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Rough Draft (To Be input into LaTeX)

Problem 4:

1.) For each $i \in [1, 100]$, $\mathbb{P}r(Y_i = y_i \mid \theta) = \theta^{y_i}(1-\theta)^{1-y_i}$ since Y_i binary random variables.

Then, with the assumption of conditional independence of the Y_i on θ, we have

$$\mathbb{P}r(Y_1 = y_1, \ldots, Y_{100} = y_{100} \mid \theta) = \prod_{i=1}^{100} \mathbb{P}r(Y_i = y_i \mid \theta)$$

$$= \prod_{i=1}^{100} \theta^{y_i}(1-\theta)^{1-y_i}$$

$$= \theta^{\sum_{i=1}^{100} y_i}(1-\theta)^{100 - \sum_{i=1}^{100} y_i}$$

Then $\mathbb{P}r\left(\sum_{i=1}^{100} Y_i = y \mid \theta\right)$ is the probability that the sum of the binary random variables is equal to y, where the sum of y of 100 binary random variables can be achieved in $\binom{100}{y}$ distinct ways.

Thus, $\mathbb{P}r\left(\sum_{i=1}^{100} Y_i = y \mid \theta\right) = \binom{100}{y} \theta^{y}(1-\theta)^{100-y}$.

$\square$

2.) (Problem 4)

* Compute $Pr\left(\sum_{i=1}^{100} y_i = 57 \mid \theta\right)$ for each θ.

* Refer to R Code for computation of $Pr\left(\sum_{i=1}^{100} y_i = 57 \mid \theta\right)$ for each θ.

- From part 1 we know that $Pr\left(\sum_{i=1}^{100} y_i = 57 \mid \theta\right) = \binom{100}{57} \theta^{57}(1-\theta)^{43}$

θ	$Pr\left(\sum_{i=1}^{100} y_i = 57 \mid \theta\right)$
0.0	0.00
0.1	4.107157×10^{-31}
0.2	3.738459×10^{-16}
0.3	1.306895×10^{-8}
0.4	2.285792×10^{-4}
0.5	3.006864×10^{-2}
0.6	6.67289×10^{-2}
0.7	1.853172×10^{-3}
0.8	1.003535×10^{-7}
0.9	9.395858×10^{-18}
1.0	0.00

Plot is attached at end of document

(3) Since $\Pr(\theta=0.0) = \Pr(\theta=0.1) = \cdots = \Pr(\theta=1.0)$

Then $\Pr(\theta=0.0) = \cdots = \Pr(\theta=1.0) = 1/11$

Since $\theta \sim$ Discrete

Then $$p(\theta \mid \sum_{i=1}^{100} y_i = 57) = \frac{\Pr(\sum_{i=1}^{100} y_i = 57 \mid \theta)\Pr(\theta)}{\Pr(\sum_{i=1}^{100} y_i = 57)}$$

$$= \frac{\binom{100}{57}\theta^{57}(1-\theta)^{43}\left(\frac{1}{11}\right)}{\Pr(\sum_{i=1}^{100} y_i = 57)}$$

* Where $1/\Pr(\sum y_i = 57)$ is just a normalization constant

Then $$\Pr(\sum y_i = 57) = \sum_{\theta} \binom{100}{57}\theta^{57}(1-\theta)^{43}\left(\frac{1}{11}\right)$$

From the R code, the values of $p(\theta \mid \sum_{i=1}^{100} y_i = 57)$ were found to be.

θ	0.0	0.1	0.2	0.3
$p(\theta \mid \sum y_i = 57)$	0.00	4.153701×10^{-30}	3.780824×10^{-15}	1.321705×10^{-7}

0.4	0.5	0.6	0.7
2.311695×10^{-3}	3.040939×10^{-1}	6.748515×10^{-01}	1.874172×10^{-2}

0.8	0.9	1.0
1.014907×10^{-6}	9.502335×10^{-17}	0.00

Plot attached.

(4) See Plot and R code.

Here,

$$p(\theta \mid \Sigma y_i = 57) = \frac{\binom{100}{57} \theta^{57} (1-\theta)^{43} \times (1)}{Pr(\Sigma y_i = 57)}$$

for $0 \leq \theta \leq 1$.

Plot attached to document.

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(5) Posterior Distribution $\propto$ Beta(58, 44)

Here, there is no normalizing constant since it Beta(58, 44) is the proportional distribution to the posterior.

See plot and R code attached.

Relationship: The plot for the sampling model in Part (2) the discrete type; indeed, it is a discrete distribution.

In Part (3) our prior was discrete, and our posterior was also discrete and was proportional to the distribution of the sampling model.

In parts (4) and (5) our prior was continuous resulting in a continuous posterior, but remained proportional to the sampling model.

Problem 5: See R Code.

According to Hoff page 6, Chapter 1,

The contour plot suggests that "lower values of n_0 are generally 90% or more certain that" $\theta > 0.5$. — Reference: A First Course in Bayesian Statistical Methods, Hoff, Chapter 1.