EN.553.732 Homework 1

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1 Problem 1

Let $p(y_1, ..., y_6|\theta)$ denote the sampling model and $\pi(\theta)$ denote the prior distribution. It is given that $y_1, ..., y_6|\theta \sim U[\theta - \frac{1}{2}, \theta + \frac{1}{2}]$ and that $\theta \sim U[10, 20]$. We then have that
$p(y_1, ..., y_6|\theta) = \dfrac{1}{(\theta + \frac{1}{2}) - (\theta - \frac{1}{2})} = 1,\ \ y_i \in (\theta - \frac{1}{2}, \theta + \frac{1}{2}),\ i = 1, ..., 6$

and $\pi(\theta) = \frac{1}{10},\ \ \theta \in [10, 20]$

Note that $\theta - \frac{1}{2} < y_i < \theta + \frac{1}{2} \implies |y - \theta| < \frac{1}{2} \implies |\theta - y_i| < \frac{1}{2} \implies y_i - \frac{1}{2} < \theta < y_i + \frac{1}{2},\ i = 1, ..., 6$

Using the available data $y_i \in \{11.0, 11.5, 11.7, 11.1, 11.4, 10.9\}$, we can find updated bounds for θ.

Indeed, since the above inequality holds for all $i = 1, ..., 6$, then $\max\limits_i\{y_i\} - \frac{1}{2} < \theta < \min\limits_i\{y_i\} + \frac{1}{2}$

$\implies 11.7 - \frac{1}{2} < \theta < 10.9 + \frac{1}{2} \implies 11.2 < \theta < 11.4$

The posterior distribution of θ is $p(\theta|y_1, ..., y_6) \propto p(y_1, ..., y_6|\theta)\pi(\theta)$, implying that the posterior distribution is also uniformly distributed. More precisely, by Bayes' rule

$$p(\theta|y_1, ..., y_6) = \frac{p(y_1, ..., y_6|\theta)\pi(\theta)}{p(y_1, ..., y_6)} = \frac{(1/10)}{p(y_1, ..., y_6)},\quad 11.2 < \theta < 11.4$$

Now,
$$1 = \int_{11.2}^{11.4} p(\theta|y_1, ..., y_6)d\theta = \int_{11.2}^{11.4} \frac{(1/10)}{p(y_1, ..., y_6)}d\theta = \frac{1}{p(y_1, ..., y_6)}\int_{11.2}^{11.4} \frac{1}{10}d\theta = \frac{0.2/10}{p(y_1, ..., y_6)}$$

$$\implies \frac{1}{p(y_1, ..., y_6)} = 50$$

Thus,
$$p(\theta|y_1, ..., y_6) = \frac{p(y_1, ..., y_6|\theta)\pi(\theta)}{p(y_1, ..., y_6)} = (\frac{1}{10})(50) = 5$$

Therefore, the posterior distribution of θ is $p(\theta|y_1, ..., y_6) = 5,\ 11.2 < \theta < 11.4$.

2 Problem 2

Let $p(y_1, ..., y_{20}|\theta)$ denote the sampling model and $\pi(\theta)$ denote the prior distribution. It is given that $y_1, ..., y_{20}|\theta \sim \text{Exponential}(\theta)$ and that $\theta \sim \text{Gamma}(\alpha,\ \beta)$ where $\text{E}[\theta] = 0.2$ and $\sigma[\theta] = 1$.

The liklihood function is then $L(\theta) = \theta^{20}e^{-(\sum\limits_{i=1}^{20} y_i)\theta}$ and prior distribution is of the form $\pi(\theta) \propto \theta^{\alpha-1}e^{-\beta\theta},\ \theta > 0$

The posterior of θ is of then of the form

$$p(\theta|y_1, ..., y_{20}) \propto L(\theta)\pi(\theta) \propto \theta^{\alpha+19} e^{-(\sum\limits_{i=1}^{20} y_i+\beta)\theta}, \quad \theta > 0$$

which is proportional to the pdf of a gamma distribution with parameters $\alpha + 20$ and $\beta + \sum\limits_{i=1}^{20} y_i$

To determine the paramaters α and β we note that since $\theta \sim \text{Gamma}(\alpha,\, \beta)$, $\text{E}[\theta] = \dfrac{\alpha}{\beta}$ and $\sigma[\theta] = \sqrt{\dfrac{\alpha}{\beta^2}}$

Then from the given information, we have that $\dfrac{\alpha}{\beta} = 0.2$ and $\sqrt{\dfrac{\alpha}{\beta^2}} = 1 \implies \alpha = 0.2\beta \implies \frac{0.2}{\beta} = 1$ $\implies \beta = 0.2 \implies \alpha = 0.04$.

Moreover, we are given that the average time to serve a customer from a sample of 20 customers is 3.8 minutes. That is, $\bar{y} = \frac{1}{20}\sum\limits_{i=1}^{20} y_i = 3.8 \implies \sum\limits_{i=1}^{20} y_i = 76$.

Thus, we have that the parameters of the posterior distribution are $\alpha + 20 = 20.04$ and $\beta + \sum\limits_{i=1}^{20} y_i = 76.2$
Therefore, $\theta|y_1, ..., y_{20} \sim \text{Gamma}(20.04,\, 76.2)$.

3 Problem 3

Proof. It is given that the sampling model, $f(y_1, ..., y_n|\, p)$, has the negative binomial distribution with unknown parameter p (and known r) and the prior distribution, $\pi(p)$, is the beta distribution with parameters α and β. That is,

$$L(p) = \prod_{i=1}^{n} f(y_i\ |p) = \prod_{i=1}^{n} \binom{y_i + r - 1}{y_i}(1-p)^r p^{y_i} \implies L(p) \propto (1-p)^{nr} p^{\sum_{i=1}^{n} y_i}$$

$$\pi(p) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} p^{\alpha-1}(1-p)^{\beta-1} \implies \pi(p) \propto p^{\alpha-1}(1-p)^{\beta-1}$$

Therefore, the posterior distribution is of the form

$$f(p\ |y_1, ..., y_n) \propto\ L(p)\pi(p) \propto\ p^{\alpha+\sum_{i=1}^{n} y_i - 1}(1-p)^{\beta+nr-1} \longrightarrow \text{Beta}(\alpha + \sum_{i=1}^{n} y_i,\ \beta + nr)$$

Thus, the posterior distribution, again, follows a beta distribution with parameters $\alpha + \sum_{i=1}^{n} y_i$ and $\beta + nr$
Therefore, the family of beta distributions is a conjugate family of prior distributions for samples from a negative binomial distribution with one unknown parameter. $\square$

4 Problem 4

Part 1: For each i $\in [1, 100]$ $\Pr(Y_i = y_i|\ \theta) = \theta^{y_i}(1-\theta)^{1-y_i}$ since each Y_i is a binary random variable. Then, with the assumption of conditional independence of the Y_i on θ, we have

$$\Pr(Y_1 = y_1, ..., Y_{100} = y_{100}|\ \theta) = \prod_{i=1}^{100} \Pr(Y_i = y_i|\ \theta) = \prod_{i=1}^{100} \theta^{y_i}(1-\theta)^{1-y_i} = \theta^{\sum_{i=1}^{100} y_i}(1-\theta)^{100-\sum_{i=1}^{100} y_i}$$

Then $\Pr(\sum_{i=1}^{100} Y_i = y_i|\ \theta)$ is the probability that the sum of the binary random variables is equal to y, where the sum of y from 100 binary random variables can be achieved in $\binom{100}{y}$ distinct ways. Thus,

$$\Pr(\sum_{i=1}^{100} Y_i = y_i |\ \theta) = \binom{100}{y}\theta^y(1-\theta)^{100-y}$$

Part 2: The R code attached computes $\Pr(\sum_{i=1}^{100} Y_i = 57|\ \theta)$ for each θ. Refer to R code for computation.

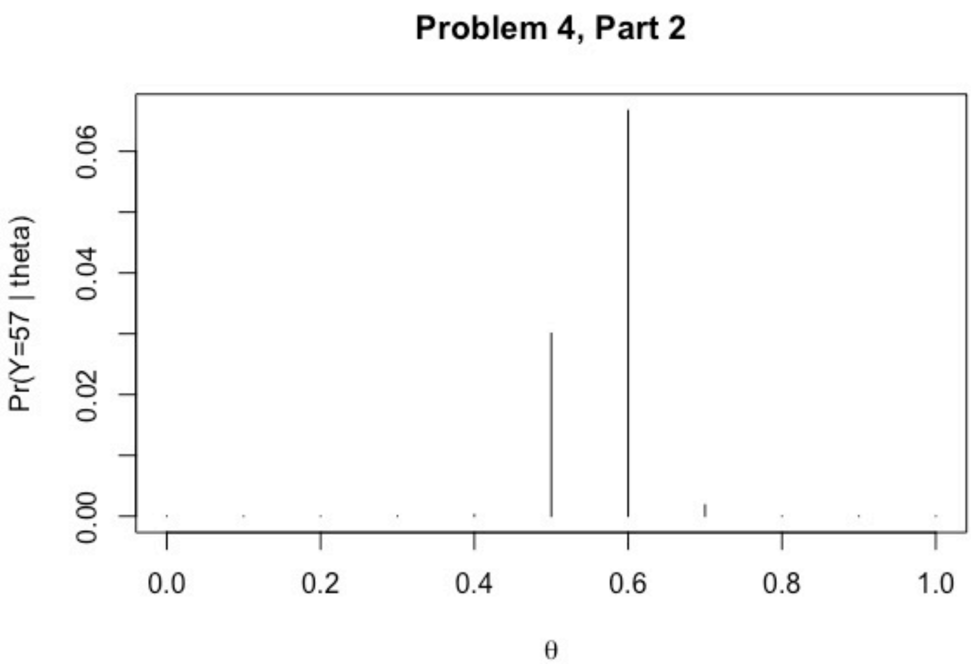
From part 1 we know that $\Pr(\sum_{i=1}^{100} Y_i = 57|\ \theta) = \binom{100}{57}\theta^{57}(1-\theta)^{43}$.
The corresponding outputs for each $\theta \in \{0.0, 0.1, ..., 0.9, 1.0\}$ are below.

θ	0.000	0.1	0.2	0.3	0.4	0.5
$\Pr(\sum_{i=1}^{100} Y_i = 57\|\ \theta)$	0.000	4.107×10^{-31}	3.738×10^{-16}	1.307×10^{-8}	2.286×10^{-4}	3.007×10^{-2}

θ	0.6	0.7	0.8	0.9	1.0
$\Pr(\sum_{i=1}^{100} Y_i = 57\|\ \theta)$	6.673×10^{-2}	1.853×10^{-3}	1.004×10^{-7}	9.396×10^{-18}	0.000

R code and Plot for Part 2

```
> #Part 2
> theta<-c(0.0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0)
> Y<-rep(1,11)
> for(i in 1:11)
+   Y[i]<-choose(100,57)*(theta[i]^57)*(1-theta[i])^43
> print(Y)
 [1] 0.000000e+00 4.107157e-31 3.738459e-16 1.306895e-08 2.285792e-04 3.006864e-02
 [7] 6.672895e-02 1.853172e-03 1.003535e-07 9.395858e-18 0.000000e+00
>
> plot(theta, Y, type = "h", main = "Problem 4, Part 2",
+       xlab = expression(paste(theta)), ylab="Pr(Y=57 | theta)")
```



Part 3: Since $p(\theta = 0.0) = p(\theta = 0.1) = ... = p(\theta = 1.0)$ then since θ discrete, we have

$p(\theta = 0.0) = p(\theta = 0.1) = ... = p(\theta = 1.0) = \dfrac{1}{11}$ Then, by Bayes' rule, we have for each $\theta \in \{0.0, 0.1, ..., 1.0\}$

$$p(\theta \mid \sum_{i=1}^{100} Y_i = 57) = \frac{p(\sum_{i=1}^{100} Y_i = 57 \mid \theta)p(\theta)}{p(\sum_{i=1}^{100} Y_i = 57)} = \frac{\binom{100}{57}\theta^{57}(1-\theta)^{43}(\frac{1}{11})}{p(\sum_{i=1}^{100} Y_i = 57)}$$

where $\dfrac{1}{p(\sum_{i=1}^{100} Y_i = 57)}$ is the normalization constant. Then,

$$\Pr(\sum_{i=1}^{100} Y_i = 57) = \sum_{\theta} \binom{100}{57}\theta^{57}(1-\theta)^{43}(\frac{1}{11})$$

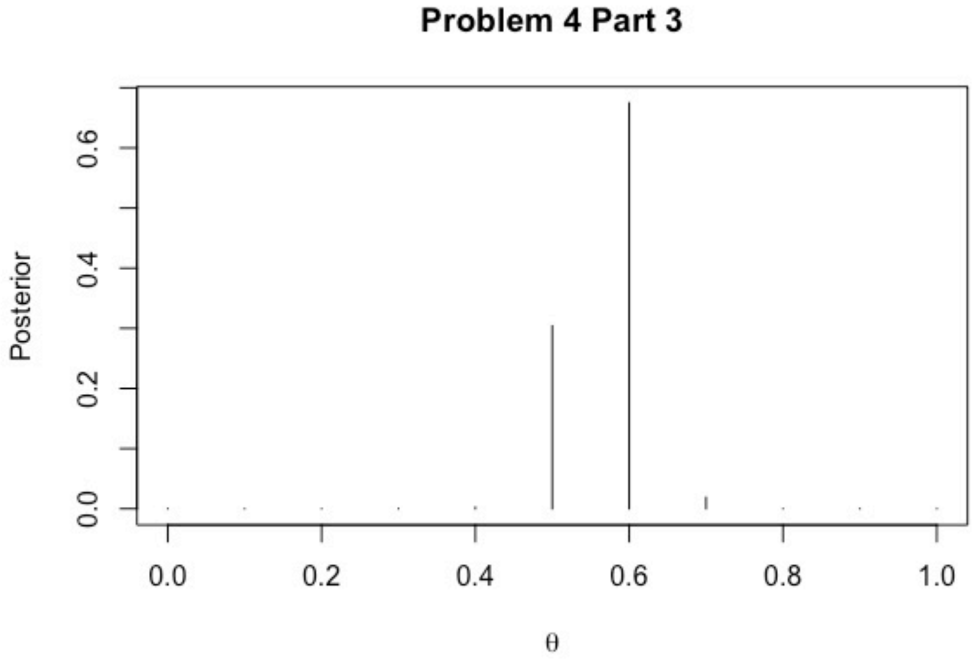
From the R code (below), the values of $p(\theta \mid \sum_{i=1}^{100} Y_i = 57)$ were found to be:

θ	0.000	0.1	0.2	0.3	0.4	0.5
$p(\theta \mid \sum_{i=1}^{100} Y_i = 57)$	0.000	4.154×10^{-30}	3.781×10^{-15}	1.322×10^{-7}	2.312×10^{-3}	3.041×10^{-1}

θ	0.6	0.7	0.8	0.9	1.0
$p(\theta \mid \sum_{i=1}^{100} Y_i = 57)$	6.749×10^{-1}	1.874×10^{-2}	1.015×10^{-6}	9.502×10^{-17}	0.000

R code and Plot for Part 3

```
> #Part 3
> x<-rep(1,11)
> x1<-rep(1,11)
> for(i in 1:11)
+   x[i]<-(choose(100,57)*(theta[i]^57)*(1-theta[i])^43)*(1/11)
> NormConstant<-1/(sum(x))
> x1<-x*NormConstant
> print(x1)
 [1] 0.000000e+00 4.153701e-30 3.780824e-15 1.321705e-07 2.311695e-03 3.040939e-01
 [7] 6.748515e-01 1.874172e-02 1.014907e-06 9.502335e-17 0.000000e+00
>
> plot(theta, x1, type = "h", main = "Problem 4 Part 3",
+       xlab = expression(paste(theta)), ylab = "Posterior")
```



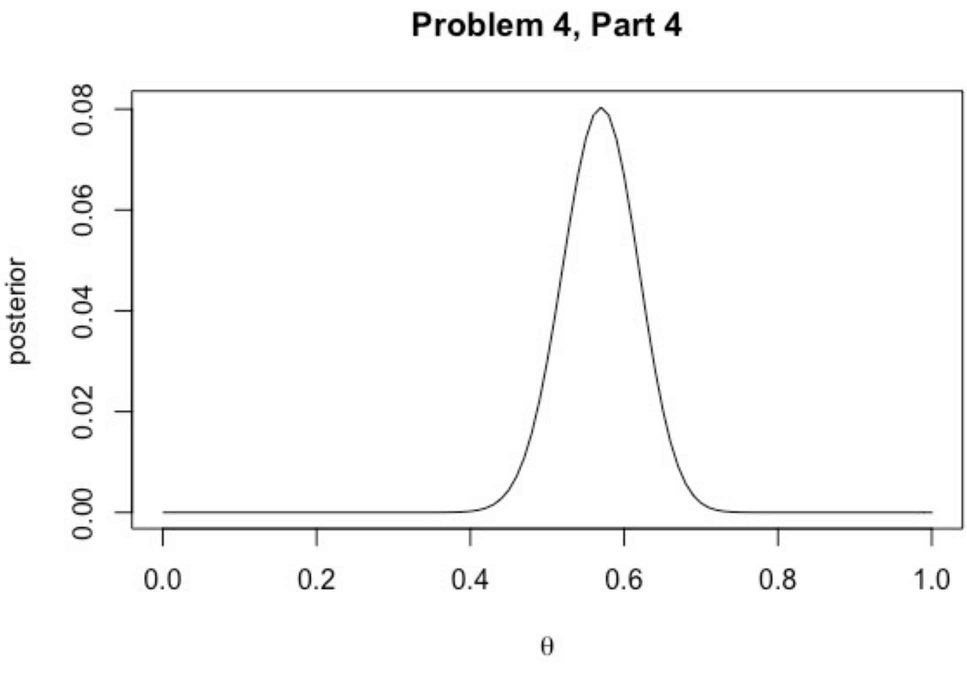
Part 4: Using the uniform density on [0,1] for the prior, the posterior density $p(\theta)\times\Pr(\sum_{i=1}^{100} Y_i = 57 \mid \theta)$ is such that

$$p(\theta \mid \sum_{i=1}^{100} Y_i = 57) = \frac{\binom{100}{57}\theta^{57}(1-\theta)^{43} \times (1)}{\Pr(\sum_{i=1}^{100} Y_i = 57)} \quad , \quad \text{for} \quad 0 \le \theta \le 1$$

This plot for this posterior density is below.

R code and Plot for Part 4

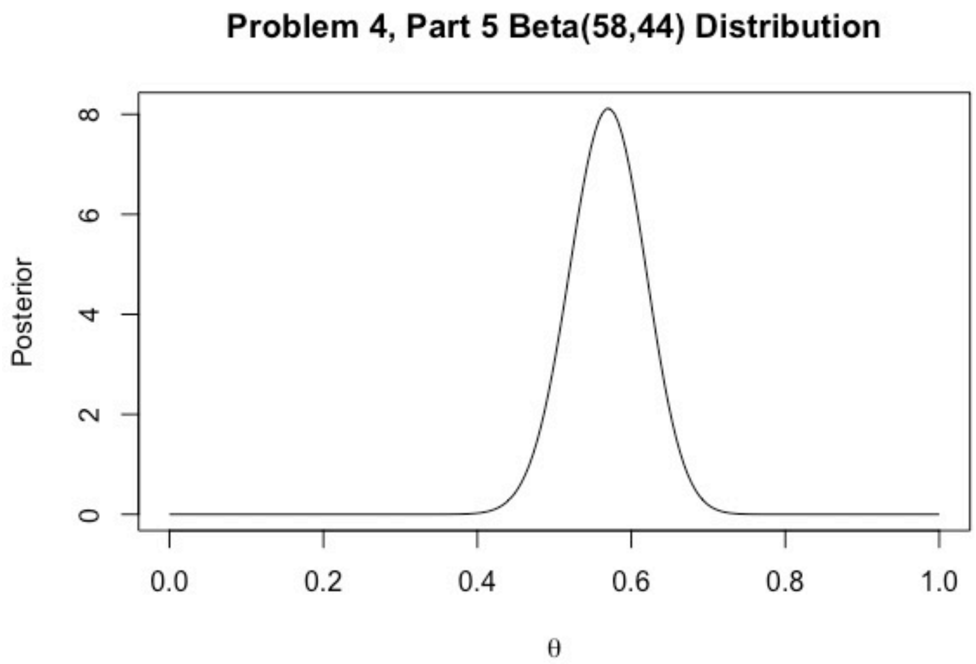
```
> #Part 4
> f<-curve(choose(100,57)*x^57*((1-x)^43), from = 0, to = 1,
+          main = "Problem 4, Part 4", xlab = expression(paste(theta)),
+          ylab = "posterior")
```



Part 5: From the above we see that the posterior distribution $\sim$ Beta(58, 44). However, we are not using the normalizing constant here. Rather, what is being shown is that the posterior maintains the same shape and proportions as the sampling model's joint distribution.
Since $p(\theta \mid \sum_{i=1}^{100} Y_i = 57) \propto \theta^{57}(1-\theta)^{43}$ then $\theta \mid \sum_{i=1}^{100} Y_i = 57 \sim \text{Beta}(58, 44)$
Below is the plot of the posterior.

R code and Plot for Part 5

```
> #Part 5
> v<-seq(0, 1, length = 200)
> z<-dbeta(v, 58, 44)
> plot(v, z, type = "l", main = "Problem 4, Part 5 Beta(58,44) Distribution",
+       xlab = expression(paste(theta)), ylab = "Posterior")
```



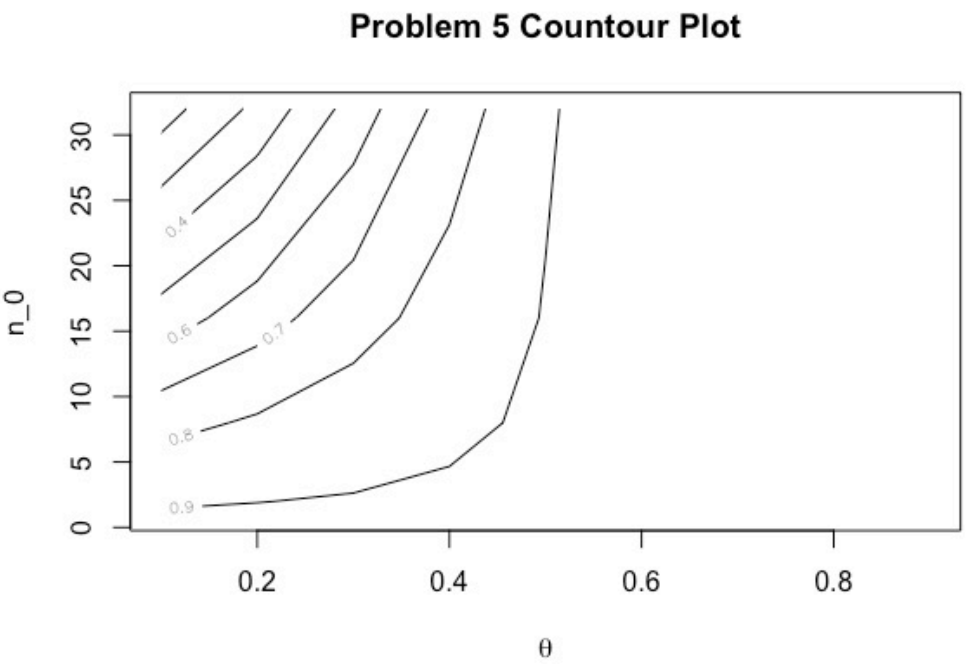
Relationship Between Plots: While the plots in parts 2 and 3 are discrete and continuous in parts 3 and 4, each set of plots is unimodal with a mode around 0.6 and has a similar shape in distribution. However, this shouldn't be too surprising since each set of plots is based on the same data/results. In parts 4 and 5 we notice that the distributions have the same shape, but the distribution in part 4 is normalized and includes the binomial coefficient. However, what is important to recognize is that the shape of the posterior distribution can be derived from the liklihood $\times$ prior, that is, without any constants or functions that are not dependent on θ. In other words, the statement poster $\propto$ liklihood $\times$ prior (or sampling model $\times$ prior) is meaningful. We are able extract sufficient information about the posterior from the sampling model and the prior.

5 Problem 5

R Code and Plot

```
> #Problem 5
> theta_0<-c(0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9)
> n_0<-c(1,2,8,16,32)
>
> a<-matrix(0L, nrow =length(theta_0), ncol =length(n_0))
> b<-matrix(0L, nrow =length(theta_0), ncol =length(n_0))
> for (i in 1:length(theta_0))
+ {for (j in 1:length(n_0))
```

```
+ {a[i,j]=theta_0[i]*n_0[j]
+ b[i,j]=(1-theta_0[i])*n_0[j]
+ }
+ }
> Pr<-matrix(0L, nrow =length(theta_0), ncol =length(n_0))
> for (i in 1:length(theta_0))
+ {for (j in 1:length(n_0))
+ {
+   f <- function(x)
+   {choose(100,57)*(x^57)*((1- x)^43)*(gamma(a[i,j]+b[i,j])/
+                    (gamma(a[i,j])*gamma(b[i,j])))*(x^(a[ i,j]-1))*(1-x)^(b[i,j]-1)}
+   bot<-integrate(f,0, 1, rel.tol=1e-10)$value
+   top<-integrate(f,0.5, 1, rel.tol=1e-10)$value
+   Pr[i,j]<-top/bot
+
+ }
+ }
> contour(theta_0, n_0, Pr,main = "Problem 5 Countour Plot",
+         xlab=expression(paste(theta)), ylab='n_0')
```



The contour plot suggests that "lower values of n_0 are generally 90% or more certain that" $\theta > 0.5$ (as cited in Hoff, pgs. 6-7). In other words, according to the countour plot, we can be at least 90% certain that $\theta > 0.5$, which is quite significant. It is reasonable to believe, then, that $\theta > 0.5$.

Reference for Response/Answer to Problem 5:

Hoff, Peter D. (2009). *A First Course in Bayesian Statistical Methods.* New York, NY: Springer.