}{2}} dx$$

$\Longrightarrow$

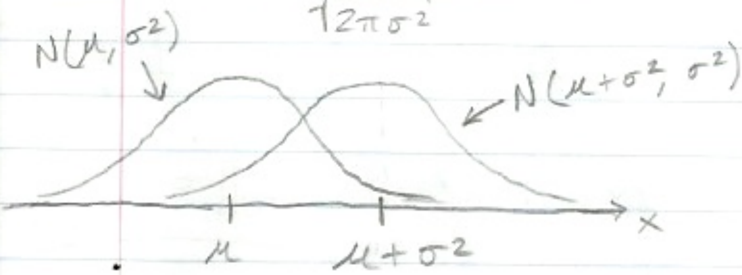
cont'd...

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2\sigma^2}[x-(\mu+\sigma^2)]^2}}{e^{\mu+\frac{1}{2}\sigma^2}} \, dx$$

(*) $$= e^{\mu+\frac{1}{2}\sigma^2} \left(\frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}[(x-\mu)-\sigma^2]^2} \, dx \right)$$ (*)

<u>Note:</u>

Where $\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}[(x-\mu)-\sigma^2]^2}$ is also a probability density with mean $\mu+\sigma^2$ and variance σ^2. Indeed, this is the $\frac{1}{\sqrt{2\pi\sigma^2}} \exp\{-\frac{1}{2\sigma^2}(x-\mu)^2\}$ translated in the positive direction by σ^2 units with respect to the domain



Since the integral is taken over all of $\mathbb{R}$, the integral is still 1.

<u>Thus</u>,

(*) $$= e^{\mu+\frac{1}{2}\sigma^2}(1) = e^{\mu+\frac{1}{2}\sigma^2}$$

$$\Rightarrow E[e^X] = e^{\mu+\frac{1}{2}\sigma^2}$$

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b.) If $W(t)$ is the standard Wiener process and $\mathcal{U}(W(t)) = \exp\left(t + \frac{W(t)}{2}\right)$, derive $E[\mathcal{U}(W(t))]$

Solution: Since $W(t)$ is the standard Wiener process then $W(t) - W(s) \sim N(0, t-s)$. for $0 \le s \le t$ and $W(0) = 0$

Then $W(t) - W(0) \sim N(0,t) \Rightarrow \underline{W(t) \sim N(0,t)}$

i.e. $E[W(t)] = 0$ and $Var[W(t)] = t$.

Now consider $E[\mathcal{U}(W(t))] = E[e^{t + \frac{1}{2}W(t)}]$

$$= E[e^t e^{\frac{1}{2}W(t)}] = e^t E[e^{\frac{1}{2}W(t)}]$$

let $Y = \frac{W(t)}{2}$; then $E[Y] = 0$, $Var[Y] = \frac{1}{4}t$ and clearly Y is a normal r.v. $\Rightarrow Y \sim N(0, \frac{1}{4}t)$

In part (a) it was found that for a r.v. $X \sim N(\mu, \sigma^2)$ $E[e^X] = e^{\mu + \sigma^2/2}$.

Then, by part (a), $E[e^Y] = e^{E[Y] + Var[Y]/2}$
$$= e^{0 + t/8} = e^{t/8}$$

Then, $E[\mathcal{U}(W(t))] = E[e^{t + \frac{1}{2}W(t)}] =$

$$= e^t E[e^Y] = e^t e^{t/8} = e^{t + t/8}$$

$$= e^{\frac{9t}{8}}$$

$$\Longrightarrow E\left[\exp\left(t + \frac{W(t)}{2}\right)\right] = e^{9t/8}$$

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(b.) the acceptance-rejection method, using the proposal density

$$g(x) = \frac{1}{2}x, \quad 0 \leq x \leq 2$$

Solution: To use the A-R method to generate random variables from $f(x)$, we first need to find an appropriate constant, c, such that $cg(x)$ majorizes $f(x)$ for all $x: 0 \leq x \leq 2$.

That is, we need to find the minimum value of c such that $\frac{f(x)}{g(x)} \leq c$.

Since $f(x)$ and $g(x)$ are both continuous functions on $[0,2]$ then

$$c = \max_{x \in [0,2]} \left(\frac{f(x)}{g(x)} \right) = \max_{x \in [0,2]} \frac{f(x)}{1/2} = \max_{x \in [0,2]} 2f(x)$$

Graph of $2f(x)$:

Since $2f(x)$ is an increasing function (as oppose to strictly increasing) the max is at an end point. Namely, $\arg\max_{x \in [0,2]} 2f(x) = 2$

Thus, $c = 2f(2) = 2\left(2 - \frac{3}{4}\right) = \frac{5}{2}$

[Note that $g(x) = 1/2$, $x \in [0,2] \Rightarrow g$ uniform dist on $[0,2]$]

The A-R Algorithm/Method is then: (1.) Generate X from $g(x) = 1/2$ $x \in [0,2]$ i.e. $X \sim U[0,2)$

(2) Generate $Y \sim U[0, 5/2]$

(3) If $Y \leq f(X)$ then Accept $\leftarrow$ True ; Return X

(4) Otherwise go back to (1).