IEOR E4007: Optimization Models and Methods

Overview of Optimization

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Asset Allocation

Problem: Allocate wealth W $(=$ \$10M$)$ over d $(= 6)$ assets.

Iteration 1

- Decision: $x_i =$ \$ invested in asset $i = 1, \ldots, d$
- Constraints: $x_i \geq 0$ (long only), $\sum_{i=1}^{d} x_i = W$
- Objective function: maximize expected return $\mathbb{E}[\tilde{r}_x] = \sum_{i=1}^{d} x_i \mathbb{E}[\tilde{r}_i]$

How does one estimate the expected return of the investment?

- Historical prices $p_i^{(t)}$ ... historical rate of return $r_i^{(t)} = \frac{p_i^{(t+1)} - p_i^{(t)}}{p_i^{(t)}}$
- Estimate of expected rate of return of asset i: $\hat{\mu}_i = \frac{1}{T} \sum_{t=1}^{T} r_i^{(t)}$
- Estimate of expected return of investment: $\sum_{i=1}^{d} \hat{\mu}_i x_i$

Note that statistics and estimated quantities enter the model.

Iteration 1 (contd.)

Optimization model

$$\begin{array}{rll} \text{maximize} & \sum_{i=1}^{d} \hat{\mu}_i x_i & \\ \text{subject to} & \sum_{i=1}^{d} x_i = W, & \\ & x_i \geq 0, & i = 1, \ldots, d. \end{array}$$

A function $f(\mathbf{x}) : \mathbb{R}^d \to \mathbb{R}$ is called linear if for all $\alpha, \beta \in \mathbb{R}$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$

$$f(\alpha \mathbf{x} + \beta \mathbf{y}) = \alpha f(\mathbf{x}) + \beta f(\mathbf{y})$$

Easy to show that $f(\mathbf{x}) = \mathbf{c}^\top \mathbf{x} = \sum_{i=1}^{d} c_i x_i$ for some $\mathbf{c} \in \mathbb{R}^d$

Above optimization model has

- linear objective function
- constraints of the form $f(\mathbf{x}) \geq$ or $\leq$ or $= b$ for some linear function f

Such an optimization problem is called a linear program.

Optimization models

Optimization = selecting best allocation of limited resources subject to constraints imposed by the problem.

Model = Mathematical approximation/abstraction of the “real” problem

- Models can be used for optimization, simulation or scenario analysis.

Components of an optimization model

- Decision variables: mathematical representation of the decisions
- Constraints: limits on the choices for the decisions
- Objective function(s): goals to optimize

Asset allocation iteration 1: Good model?

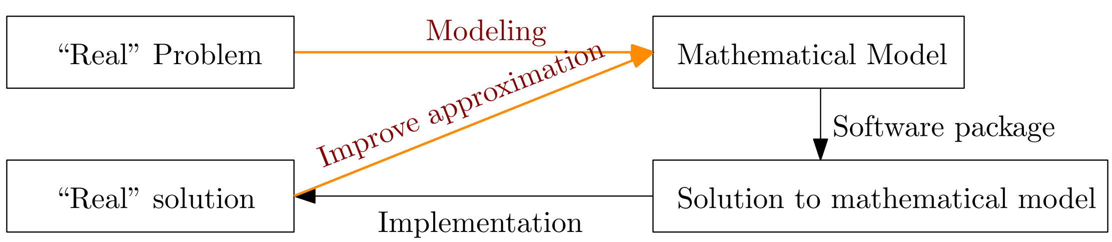
Do the constraints make sense?

- Without side constraints, all the investment in one asset!
- Not surprising, given the objective function.
- Is it always bad to have no diversification?

Are the decisions robust with respect to statistical errors?

- Want to identify $i^* = \mathrm{argmax}_{1 \leq i \leq d}\{\mu_i\}$ – notice no hat!
- What if there are two mean returns for two assets are very close?

Typical Modeling Cycle



Most value added in modeling step: "real" problem $\to$ math problem

Two opposing forces:

- Math problem valid (i.e. close to reality): Analysis hard
- Math problem tractable: Poor performance

Iteration 2: Penalize variability!

Variance of the investment return

$$\boldsymbol{var}(\tilde{r}_x) \quad = \quad \boldsymbol{var}\Big(\sum_{i=1}^{d} x_i \tilde{r}_i\Big) \quad = \quad \sum_{i=1}^{d}\sum_{j=1}^{d} x_i x_j \boldsymbol{cov}(\tilde{r}_i, \tilde{r}_j)$$

How does one estimate the covariance?

- Historical estimate

$$\boldsymbol{cov}(\tilde{r}_i, \tilde{r}_j) \approx \hat{Q}_{ij} = \frac{1}{T-1}\sum_{t=1}^{T}(r_i^{(t)} - \hat{\mu}_i)(r_j^{(t)} - \hat{\mu}_j)$$

  Problems? Can one do better?

- Use Capital Asset Pricing Model and other forward looking methods

Iteration 2 (contd)

Optimization model

$$\begin{array}{rl} \text{maximize} & \sum_{i=1}^{d} \hat{\mu}_i x_i - \lambda \sum_{i=1}^{d}\sum_{j=1}^{d} x_i x_j \hat{Q}_{ij} = \hat{\mu}^\top x - \lambda x^\top \hat{\mathbf{Q}} x \\ \text{subject to} & \mathbf{1}^\top x = W, \\ & x \geq \mathbf{0}. \end{array}$$

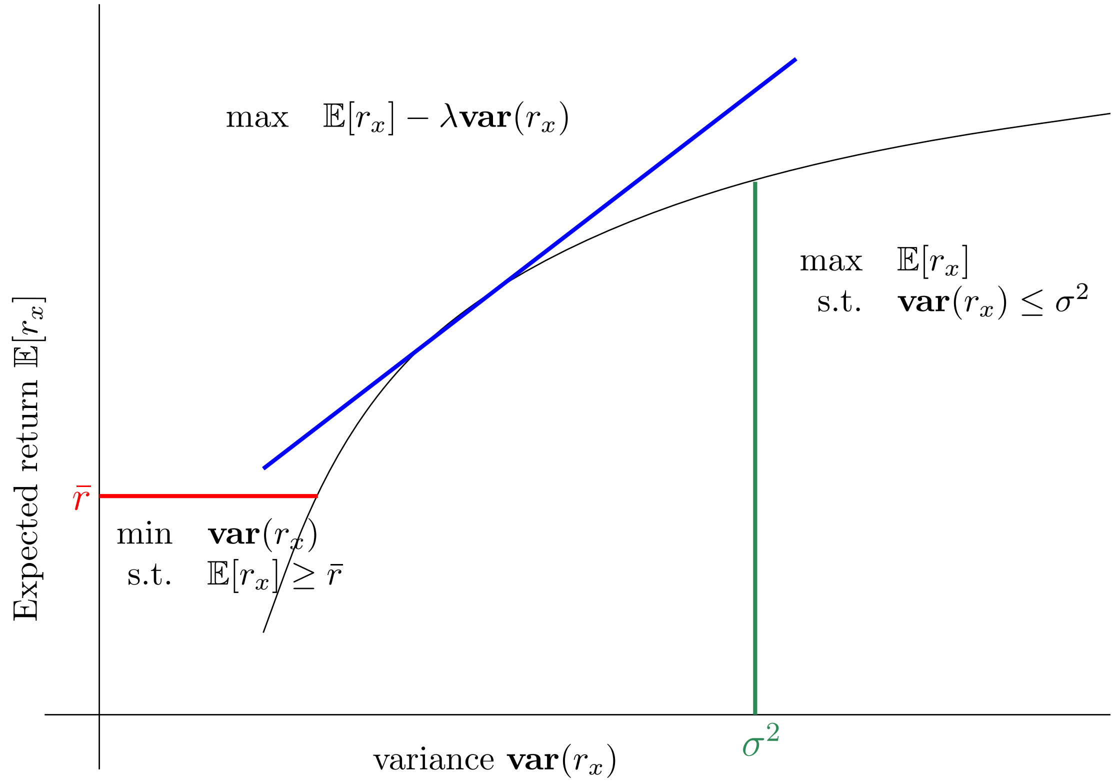
- quadratic objective function: $\lambda =$ risk aversion parameter
- linear constraints

Such a model is called a quadratic program.

Really a two objective problem: maximize $\mathbb{E}[\tilde{r}_x]$ and minimize $\boldsymbol{var}(\tilde{r}_x)$

- Characterize the Pareto frontier or the Efficient frontier for the two quantities
- Can be done in three different ways

Iteration 2 (contd)



Iteration 2: Stress testing the model

Statistical issues

- Variance is a good measure only for elliptical distributions. Particularly bad for heavy tailed distributions.
- Number of correlations grow as $\mathcal{O}(d^2)$. Never have enough data!
- Errors in correlations have serious impact on portfolio holdings.

Operational issues

- How does one estimate the risk aversion?
- Many assets have very small holdings. Either $x_i = 0$ or $|x_i| \geq L$
- How does one handle trading costs? May be add a penalty term

$$
\begin{array}{rl}
\text{maximize} & \hat{\mu}^\top x - \lambda x^\top \hat{\mathbf{Q}} x - \eta \|x - x^{\text{old}}\| \\
\text{subject to} & \mathbf{1}^\top x = W, \\
& x \geq \mathbf{0}.
\end{array}
$$

What does this miss?

Identify hard/easy constraints

No more than 75% of the total capital in any set of $n/2$ assets

$$\max_{S:|S|\leq n/2} \sum_{i\in S} x_i \leq 0.75$$

Convex constraint: In fact, a **linear** constraint. Easy!

$$\frac{n}{2}\alpha + \sum_{i=1}^{n} \beta_i \leq 0.75, \quad \alpha + \beta_i \geq x_i,\ i = 1, \ldots, n.$$

No more than $n/2$ assets in the portfolio?

Integer constraint: much harder!

$$x_i \leq M y_i,\ y_i \in \{0, 1\},\ i = 1, \ldots, n, \quad \sum_{i=1}^{n} y_i \leq \frac{n}{2}$$

Iteration 3: General risk measures

Risk measure ρ: Random variables $\mapsto$ Real number

Examples: Variance, Value-at-Risk, Conditional Value-at-Risk

Optimization problem

$$\begin{array}{rl} \text{minimize} & \rho(\tilde{r}_x) \\ \text{subject to} & \hat{\mu}^\top x \geq \bar{r} \\ & \mathbf{1}^\top x = W, \\ & x \geq 0. \end{array}$$

New issues

- Analytical expression for ρ is hard.
- Have to resort to samples ... Stochastic programming.
- New statistical issues: bias, out-of-sample performance.

Iteration 4: Dynamics

Asset allocation problems are never 1-period problems!

- Long term goals: Financing retirement, purchasing the first home, education
- Multiple re-balancing

Model

- Position at time n: $y^{(n)} = (y_1^{(n)}, \ldots, y_d^{(n)})^\top$
- New position after trade at time n: x
- Trading cost: $c(x, y)$
- Market returns realized ... positions at time $n+1$:

$$y^{(n+1)} = \Big((1 + \tilde{r}_1^{(n)})x_1, \ldots, (1 + \tilde{r}_d^{(n)})x_d\Big) = R^{(n)} \circ x$$

Iteration 4 (contd.)

$V_n(y) =$ "value" of holding position y at time n. Then

$$V_n(y) = \max_x \left\{ c(x, y) + \frac{1}{1 + \iota_n} \mathbb{E} V_{n+1}(R^{(n)} \circ x) \right\}$$

Bellman equation ... Dynamic programming

Is the expectation with respect to "Real world" or risk neutral probability?

Computational issues:

- Have to recursively compute the value function V_n. Computationally intractable!
- New idea: approximate $V_n(x) = \sum_k \beta_k f_k(x)$ for some basis functions f_k. Computation β using regression.
- Method called Approximate Dynamic Progamming.

Course structure

This course is about modeling and solving optimization models

- approximate problems by a standard model
- solve a set of standard models using standard solvers

Other issues

- Robustness of the model
- Sensitivity of the solution
- Large scale models and iterative solutions

Theory supports the last set of questions

Theory supports applications

- If you do not see a connection, interrupt me!