Chapter 2

Linear Programming: Theory and Algorithms

2.1 The Linear Programming Problem

One of the most common and fundamental optimization problems is the linear optimization, or *linear programming* (LP) problem. LP is the problem of optimizing a linear objective function subject to linear equality and inequality constraints. A generic linear optimization problem has the following form:

$$\begin{array}{lrcl} \min_x & c^T x & & \\ & a_i^T x & = & b_i, \ i \in \mathcal{E} \\ & a_i^T x & \geq & b_i, \ i \in \mathcal{I}, \end{array} \tag{2.1}$$

where $\mathcal{E}$ and $\mathcal{I}$ are the index sets for equality and inequality constraints, respectively. Linear programming is arguably the best known and the most frequently solved optimization problem. It owes its fame mostly to its great success; real world problems coming from as diverse disciplines as sociology, finance, transportation, economics, production planning, and airline crew scheduling have been formulated and successfully solved as LPs.

For algorithmic purposes, it is often desirable to have the problems structured in a particular way. Since the development of the simplex method for LPs the following form has been a popular standard and is called the *standard form LP*:

$$\begin{array}{lrcl} \min_x & c^T x & & \\ & Ax & = & b \\ & x & \geq & 0. \end{array} \tag{2.2}$$

Here $A \in I\!R^{m \times n}$, $b \in I\!R^m$, $c \in I\!R^n$ are given, and $x \in I\!R^n$ is the variable vector to be determined as the solution of the problem.

The standard form is not restrictive: Inequalities other than nonnegativity constraints can be rewritten as equalities after the introduction of a so-called *slack* or *surplus* variable that is restricted to be nonnegative. For example,

$$\begin{array}{lrcrcl} \min & -x_1 & - & x_2 & & \\ & 2x_1 & + & x_2 & \leq & 12 \\ & x_1 & + & 2x_2 & \leq & 9 \\ & \multicolumn{5}{l}{x_1 \geq 0,\ x_2 \geq 0} \end{array} \tag{2.3}$$

can be rewritten as

$$\begin{array}{lrcrcrcrcl} \min & -x_1 & - & x_2 & & & & & & \\ & 2x_1 & + & x_2 & + & x_3 & & & = & 12 \\ & x_1 & + & 2x_2 & & & + & x_4 & = & 9 \\ & \multicolumn{9}{l}{x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ x_4 \geq 0.} \end{array} \tag{2.4}$$

Variables that are unrestricted in sign can be expressed as the difference of two new nonnegative variables. Maximization problems can be written as minimization problems by multiplying the objective function by a negative constant. Simple transformations are available to rewrite any given LP in the standard form above. Therefore, in the rest of our theoretical and algorithmic discussion we assume that the LP is in the standard form.

Exercise 2.1 Write the following linear program in standard form.

$$\begin{array}{lrcrcl} \min & & & x_2 & & \\ & x_1 & + & x_2 & \geq & 1 \\ & x_1 & - & x_2 & \leq & 0 \\ & \multicolumn{5}{l}{x_1, x_2 \text{ unrestricted in sign.}} \end{array}$$

Answer:

After writing $x_i = y_i - z_i$, $i = 1, 2$ with $y_i \geq 0$ and $z_i \geq 0$ and introducing surplus variable s_1 for the first constraint and slack variable s_2 for the second constraint we obtain:

$$\begin{array}{lrcrcrcrcrcrcl} \min & & & & & y_2 & - & z_2 & & & & & & \\ & y_1 & - & z_1 & + & y_2 & - & z_2 & - & s_1 & & & = & 1 \\ & y_1 & - & z_1 & - & y_2 & + & z_2 & & & + & s_2 & = & 0 \\ & \multicolumn{13}{l}{y_1 \geq 0,\ z_1 \geq 0,\ y_2 \geq 0,\ z_2 \geq 0,\ s_1 \geq 0,\ s_2 \geq 0.} \end{array}$$

Exercise 2.2 Write the following linear program in standard form.

$$\begin{array}{lrcrcrcl} \max & 4x_1 & + & x_2 & - & x_3 & & \\ & x_1 & & & + & 3x_3 & \leq & 6 \\ & 3x_1 & + & x_2 & + & 3x_3 & \geq & 9 \\ & \multicolumn{7}{l}{x_1 \geq 0,\ x_2 \geq 0,\ x_3 \text{ unrestricted in sign.}} \end{array}$$

Recall the following definitions from the Chapter 1: The LP (2.2) is said to be *feasible* if its constraints are consistent and it is called *unbounded* if there exists a sequence of feasible vectors $\{x^k\}$ such that $c^T x^k \to -\infty$. When we talk about a *solution* (without any qualifiers) to (2.2) we mean any candidate vector $x \in I\!R^n$. A *feasible solution* is one that satisfies the constraints, and an *optimal solution* is a vector x that satisfies the constraints and minimizes the objective value among all feasible vectors. When $\mathcal{LP}$ is feasible but not unbounded it has an optimal solution.

Exercise 2.3
(a) Write a 2-variable linear program that is unbounded.
(b) Write a 2-variable linear program that is infeasible.

Exercise 2.4 Draw the feasible region of the following 2-variable linear program.

$$\begin{array}{rrcrcl} \max & 2x_1 & - & x_2 & & \\ & x_1 & + & x_2 & \geq & 1 \\ & x_1 & - & x_2 & \leq & 0 \\ & 3x_1 & + & x_2 & \leq & 6 \\ & \multicolumn{5}{l}{x_1 \geq 0,\ x_2 \geq 0.} \end{array}$$

Determine the optimal solution to this problem by inspection.

The most important questions we will address in this chapter are the following: How do we recognize an optimal solution and how do we find such solutions? One of the most important tools in optimization to answer these questions is the notion of a dual problem associated with the LP problem (2.2). We describe the dual problem in the next subsection.

2.2 Duality

Consider the standard form LP in (2.4) above. Here are a few alternative feasible solutions:

$$\begin{array}{rll} (x_1, x_2, x_3, x_4) = (0, \frac{9}{2}, \frac{15}{2}, 0) & \text{Objective value} & = -\frac{9}{2} \\ (x_1, x_2, x_3, x_4) = (6, 0, 0, 3) & \text{Objective value} & = -6 \\ (x_1, x_2, x_3, x_4) = (5, 2, 0, 0) & \text{Objective value} & = -7 \end{array}$$

Since we are minimizing, the last solution is the best among the three feasible solutions we found, but is it the optimal solution? We can make such a claim if we can, somehow, show that there is no feasible solution with a smaller objective value.

Note that the constraints provide some bounds on the value of the objective function. For example, for any feasible solution, we must have

$$-x_1 - x_2 \geq -2x_1 - x_2 - x_3 = -12$$

using the first constraint of the problem. The inequality above must hold for all feasible solutions since x_i's are all nonnegative and the coefficient of each variable on the LHS are at least as large as the coefficient of the corresponding variable on the RHS. We can do better using the second constraint:

$$-x_1 - x_2 \geq -x_1 - 2x_2 - x_4 = -9$$

and even better by adding a negative third of each constraint:

$$\begin{array}{rcl} -x_1 - x_2 & \geq & -x_1 - x_2 - \frac{1}{3}x_3 - \frac{1}{3}x_4 \\ & = & -\frac{1}{3}(2x_1 + x_2 + x_3) - \frac{1}{3}(x_1 + 2x_2 + x_4) = -\frac{1}{3}(12 + 9) = -7. \end{array}$$

This last inequality indicates that for any feasible solution, the objective function value cannot be smaller than -7. Since we already found a feasible solution achieving this bound, we conclude that this solution, namely $(x_1, x_2, x_3, x_4) = (5, 2, 0, 0)$ must be an optimal solution of the problem.

This process illustrates the following strategy: If we find a feasible solution to the LP problem, and a bound on the optimal value of problem such that the bound and the objective value of the feasible solution coincide, then we can conclude that our feasible solution is an optimal solution. We will comment on this strategy shortly. Before that, though, we formalize our approach for finding a bound on the optimal objective value.

Our strategy was to find a linear combination of the constraints, say with multipliers y_1 and y_2 for the first and second constraint respectively, such that the combined coefficient of **each** variable forms a lower bound on the objective coefficient of that variable. Namely, we tried to choose multipliers y_1 and y_2 associated with constraints 1 and 2 such that

$$y_1(2x_1+x_2+x_3)+y_2(x_1+2x_2+x_4) = (2y_1+y_2)x_1+(y_1+2y_2)x_2+y_1x_3+y_2x_4$$

provides a lower bound on the optimal objective value. Since x_i's must be nonnegative, the expression above would necessarily give a lower bound if the coefficient of each x_i is less than or equal to the corresponding objective function coefficient, or if:

$$\begin{array}{rcrcl} 2y_1 & + & y_2 & \leq & -1 \\ y_1 & + & 2y_2 & \leq & -1 \\ y_1 & & & \leq & 0 \\ & & y_2 & \leq & 0. \end{array}$$

Note that the objective coefficients of x_3 and x_4 are zero. Naturally, to obtain the largest possible lower bound, we would like to find y_1 and y_2 that achieve the maximum combination of the right-hand-side values:

$$\max 12y_1 + 9y_2.$$

This process results in a linear programming problem that is strongly related to the LP we are solving. We want to

$$\begin{array}{lrcrcl} \max & 12y_1 & + & 9y_2 & & \\ & 2y_1 & + & y_2 & \leq & -1 \\ & y_1 & + & 2y_2 & \leq & -1 \\ & y_1 & & & \leq & 0 \\ & & & y_2 & \leq & 0. \end{array} \tag{2.5}$$

This problem is called the *dual* of the original problem we considered. The original LP in (2.2) is often called the *primal* problem. For a generic primal LP problem in standard form (2.2) the corresponding dual problem can be written as follows:

$$(\mathcal{LD}) \qquad \begin{array}{lrcl} \max_y & b^T y & & \\ & A^T y & \leq & c, \end{array} \tag{2.6}$$

where $y \in I\!R^m$. Rewriting this problem with explicit *dual slacks*, we obtain the standard form dual linear programming problem:

$$
\begin{array}{llrcrcl}
(\mathcal{LD}) & \max_{y,s} & b^T y & & & & \\
 & & A^T y & + & s & = & c \\
 & & & & s & \geq & 0,
\end{array}
\tag{2.7}
$$

where $s \in I\!R^n$.

Exercise 2.5 Consider the following LP:

$$
\begin{array}{lrcrcl}
\min & 2x_1 & + & 3x_2 & & \\
 & x_1 & + & x_2 & \geq & 5 \\
 & x_1 & & & \geq & 1 \\
 & & & x_2 & \geq & 2.
\end{array}
$$

Prove that $x^* = (3, 2)$ is the optimal solution by showing that the objective value of any feasible solution is at least 12.

Next, we make some observations about the relationship between solutions of the primal and dual LPs. The objective value of any primal feasible solution is at least as large as the objective value of any feasible dual solution. This fact is known as the *weak duality theorem*:

Theorem 2.1 (Weak Duality Theorem) *Let x be any feasible solution to the primal LP (2.2) and y be any feasible solution to the dual LP (2.6). Then*

$$c^T x \geq b^T y.$$

Proof:
Since $x \geq 0$ and $c - A^T y \geq 0$, the inner product of these two vectors must be nonnegative:

$$(c - A^T y)^T x = c^T x - y^T A x = c^T x - y^T b \geq 0.$$

$\square$

The quantity $c^T x - y^T b$ is often called the *duality gap*. The following three results are immediate consequences of the weak duality theorem.

Corollary 2.1 *If the primal LP is unbounded, then the dual LP must be infeasible.*

Corollary 2.2 *If the dual LP is unbounded, then the primal LP must be infeasible.*

Corollary 2.3 *If x is feasible for the primal LP, y is feasible for the dual LP, and $c^T x = b^T y$, then x must be optimal for the primal LP and y must be optimal for the dual LP.*

Exercise 2.6 Show that the dual of the linear program

$$\begin{array}{rrcl} \min_x & c^T x & & \\ & Ax & \geq & b \\ & x & \geq & 0 \end{array}$$

is the linear program

$$\begin{array}{rrcl} \max_y & b^T y & & \\ & A^T y & \leq & c \\ & y & \geq & 0. \end{array}$$

Exercise 2.7 We say that two linear programming problems are equivalent if one can be obtained from the other by (i) multiplying the objective function by -1 and changing it from min to max, or max to min, and/or (ii) multiplying some or all constraints by -1. For example, $\min\{c^T x : \ Ax \geq b\}$ and $\max\{-c^T x : \ -Ax \leq -b\}$ are equivalent problems. Find a linear program which is equivalent to its own dual.

Exercise 2.8 Give an example of a linear program such that it and its dual are both infeasible.

Exercise 2.9 For the following pair of primal-dual problems, determine whether the listed solutions are optimal.

$$\begin{array}{rrcrcl} \min & 2x_1 & + & 3x_2 & & \\ & 2x_1 & + & 3x_2 & \leq & 30 \\ & x_1 & + & 2x_2 & \geq & 10 \\ & x_1 & - & x_2 & \geq & 0 \\ & x_1, & & x_2 & \geq & 0 \end{array} \qquad \begin{array}{rrcrcrcl} \max & -30y_1 & + & 10y_2 & & & & \\ & -2y_1 & + & y_2 & + & y_3 & \leq & 2 \\ & -3y_1 & + & 2y_2 & - & y_3 & \leq & 3 \\ & y_1, & & y_2, & & y_3 & \geq & 0. \end{array}$$

(a) $x_1 = 10$, $x_2 = \frac{10}{3}$; $y_1 = 0$, $y_2 = 1$, $y_3 = 1$.
(b) $x_1 = 20$, $x_2 = 10$; $y_1 = -1$, $y_2 = 4$, $y_3 = 0$.
(c) $x_1 = \frac{10}{3}$, $x_2 = \frac{10}{3}$; $y_1 = 0$, $y_2 = \frac{5}{3}$, $y_3 = \frac{1}{3}$.

2.3 Optimality Conditions

Corollary 2.3 in the previous section identified a sufficient condition for optimality of a primal-dual pair of feasible solutions, namely that their objective values coincide. One natural question to ask is whether this is a necessary condition. The answer is yes, as we illustrate next.

Theorem 2.2 (Strong Duality Theorem) *If the primal (dual) problem has an optimal solution x (y), then the dual (primal) has an optimal solution y (x) such that $c^T x = b^T y$.*

The reader can find a proof of this result in most standard linear programming textbooks (see Chvátal [19] for example). A consequence of the Strong Duality Theorem is that, if both the primal LP problem and the dual LP have feasible solutions then they both have optimal solutions and for any primal optimal solution x and dual optimal solution y we have that $c^T x = b^T y$.

The strong duality theorem provides us with conditions to identify optimal solutions (called *optimality conditions*): $x \in I\!R^n$ is an optimal solution of (2.2) if and only if

1. x is primal feasible: $Ax = b$, $x \geq 0$, and there exists a $y \in I\!R^m$ such that

2. y is dual feasible: $A^T y \leq c$, and

3. there is no duality gap: $c^T x = b^T y$.

Further analyzing the last condition above, we can obtain an alternative set of optimality conditions. Recall from the proof of the weak duality theorem that $c^T x - b^T y = (c - A^T y)^T x \geq 0$ for any feasible primal-dual pair of solutions, since it is given as an inner product of two nonnegative vectors. This inner product is 0 ($c^T x = b^T y$) if and only if the following statement holds: For each $i = 1, \ldots, n$, either x_i or $(c - A^T y)_i = s_i$ is zero. This equivalence is easy to see. All the terms in the summation on the RHS of the following equation are nonnegative:

$$0 = (c - A^T y)^T x = \sum_{i=1}^{n} (c - A^T y)_i x_i$$

Since the sum is zero, each term must be zero. Thus we found an alternative set of optimality conditions: $x \in I\!R^n$ is an optimal solution of (2.2) if and only if

1. x is primal feasible: $Ax = b$, $x \geq 0$, and there exists a $y \in I\!R^m$ such that

2. y is dual feasible: $s := c - A^T y \geq 0$, and

3. complementary slackness: for each $i = 1, \ldots, n$ we have $x_i s_i = 0$.

Exercise 2.10 Consider the linear program

$$\begin{array}{rrcrcrcl}
\min & 5x_1 & + & 12x_2 & + & 4x_3 & & \\
 & x_1 & + & 2x_2 & + & x_3 & = & 10 \\
 & 2x_1 & - & x_2 & + & 3x_3 & = & 8 \\
 & \multicolumn{7}{l}{x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0.}
\end{array}$$

You are given the information that x_2 and x_3 are positive in the optimal solution. Use the complementary slackness conditions to find the optimal dual solution.

Exercise 2.11 Consider the following linear programming problem:

$$\begin{array}{rrcrcrcrcrcr}
\max & 6x_1 & + & 5x_2 & + & 4x_3 & + & 5x_4 & + & 6x_5 & & \\
 & x_1 & + & x_2 & + & x_3 & + & x_4 & + & x_5 & \leq & 3 \\
 & 5x_1 & + & 4x_2 & + & 3x_3 & + & 2x_4 & + & x_5 & \leq & 14
\end{array}$$

$$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ x_4 \geq 0,\ x_5 \geq 0$$

Solve this problem using the following strategy:

a) Find the dual of the above LP. The dual has only two variables. Solve the dual by inspection after drawing a graph of the feasible set.

b) Now using the optimal solution to the dual problem, and complementary slackness conditions, determine which primal constraints are active, and which primal variables must be zero at an optimal solution. Using this information determine the optimal solution to the primal problem.

Exercise 2.12 Using the optimality conditions for

$$\begin{array}{rrcl} \min_x & c^T x & & \\ & Ax & = & b \\ & x & \geq & 0, \end{array}$$

deduce that the optimality conditions for

$$\begin{array}{rrcl} \max_x & c^T x & & \\ & Ax & \leq & b \\ & x & \geq & 0 \end{array}$$

are $Ax \leq b$, $x \geq 0$ and there exists y such that $A^T y \geq c$, $y \geq 0$, $c^T x = b^T y$.

Exercise 2.13 Consider the following investment problem over T years, where the objective is to maximize the value of the investments in year T. We assume a perfect capital market with the same annual lending and borrowing rate $r > 0$ each year. We also assume that exogenous investment funds b_t are available in year t, for $t = 1, \ldots, T$. Let n be the number of possible investments. We assume that each investment can be undertaken fractionally (between 0 and 1). Let a_{tj} denote the cash flow associated with investment j in year t. Let c_j be the value of investment j in year T (including all cash flows subsequent to year T discounted at the interest rate r).

The linear program that maximizes the value of the investments in year T is the following. Denote by x_j the fraction of investment j undertaken, and let y_t be the amount borrowed (if negative) or lent (if positive) in year t.

$$\begin{array}{rrcll} \max & \sum_{j=1}^n c_j x_j + y_T & & & \\ & -\sum_{j=1}^n a_{1j} x_j + y_1 & \leq & b_1 & \\ -\sum_{j=1}^n a_{tj} x_j - (1+r) y_{t-1} + y_t & & \leq & b_t & \text{for } t = 2, \ldots, T \\ & 0 \leq x_j & \leq & 1 & \text{for } j = 1, \ldots, n. \end{array}$$

(i) Write the dual of the above linear program.

(ii) Solve the dual linear program found in (i). [Hint: Note that some of the dual variables can be computed by backward substitution.]

(iii) Write the complementary slackness conditions.

(iv) Deduce that the first T constraints in the primal linear program hold as equalities.

(v) Use the complementary slackness conditions to show that the solution obtained by setting $x_j = 1$ if $c_j + \sum_{t=1}^T (1+r)^{T-t} a_{tj} > 0$, and $x_j = 0$ otherwise, is an optimal solution.

(vi) Conclude that the above investment problem always has an optimal solution where each investment is either undertaken completely or not at all.

2.4 The Simplex Method

The best known and most successful methods for solving LPs are *interior-point methods* (IPMs) and the *simplex method*. We discuss the simplex method here and postpone our discussion IPMs till we study quadratic programming problems, as IPMs are also applicable to quadratic programs and other more general classes of optimization problems.

We introduce the essential elements of the simplex method using a simple bond portfolio selection problem.

Example 2.1 *A bond portfolio manager has \$100,000 to allocate to two different bonds; one corporate and one government bond. The corporate bond has a yield of 4%, a maturity of 3 years and an* `A` *rating from a rating agency that is translated into a numerical rating of 2 for computational purposes. In contrast, the government bond has a yield of 3%, a maturity of 4 years and rating of* `Aaa` *with the corresponding numerical rating of 1 (lower numerical ratings correspond to higher quality bonds). The portfolio manager would like to allocate her funds so that the average rating for the portfolio is no worse than* `Aa` *(numerical equivalent 1.5) and average maturity of the portfolio is at most 3.6 years. Any amount not invested in the two bonds will be kept in a cash account that is assumed to earn no interest for simplicity and does not contribute to the average rating or maturity computations[1]. How should the manager allocate her funds between these two bonds to achieve her objective of maximizing the yield from this investment?*

Letting variables x_1 and x_2 denote the allocation of funds to the corporate and government bond respectively (in thousands of dollars) we obtain the following formulation for the portfolio manager's problem:

$$\begin{array}{lrcl} \max & Z = & 4x_1 + 3x_2 & \\ \textit{subject to:} & & & \\ & x_1 + x_2 & \leq & 100 \\ & \frac{2x_1+x_2}{100} & \leq & 1.5 \\ & \frac{3x_1+4x_2}{100} & \leq & 3.6 \\ & x_1, x_2 & \geq & 0. \end{array}$$

We first multiply the second and third inequalities by 100 to avoid fractions. After we add slack variables to each of the functional inequality constraints we obtain a representation of the problem in the standard form, suitable for the simplex method[2]. For example, letting x_3 denote the amount we keep

[1] In other words, we are assuming a quality rating of 0–"perfect" quality, and maturity of 0 years for cash.

[2] This representation is not exactly in the standard form since the objective is maximization rather than minimization. However, any maximization problem can be transformed into a minimization problem by multiplying the objective function by -1. Here, we avoid

as cash, we can rewrite the first constraint as $x_1 + x_2 + x_3 = 100$ *with the additional condition of* x_3*. Continuing with this strategy we obtain the following formulation:*

$$\begin{array}{llcccccccccc} \max & Z = & 4x_1 & + & 3x_2 & & & & & & & \\ \textit{subject to:} & & & & & & & & & & & \\ & & x_1 & + & x_2 & + & x_3 & & & & = & 100 \\ & & 2x_1 & + & x_2 & & & + & x_4 & & = & 150 \\ & & 3x_1 & + & 4x_2 & & & & & + \; x_5 & = & 360 \\ & & x_1 & , & x_2 & , & x_3 & , & x_4 & , \; x_5 & \geq & 0. \end{array} \tag{2.8}$$

2.4.1 Basic Solutions

Let us consider a general LP problem in the following form:

$$\max \; \mathbf{c}\,\mathbf{x} \tag{2.9}$$
$$\mathbf{A}\mathbf{x} \leq \mathbf{b} \tag{2.10}$$
$$\mathbf{x} \geq \mathbf{0}, \tag{2.11}$$

where $\mathbf{A}$ is an $m \times n$ matrix, $\mathbf{b}$ is an m-dimensional column vector and $\mathbf{c}$ is an n-dimensional row vector. The n-dimensional column vector $\mathbf{x}$ represents the variables of the problem. (In the bond portfolio example we have $m = 3$ and $n = 2$.) Here is how we can represent these vectors and matrices:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} c_1 & c_2 & \dots & c_n \end{bmatrix},$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Next, we add slack variables to each of the functional constraints to get the augmented form of the problem. Let $\mathbf{x}_s$ denote the vector of slack variables

$$\mathbf{x}_s = \begin{bmatrix} x_{n+1} \\ x_{n+2} \\ \vdots \\ x_{n+m} \end{bmatrix}$$

and let $\mathbf{I}$ denote the $m \times m$ identity matrix. Now, the constraints in the augmented form can be written as

$$\begin{bmatrix} \mathbf{A}, & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{x_s} \end{bmatrix} = \mathbf{b}, \quad \begin{bmatrix} \mathbf{x} \\ \mathbf{x_s} \end{bmatrix} \geq \mathbf{0}. \tag{2.12}$$

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such a transformation to leave the objective function in its natural form–it should be straightforward to adapt the steps of the algorithm in the following discussion to address minimization problems.

There are many potential solutions to system (2.12). Let us focus on the equation $\begin{bmatrix} \mathbf{A}, & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{x_s} \end{bmatrix} = \mathbf{b}$. By choosing $\mathbf{x} = \mathbf{0}$ and $\mathbf{x_s} = \mathbf{b}$, we immediately satisfy this equation–but not necessarily all the inequalities. More generally, we can consider partitions of the augmented matrix $[A, I]$[3]:

$$\begin{bmatrix} \mathbf{A}, & \mathbf{I} \end{bmatrix} \equiv \begin{bmatrix} \mathbf{B}, & \mathbf{N} \end{bmatrix},$$

where $\mathbf{B}$ is an $m \times m$ square matrix that consists of linearly independent columns of $[\mathbf{A}, \mathbf{I}]$. If we partition the variable vector $\begin{bmatrix} \mathbf{x} \\ \mathbf{x_s} \end{bmatrix}$ in the same way

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{x_s} \end{bmatrix} \equiv \begin{bmatrix} \mathbf{x_B} \\ \mathbf{x_N} \end{bmatrix},$$

we can rewrite the equality constraints in (2.12) as

$$\begin{bmatrix} \mathbf{B}, \mathbf{N} \end{bmatrix} \begin{bmatrix} \mathbf{x_B} \\ \mathbf{x_N} \end{bmatrix} = \mathbf{B}\mathbf{x_B} + \mathbf{N}\mathbf{x_N} = \mathbf{b},$$

or by multiplying both sides by B^{-1} from left,

$$\mathbf{x_B} + \mathbf{B^{-1}}\mathbf{N}\mathbf{x_N} = \mathbf{B^{-1}}\mathbf{b}.$$

By our construction, the following three systems of equations are equivalent in the sense that any solution to one is a solution for the other two:

$$\begin{gathered} \begin{bmatrix} \mathbf{A}, & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{x_s} \end{bmatrix} = \mathbf{b}, \\ \mathbf{B}\mathbf{x_B} + \mathbf{N}\mathbf{x_N} = \mathbf{b} \\ \mathbf{x_B} + \mathbf{B^{-1}}\mathbf{N}\mathbf{x_N} = \mathbf{B^{-1}}\mathbf{b} \end{gathered}$$

Indeed, the second and third linear systems are just other representations of the first one in terms of the matrix $\mathbf{B}$. As we observed above, an obvious solution to the last system (and therefore, to the other two) is $\mathbf{x_N} = \mathbf{0}$, $\mathbf{x_B} = \mathbf{B^{-1}}\mathbf{b}$. In fact, for any fixed values of the components of $\mathbf{x_N}$ we can obtain a solution by simply setting

$$\mathbf{x_B} = \mathbf{B^{-1}}\mathbf{b} - \mathbf{B^{-1}}\mathbf{N}\mathbf{x_N}. \tag{2.13}$$

One can think of $\mathbf{x_N}$ as the *independent* variables that we can choose freely, and once they are chosen, the *dependent* variables $\mathbf{x_B}$ are determined uniquely. We call a solution of the systems above a *basic solution* if it is of the form

$$\mathbf{x_N} = \mathbf{0},\ \mathbf{x_B} = \mathbf{B^{-1}}\mathbf{b},$$

[3]Here, we are using the notation $U \equiv V$ to indicate that the matrix V is obtained from the matrix U by permuting its columns. Similarly, for column vectors u and v, $u \equiv v$ means that v is obtained from u by permuting its elements.

for some *basis matrix* $\mathbf{B}$. If in addition, $\mathbf{x_B} = \mathbf{B^{-1}b} \geq \mathbf{0}$, the solution $\mathbf{x_B} = \mathbf{B^{-1}b}$, $\mathbf{x_N} = \mathbf{0}$ is a *basic feasible solution* of the LP problem above. The variables $\mathbf{x_B}$ are called the *basic variables*, while $\mathbf{x_N}$ are the *nonbasic variables*. Geometrically, basic feasible solutions correspond to extreme points of the feasible set $\{x : Ax \leq b, x \geq 0\}$. Extreme points of a set are those that cannot be written as a convex combination of two other points in the set.

The objective function $\mathbf{Z} = \mathbf{c}\ \mathbf{x}$ can be represented similarly using the basis partition. Let $\mathbf{c} = \left[\begin{array}{cc} \mathbf{c_B}, & \mathbf{c_N} \end{array}\right]$ represent the partition of the objective vector. Now, we have the following sequence of equivalent representations of the objective function equation:

$$\mathbf{Z} = \mathbf{c}\ \mathbf{x} \Leftrightarrow \mathbf{Z} - \mathbf{c}\ \mathbf{x} = \mathbf{0}$$

$$\mathbf{Z} - \left[\begin{array}{cc} \mathbf{c_B}, & \mathbf{c_N} \end{array}\right] \left[\begin{array}{c} \mathbf{x_B} \\ \mathbf{x_N} \end{array}\right] = \mathbf{0}$$

$$\mathbf{Z} - \mathbf{c_B}\ \mathbf{x_B} - \mathbf{c_N}\ \mathbf{x_N} = \mathbf{0}$$

Now substituting $\mathbf{x_B} = \mathbf{B^{-1}b} - \mathbf{B^{-1}Nx_N}$ from (2.13) we obtain

$$\mathbf{Z} - \mathbf{c_B}\ (\mathbf{B^{-1}b} - \mathbf{B^{-1}Nx_N}) - \mathbf{c_N}\ \mathbf{x_N} = \mathbf{0}\mathbf{Z} - (\mathbf{c_N} - \mathbf{c_B B^{-1}N})\ \mathbf{x_N} = \mathbf{c_B B^{-1}b}$$

Note that the last equation does not contain the basic variables. This representation allows us to determine the *net* effect on the objective function of changing a nonbasic variable. This is an essential property used by the simplex method as we discuss in the following subsection. The vector of objective function coefficients $\mathbf{c_N} - \mathbf{c_B B^{-1}N}$ corresponding to the nonbasic variables is often called the vector of *reduced costs* since they contain the cost coefficients $\mathbf{c_N}$ “reduced” by the cross effects of the basic variables given by $\mathbf{c_B B^{-1}N}$.

Exercise 2.14 Consider the following linear programming problem:

$$\begin{array}{lrcrcl} \max & 4x_1 & + & 3x_2 & & \\ & 3x_1 & + & x_2 & \leq & 9 \\ & 3x_1 & + & 2x_2 & \leq & 10 \\ & x_1 & + & x_2 & \leq & 4 \\ & \multicolumn{5}{l}{x_1 \geq 0,\ x_2 \geq 0.} \end{array}$$

First, transform this problem into the standard form. How many basic solutions does the standard form problem have? What are the basic feasible solutions and what are the extreme points of the feasible region?

Exercise 2.15 A plant can manufacture five products P_1, P_2, P_3, P_4 and P_5. The plant consists of two work areas: the job shop area A_1 and the assembly area A_2. The time required to process one unit of product P_j in work area A_i is p_{ij} (in hours), for $i = 1, 2$ and $j = 1, \ldots, 5$. The weekly capacity of work area A_i is C_i (in hours). The company can sell all it produces of product P_j at a profit of s_j, for $i = 1, \ldots, 5$.

The plant manager thought of writing a linear program to maximize profits, but never actually did for the following reason: From past experience, he observed that the plant operates best when at most two products are manufactured at a time. He believes that if he uses linear programming, the optimal solution will consist of producing all five products and therefore it will not be of much use to him. Do you agree with him? Explain, based on your knowledge of linear programming.

Answer: The linear program has two constraints (one for each of the work areas). Therefore, at most two variables are positive in a basic solution. In particular, this is the case for an optimal basic solution. So the plant manager is mistaken in his beliefs. There is always an optimal solution of the linear program in which at most two products are manufactured.

2.4.2 Simplex Iterations

A key result of linear programming theory is that when a linear programming problem has an optimal solution, it **must** have an optimal solution that is an extreme point. The significance of this result lies in the fact that when we are looking for a solution of a linear programming problem we can focus on the objective value of extreme point solutions only. There are only finitely many of them, so this reduces our search space from an infinite space to a finite one.

The simplex method solves a linear programming problem by moving from one extreme point to an adjacent extreme point. Since, as we discussed in the previous section, extreme points of the feasible set correspond to basic feasible solutions (BFSs), algebraically this is achieved by moving from one BFS to another. We describe this strategy in detail in this section.

The process we mentioned in the previous paragraph must start from an initial BFS. How does one find such a point? While finding a basic solution is almost trivial, finding feasible basic solutions can be difficult. Fortunately, for problems of the form (2.9), such as the bond portfolio optimization problem (2.8) there is a simple strategy. Choosing

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \mathbf{x_B} = \begin{bmatrix} x_3 \\ x_4 \\ x_5 \end{bmatrix}, \mathbf{N} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 5 & 10 \end{bmatrix}, \mathbf{x_N} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

we get an initial basic feasible solution (BFS) with $\mathbf{x_B} = \mathbf{B^{-1}b} = [100, 150, 360]^T$. The objective value for this BFS is $4 \cdot 0 + 3 \cdot 0 = 0$.

Once we obtain a BFS, we first need to determine whether this solution is optimal or whether there is a way to improve the objective value. Recall that the basic variables are uniquely determined once we choose to set the nonbasic variables to a specific value, namely zero. So, the only way to obtain alternative solutions is to modify the values of the nonbasic variables. We observe that both the nonbasic variables x_1 and x_2 would improve the objective value if they were introduced into the basis. Why? The initial basic feasible solution has $x_1 = x_2 = 0$ and we can get other feasible solutions by increasing the value of one of these two variables. To preserve feasibility of the equality constraints, this will require adjusting the values of the basic variables x_3, x_4, and x_5. But since all three are strictly positive in the initial basic feasible solution, it is possible to make x_1 strictly positive without violating any of the constraint, including the nonnegativity requirements.

None of the variables x_3, x_4, x_5 appear in the objective row. Thus, we only have to look at the coefficient of the nonbasic variable we would increase to see what effect this would have on the objective value. The rate of improvement in the objective value for x_1 is 4 and for x_2 this rate is only 3. While a different method may choose the increase both of these variables simultaneously, the simplex method requires that only one nonbasic variable is modified at a time. This requirement is the algebraic equivalent of the geometric condition of moving from one extreme point to an *adjacent* extreme point. Between x_1 and x_2, we choose the variable x_1 to enter the basis since it has a faster rate of improvement.

The basis holds as many variables as there are equality constraints in the standard form formulation of the problem. Since x_1 is to enter the basis, one of x_3, x_4, and x_5 must leave the basis. Since nonbasic variables have value zero in a basic solution, we need to determine how much to increase x_1 so that one of the current basic variables becomes zero and can be designated as nonbasic. The important issue here is to maintain the nonnegativity of all basic variables. Because each basic variable only appears in one row, this is an easy task. As we increase x_1, all current basic variables will decrease since x_1 has positive coefficients in each row[4]. We guarantee the nonnegativity of the basic variables of the next iteration by using the ratio test. We observe that

$$\begin{array}{lcl} \text{increasing } x_1 \text{ beyond } 100/1{=}100 & \Rightarrow & x_3 < 0, \\ \text{increasing } x_1 \text{ beyond } 150/2{=}75 & \Rightarrow & x_4 < 0, \\ \text{increasing } x_1 \text{ beyond } 360/3{=}120 & \Rightarrow & x_5 < 0, \end{array}$$

so we should not increase x_1 more than $\min\{100, 75, 120\} = 75$. On the other hand if we increase x_1 exactly by 75, x_4 will become zero. The variable x_4 is said to *leave the basis*. It has now become a nonbasic variable.

Now we have a new basis: $\{x_3, x_1, x_5\}$. For this basis we have the following basic feasible solution:

$$\mathbf{B} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 3 & 1 \end{bmatrix}, \mathbf{x}_B = \begin{bmatrix} x_3 \\ x_1 \\ x_5 \end{bmatrix} = \mathbf{B^{-1}b} = \begin{bmatrix} 1 & -1/2 & 0 \\ 0 & 1/2 & 0 \\ 0 & -3/2 & 1 \end{bmatrix} \begin{bmatrix} 100 \\ 150 \\ 360 \end{bmatrix} = \begin{bmatrix} 25 \\ 75 \\ 135 \end{bmatrix},$$

$$\mathbf{N} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 4 & 0 \end{bmatrix}, \mathbf{x_N} = \begin{bmatrix} x_2 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

[4] If x_1 had a zero coefficient in a particular row, then increasing it would not effect the basic variable in that row. If, x_1 had a negative coefficient in a row, then as x_1 was being increased the basic variable of that row would need to be increased to maintain the equality in that row; but then we would not worry about that basic variable becoming negative.

After finding a new feasible solution, we always ask the question 'Is this the optimal solution, or can we still improve it?'. Answering that question was easy when we started, because none of the basic variables were in the objective function. Now that we have introduced x_1 into the basis, the situation is more complicated. If we now decide to increase x_2, the objective row coefficient of x_2 does not tell us how much the objective value changes per unit change in x_2, because changing x_2 requires changing x_1, a basic variable that appears in the objective row. It may happen that, increasing x_2 by 1 unit does not increase the objective value by 3 units, because x_1 may need to be decreased, pulling down the objective function. It could even happen that increasing x_2 actually decreases the objective value even though x_2 has a positive coefficient in the objective function. So, what do we do? We could still do what we did with the initial basic solution **if** x_1 did not appear in the objective row and the rows where it is not the basic variable. To achieve this, all we need to do is to use the row where x_1 is the basic variable (in this case the second row) to solve for x_1 in terms of the nonbasic variables and then substitute this expression for x_1 in the objective row and other equations. So, the second equation

$$2x_1 + x_2 + x_4 = 150$$

would give us:

$$x_1 = 75 - \frac{1}{2}x_2 - \frac{1}{2}x_4.$$

Substituting this value in the objective function we get:

$$Z = 4x_1 + 3x_2 = 4(75 - \frac{1}{2}x_2 - \frac{1}{2}x_4) + 3x_2 = 300 + x_2 - 2x_4.$$

Continuing the substitution we get the following representation of the original bond portfolio problem:

$$\begin{array}{rlllllllllll}
\max & Z & & & & & & & & & & \\
\text{subject to:} & & & & & & & & & & & \\
& Z & -x_2 & + & 2x_4 & & & & & & = & 300 \\
& & \frac{1}{2}x_2 & - & \frac{1}{2}x_4 & + & x_3 & & & & = & 25 \\
& & \frac{1}{2}x_2 & + & \frac{1}{2}x_4 & & & + & x_1 & & = & 75 \\
& & \frac{5}{2}x_2 & - & \frac{3}{2}x_4 & & & & & + \; x_5 & = & 135 \\
& & x_2 & , & x_4 & , & x_3 & , & x_1 & , \; x_5 & \geq & 0.
\end{array}$$

This representation looks exactly like the initial system. Once again, the objective row is free of basic variables and basic variables only appear in the row where they are basic, with a coefficient of 1. Therefore, we now can tell how a change in a nonbasic variables would effect the objective function: increasing x_2 by 1 unit will increase the objective function by 1 unit (not 3!) and increasing x_4 by 1 unit will decrease the objective function by 2 units.

Now that we represented the problem in a form identical to the original, we can repeat what we did before, until we find a representation that gives the optimal solution. If we repeat the steps of the simplex method , we find that x_2 will be introduced into the basis next, and the leaving variable will be x_3. If we solve for x_1 using the first equation and substitute for it in the remaining ones, we get the following representation:

$$
\begin{array}{rlrrrrrrrrrrrl}
\max & Z \\
\text{subject to:} \\
& Z & + & 2x_3 & + & x_4 & & & & & & & = & 350 \\
& & & 2x_3 & - & x_4 & + & x_2 & & & & & = & 50 \\
& & & -x_3 & + & x_4 & & & + & x_1 & & & = & 50 \\
& & & -5x_3 & + & x_4 & & & & & + & x_5 & = & 10 \\
& & & x_3 & , & x_4 & , & x_2 & , & x_1 & , & x_5 & \geq & 0.
\end{array}
$$

Once again, notice that this representation is very similar to the tableau we got at the end of the previous section. The basis and the basic solution that corresponds to the system above is:

$$
\mathbf{B} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 4 & 3 & 1 \end{bmatrix}, \mathbf{x_B} = \begin{bmatrix} x_2 \\ x_1 \\ x_5 \end{bmatrix} = B^{-1}b = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & 0 \\ -5 & 1 & 1 \end{bmatrix} \begin{bmatrix} 100 \\ 150 \\ 360 \end{bmatrix} = \begin{bmatrix} 50 \\ 50 \\ 10 \end{bmatrix},
$$

$$
\mathbf{N} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \mathbf{x_N} = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
$$

At this point we can conclude that this basic solution is the optimal solution. Let us try to understand why. From the objective function row of our final representation of the problem we have that for any feasible solution $x = (x_1, x_2, x_3, x_4, x_5)$, the objective function $\mathbf{Z}$ satisfies

$$\mathbf{Z} + 2x_3 + x_4 = 350.$$

Since $x_3 \geq 0$ and $x_4 \geq 0$ is also required, this implies that in every feasible solution

$$\mathbf{Z} \leq 350.$$

But we just found a basic feasible solution with value 350. So this is the optimal solution.

More generally, recall that for any BFS $\mathbf{x} = (\mathbf{x_B}, \mathbf{x_N})$, the objective value $\mathbf{Z}$ satisfies

$$\mathbf{Z} - (\mathbf{c_N} - \mathbf{c_B B^{-1} N})\ \mathbf{x_N} = \mathbf{c_B B^{-1} b}$$

If for a BFS $\mathbf{x_B} = \mathbf{B^{-1}b} \geq \mathbf{0}$, $\mathbf{x_N} = \mathbf{0}$, we have

$$\mathbf{c_N} - \mathbf{c_B B^{-1} N} \leq \mathbf{0},$$

then this solution is an optimal solution since it has objective value $\mathbf{Z} = \mathbf{c_B B^{-1} b}$ whereas, for all other solutions, $\mathbf{x_N} \geq \mathbf{0}$ implies that $\mathbf{Z} \leq \mathbf{c_B B^{-1} b}$.

Exercise 2.16 What is the solution to the following linear programming problem:

$$\begin{array}{ll} \text{Max} & z = c_1x_1 + c_2x_2 + \cdots + c_nx_n \\ \text{s.t.} & a_1x_1 + a_2x_2 + \cdots + a_nx_n \leq \quad b, \\ & 0 \leq x_i \leq u_i \ (i = 1, 2, \ldots, n), \end{array}$$

Assume that all the data elements (c_i, a_i, and u_i) are strictly positive and the coefficients are arranged such that:

$$\frac{c_1}{a_1} \geq \frac{c_2}{a_2} \geq \ldots \geq \frac{c_n}{a_n}.$$

Write the problem in standard form and apply the simplex method to it. What will be the steps of the simplex method when applied to this problem, i.e., in what order will the variables enter and leave the basis?

2.4.3 The Tableau Form of the Simplex Method

In most linear programming textbooks, the simplex method is described using *tableaus* that summarize the information in the different representations of the problem we saw above. Since the reader will likely encounter simplex tableaus elsewhere, we include a brief discussion for the purpose of completeness. To study the tableau form of the simplex method, we recall the bond portfolio example of the previous subsection. We begin by rewriting the objective row as

$$Z - 4\,x_1 - 3\,x_2 = 0$$

and represent this system using the following tableau:

		⇓					
	Basic var.	x_1	x_2	x_3	x_4	x_5	
	Z	-4	-3	0	0	0	0
	x_3	1	1	1	0	0	100
⇐	x_4	2^*	1	0	1	0	150
	x_5	3	4	0	0	1	360

This tableau is often called the *simplex tableau*. The columns labeled by each variable contain the coefficients of that variable in each equation, including the objective row equation. The leftmost column is used to keep track of the basic variable in each row. The arrows and the asterisk will be explained below.

Step 0. *Form the initial tableau.*

Once we have formed this tableau we look for an *entering variable*, i.e., a variable that has a negative coefficient in the objective row and will improve the objective function if it is introduced into the basis. In this case, two of the variables, namely x_1 and x_2, have negative objective row coefficients.

Since x_1 has the most negative coefficient we will pick that one (this is indicated by the arrow pointing down on x_1), but in principle any variable with a negative coefficient in the objective row can be chosen to enter the basis.

Step 1. *Find a variable with a negative coefficient in the first row (the objective row). If all variables have nonnegative coefficients in the objective row, STOP, the current tableau is optimal.*

After we choose x_1 as the entering variable, we need to determine a *leaving variable*. The leaving variable is found by performing a *ratio test*. In the ratio test, one looks at the column that corresponds to the entering variable, and for each *positive* entry in that column computes the ratio of that positive number to the right hand side value in that row. The minimum of these ratios tells us how much we can increase our entering variable without making any of the other variables negative. The basic variable in the row that gives the minimum ratio becomes the leaving variable. In the tableau above the column for the entering variable, the column for the right-hand-side values, and the ratios of corresponding entries are

$$\begin{array}{c} x_1 \\ \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix} \end{array}, \quad \begin{array}{c} \text{RHS} \\ \begin{bmatrix} 100 \\ 150 \\ 360 \end{bmatrix} \end{array}, \quad \begin{array}{c} \text{ratio} \\ 100/1 \\ 150/2 \\ 360/3 \end{array}, \quad \min\{\frac{100}{1}, \frac{150}{2}^*, \frac{360}{3}\} = 75,$$

and therefore x_4, the basic variable in the second row, is chosen as the leaving variable, as indicated by the left-pointing arrow in the tableau.

One important issue here is that, we only look at the positive entries in the column when we perform the ratio test. Notice that if some of these entries were negative, then increasing the entering variable would only increase the basic variable in those rows, and would not force them to be negative, therefore we need not worry about those entries. Now, if all of the entries in a column for an entering variable turn out to be zero or negative, then we conclude that the problem must be *unbounded*; we can increase the entering variable (and the objective value) indefinitely, the equalities can be balanced by *increasing* the basic variables appropriately, and none of the nonnegativity constraints will be violated.

Step 2. *Consider the column picked in Step 1. For each positive entry in this column, calculate the ratio of the right-hand-side value to that entry. Find the row that gives minimum such ratio and choose the basic variable in that row as the leaving variable. If all the entries in the column are zero or negative, STOP, the problem is unbounded.*

Before proceeding to the next iteration, we need to update the tableau to reflect the changes in the set of basic variables. For this purpose, we choose a *pivot element*, which is the entry in the tableau that lies in the intersection of the column for the entering variable (the *pivot column*), and the row for the leaving variable (the *pivot row*). In the tableau above, the pivot element is the number 2, marked with an asterisk. The next job is *pivoting*. When we pivot, we aim to get the number 1 in the position of the pivot element (which can be achieved by dividing the entries in the pivot row by the pivot element), and zeros elsewhere in the pivot column (which can be achieved by adding suitable multiples of the pivot row to the other rows, including the objective row). All these operations are row operations on the matrix that consists of the numbers in the tableau, and what we are doing is essentially Gaussian elimination on the pivot column. Pivoting on the tableau above yields:

	Basic var.	x_1	x_2 ⇓	x_3	x_4	x_5	
	Z	0	-1	0	2	0	300
⇐	x_3	0	1/2*	1	-1/2	0	25
	x_1	1	1/2	0	1/2	0	75
	x_5	0	5/2	0	-3/2	1	135

Step 3. *Find the entry (the pivot element) in the intersection of the column picked in Step 1 (the pivot column) and the row picked in Step 2 (the pivot row). Pivot on that entry, i.e., divide all the entries in the pivot row by the pivot element, add appropriate multiples of the pivot row to the others in order to get zeros in other components of the pivot column. Go to Step 1.*

If we repeat the steps of the simplex method, this time working with the new tableau, we first identify x_2 as the only candidate to enter the basis. Next, we do the ratio test:

$$\min\{\frac{25^*}{1/2}, \frac{75}{1/2}, \frac{135}{5/2}\} = 50,$$

so x_3 leaves the basis. Now, one more pivot produces the optimal tableau:

Basic var.	x_1	x_2	x_3	x_4	x_5	
Z	0	0	2	1	0	350
x_2	0	1	2	-1	0	50
x_1	1	0	-1	1	0	50
x_5	0	0	-5	1	1	10

This solution is optimal since all the coefficients in the objective row are nonnegative.

Exercise 2.17 Solve the following linear program by the simplex method.

$$\begin{array}{lrcrcrcl}
\max & 4x_1 & + & x_2 & - & x_3 & & \\
 & x_1 & & & + & 3x_3 & \leq & 6 \\
 & 3x_1 & + & x_2 & + & 3x_3 & \leq & 9 \\
 & x_1 \geq 0, & & x_2 \geq 0, & & x_3 \geq 0. & &
\end{array}$$

Answer:

	x_1	x_2	x_3	s_1	s_2	
Z	-4	-1	1	0	0	0
s_1	1	0	3	1	0	6
s_2	3	1	3	0	1	9
Z	0	$\frac{1}{3}$	5	0	$\frac{4}{3}$	12
s_1	0	$\frac{-1}{3}$	2	1	$\frac{-1}{3}$	3
x_1	1	$\frac{1}{3}$	1	0	$\frac{1}{3}$	3

The optimal solution is $x_1 = 3$, $x_2 = x_3 = 0$.

Exercise 2.18 Solve the following linear program by the simplex method.

$$
\begin{array}{lrcrcrcl}
\max & 4x_1 & + & x_2 & - & x_3 & & \\
 & x_1 & & & + & 3x_3 & \leq & 6 \\
 & 3x_1 & + & x_2 & + & 3x_3 & \leq & 9 \\
 & x_1 & + & x_2 & - & x_3 & \leq & 2 \\
 & x_1 \geq 0, & & x_2 \geq 0, & & x_3 \geq 0. & &
\end{array}
$$

Exercise 2.19 Suppose the following tableau was obtained in the course of solving a linear program with nonnegative variables x_1, x_2, x_3 and two inequalities. The objective function is maximized and slack variables x_4 and x_5 were added.

Basic var.	x_1	x_2	x_3	x_4	x_5	
Z	0	a	b	0	4	82
x_4	0	-2	2	1	3	c
x_1	1	-1	3	0	-5	3

Give conditions on a, b and c that are required for the following statements to be true:

(i) The current basic solution is a basic feasible solution.

Assume that the condition found in (i) holds in the rest of the exercise.

(ii) The current basic solution is optimal.

(iii) The linear program is unbounded (for this question, assume that $b > 0$).

(iv) The current basic solution is optimal and there are alternate optimal solutions (for this question, assume that $a > 0$).

2.4.4 Graphical Interpretation

Figure 2.1 shows the feasible region for Example 2.1. The five inequality constraints define a convex pentagon. The five corner points of this pentagon (the black dots on the figure) are the basic feasible solutions: each such solution satisfies two of the constraints with equality.

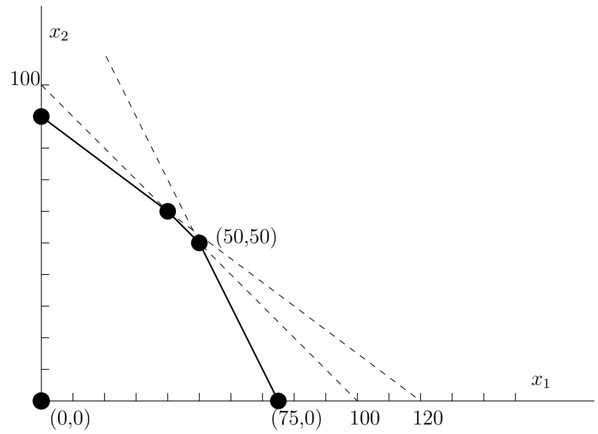

Figure 2.1: Graphical interpretation of the simplex iterations

Which are the solutions explored by the simplex method? The simplex method starts from the basic feasible solution $(x_1 = 0, x_2 = 0)$ (in this solution, x_1 and x_2 are the nonbasic variables. The basic variables $x_3 = 100$, $x_4 = 150$ and $x_5 = 360$ correspond to the constraints that are not satisfied with equality). The first iteration of the simplex method makes x_1 basic by increasing it along an edge of the feasible region until some other constraint is satisfied with equality. This leads to the new basic feasible solution $(x_1 = 75, x_2 = 0)$ (in this solution, x_2 and x_4 are nonbasic, which means that the constraints $x_2 \geq 0$ and $2x_1 + x_2 \leq 150$ are satisfied with equality). The second iteration makes x_2 basic while keeping x_4 nonbasic. This correspond to moving along the edge $2x_1 + x_2 = 150$. The value x_2 is increased until another constraint becomes satisfied with equality. The new solution is $x_1 = 50$ and $x_2 = 50$. No further movement from this point can increase the objective, so this is the optimal solution.

Exercise 2.20 Solve the linear program of Exercise 2.14 by the simplex method. Give a graphical interpretation of the simplex iterations.

Exercise 2.21 Find basic solutions of Example 2.1 that are not feasible. Identify these solutions in Figure 2.1.

2.4.5 The Dual Simplex Method

The previous sections describe the *primal* simplex method, which moves from a basic feasible solution to another until all the reduced costs are nonpositive. There are certain applications where the *dual simplex method* is faster. In contrast to the primal simplex method, this method keeps the reduced costs nonpositive and moves from a basic (infeasible) solution to another until a basic feasible solution is reached.

We illustrate the dual simplex method on an example. Consider Example 2.1 with the following additional constraint.

$$6x_1 + 5x_2 \leq 500$$

Adding a slack variable x_6, we get $6x_1+5x_2+x_6 = 500$. To initialize the dual simplex method, we can start from any basic solution with nonpositive reduced costs. For example, we can start from the optimal solution that we found in Section 2.4.3, without the additional constraint, and make x_6 basic. This gives the following tableau.

Basic var.	x_1	x_2	x_3	x_4	x_5	x_6	
Z	0	0	2	1	0	0	350
x_2	0	1	2	-1	0	0	50
x_1	1	0	-1	1	0	0	50
x_5	0	0	-5	1	1	0	10
x_6	6	5	0	0	0	1	500

Actually, this tableau is not yet in the right format. Indeed, x_1 and x_2 are basic and therefore their columns in the tableau should be unit vectors. To restore this property, it suffices to eliminate the 6 and 5 in the row of x_6 by subtracting appropriate multiples of the rows of x_1 and x_2. This now gives the tableau in the correct format.

Basic var.	x_1	x_2	x_3	x_4	x_5	x_6	
Z	0	0	2	1	0	0	350
x_2	0	1	2	-1	0	0	50
x_1	1	0	-1	1	0	0	50
x_5	0	0	-5	1	1	0	10
x_6	0	0	-4	-1	0	1	-50

Now we are ready to apply the dual simplex algorithm. Note that the current basic solution $x_1 = 50$, $x_2 = 50$, $x_3 = x_4 = 0$, $x_5 = 10$, $x_6 = -50$ is infeasible since x_6 is negative. We will pivot to make it nonnegative. As a result, variable x_6 will *leave the basis*. The pivot element will be one of the negative entry in the row of x_6, namely -4 or -1. Which one should we choose in order to keep all the reduced costs nonnegative? The minimum ratio between $\frac{2}{|-4|}$ and $\frac{1}{|-1|}$ determines the variable that *enters the basis*. Here the minimum is $\frac{1}{2}$, which means that x_3 enters the basis. After pivoting on -4, the tableau becomes:

Basic var.	x_1	x_2	x_3	x_4	x_5	x_6	
Z	0	0	0	0.5	0	0.5	325
x_2	0	1	0	-1.5	0	0.5	25
x_1	1	0	0	1.25	0	-0.25	62.5
x_5	0	0	0	2.25	1	-1.25	72.5
x_3	0	0	1	0.25	0	-0.25	12.5

The corresponding basic solution is $x_1 = 62.5$, $x_2 = 25$, $x_3 = 12.5$, $x_4 = 0$, $x_5 = 72.5$, $x_6 = 0$. Since it is feasible and all reduced costs are nonpositive, this is the optimum solution. If there had still been negative basic variables in the solution, we would have continued pivoting using the rules outlined above: the variable that leaves the basis is one with a negative value, the pivot element is negative, and the variable that enters the basis is chosen by the minimum ratio rule.

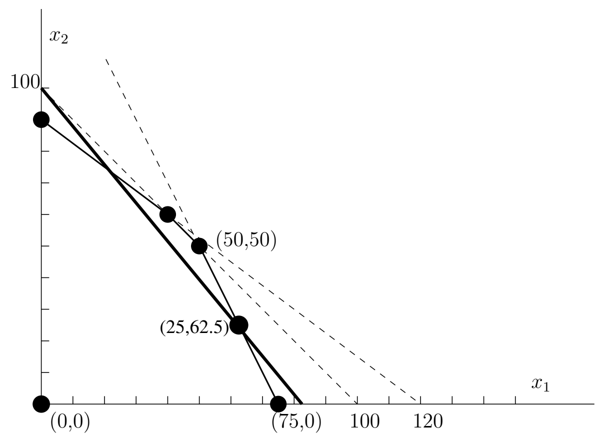

Figure 2.2: Graphical interpretation of the dual simplex iteration

Exercise 2.22 Solve the following linear program by the dual simplex method, starting from the solution found in Exercise 2.17.

$$\begin{array}{llllllll}
\max & 4x_1 & + & x_2 & - & x_3 & & \\
 & x_1 & & & + & 3x_3 & \leq & 6 \\
 & 3x_1 & + & x_2 & + & 3x_3 & \leq & 9 \\
 & x_1 & + & x_2 & - & x_3 & \leq & 2 \\
 & x_1 \geq 0, & & x_2 \geq 0, & & x_3 \geq 0. & &
\end{array}$$

2.4.6 Alternatives to the Simplex Method

Performing a pivot of the simplex method is extremely fast on today's computers, even for problems with thousands of variables and hundreds of constraints. This explains the success of the simplex method. However, for large problems, the number of iterations also tends to be large. At the time of this writing, LPs with tens of thousands of constraints and 100,000 or more variables are generally considered large problems. Such models are not uncommon in financial applications and can often be handled by the simplex method.

Although the simplex method demonstrates satisfactory performance for the solution of most practical problems, it has the disadvantage that, in the worst case, the amount of computing time (the so-called *worst-case complexity*) can grow exponentially in the size of the problem. Here *size* refers to the space required to write all the data in binary. If all the numbers are bounded (say between 10^{-6} and 10^{6}), a good proxy for the size of a linear program is the number of variables times the number of constraints. One of the important concepts in the theoretical study of optimization algorithms is the concept of *polynomial-time algorithms*. This refers to an algorithm whose running time can be bounded by a polynomial function of the input size for all instances of the problem class that it is intended for. After it was discovered in the 1970s that the worst case complexity of the simplex method is exponential (and, therefore, that the simplex method is not a polynomial-time algorithm) there was an effort to identify alternative methods for linear programming with polynomial-time complexity. The first such method, called the *ellipsoid method* was developed by Yudin and Nemirovski in 1979. The same year Khachiyan [41] proved that the ellipsoid method is a polynomial-time algorithm for linear programming. But the more exciting and enduring development was the announcement by Karmarkar in 1984 that an *Interior Point Method* (IPM) can solve LPs in polynomial time. What distinguished Karmarkar's IPM from the ellipsoid method was that, in addition to having this desirable theoretical property, it could solve some real-world LPs much faster than the simplex method. These methods use a different strategy to reach the optimum, generating iterates in the interior of the feasible region rather than at its extreme points. Each iteration is fairly expensive, but the number of iterations needed does not depend much on the size of the problem and is often less than 50. As a result, interior point methods can be faster than the simplex method for large scale problems. Most state-of-the-art linear programming packages (Cplex, Xpress, OSL, etc.) give you the option to solve your linear programs by either method.

We present interior point methods in Chapter 7, in the context of solving quadratic programs.