IEOR E4007: Optimization Models and Methods

Integer Programming

Garud Iyengar
Columbia University
Industrial Engineering and Operations Research

Integer linear programs

- Single linear objective function
- Linear constraints
- Some/all variables take only integer values

Example: Minimum investment requirement constraints

- **either** $x_i = 0$ **or** $x_i \geq m_i > 0$, $i = 1, \ldots, d$
- $\sum_{i=1}^{d} x_i \leq W$

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$$y_i = \begin{cases} 1 & x_i \geq m_i \\ 0 & x_i = 0 \end{cases} \qquad \Rightarrow \qquad \begin{array}{c} x_i \geq m_i y_i \\ x_i \leq M y_i \end{array}$$

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How should M be set? Need to ensure that $x_i \leq M$. So, set $M = W$

Either-or examples

Example: Either $|x_i| = 0$ or $|x_i| \geq m_i$, x_i free

Naive (incorrect) approach: $|x_i| \leq My_i$, $|x_i| \geq m_i y_i$. Problem? why?

There are three option here: $x_i \geq m_i$, or $x_i = 0$, or $x_i \leq -m_i$

$$y_i = \begin{cases} 1 & x_i \geq m_i \\ 0 & \text{otherwise} \end{cases} \qquad z_i = \begin{cases} 1 & x_i \leq -m_i \\ 0 & \text{otherwise} \end{cases}$$

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Have to represent these relationships as linear constraints.

$$m_i y_i \leq x_i \leq -m_i z_i$$

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$$-Mz_i + m_i y_i \leq x_i \leq -m_i z_i + My_i$$

How does one set M? Need to ensure that $|x_i| \leq M$.

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Margin constraint: $\sum_{i=1}^{d} |x_i| \leq \beta W$, $\beta > 1 \Rightarrow M = \beta W$

Another version of the long-short investment

Define

- $u_i = x_i^+ \equiv$ long position
- $v_i = x_i^- = \max\{-x_i, 0\} \equiv$ (absolute value of) short position
- $x_i = u_i - v_i$ and $|x_i| = u_i + v_i$

Therefore

$$\sum_{i=1}^{d} |x_i| = \sum_{i=1}^{d} (u_i + v_i) \leq \beta W$$

The constraint **either** $|x_i| \geq m_i$ **or** $|x_i| = 0$ translates to the following collection of constraints

- $m_i y_i \leq u_i \leq (\beta W) y_i$
- $m_i z_i \leq v_i \leq (\beta W) z_i$
- $y_i + z_i \leq 1$

Fixed trading costs

Initial portfolio $y = (y_1, \ldots, y_d)$

New portfolio $x = (x_1, \ldots, x_d)$

Goal: Write the transaction cost as a linear function of x

$$c = \alpha \sum_{i=1}^{d} \mathbf{1}(x_i \neq y_i) + \beta \sum_{i=1}^{d} |x_i - y_i|$$

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New variables/constraints

- $u_i \geq |x_i - y_i|$
- $z_i \in \{0, 1\}, 0 \leq u_i \leq M z_i$ ... what M to use?

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Need to ensure that $|x_i - y_i| \le M$ for all feasible x.

$$\sum_{i=1}^{d} |x_i| \le \beta \sum_{i=1}^{d} y_i \quad \Rightarrow \quad |x_i - y_i| \le |x_i| + |y_i| \le \beta\big(\mathbf{1}^\top y\big) + \max_{1 \le i \le d} |y_i|$$

Portfolio liquidation problem

Problem setup: d assets

- current positions (number of shares): $w = (w_1, w_2, \ldots, w_d)$
- cost basis for the shares: $q = (q_1, \ldots, q_d)$
- price per share: $p = (p_1, \ldots, p_d)$
- future expected payoff per share: $r = (r_1, \ldots, r_d)$

Goal: Compute the new positions that maximize future payoff while raising a capital K (net of capital gains tax).

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New position $x = (x_1, \ldots, x_d)$. The capital gains tax

$$T(x) = \beta\Big(\sum_{i=1}^{d}(p_i - q_i)(w_i - x_i)^+\Big)^+$$

Is $T(x)$ a convex function of x? concave function of x?

Portfolio liquidation problem

Portfolio selection problem

$$\begin{array}{rl} \text{max} & r^\top x \\ \text{s.t.} & p^\top w - p^\top x - t \geq K \\ & t - \beta \sum_{i=1}^{d} (p_i - q_i)(w_i - x_i)^+ \geq 0 \\ & t, x \geq 0 \text{ (only long positions allowed)} \end{array}$$

New variables/constraints

- $s_i = \#$ shares of asset i sold
- $b_i = \#$ shares of asset i bought
- $x_i = w_i + b_i - s_i$ and $b_i \cdot s_i = 0$ ... why do we need this?

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Binary variable

$$
y_i = \begin{cases} 1 & b_i \geq 0 \\ 0 & \text{otherwise} \end{cases} \qquad \begin{array}{l} 0 \leq b_i \leq \left(\frac{p^\top w}{p_i}\right) y_i \\ 0 \leq s_i \leq w_i \,(1 - y_i) \end{array} \qquad (*)
$$

Portfolio liquidation problem

Portfolio selection problem

$$\begin{array}{rl} \text{max} & r^\top (w + b - s) \\ \text{s.t.} & p^\top (s - b) - t \geq K \\ & t - \beta(p - q)^\top s \geq 0 \\ & (*) \\ & t, b, s \geq 0 \end{array}$$

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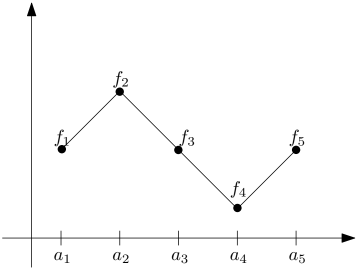
Piecewise linear functions

Consider the following piecewise linear function



For $x \in [a_i, a_{i+1}]$, $i = 1, \ldots, n-1$

$$
\begin{aligned}
f(x) &= f_i + \left(\frac{f_{i+1}-f_i}{a_{i+1}-a_i}\right)(x - a_i) \\
&= \underbrace{\left(\frac{a_{i+1}-x}{a_{i+1}-a_i}\right)}_{\lambda_i} f_i \\
&\quad + \underbrace{\left(\frac{x-a_i}{a_{i+1}-a_i}\right)}_{\lambda_{i+1}} f_{i+1}
\end{aligned}
$$

where $\lambda_i, \lambda_{i+1} \geq 0$ and $\lambda_i + \lambda_{i+1} = 1$

Also note that $x = \lambda_i a_i + \lambda_{i+1} a_{i+1}$

If the interval is known, the function can be linearized using the variables λ

Piecewise linear functions (contd)

Define

$$z_i = \begin{cases} 1 & x \in [a_i, a_{i+1}] \\ 0 & \text{otherwise} \end{cases}$$

In order to ensure that x belongs only one interval we set $\sum_{i=1}^{n-1} z_i = 1$

Write x and $f(x)$ as convex combination of $\{a_i\}$ and $\{f_i\}$, respectively:

$$\begin{array}{rcl} x & = & \sum_{i=1}^{n} \lambda_i a_i, \\ f(x) & = & \sum_{i=1}^{n} \lambda_i f_i \\ 1 & = & \sum_{i=1}^{n} \lambda_i \\ \lambda & \geq & 0 \end{array}$$

This formulation is exact only if only a pair of consecutive λ's is non-zero

$$\begin{array}{rcll} \lambda_1 & \leq & z_1 & \\ \lambda_i & \leq & z_i + z_{i-1} & i = 2, \ldots, n-1, \\ \lambda_n & \leq & z_{n-1} & \end{array}$$

Value at Risk constraint

$L_k \in \mathbb{R}^d =$ rate of loss of d instruments in scenario k

Probability of scenario $k = p_k$

Rate of loss on portfolio x in scenario $k = L_k^\top x$.

Value-at-risk of the portfolio

$$\mathsf{VaR}_p(x) = \inf \left\{ y : \sum_{k=1}^{N} p_k \mathbf{1}(L_k^\top x > y) \leq 1 - p \right\}$$

Goal: Reformulate the constraint $\mathsf{VaR}_p(x) \leq \nu$ as a set of linear constraints.

$$\mathsf{VaR}_p(x) \leq \nu \quad \Leftrightarrow \quad \sum_{k=1}^{N} p_k \mathbf{1}(L_k^\top x > \nu) \leq 1 - p$$

Value at Risk constraint

Let $z_k \geq \mathbf{1}(L_k^\top x > \nu)$ and $z_k \in \{0, 1\}$. Then

$$L_k^\top x \leq \nu + M_k z_k$$

for a suitably large M_k.

The VaR constraint in terms of z: $\sum_{k=1}^{N} p_k z_k \leq (1 - p)$. There is a subtle slippage here ...

How does one set M_k? Suppose feasible x belong to a bounded set $\mathcal{X}$.

$$M_k = \max_{y \in \mathcal{X}} \{L_k^\top y\} - \nu$$

mean-VaR portfolio selection problem

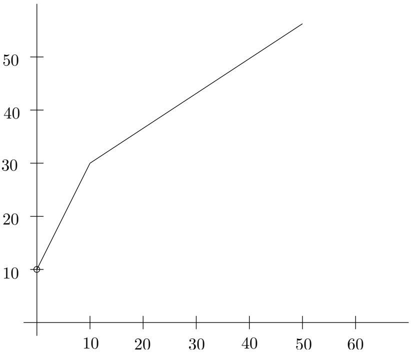
Portfolio selection problem

$$\begin{array}{rl} \max & \mu^\top x \\ \text{s.t.} & \mathrm{VaR}_p(x) \le \nu \\ & \mathbf{1}^\top x = 1 \\ & x \in \mathcal{X} \end{array}$$

Reformulation

$$\begin{array}{rll} \max & \mu^\top x & \\ \text{s.t.} & \sum_{k=1}^N p_k z_k \le (1-p) & \\ & L_k^\top x - M_k z_k \le \nu & k = 1, \ldots, N \\ & \mathbf{1}^\top x = 1 & \\ & x \in \mathcal{X}, z \in \{0,1\}^N & \end{array}$$

Concave trading cost



$$f(x) = \begin{cases} 0 & x = 0 \\ 10 + 2x & x \in (0, 10] \\ 20 + x & x \in (10, 100] \end{cases}$$

Three cases

- $x = 0$: $y = 0$ and $z = 0$
- $0 < x \leq 10$: $y = 1$ and $z = 0$
- $10 < x \leq 100$: $y = 0$ and $z = 1$

Split the contribution from each part:

$$\begin{aligned} & x = x_1 + x_2 \\ & 0 \leq x_1 \leq 10y \\ & 10z \leq x_2 \leq 100z \\ & y + z \leq 1 \end{aligned}$$

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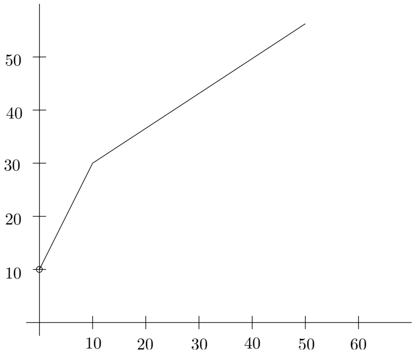

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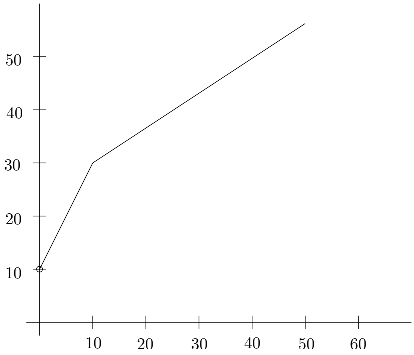

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Split the contribution from each part: there is a subtle "error" here!

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Set covering, packing, and partitioning

n strategies

$$z_i = \begin{cases} 1 & \text{run strategy } i \\ 0 & \text{do not run strategy } i \end{cases}$$

Can only run at most d strategies:

Set covering, packing, and partitioning

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If strategy k and ℓ are run together, the amount invested in the two strategies must be equal.

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$$|w_k - w_l| \leq W(1 - y), \quad y = z_k z_\ell$$

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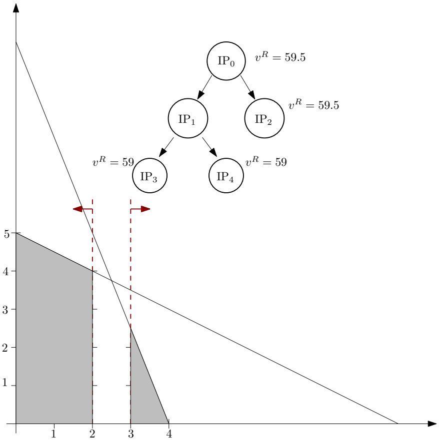
$$|w_k - w_l| \leq W(1-y), \quad y \leq z_1, \quad y \leq z_2, \quad y \geq 0, \quad y \geq z_1 + z_2 - 1$$

Branch and Bound (BB)

An intelligent enumeration technique. Main steps

- Solve the LP relaxation of the IP. Gives a bound v^R for the IP.
- If the solution is feasible for the IP. Stop. IP optimal found.
- If not, branch, i.e. split the feasible region. Splitting rule
  - Choose a node that has the best bound
  - Branch on the variable that is most fractional
- Remove (fathom) all infeasible nodes from BB tree.
- Fathom nodes where the LP optimal is feasible for IP.
- Fathom all nodes with LP value v^R worse than best IP solution.

BB example



$$
\begin{array}{ll}
\text{max} & 13x_1 + 8x_2 \\
\text{s.t.} & x_1 + 2x_2 \le 10 \\
 & 5x_1 + 2x_2 \le 20 \\
 & x \in \mathbb{Z}^2_+
\end{array}
$$

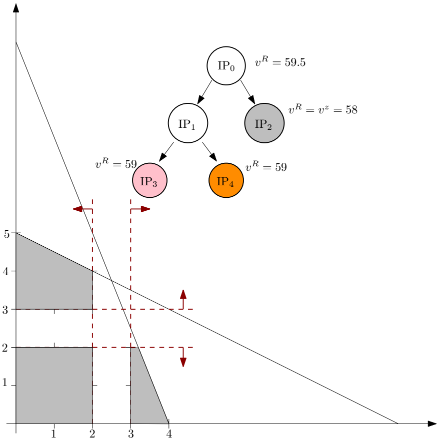
LP$_0$ $x_0^* = (2.5, 3.75)$ and $v^R = 59.5$

Branch on x_1: $\underbrace{x_1 \ge 3}_{\text{IP}_1}$ or $\underbrace{x_1 \le 2}_{\text{IP}_2}$

LP$_1$: $x_1^* = (3, 2.5)$ and $v^R = 59$

Branch on x_2: $\underbrace{x_2 \ge 3}_{\text{IP}_3}$ or $\underbrace{x_2 \le 2}_{\text{IP}_4}$

BB example



LP$_2$: $x^* = (2,4)$ $v^R = v^Z = 58$. Fathomed.

LP$_3$: Infeasible. Fathomed.

LP$_4$: $x^* = (3.2, 2)$, $v^R = 57.6 < 58$. Fathomed.

Optimal IP solution: $x^* = (2,4)$

Cutting planes

Add new linear constraints so that

- no feasible integer points violate these constraints
- all extreme point of new polyhedron satisfy integer constraints

why?

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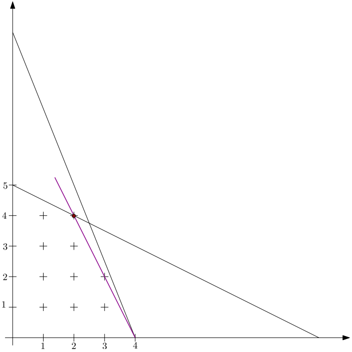
why?

LP solution will be IP optimal.



Add new linear constraint

$$2x_1 + x_2 \le 8$$

Satisfies both requirements above.

LP optimal solution $x^* = (2, 4)$

Use cutting planes and BB together.