553.633/433

Homework #6

Due Wed. 10/11/17

Three problems:

A. For a special case of SDE,

$$dX(\mathrm{t}) = \lambda X(t)dt + \mu X(t)dW(t), X(0) = 1$$

we have the closed form solution:

$$X(t) = \mathrm{e}^{\left(\lambda - \frac{\mu^2}{2}\right)t + \mu W(t)}.$$

Find the following expectations:

(a) $E[X(t)]$

(b) $E\left[\lambda \int_0^t X(s)ds\right]$

(c) $E\left[\mu \int_0^t X(s)dW(s)\right]$

(Hint: $E\left[\lambda \int_0^t X(s)ds\right] = \lambda \int_0^t E[X(s)]ds$.)

B. Consider the Ornstein–Uhlenbeck process:

$$dX(t) = \theta(\mu - X(t))dt + \sigma dW(t),$$

where $\theta = 1$, $\mu = 20$, and $\sigma = 10$, and $W(t)$ denotes the (standard) Wiener process. Simulate the stochastic process $X(t)$ using the Euler–Maruyama method described in the slide "Numerical Solution of Itô SDE" from the class notes (see also Sect. 4 of Higham, 2001). (That is, use the *full* E-M form in the class notes as if no solution to the SDE were available, rather than the solution of the O-U process discussed on the slide "Example 2 of Itô SDE: Ornstein–Uhlenbeck Process" in the class notes.) Run the E-M process 50 separate (statistically independent) times over the time interval [0, 5] with a $\Delta t = 0.01$ and $X(0) = 0$ (i.e., carry out the E-M process 50 independent times, each beginning at the same initial condition $X(0) = 0$). From these 50 runs, do the following:

(a) Show the first 5 (of 50) solution paths on one plot (one solution path is the sequence of X_j from the E-M process over all j, representing time from 0 to 5).

(b) Produce a separate line that represents the sample mean of the 50 paths and comment on how the sample mean differs from its limiting value as a function of t (this line may be on the plot in part (a) or on a separate plot). It is not necessary to run any formal statistical (or other) tests for analyzing the difference; a brief "words-only" discussion is sufficient.

(c) Perform a statistical t-test on whether the (unknown) true mean of the value of X_j (from the E-M process) that represents $X(2)$ is μ. That is, report a P-value and provide some brief interpretation. Do the same for $X(5)$. (Note: This part of the problem uses the basic statistical material from slides 24 and 25 in Chap1_633_handout.pdf; you should be familiar with that material from course prerequisites. The material will also be briefly reviewed in class on Monday, 10/9/17.)

C. Consider the setting of Figure 4 in Higham (2001). This problem will be a statistical test of strong convergence (5.2). Using the linear SDE in (4.5) of Higham and the same coefficient settings, perform a statistical test of the accuracy of $\Delta t = 2^{-9}$ versus $\Delta t = 16\times 2^{-9}$ in the E-M method. In particular, compute the P-value from a two-sample matched-pairs statistical t-test using 100 independent runs of the process at each of the above two values of Δt, with each run starting from the same X_0. Each of the 100 values in the statistical test for each Δt will be the absolute value of the difference between the terminal (endpoint) E-M solution and endpoint "truth" as approximated by substituting the values for the discretized Brownian path (using $\delta t = 16\times 2^{-9}$) into the exact solution (4.6), as described in Higham. The P-value in the statistical test will be based on a null hypothesis of no difference in accuracy. Because this is a matched-pairs test, the exact solution used in the absolute value of the difference at each pair of Δt will be the same. (Summary: In run 1, you compute "truth" via (4.6) and use that truth to compute the two absolute differences with E-M from the two values of Δt; then in run 2, you generate a new independent "truth" and repeat. This is done 100 times to get the data for the statistical test, comprising two columns of numbers, each column representing the absolute differences.) You may use the Matlab code of Higham if you wish. Two-sample statistical tests are described in the file, Spall_ISSO_excerpt_Appendix B.pdf, at the Materials link of the course page.