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EN.553.732 Homework 2

Problem 1.) Suppose $x_1, \ldots, x_n$ is a random sample from an exponential distribution with mean $1/\theta$.

a.) Derive the Jeffrey's prior for θ.

Solution: We are given that $x|\theta \sim Exp(\theta)$ with mean $1/\theta$. then,

$$p(x|\theta) = \theta e^{-\theta x}, \quad \theta \geq 0$$

Moreover, Jeffrey's prior is such that $\pi(\theta) \propto \sqrt{I(\theta)}$

where $I(\theta) = -E\left[\frac{\partial^2}{\partial\theta^2}[\log p(x|\theta)] \,\middle|\, \theta\right]$.

then, $\log p(x|\theta) = \log\theta - \theta x$; $\frac{\partial}{\partial\theta}[\log p(x|\theta)] = \frac{1}{\theta} - x$

$$\Rightarrow \frac{\partial^2}{\partial\theta^2}[\log p(x|\theta)] = \frac{-1}{\theta^2}$$

$$\Rightarrow I(\theta) = -E[-1/\theta^2 | \theta] = 1/\theta^2$$

$$\Rightarrow \text{Jeffrey's prior } \pi(\theta) \propto \sqrt{1/\theta^2} = 1/\theta, \quad \theta \geq 0$$

(b.) Derive the posterior distribution of θ using Jeffrey's prior.

Solution: $p(\theta|x_1, \ldots, x_n) \propto \pi(\theta)\, p(x_1, \ldots, x_n|\theta)$

where $\pi(\theta) = 1/\theta$ and $p(x_1, \ldots, x_n|\theta) = \prod_{i=1}^{n} p(x_i|\theta) =$

$$= \prod_{i=1}^{n} \theta e^{-\theta x_i} = \theta^n e^{-\theta \sum_{i=1}^{n} x_i}$$

$\Longrightarrow$

$$\Longrightarrow p(\theta|x_1,\dots,x_n) \propto \frac{1}{\theta}\left(\theta^n e^{-\theta\sum_{i=1}^n x_i}\right)$$

$$= \theta^{n-1} e^{-\theta\sum_{i=1}^n x_i}$$

and $\theta^{n-1} e^{-\theta\sum_{i=1}^n x_i} \longrightarrow \text{Gamma}(n, \sum_{i=1}^n x_i)$

i.e. $\theta | x_1,\dots,x_n \sim \text{Gamma}(n, \sum_{i=1}^n x_i)$

$$\Longrightarrow p(\theta|x_1,\dots,x_n) = \frac{(\sum x_i)^n}{\Gamma(n)} \cdot \theta^{n-1} e^{-\theta\sum_{i=1}^n x_i} \quad , \theta \geq 0$$ (By def. of the Gamma dist.)

(c) Derive the predictive distribution of a future observation z.

Solution: Clearly $z \sim \text{Exp}(\theta)$ since z is from the same population as the $x_1,\dots,x_n$. Now, using $p(\theta|x_1,\dots,x_n)$ from part (b) and assuming the $x_1,\dots,x_n$ iid, the predictive distribution is

$$p(z|x_1,\dots,x_n) = \int_0^\infty p(z,\theta|x_1,\dots,x_n)\,d\theta$$

$$= \int_0^\infty p(z|\theta,x_1,\dots,x_n)\,p(\theta|x_1,\dots,x_n)\,d\theta$$ (By Bayes' Rule)

$$= \int_0^\infty p(z|\theta)\,p(\theta|x_1,\dots,x_n)\,d\theta$$ Assuming independence of z and the $x_1,\dots,x_n$ (Lecture 4)

$$= \int_0^\infty \theta e^{-\theta z} \cdot \frac{(\sum_{i=1}^n x_i)^n}{\Gamma(n)} \theta^{n-1} e^{-\theta\sum x_i}\,d\theta$$ (from part (b))

$$= \frac{(\sum x_i)^n}{\Gamma(n)} \int_0^\infty \theta^n e^{-\theta(z+\sum x_i)}\,d\theta \,;$$

$$\Longrightarrow \text{let } u = \theta(z + \sum_{i=1}^n x_i) \Rightarrow du = (z+\sum x_i)\,d\theta$$

$$\Longrightarrow d\theta = \frac{du}{z+\sum x_i} \quad \text{and} \quad \theta = \frac{u}{z+\sum x_i} \quad \Longrightarrow$$

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$\Rightarrow p(z \mid x_1, \ldots, x_n)$ is then

$$= \frac{\left(\sum x_i\right)^n}{\Gamma(n)} \int_0^\infty \left(\frac{u}{z + \sum x_i}\right)^n e^{-u} \cdot \frac{du}{z + \sum x_i} =$$

$$= \frac{\left(\sum x_i\right)^n}{\Gamma(n)\left(z + \sum x_i\right)^n} \int_0^\infty u^n e^{-u}\, du = \frac{\left(\sum x_i\right)^n \Gamma(n+1)}{\Gamma(n)\left(z + \sum x_i\right)^n}$$

$$= \frac{\Gamma(n+1)}{\Gamma(n)\Gamma(1)} \cdot \left(\frac{\sum x_i}{z + \sum x_i}\right)^n \left(1 - \frac{\sum x_i}{z + \sum x_i}\right)^{1-1} \quad \left(\sim \text{Beta}(n+1, 1)\right)$$

Since $\Gamma(1) = 1$ and $\left(1 - \frac{\sum x_i}{z + \sum x_i}\right)^{1-1} = 1$

$\Rightarrow$ Thus, $z \mid x_1, \ldots, x_n \sim \text{Beta}(n+1, 1)$

(d) Derive a 95% credible interval for z.

Solution:

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Problem 2

(a.) Let $y \sim \text{binomial}(n, \theta)$. Obtain Jeffrey's prior distribution $p_J(\theta)$ for this model.

Solution: Since $y \sim \text{binomial}(n, \theta)$, then letting $p(y|\theta)$ be the sampling model distribution,

$$p(y|\theta) = \binom{n}{y} \theta^y (1-\theta)^{n-y} \quad , \quad \theta \in [0,1].$$

Moreover, $p_J(\theta) \propto \sqrt{I(\theta)}$

where $I(\theta) = -E\left[\frac{\partial^2}{\partial\theta^2}[\log p(y|\theta)] \,\middle|\, \theta\right]$

$$\log p(y|\theta) = \log \binom{n}{y}\theta^y(1-\theta)^{n-y} = \log\binom{n}{y} + y\log\theta + (n-y)\log(1-\theta)$$

then, $\frac{\partial}{\partial\theta}[\log p(y|\theta)] = \frac{y}{\theta} - \frac{n-y}{1-\theta}$;

$$\frac{\partial^2}{\partial\theta^2}[\log p(y|\theta)] = \frac{-y}{\theta^2} - \frac{n-y}{(1-\theta)^2}$$

$$\Rightarrow I(\theta) = -E\left[-\frac{y}{\theta^2} - \frac{n-y}{(1-\theta)^2} \,\middle|\, \theta\right] = E\left[\frac{y}{\theta^2} \,\middle|\, \theta\right] + E\left[\frac{n-y}{(1-\theta)^2} \,\middle|\, \theta\right]$$

$$= \frac{1}{\theta^2}E[y|\theta] + \frac{1}{(1-\theta)^2}\left(n - E[y|\theta]\right)$$

since $y|\theta \sim \text{Bin}(n,\theta) \Rightarrow E[y|\theta] = n\theta$

$$\Rightarrow \quad I(\theta) = \frac{n\theta}{\theta^2} + \frac{(n-n\theta)}{(1-\theta)^2} = \frac{n}{\theta} + \frac{n(1-\theta)}{(1-\theta)^2} =$$

$$= \frac{n}{\theta} + \frac{n}{1-\theta} = \frac{n-n\theta+n\theta}{\theta(1-\theta)} = \frac{n}{\theta(1-\theta)}$$

$\Longrightarrow$

Thus, $p_J(\theta) \propto \sqrt{I(\theta)} = \sqrt{\frac{n}{\theta(1-\theta)}}$

(b.) Reparameterize the binomial sampling model with $\phi = \log\left[\frac{\theta}{1-\theta}\right]$, so that $p(y|\phi) = \binom{n}{y} e^{\phi y}(1+e^{\phi})^{-n}$.

Obtain Jeffrey's prior distribution $p_J(\phi)$ for this model.

<u>Solution</u>: We have $p(y|\phi) = \binom{n}{y} e^{\phi y}(1+e^{\phi})^{-n}$

where $\phi = \log\left(\frac{\theta}{1-\theta}\right)$.

$$\log p(y|\phi) = \log\binom{n}{y} + \phi y - n\left(\log(1+e^{\phi})\right)$$

$$\frac{\partial}{\partial\phi}\left[\log p(y|\phi)\right] = y - \frac{n}{1+e^{\phi}}e^{\phi}$$

$$\Rightarrow \frac{\partial^2}{\partial\phi^2}\left[\log p(y|\phi)\right] = \frac{ne^{\phi}}{(1+e^{\phi})^2}e^{\phi} - \frac{ne^{\phi}}{1+e^{\phi}} =$$

$$= \frac{ne^{2\phi} - ne^{\phi}(1+e^{\phi})}{(1+e^{\phi})^2} = \frac{-ne^{\phi}}{(1+e^{\phi})^2}$$

$$\Rightarrow I(\theta) = -E\left[\frac{-ne^{\phi}}{(1+e^{\phi})^2}\,\middle|\,\phi\right] = nE\left[\frac{e^{\phi}}{(1+e^{\phi})^2}\,\middle|\,\phi\right] =$$

$$= \frac{ne^{\phi}}{(1+e^{\phi})^2} \Longrightarrow p_J(\phi) \propto \sqrt{\frac{ne^{\phi}}{(1+e^{\phi})^2}} = \frac{\sqrt{ne^{\phi}}}{1+e^{\phi}}$$

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(c) <u>Solution</u>: From the supposition in part (b) we know that

$$\phi = \log \frac{\theta}{1-\theta} = g(\theta).$$

The change of variables formula: $p_J(\phi) = p_J(\theta) \left| \frac{d\theta}{d\phi} \right|$

From (a) we have that $p_J(\theta) \propto \sqrt{\frac{n\theta}{1-\theta}}$

and, $\log \frac{\theta}{1-\theta} = \phi \Rightarrow \frac{\theta}{1-\theta} = e^{\phi} \Rightarrow \frac{1-\theta}{\theta} = \frac{1}{e^{\phi}}$

$$\Rightarrow \frac{1}{\theta} = \frac{1}{e^{\phi}} + 1 = \frac{1+e^{\phi}}{e^{\phi}} \Rightarrow \boxed{\theta = \frac{e^{\phi}}{1+e^{\phi}}}$$

Then, $\left| \frac{d\theta}{d\phi} \right| = \left| -e^{\phi}(1+e^{\phi})^{-2} e^{\phi} + e^{\phi}(1+e^{\phi})^{-1} \right| =$

$$= \left| \frac{-e^{2\phi}}{(1+e^{\phi})^2} + \frac{e^{\phi}(1+e^{\phi})}{(1+e^{\phi})^2} \right| = \left| \frac{e^{\phi}}{(1+e^{\phi})^2} \right| = \frac{e^{\phi}}{(1+e^{\phi})^2}$$

Then $p_J(\phi) \propto \sqrt{\frac{n}{\theta(1-\theta)}} \cdot \frac{e^{\phi}}{(1+e^{\phi})^2} =$

$$= \sqrt{\frac{n}{\frac{e^{\phi}}{1+e^{\phi}}\left(1-\frac{e^{\phi}}{1+e^{\phi}}\right)}} \cdot \frac{e^{\phi}}{(1+e^{\phi})^2} = \sqrt{\frac{n}{\left(\frac{e^{\phi}}{1+e^{\phi}}\right)^2\left(\frac{1+e^{\phi}}{e^{\phi}}-1\right)}} \cdot \frac{e^{\phi}}{(1+e^{\phi})^2} =$$

$$= \frac{1+e^{\phi}}{e^{\phi}} \sqrt{\frac{n}{\frac{1}{e^{\phi}}}} \cdot \frac{e^{\phi}}{(1+e^{\phi})^2} = \sqrt{ne^{\phi}} \cdot \frac{1}{(1+e^{\phi})}$$

$\Rightarrow p_J(\phi) \propto \frac{\sqrt{ne^{\phi}}}{(1+e^{\phi})}$. This is the same density found in part (b). Hence, consistency under reparameterization. $\square$

Problem 3) Suppose that $f(x|\theta,\sigma)$ is normal with mean θ and standard deviation σ.

(a) Suppose that σ is known, and show that the Jeffrey's prior for θ is given by $p(\theta) = 1$, $\theta \in \mathbb{R}$

Proof: Since σ is known, without loss of generality let $\sigma = 1$ so that $X|\theta \sim N(\theta, 1)$.

we have, $f(x|\theta) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\theta)^2}{2}}$, $-\infty < x < \infty$
$-\infty < \theta < \infty$

Then,

$$\log f(x|\theta) = \log \frac{1}{\sqrt{2\pi}} - \frac{(x-\theta)^2}{2}$$

$$\Rightarrow \frac{\partial \log f(x|\theta)}{\partial \theta} = x - \theta \quad ; \quad \frac{\partial^2 \log f(x|\theta)}{\partial \theta^2} = -1$$

$$\Rightarrow I(\theta) = -E\left[\frac{\partial^2 \log f(x|\theta)}{\partial \theta^2} \Big| \theta\right] = -E[-1|\theta] = 1$$

$$\Rightarrow p_J(\theta) = \sqrt{I(\theta)} = \sqrt{1} = 1$$

If instead we let $\sigma = \sigma_c$ for some known σ_c, then $I(\theta) = 1/\sigma_c \propto 1 \Rightarrow p_J(\theta) \propto 1$

(so generality is not lost by choosing a σ) $\square$

(b.) Next, suppose that θ is known, and show that the Jeffrey's prior for σ is given by $p(\sigma) = 1/\sigma$, $\sigma > 0$.

Pf: Without loss of generality let $\theta = 0$ (since θ known) so that $X|\sigma \sim N(0, \sigma^2)$

Then,

$$f(x|\sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}}, \quad -\infty < x < \infty, \ \sigma > 0$$

$\Rightarrow$

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$$\Rightarrow \log f(x|\sigma) = \log \frac{1}{\sqrt{2\pi\sigma^2}} - \frac{x^2}{2\sigma^2}$$

$$\frac{\partial \log f(x|\sigma)}{\partial\sigma} = -\sqrt{2\pi\sigma^2}\left(\frac{1}{2}\right)(2\pi\sigma^2)^{-3/2}(4\pi\sigma) + \frac{x^2}{\sigma^3}$$

$$= -\frac{4\pi\sigma}{2\pi\sigma^2} + \frac{x^2}{\sigma^3} = \frac{-2}{\sigma} + \frac{x^2}{\sigma^3}$$

$$\frac{\partial^2 \log f(x|\sigma)}{(\partial\sigma)^2} = \frac{2}{\sigma^2} - \frac{3x^2}{\sigma^4}$$

$$\Rightarrow -E\left[\frac{\partial^2 \log f(x|\sigma)}{(\partial\sigma)^2}\Big|\sigma\right] = -E\left[\frac{2}{\sigma^2} - \frac{3x^2}{\sigma^4}\Big|\sigma\right] =$$

$$= \frac{-2}{\sigma^2} + \frac{3}{\sigma^4}E[x^2|\sigma] = \frac{-2}{\sigma^2} + \frac{3}{\sigma^4}\sigma^2 = \frac{1}{\sigma^2}$$

$$\Rightarrow p_J(\sigma) \propto \sqrt{I(\sigma)} = \sqrt{1/\sigma^2} = \frac{1}{\sigma}$$

If we had used some arbitrary $\theta = \mu$, then

$$\frac{\partial^2 \log f(x|\sigma)}{(\partial\sigma)^2} = \frac{2}{\sigma^2} - \frac{3(x-\mu)^2}{\sigma^4}$$

$$\Rightarrow p_J(\sigma) \propto \sqrt{I(\sigma)} = \sqrt{-\frac{2}{\sigma^2} + \frac{3}{\sigma^4}E[(x-\mu)^2|\sigma]}$$

$$= \sqrt{-\frac{2}{\sigma^2} + \frac{3}{\sigma^4}\sigma^2} = \sqrt{\frac{1}{\sigma^2}} = 1/\sigma$$

(Just to illustrate that we indeed do not lose generality by choosing a θ.)

$$\Rightarrow p_J(\sigma) \propto \frac{1}{\sigma}$$

$\square$

(c) Finally, assume that θ and σ are both unknown, and show that the bivariate Jeffrey's prior is given by $p(\theta, \sigma) = 1/\sigma^2$, $\theta \in \mathbb{R}$, $\sigma > 0$.

Pf: We have that $f(x \mid \theta, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\theta)^2}{2\sigma^2}}$; $x, \theta \in \mathbb{R}$, $\sigma > 0$

Let $\ell(\theta, \sigma) = \log f(x \mid \theta, \sigma)$ (for convenience).

Then, $\ell(\theta, \sigma) = \log \frac{1}{\sigma\sqrt{2\pi}} - \frac{(x-\theta)^2}{2\sigma^2}$

and $p_J(\theta, \sigma) \propto \sqrt{I(\theta, \sigma)} \underset{=}{=} -E\left[|\nabla^2 \ell(\theta, \sigma)| \mid \theta, \sigma\right]$

where $\nabla^2 \ell(\theta, \sigma)$ is the Hessian of $\ell(\theta, \sigma)$ and $|\cdot|$ its determinant.

i.e. $\nabla^2 \ell(\theta, \sigma) = \begin{bmatrix} \frac{\partial^2 \ell}{\partial \theta^2} & \frac{\partial^2 \ell}{\partial \theta \partial \sigma} \\ \frac{\partial^2 \ell}{\partial \theta \partial \sigma} & \frac{\partial^2 \ell}{(\partial \sigma)^2} \end{bmatrix}$

Then, $\frac{\partial \ell}{\partial \theta} = \frac{(x-\theta)}{\sigma^2}$; $\frac{\partial^2 \ell}{(\partial \theta)^2} = \frac{-1}{\sigma^2}$;

$\frac{\partial^2 \ell}{\partial \theta \partial \sigma} = \frac{-2(x-\theta)}{\sigma^3}$; $\frac{\partial \ell}{\partial \sigma} = \frac{-\sigma\sqrt{2\pi}}{\sigma^2\sqrt{2\pi}} + \frac{(x-\theta)^2}{\sigma^3}$

$= -\frac{1}{\sigma} + \frac{(x-\theta)^2}{\sigma^3}$

$\Rightarrow \frac{\partial^2 \ell}{(\partial \sigma)^2} = \frac{1}{\sigma^2} - \frac{3(x-\theta)^2}{\sigma^4}$

$$\Rightarrow \quad \nabla^2 \ell(\theta, \mu) = \begin{bmatrix} -\frac{1}{\sigma^2} & \frac{-2(x-\theta)}{\sigma^3} \\ \frac{-2(x-\theta)}{\sigma^3} & \frac{1}{\sigma^2} - \frac{3(x-\theta)^2}{\sigma^4} \end{bmatrix} \Longrightarrow$$

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Now, $|\nabla^2 l(\theta,\sigma)| = -\frac{1}{\sigma^2}\left(\frac{1}{\sigma^2} - \frac{3(x-\theta)^2}{\sigma^4}\right) - \left(\frac{-2(x-\theta)}{\sigma^3}\right)^2$

$$= -\frac{1}{\sigma^4} + \frac{3(x-\theta)^2}{\sigma^6} - \frac{4(x-\theta)^2}{\sigma^6}$$

$$= \frac{-1}{\sigma^4} - \frac{(x-\theta)^2}{\sigma^6}$$

Then,

$$-E\left[|\nabla^2 l(\theta,\sigma)| \,\middle|\, \theta,\sigma\right] = \frac{1}{\sigma^4} + \frac{1}{\sigma^6} E[(x-\theta)^2 | \mu,\sigma]$$

$$= \frac{1}{\sigma^4} + \frac{1}{\sigma^6}\sigma^2 = \frac{2}{\sigma^4}$$

Then, $p_J(\theta,\sigma) \propto \sqrt{I(\theta,\sigma)} = \sqrt{\frac{2}{\sigma^4}} = \frac{\sqrt{2}}{\sigma^2} \propto \frac{1}{\sigma^2}$

Problem 4: Solution: Let $\tilde{y} = (y_1, y_2, \dots, y_n)$

To compute $p(\theta|\tilde{y})$ consider the following relationships

(1) $p(\theta|\tilde{y}) = \int p(\theta,\sigma^2|\tilde{y})\,d\sigma^2 \propto \int p(\tilde{y}|\theta,\sigma^2)\,p(\theta,\sigma^2)\,d\sigma^2$

(2.) $p(\theta,\sigma^2) = p(\theta|\sigma^2)\,p(\sigma^2)$

(3) $\tilde{y}|\theta,\sigma^2 \sim N(\theta,\Sigma)$, since $\{y_i\}_{i=1}^{n} \sim$ i.i.d. $N(\theta,\sigma^2)$

(4.) $\frac{1}{\sigma^2} \sim \text{Gamma}\left(\frac{\nu_0}{2}, \frac{\nu_0}{2}\sigma_0^2\right)$ and $\theta|\sigma^2 \sim N(\mu_0, \sigma^2/\kappa_0)$
(from Chapter 5, pg. 74-75, Hoff)

We can then proceed as follows $\Longrightarrow$ (continued... $\Rightarrow$)

From (2) and (4) we have

$$p(\theta, \sigma^2) = p(\theta|\sigma^2)p(\sigma^2) = \left(\frac{1}{\sqrt{2\pi\frac{\sigma^2}{\kappa_0}}} e^{-\frac{(\theta-\theta_0)^2}{2\sigma^2/\kappa_0}}\right) \cdot \left(\frac{\left(\frac{\nu_0\sigma_0^2}{2}\right)^{\nu_0/2}\left(\frac{1}{\sigma^2}\right)^{\frac{\nu_0}{2}-1} e^{-\frac{\nu_0\sigma_0^2}{2\sigma^2}}}{\Gamma(\nu_0/2)}\right)$$

$$= \frac{\sqrt{\frac{\kappa_0}{2\pi}}\left(\frac{\nu_0\sigma_0^2}{2}\right)^{\nu_0/2}}{\Gamma(\nu_0/2)} \left(\frac{1}{\sigma^2}\right)^{1/2}\left(\frac{1}{\sigma^2}\right)^{\frac{\nu_0}{2}-1} e^{-\frac{1}{2\sigma^2}\left[(\theta-\theta_0)^2\kappa_0 + \nu_0\sigma_0^2\right]}$$

$$(5) \quad \propto \left(\frac{1}{\sigma^2}\right)^{\frac{\nu_0-3}{2}} e^{-\frac{\kappa_0}{2\sigma^2}\left[(\theta-\theta_0)^2 + \frac{\nu_0\sigma_0^2}{\kappa_0}\right]} \propto p(\theta, \sigma^2)$$

By (3) we have

$$p(\tilde{y}|\theta, \sigma^2) = p(y_1, \dots, y_n|\theta, \sigma^2) = \prod_{i=1}^{n} (2\pi\sigma^2)^{-1/2} e^{-\frac{(y_i-\theta)^2}{2\sigma^2}}$$

$$(6) \quad = (2\pi\sigma^2)^{-n/2} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i-\theta)^2}$$

By (1), (5), and (6) we have that

$$p(\theta|\tilde{y}) = \int_0^\infty p(\theta, \sigma^2|\tilde{y})\, d\sigma^2 \propto \int_0^\infty p(\tilde{y}|\theta, \sigma^2)\, p(\theta, \sigma^2)\, d\sigma^2$$

$$= \int_0^\infty (2\pi\sigma^2)^{-n/2} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i-\theta)^2} \left(\frac{1}{\sigma^2}\right)^{\frac{\nu_0-3}{2}} e^{-\frac{\kappa_0}{2\sigma^2}\left[(\theta-\theta_0)^2 + \frac{\nu_0\sigma_0^2}{\kappa_0}\right]} d\sigma^2$$

$$\propto \int_0^\infty (\sigma^2)^{-\left(\frac{\nu_0+n-3}{2}\right)} e^{-\frac{\kappa_0}{2\sigma^2}\left[\frac{1}{\kappa_0}\sum(y_i-\theta)^2 + (\theta-\theta_0)^2 + \frac{\nu_0\sigma_0^2}{\kappa_0}\right]} d\sigma^2$$

Using the change of variables presented in Lecture 4 slides and on page 74-75 in Hoff:

Let

$$\kappa_n = \kappa_0 + n, \quad \theta_n = \frac{\kappa_0\theta_0 + n\bar{y}}{\kappa_n}, \quad \nu_n = \nu_0 + n$$

and $$SS_n = \frac{1}{\nu_n}\left[\nu_0\sigma_0^2 + \sum(y_i-\bar{y})^2 + \frac{\kappa_0 n}{\kappa_n}(\bar{y}-\theta_0)^2\right] \Rightarrow$$

$$\Rightarrow \propto \int_0^\infty (\sigma^2)^{-\left(\frac{\nu_n+1}{2}+1\right)} \exp\left\{-\frac{\kappa_n}{2\sigma^2}\left(\frac{SS_n^2}{\kappa_n} + (\theta-\theta_n)^2\right)\right\} d\sigma^2$$

(from Lecture Slides and Hoff)

$$\left(\text{Let } A = \kappa_n\left[\frac{SS_n^2}{\kappa_n} + (\theta-\theta_n)^2\right] \quad \text{and } z = \frac{A}{2\sigma^2} \right)$$

$$\Rightarrow \propto A^{-\frac{\nu_n+1}{2}} \int_0^\infty z^{\frac{(\nu_n-1)}{2}} e^{-z} dz \qquad \text{(Gamma Integral)}$$

$$\propto \left[1 + \left(\frac{\theta-\theta_n}{SS_n/\sqrt{\kappa_n}}\right)^2\right]^{-\frac{(\nu_n+1)}{2}}$$

Thus, $\theta \mid \tilde{y} \sim t\left(\nu_n, \theta_n, \frac{SS_n^2}{\kappa_n \nu_n}\right)$ //

Similarly, $p(\sigma^2 \mid \tilde{y}) = \int_{-\infty}^{\infty} p(\theta, \sigma^2 \mid \tilde{y}) d\theta \propto \int_{-\infty}^{\infty} p(\tilde{y} \mid \theta, \sigma^2) p(\sigma^2) d\theta$

$$\propto \int_{-\infty}^{\infty} (\sigma^2)^{-\left(\frac{\nu_n+1}{2}+1\right)} \exp\left\{-\frac{\kappa_n}{2\sigma^2}\left[\frac{SS_n^2}{\kappa_n} + (\theta-\theta_n)^2\right]\right\} d\theta$$

$$= (\sigma^2)^{-\left(\frac{\nu_n}{2}+1\right)} \exp\left\{-\frac{1}{2\sigma^2} SS_n^2\right\} \cdot \underbrace{\frac{1}{\sqrt{\sigma^2}} \int_{-\infty}^{\infty} \exp\left\{-\frac{\kappa_n}{2\sigma^2}(\theta-\theta_n)^2\right\} d\theta}_{\propto 1}$$

$$\propto (\sigma^2)^{-\left(\frac{\nu_n}{2}+1\right)} \exp\left\{-\frac{1}{2\sigma^2} SS_n^2\right\} \longrightarrow \text{Inv-Gamma}\left(\frac{\nu_n}{2}, \frac{\nu_n}{2} SS_n^2\right)$$

$$\Longrightarrow \frac{1}{\sigma^2} \sim \text{Inv-Gamma}\left(\frac{\nu_n}{2}, \frac{\nu_n SS_n^2}{2}\right)$$

See R Code for Monte Carlo Simulation. //

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Problem 4 R Code and Plot

```
> #Problem 4
>
> n<-5000
>
> #Chosen values and Prior Parameters:
> theta0 = 1
> sig0 = 0.5
> v0 = 1
> k0 = 1
> kn<-k0+n
> vn<-v0+n
>
> y<-rnorm(n, mean=0, sd = 1)
> yb<-mean(y)
> SS<-sum((y-yb)^2)
> theta_n<-(k0*theta0+n*yb)/kn
> SSn<-(v0*sig0+SS+(k0*n)*(y-theta0)^2/kn)/vn
> sig <- 1/rgamma(5000,vn/2,vn*SSn/2)
> theta<-rnorm(5000, theta_n, sqrt(sig/kn))
> t<-rt(5000, df=vn)*sqrt(SSn/kn)+theta_n
> theta_density<-density(theta)
> t_dist<-density(t)
> plot(theta_density)
> lines(t_dist, col="red")
```