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553.633 HW # 6

Problem A :

(a.) $E[X(t)]$: $X(t) = e^{(\lambda - \frac{\mu^2}{2})t + \mu W(t)}$

Then since $W(t) \sim N(0, t)$ (standard Wiener process)
for $Z \sim N(0,1)$, $W(t) = \sqrt{t}\, Z$.
we now proceed,

$$E[X(t)] = E[\exp[(\lambda - \tfrac{\mu^2}{2})t + \mu W(t)]] =$$

$$= E[e^{(\lambda - \frac{\mu^2}{2})t} e^{\mu W(t)}] = e^{(\lambda - \frac{\mu^2}{2})t} E[e^{\mu W(t)}]$$

$$= e^{(\lambda - \frac{\mu^2}{2})t} E[e^{\mu\sqrt{t} Z}] \quad \text{(from above)}$$

$$= e^{(\lambda - \frac{\mu^2}{2})t} \int_{-\infty}^{\infty} e^{\mu\sqrt{t} z} \frac{1}{\sqrt{2\pi}} e^{-(z^2/2)}\, dz \quad (\text{Since } Z \sim N(0,1))$$

$$= \frac{e^{(\lambda - \frac{\mu^2}{2})t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(z^2 - 2\mu\sqrt{t} z)}{2}}\, dz$$

$$= \frac{e^{(\lambda - \frac{\mu^2}{2})t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(z-\mu\sqrt{t})^2}{2} + \frac{\mu^2 t}{2}}\, dz$$

$$= \frac{e^{(\lambda - \frac{\mu^2}{2})t} e^{\mu^2 t/2}}{\sqrt{2\pi}} \int e^{-\frac{(z-\mu\sqrt{t})^2}{2}}\, dz$$

$\Longrightarrow$

$$= e^{\lambda t} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-(z-\mu\sqrt{t})^2}{2}} dz = e^{\lambda t} \cdot (1)$$

$= e^{\lambda t}$. Thus, $E[X(t)] = e^{\lambda t}$ //

(b) $E\left[\lambda \int_0^t X(s)\,ds\right]$

$$E\left[\lambda \int_0^t X(s)\,ds\right] = \lambda \int_0^t E[X(s)]\,ds$$ by linearity of expectation

(and by part (a) we showed that $E[X(t)] = e^{\lambda t}$)

$$= \lambda \int_0^t e^{\lambda s}\,ds = e^{\lambda s}\Big|_0^t = e^{\lambda t} - e^0 = e^{\lambda t} - 1$$

$$\Rightarrow \quad E\left[\lambda \int_0^t X(s)\,ds\right] = e^{ts} - 1$$ //

(c) $E\left[\mu \int_0^t X(s)\,dW(s)\right]$

From $dX(t) = \lambda X(t)dt + \mu X(t)\,dW(t)$ and $X(0) = 1$

Then, using the Ito integral equation (from slide 8)

$$X(t) = X(0) + \int_0^t \lambda X(s)\,ds + \int_0^t \mu X(s)\,dW(s)$$

We can use the values from (a) and (b) and solve. $\Longrightarrow$

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$$\Rightarrow E[X(t)] = E[X(0)] + E\left[\lambda \int_0^t X(s)\,ds\right] + $$
$$+ E\left[\mu \int_0^t X(s)\,dW(s)\right]$$

$$\Rightarrow e^{\lambda t} = 1 + (e^{ts} - 1) + E\left[\mu \int_0^t X(s)\,dW(s)\right]$$

$$\Rightarrow E\left[\mu \int_0^t X(s)\,dW(s)\right] = e^{\lambda t} - e^{\lambda t} = 0$$

$$\Rightarrow E\left[\mu \int_0^t X(s)\,dW(s)\right] = 0$$

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Problem B

In[202]:=
```
sigma = 10;
mu = 20;
theta = 1;
t = 0.01;
x0 = 0;
n = 50;
proc = OrnsteinUhlenbeckProcess[mu, sigma, theta, x0];
```

In[238]:=
```
x = ItoProcess[proc];
```

In[246]:=
```
data = RandomFunction[x, {0, 5, t}, n]
```

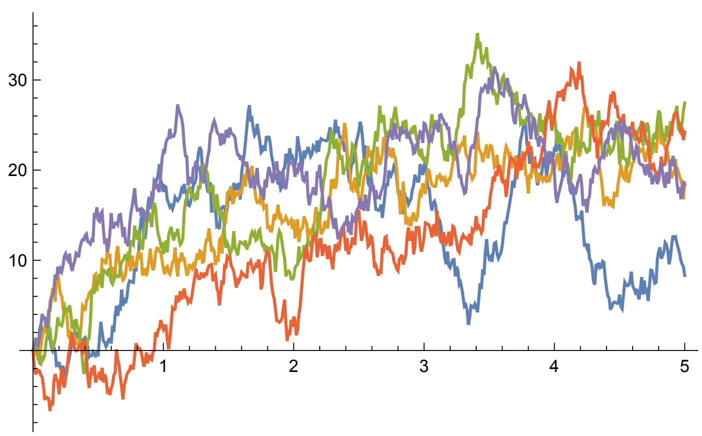
Out[246]=
```
TemporalData[ Time: 0. to 5.  Data points: 25050  Paths: 50 ]
```

In[247]:=
```
alldata = Table[data, n];
```

Part (a)

In[248]:=
```
ListLinePlot[data["Path", {1, 2, 3, 4, 5}], PlotRange → All]
```

Out[248]=

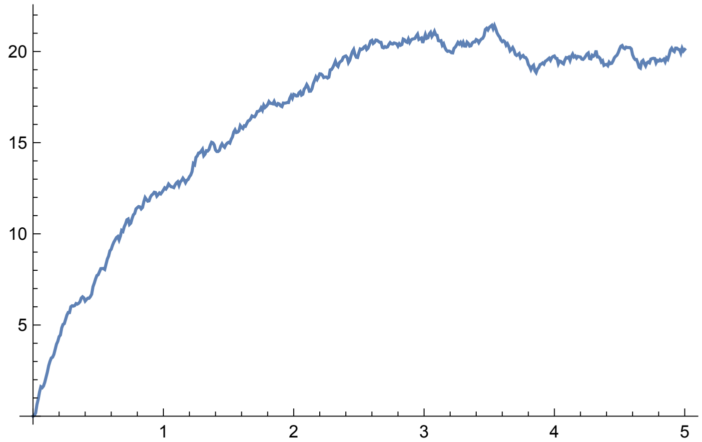


Part (b)

In[249]:=
```
meanpath = Mean[data["Path", {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16,
      17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33,
      34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50}]];
```

In[250]:=
```
ListLinePlot[meanpath, PlotRange → All]
```

Out[250]=



From Slide 10 “Example 2 of Ito SDE: Ornstein-Uhlenbeck Process” , E[X(t)] ⟶ $\mu = 20$. It can be seen in the plot of the mean path above that the sample mean of the 50 simulated paths converge to 20 as t gets larger.

Part (c)

In[258]:=
```
meanx2 = Mean[data[2]];
```

In[259]:=
```
stdevx2 = StandardDeviation[data[2]];
```

Hypothesis Test: $H_0 : \mu = 20$, $H_1 : \mu \neq 20$

In[260]:=
```
tstat2 = ((meanx2 - mu) / (stdevx2)) * Sqrt[n]
```

Out[260]=
```
-2.59505
```

Thus, our t-statistics is -2.59505

In[271]:=
```
pvalue2 = CDF[StudentTDistribution[n - 1], tstat2] +
   (1 - CDF[StudentTDistribution[n - 1], -tstat2])
```

Out[271]=
```
0.0124431
```

Thus, our P-value is 0.0124431. At the 0.05 significance level, since our p-value < 0.05/2 = 0.025, then we reject the null hypothesis. That is, there is not enough evidence to claim that the mean of X(2) approaches the true mean.

```
meanx5 = Mean[data[5]];
stdevx5 = StandardDeviation[data[5]];
```

Hypothesis Test: $H_0 : \mu = 20$, $H_1 : \mu \neq 20$

In[263]:=
```
tstat5 = ((meanx5 - mu) / (stdevx5)) * Sqrt[n]
```

Out[263]=
```
0.101619
```

Thus, our t-statistics is 0.101619

In[272]:=
```
pvalue5 = CDF[StudentTDistribution[n - 1], -tstat5] +
   (1 - CDF[StudentTDistribution[n - 1], tstat5])
```

Out[272]=
```
0.919473
```

For X(5) we get a much larger P-Value = 0.919473. Then, at any significance level (say 0.05) since 0.919 < 0.05/2 = 0.025, we conclude that there is not enough evidence to reject the null hypothesis that the 50 simulations at X(5) approaches the true mean.