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EN.553.633 Homework #3

Problem 1.19) Let $Y = e^X$, where $X \sim N(0,1)$.

(a) Determine the pdf of Y

Solution: Let $g(X) = y = e^x$.

Now, since e^x is a monotonically increasing function $\forall x \in (-\infty, \infty)$, then by eq'n (1.16) in the course text we have

$$f_Y(y) = f_X(g^{-1}(y)) \frac{dg^{-1}(y)}{dy}$$

It is given that $X \sim N(0,1)$

$$\Rightarrow f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \quad ; -\infty < x < \infty.$$

Moreover, $g(x) = y = e^x \Rightarrow g^{-1}(y) = \ln y$

and $\frac{d}{dy}[g^{-1}(y)] = \frac{d}{dy}[\ln y] = \frac{1}{y}$

Therefore, $f_Y(y) = f_X(\ln y) \cdot \frac{1}{y} = \frac{1}{y\sqrt{2\pi}} e^{-(\ln y)^2/2}$

and since $y = e^x$ for $x \in \mathbb{R}$
$\Rightarrow y > 0$

Thus, $f_Y(y) = \frac{1}{y\sqrt{2\pi}} \cdot e^{-(\ln y)^2/2} \quad ; y > 0$ //

$\longrightarrow$

(b.) Determine the expected value of Y.

Solution: In part (a) we determined that

$$f_Y(y) = \frac{1}{y\sqrt{2\pi}} e^{-(\ln y)^2/2} \quad ; y > 0$$

Then,

$$E[Y] = \frac{1}{\sqrt{2\pi}} \int_0^\infty y \cdot \frac{e^{-(\ln y)^2/2}}{y} dy = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-(\ln y)^2/2} dy$$

Then since $y = e^x \Rightarrow dy = d(e^x) = e^x dx$

$$= \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-(\ln e^x)^2/2} d(e^x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{-x^2/2} \cdot e^x dx$$

[Note: $y > 0 \Rightarrow x \in (-\infty, \infty)$ since $\ln y = x$]

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{-\frac{1}{2}[x^2 - 2x]} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{-\frac{1}{2}[(x^2-2x+1)-1]} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{-\frac{1}{2}[(x-1)^2-1]} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{\frac{-(x-1)^2}{2}} e^{\frac{1}{2}} dx$$

$$= e^{1/2} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{\frac{-(x-1)^2}{2}} dx \right) = e^{1/2} \cdot (1) = e^{1/2}$$

$\square$

[Indeed, $\frac{1}{\sqrt{2\pi}} e^{-(x-1)^2/2}$, $x \in (-\infty, \infty)$ is the pdf for a r.v. with mean $= 1$ and variance $= 1$. Thus, the integration over $x \in (-\infty, \infty)$ is 1.

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(A.) Given: The components of a random vector are jointly Gaussian if the vector is multivariate normally (MVN) distributed. Suppose that two random variables X and Y are jointly Gaussian and satisfy $\operatorname{cov}(X,Y)=0$. It is then well known that X and Y are independent.

Question: Now, consider the case where X is normally distributed and Y is ~~normally~~ distributed and $\operatorname{cov}(X,Y)=0$. Show that X and Y are not necessarily independent.

Proof: Suppose that $X \sim N(0,1)$ and that

$$Y = \begin{cases} X, & \text{probability} = 1/2 \\ -X, & \text{probability} = 1/2 \end{cases}$$

Then since $f_Y(y) = 0.5 f_X(x) + 0.5 f_X(-x)$

$$= 0.5 f_X(x) + 0.5 f_X(x) \quad \left(\text{by symmetry of normal dist.}\right)$$

$$= f_X(x)$$

$\Longrightarrow Y \sim N(0,1)$

Thus, both X and Y are normally distributed with mean 0 and variance 1.

Then,

$$\operatorname{Cov}(X,Y) = E[XY] - E[X]E[Y] = E[XY]$$

(since $E[X] = 0 = E[Y]$)

Then,

$$\operatorname{Cov}(X,Y) = E[XY] = \int_{-\infty}^{0} -x^2 f_X(x)\,dx + \int_0^{\infty} x^2 f_X(x)\,dx$$

$$= \int_0^{\infty} -(-x)^2 f_X(-x)\,dx + \int_0^{\infty} x^2 f_X(x)\,dx$$

$$= -\int_0^{\infty} x^2 f_X(x)\,dx + \int x^2 f_X(x)\,dx = 0$$

(By symmetry of the normal pdf.)

$\Longrightarrow$

so we have that $Cov(X,Y)=0$

but that $Y = \begin{cases} X, & p=1/2 \\ -X, & p=1/2 \end{cases}$

so that Y clearly depends on X

$\Rightarrow$ X and Y <u>NOT</u> independent.

Thus, we have $Cov(X,Y)=0$, but X and Y not independent where X and Y are normally distributed random variables.

$\square$

(B.) Consider an LCG with $c=0$, $X_0=1$, and modulus, $m=13$. Suppose we consider 12 possible values of a, namely $a \in \{1,2,\dots,12\}$. Which values of a in the set of 12 possible values will yield a generator that produces all possible outcomes $X_k \in \{1,2,\dots,12\}$? (Note that X_k cannot equal 0, or else the algorithm will get stuck at 0.).

<u>Solution</u>: From Ch. 2 handout, the LCG with $c=0$ is a MRG defined by $X_k = aX_{k-1} \bmod 13$

Since $m=13$ is prime, the maximal period is $13^1-1=12$ for properly chosen a:

For $a=1$:

k	0	1	2	...	N	...
X_k	1	1	1	...	1	

X

✓ For $a=2$:

k	0	1	2	3	4	5	6	7	8	9	10	11	12
X_k	1	2	4	8	3	6	12	11	9	5	10	7	1

Thus, for $a=2$ $X_k \in \{1,2,\dots,12\}$ generates all possible outcomes.

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For a=3: $X_k = 3X_{k-1} \bmod 13$

k	0	1	2	3	4	5	6	7	...
X_k	1	3	4	12	10	4	12	10	...

X

For a=4: $X_k = 4X_{k-1} \bmod 13$

k	0	1	2	3	4	5	6	7	...
X_k	1	4	3	12	9	10	1	4	...

X

For a=5 $X_k = 5X_{k-1} \bmod 13$

k	0	1	2	3	4	5	6	7
X_k	1	5	12	8	1	5	12	...

X

For a=6: $X_k = 6X_{k-1} \bmod 13$

k	0	1	2	3	4	5	6	7	8	9	10	11	12
X_k	1	6	10	8	9	2	12	7	3	5	4	11	1

✓

* Thus, when a=6, $X_k \in \{1, 2, \ldots, 12\}$ generates all possible outcomes

For a=7: $X_k = 7X_{k-1} \bmod 13$

k	0	1	2	3	4	5	6	7	8	9	10	11	12
X_k	1	7	10	5	9	11	12	6	3	8	4	2	1

✓

Thus, when a=7, $X_k \in \{1, 2, \ldots, 12\}$ generates all possible outcomes

For a=8: $X_k = 8X_{k-1} \bmod 13$

k	0	1	2	3	4	5	6
X_k	1	8	12	5	1	8	...

X

$\Longrightarrow$

cont'd...

For $a=9$: $X_k = 9X_{k-1} \bmod 13$

k	0	1	2	3	4
X_k	1	9	3	1	...

For $a=10$ $X_k = 10X_{k-1} \bmod 13$

k	0	1	2	3	4	5	6	
X_k	1	10	9	12	3	4	1	...

For $a=11$ $X_k = 11X_{k-1} \bmod 13$

k	0	1	2	3	4	5	6	7	8	9	10	11	12
X_k	1	11	4	5	3	7	12	2	9	8	10	6	1

so when $a=11$ $X_k \in \{1, 2, \ldots, 12\}$ generates all possible outcomes.

For $a=12$ $X_k = 12X_{k-1} \bmod 13$

k	0	1	2	3	4	
X_k	1	12	1	12	1	...

Thus, the values of a from the set $\{1, 2, \ldots, 12\}$ that yield a generator that produces all possible outcomes $X_k \in \{1, 2, \ldots, 12\}$

are $a = 2, 6, 7,$ and 11