IEOR E4007: Optimization Models and Methods

Linear Programming

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Overview

- Examples of linear programming models
- General form of linear programs
- Solution methods
- Geometry of linear programs and duality
- Linearizable models

Portfolio selection using scenarios

	Asset 1	Asset 2	Asset 3	Asset 4	Req
Cost	2.0	3.0	$1.0 (= c_3)$	0.5	
Scenario 1	0.2	1.0	0.1	0.5	$10 (= r_1)$
Scenario 2	0.5	1.2	1.0	$0.8 (= S_{24})$	20
Scenario 3	1.0	0.2	1.3	1.2	15

Long-only investment in the four assets that meets all the requirements.

$$
\begin{array}{rl}
\text{min} & c_1 x_1 + c_2 x_2 + c_3 x_3 + c_4 x_4 \\
\text{subject to} & S_{11} x_1 + S_{12} x_2 + S_{13} x_3 + S_{14} x_4 \geq r_1 \\
& S_{21} x_1 + S_{22} x_2 + S_{23} x_3 + S_{24} x_4 \geq r_2 \\
& S_{31} x_1 + S_{32} x_2 + S_{33} x_3 + S_{34} x_4 \geq r_3 \\
& x \geq 0
\end{array}
$$

Key features:

- Recall: linear function $\equiv a_1 x_1 + a_2 x_2 + \ldots + a_n x_n$
- Single linear objective
- Linear inequality constraints
- Decision variables take continuous values

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Long-only investment in the four assets that meets all the requirements.

$$
\begin{aligned}
\text{min} \quad & \begin{bmatrix} c_1 & c_2 & c_3 & c_4 \end{bmatrix} x \\
\text{subject to} \quad & \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} \\ S_{21} & S_{22} & S_{23} & S_{24} \\ S_{31} & S_{32} & S_{33} & S_{34} \end{bmatrix} x \geq \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} \\
& x \geq 0
\end{aligned}
$$

Portfolio selection using scenarios

	Asset 1	Asset 2	Asset 3	Asset 4	Req
Cost	2.0	3.0	$1.0(= c_3)$	0.5	
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Long-only investment in the four assets that meets all the requirements.

$$\begin{array}{ll} \text{min} & \begin{bmatrix} c_1 & c_2 & c_3 & c_4 \end{bmatrix} x \\ \text{subject to} & \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} \\ S_{21} & S_{22} & S_{23} & S_{24} \\ S_{31} & S_{32} & S_{33} & S_{34} \end{bmatrix} x \geq \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} \\ & x \geq 0 \end{array}$$

Sensitivity analysis: why?

- What if the requirement in Scenario 2 goes up by 10%?
- What if the cost of asset 2 drops by 5%?

Can we estimate this without solving a new LP?

Solving LPs

MATLAB command `x = linprog(f,A,b,Aeq,beq,l,u)` solves

$$\begin{array}{ll} \text{min} & f^\top x \\ \text{s.t.} & A_{eq}x = b_{eq} \\ & Ax \le b \\ & \ell \le x \le u \end{array} \quad \Rightarrow \quad \begin{array}{ll} f = c, & \\ A = -S, & b = -r, \\ A_{eq} = [\,], & b_{eq} = [\,], \\ \ell = 0 & \end{array}$$

```
x = linprog(c,-S,-r,[],[],zeros(4,1))
```

CVX code: MATLAB based modeling language

```
variables x(n)
minimize (c'*x)
subject to
   S*x >= r;
    x >= 0;
```

EXCEL: Not scalable but still useful in practice.

Solving LPs

R

- `linprog`
- `lpsolveAPI`: A good API to the lpsolve solver. Allows you to iteratively construct the LP. Can also solve mixed-integer linear programs.

Python

- `CVXOPT`: Analog of CVX. Very good for convex optimization problems. Does not allow for integer or binary variables.
- `PICOS`: Another version of CVX. Allows one to interact with multiple solvers. But not lpsolve.
- `lpsolve`: Free solver for linear and mixed-integer linear programs. But don't know of a python API for this solver.

Linear program

Optimization problem with

- a single linear objective (max/min)
- linear equality or inequality constraints
- continuous variables

All three requirements are necessary.

General form of a linear program

$$
\begin{array}{ll}
\text{max/min}_x & \sum_{j=1}^{n} c_j x_j \\
\text{subject to} & \left.\begin{array}{ll} \sum_{j=1}^{n} a_{ij} x_j = b_i, & i \in E \\ \sum_{j=1}^{n} a_{ij} x_j \leq b_i, & i \in L \\ \sum_{j=1}^{n} a_{ij} x_j \geq b_i, & i \in G \end{array}\right\} \text{Main constraints} \\
& \left.\begin{array}{ll} x_j \geq 0, & i \in J_{pos} \\ x_j \leq 0, & i \in J_{neg} \end{array}\right\} \text{Variable constraints}
\end{array}
$$

Difference between main / variable constraints becomes relevant later.

Short term cash flow management (Section 3.1)

A company faces the following cash flow (+ve = income, -ve=liability)

Month	Jan	Feb	Mar	Apr	May	Jun
Cash	-150K	-100K	+200K	-200K	+50K	+350K

Assume cashflow c_t occurs on the first day of the month.

Cash flow problem!

- Net cash flow $= \sum_{t=1}^{6} c_t = +150K > 0$
- But ... $\sum_{t=1}^{\tau} c_t < 0$ for $\boldsymbol{\tau} = 1, 2, 4, 5$

Use financial products to get flexibility

- Line of credit: 100K with $r_l = 1\%$ p.m. compounded monthly
  - 30K in Jan & 20K in Feb $\Rightarrow$ Total $= 30(1 + r_L)^2 + 20(1 + r_L)$
  - LOC remaining in Mar: $100 - \big(30(1 + r_L)^2 + 20(1 + r_L)\big)$

Short term cash flows (contd)

- 90-day commercial paper
  - Can be issued in Jan, Feb and Mar *only* (why?)
  - $r_p = 2\%$ for the entire 90 day period
  - $100K$ paper issued in Jan $\Rightarrow$ $+100$K in Jan, $-100(1 + r_p)$K in Apr
- Risk-free reinvestment rate $r_f = 0.3\%$ p.m.

Goal: Maximize cash position in June ($\equiv$ Month 6) meeting all liabilities

Short term financing: LP model

Decision variables

- x_t = total amount owed on line of credit in month $t \leq T-1$

  Assume that we first return $(1+r_l)x_{t-1}$ before we borrow x_t. Why is this okay? When will it not be okay?
- y_t = paper issued in month $t = 1, \ldots, T-3$
- z_t = excess cash in month $t = 1, \ldots, T$

Objective function: max z_T

Constraints

- Inflow = Outflow constraints: Generic constraint

$$x_t + y_t + (1+r_f)z_{t-1} + c_t = z_t + (1+r_l)x_{t-1} + (1+r_p)y_{t-3}$$

$$\Updownarrow$$

$$x_t - (1+r_l)x_{t-1} + y_t - (1+r_p)y_{t-3} - z_t + (1+r_f)z_{t-1} = -c_t$$

Need to make sure to drop variables when the index is not in range

Short term financing: LP model (contd)

- Credit limit upper bounds: $x_t \leq U$
- Non-negative decision variables: $x_t, y_t, z_t \geq 0$, $t = 1, \ldots T$

Assumptions?

Short term financing: LP model (contd)

- Credit limit upper bounds: $x_t \leq U$
- Non-negative decision variables: $x_t, y_t, z_t \geq 0$, $t = 1, \ldots T$

Assumptions?

- No transaction costs
- Completely inelastic market for commercial paper
- Deterministic cash flows

Short term financing: LP model (contd)

- Credit limit upper bounds: $x_t \leq U$
- Non-negative decision variables: $x_t, y_t, z_t \geq 0$, $t = 1, \ldots T$

Assumptions?

- No transaction costs
- Completely inelastic market for commercial paper
- Deterministic cash flows

Sensitivity Analysis

- What is the impact if the liability in January goes *up* by 50K?
- Would a higher credit limit help?
- Suppose the vendor could move all/part of the liability in Jan to Jun by paying a total interest r. When is this profitable?

General modeling of credit line

Variables:

- x_t = amount **owed** on line of credit at time t
- u_t = amount **borrowed** from the line of credit at time t
- v_t = amount **repaid** to the line of credit at time t

Dynamics

$$x_t = (1 + r_l)x_{t-1} + u_t - v_t \qquad x_T = 0$$

$$x_t \geq 0, u_t \geq 0, v_t \geq 0$$

Bounds on the total amount owed: $x_t \leq U$

Proportional transaction cost: $\sum_{t=1}^{T} \alpha(u_t + v_t)$, $\alpha \geq 0$

Dedication (Section 3.2)

Liability stream for a municipality

	Yr 1	Yr 2	Yr 3	Yr 4	Yr 5	Yr 6	Yr 7	Yr 8
Liability	12K	18K	20K	20K	16K	15K	12K	10K

Bonds available for hedging: annual coupon and face value $F = 100$

	1	2	3	4	5	6	7	8	9	10
Price	102	99	101	98	98	104	100	101	102	94
Coupon	5	3.5	5	3.5	4	9	6	8	9	7
Maturity	1	2	2	3	4	5	5	6	7	8

$$c_t = \begin{cases} c \cdot F & t < T, \\ (1+c)\cdot F & t = T \\ 0 & \text{otherwise} \end{cases} \qquad c^{(k)} = \text{ cash flow vector for bond } k$$

Reinvestment interest rate $r_f = 0$ – minimize exposure to interest rate risk

Goal: Minimum cost portfolio that covers the liabilities.

Dedication (contd)

Variables

- $x_k = \#$ units of bond k purchased now, i.e. year 0. Note that x_i is continuous but in reality x_i is discrete. Will have to deal with this!
- $z_t =$ excess cash in year t, $t = 0, \ldots, T$

Objective: min $\sum_k P_k x_k + z_0$

Constraints

- inflow = outflow: $\sum_{k=1}^{n} c_t^{(k)} x_k + z_{t-1} - z_t = \ell_t$
- Diversification constraints: $S =$ class of bonds, e.g. municipal bonds

$$l_S \leq \sum_{k \in S} P_k x_k \leq u_S$$

- Non-negativity constraints: $x \geq 0$, $z \geq 0$

Sensitivity

What is the impact on the cost of the portfolio if the liability in year 4 goes up to 22K?

How will the portfolio change if the price p_6 of bond 6 increases by 20¢?

What is the implied yield curve faced by the portfolio manager?

At what interest rate would the portfolio manager hold cash?

Network LPs

$$\begin{aligned} c_{ji} &= \text{exchange rate between currencies } j \text{ and } i \\ &= 1 \text{ unit of currency } j \text{ can be exchanged for } c_{ji} \text{ units of currency } i \end{aligned}$$

Is there an arbitrage opportunity in a set of N currencies?

Network LPs

$$
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Is there an arbitrage opportunity in a set of N currencies?

Yes, if the product over a cycle of conversions is strictly greater than 1

Network LPs

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Is there an arbitrage opportunity in a set of N currencies?

Yes, if the product over a cycle of conversions is strictly greater than 1

Define

$$x_{ji} = \text{units of currency } j \text{ converted into currency } i$$

Linear program

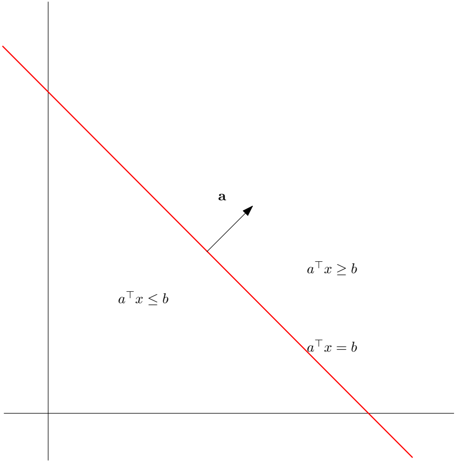
$$\begin{array}{rl} \max & \sum_{j=1}^{N} c_{j1} x_{j1} \\ \text{s.t.} & \sum_{k=1}^{N} c_{kj} x_{kj} - \sum_{k=1}^{N} x_{jk} = 0, \quad j \neq 1 \\ & \sum_{k=1}^{N} x_{1k} = 1 \\ & x \geq 0 \end{array}$$

What is this LP doing?

Solution methods and sensitivity analysis

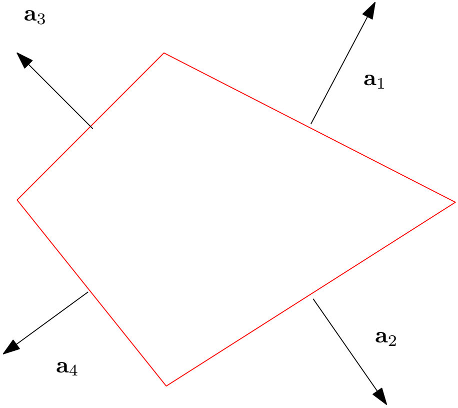
- Geometry of linear programs
- Optimality conditions
- “Corner hopping algorithm”: simplex algorithm
- Sensitivity analysis of linear programs
- Bounds on the optimal value: dual linear program

Level sets of linear constraint



- $\{x : a^\top x \leq b\}$: halfspace
- $\{x : a^\top x = b\}$: hyperplane

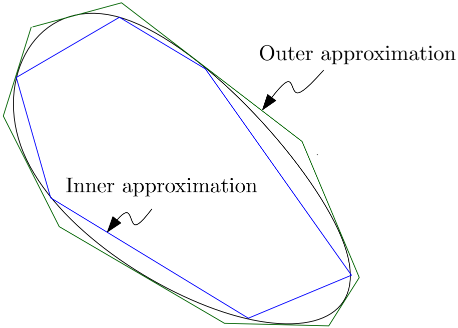
Feasible set of a linear program



- Feasible set of linear inequalities $\equiv$ intersection of halfspaces
- Why did we not consider linear equalities?

Why bother with linear programs?

A large number of optimization problems are either linear programs or can be approximated by linear programs.



Very large scale LPs can be solved efficiently – so linear approximation can be sometimes be the most efficient way to solve non-linear problems.

Geometry of linear programs

2 variable 2 constraint LP:

$$\begin{array}{rl} \text{max} & 13x_1 + 23x_2 \\ \text{s.t.} & 5x_1 + 15x_2 \leq 480 \\ & 4x_1 + 4x_2 \leq 160 \\ & x_1, x_2 \geq 0. \end{array} \qquad c = \begin{bmatrix} 13 \\ 23 \end{bmatrix}, a_1 = \begin{bmatrix} 5 \\ 15 \end{bmatrix}, a_2 = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

x feasible solution $\Leftrightarrow$ x satisfies all the constraints

Feasible region $\mathcal{F} = \{x : x \text{ is a feasible solution}\}$

Level set of a linear function:

- Fix $x_0 \in \mathbb{R}^2$: Let $z_0 = c^\top x_0$.
- Line through x_0 perpendicular to c: $\ell = \{x : c^\top (x - x_0) = 0\}$
- Objective value $c^\top x$ for any $x \in \ell$?

Geometry of linear programs

2 variable 2 constraint LP:

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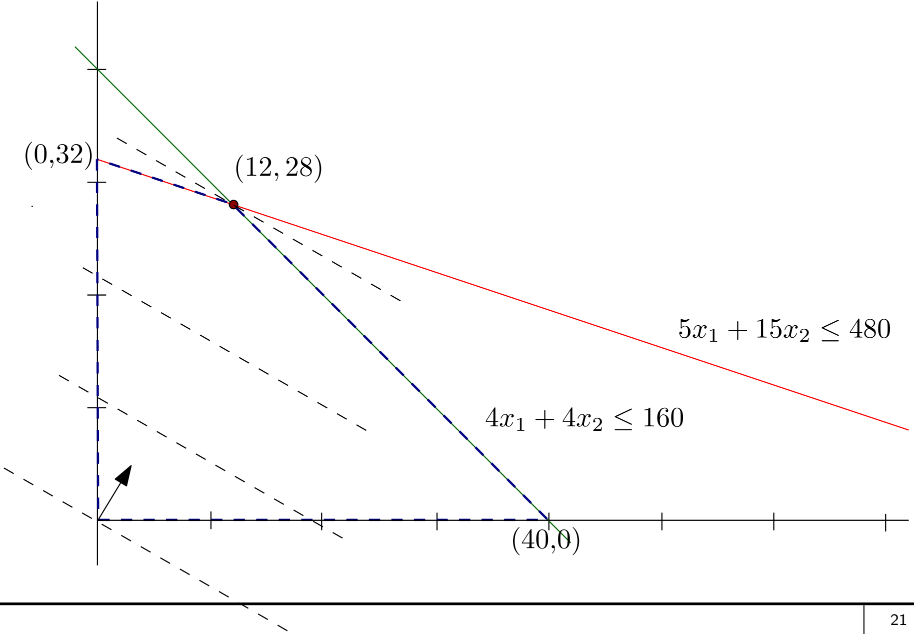
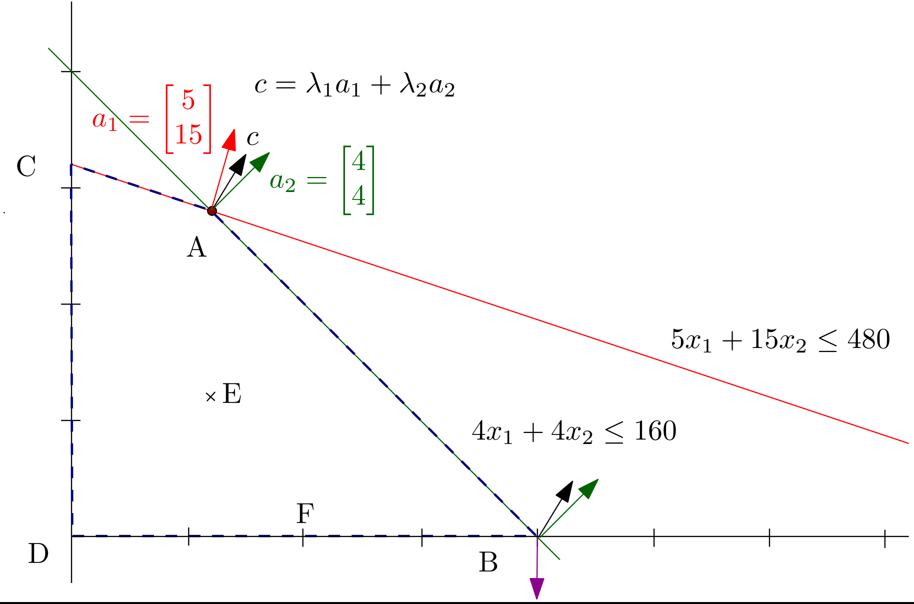
- Line through x_0 perpendicular to c: $\ell = \{x : c^\top (x - x_0) = 0\}$
- Objective value $c^\top x$ for any $x \in \ell$? $c^\top x = z_0$ for all $x \in \ell$

"Algorithm" for computing optimal solution



Optimal Corners



Algebraic definition of a corner

A constraint $a^\top x \leq b$ is active or tight at a feasible point x_0 if $a^\top x_0 = b$.

A feasible point x_0 is a corner if it is the unique solution of the tight constraints at x_0.

A	$(12, 28)$	$5x_1 + 15x_2 \leq 480, 4x_1 + 4x_2 \leq 160$	corner
B	$(40, 0)$	$4x_1 + 4x_2 \leq 160, x_2 \geq 0$	corner
C	$(0, 32)$	$5x_1 + 15x_2 \leq 480, x_1 \geq 0$	corner
D	$(0, 0)$	$x_1 \geq 0, x_2 \geq 0$	corner
E	$(12, 12)$		not a corner
F	$(20, 0)$	$x_2 \geq 0$	not a corner

Standard form LPs allow one to characterize corners and adjacent corners.

Standard form LPs

- All main constraints are equality constraints
- All variables are non-negative

$$\begin{aligned} \text{min/max} \quad & c^\top x \\ \text{s.t.} \quad & Ax = b, \\ & x \geq 0. \end{aligned}$$

Any LP can be transformed into standard form LP using new variables.

$$\begin{aligned} \text{max} \quad & 13x_1 + 23x_2 + 20x_3 \\ \text{s.t.} \quad & 5x_1 + 15x_2 + 12x_3 \geq 480 \\ & 4x_1 + 4x_2 + 3x_3 = 160 \\ & x_1 \geq 0, x_2 \leq 0, x_3 \text{ free} \end{aligned}$$

New variables

- $s = (5x_1 + 15x_2 + 12x_3) - 480 \geq 0$
- $y = -x_2 \geq 0$
- $x_3 = z_1 - z_2,\ z_1, z_2 \geq 0$

Standard form LPs

- All main constraints are equality constraints
- All variables are non-negative

$$\begin{aligned} \text{min/max} \quad & c^\top x \\ \text{s.t.} \quad & Ax = b, \\ & x \geq 0. \end{aligned}$$

Any LP can be transformed into standard form LP using new variables.

$$\begin{aligned} \text{max} \quad & 13x_1 - 23y + 20(z_1 - z_2) \\ \text{s.t.} \quad & 5x_1 - 15y + 12(z_1 - z_2) - s = 480 \\ & 4x_1 - 4y + 3(z_1 - z_2) = 160 \\ & x_1 \geq 0, y, z_1, z_2, s \geq 0. \end{aligned}$$

New variables

- $s = (5x_1 + 15x_2 + 12x_3) - 480 \geq 0$
- $y = -x_2 \geq 0$
- $x_3 = z_1 - z_2,\ z_1, z_2 \geq 0$

Corners in Standard form LPs

$$\begin{aligned} \text{min/max} \quad & c^\top x \\ \text{s.t.} \quad & Ax = b \quad (\in \mathbb{R}^m) \\ & x \geq 0 \quad (\in \mathbb{R}^d) \end{aligned}$$

Corner $\equiv$ unique solution of active constraints.

- m active constraints from the equality constraints
- $d - m$ active constraints from the inequality constraints, i.e. $d - m$ components $x_j = 0$

Algorithm to compute a corner:

- Pick a set B of m indices from $\{1, \ldots, d\}$. Set $N = B^c$.
- For all $j \in N$, set $x_j = 0$. These are called non-basic variables.
- Solve for $j \in B$ using the m equations

$$\sum_{j \in B} A_{ij} x_j = b_i, \quad i = 1, \ldots, m.$$

- If a unique solution exists and is non-negative, it is a corner.

Corners in standard form LPs

Optimal solutions of LPs tend to be very sparse. Good?

Adjacent corners differ in exactly 1 basic variable.

Simplex algorithm:

- Pick an initial corner
- Move to a better adjacent corner
- Stop when if current corner is optimal or problem unbounded.

Some definitions and algebra

- $x_B = (x_j)_{j \in B} =$ basic variables, $x_N = (x_j)_{j \in N} =$ non-basic variables
- $A_j = j$-th column of $A =$ column that multiplies variable x_j
- $A_B = (A_j)_{j \in B}$ and $A_N = (A_j)_{j \in N}$
- $Ax = A_N x_N + A_B x_B = b \Rightarrow x_B = A_B^{-1} b$

Simple LP in standard form

$$\begin{aligned} \text{max} \quad & 13x_1 + 23x_2 & \equiv \quad & \text{max} \quad 13x_1 + 23x_2 \\ \text{s.t.} \quad & 5x_1 + 15x_2 \le 480 & & \text{s.t.} \quad 5x_1 + 15x_2 + x_3 = 480 \\ & 4x_1 + 4x_2 \le 160 & & \qquad\; 4x_1 + 4x_2 + x_4 = 160 \\ & x_1, x_2 \ge 0. & & \qquad\; x_1, x_2, x_3, x_4 \ge 0. \end{aligned}$$

$$\equiv \quad \begin{aligned} \text{max} \quad & \begin{bmatrix} 13 & 23 & 0 & 0 \end{bmatrix} x \\ \text{s.t.} \quad & \begin{bmatrix} 5 & 15 & 1 & 0 \\ 4 & 4 & 0 & 1 \end{bmatrix} x = \begin{bmatrix} 480 \\ 160 \end{bmatrix} \\ & x \ge 0. \end{aligned}$$

Initial corner: $B = \{3, 4\}$ and $N = \{1, 2\}$

$$A_B = \begin{bmatrix} A_3 & A_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow x_B = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = A_B^{-1} b = \begin{bmatrix} 480 \\ 160 \end{bmatrix}$$

This is Corner D on page 19.

Move to adjacent corners

Suppose we are at the corner (B, N). Suppose $\bar{x}_B = A_B^{-1} b > 0$.

$$
\begin{aligned}
\mathcal{F} &= \left\{ x : x_B = A_B^{-1} b - A_B^{-1} A_N x_N \geq 0, x_N \geq 0 \right\} \\
&= \left\{ x : x_B = \bar{x}_B - \sum_{j \in N} A_B^{-1} A_j x_j \geq 0, x_N \geq 0 \right\}
\end{aligned}
$$

Since $\bar{x}_B > 0$, all small enough values of x_N are feasible. Thus,

$$
c^\top x = c_B^\top \bar{x}_B + \sum_{j \in N} \underbrace{(c_j - c_B^\top A_B^{-1} A_j)}_{\bar{c}_j \equiv \text{reduced cost}} x_j
$$

As one increase a non-basic variable x_j two things happen

- x_j increase: objective changes by $c_j x_j$
- basic variables change by $-A_B^{-1} A_j$: objective changes by $-c_B^\top A_B^{-1} A_j$

Reduced cost is the cumulative effect

Move to adjacent corners

Suppose we are at the corner (B, N). Suppose $\bar{x}_B = A_B^{-1} b > 0$.

$$\begin{aligned} \mathcal{F} &= \Big\{ x : x_B = A_B^{-1} b - A_B^{-1} A_N x_N \geq 0, x_N \geq 0 \Big\} \\ &= \Big\{ x : x_B = \bar{x}_B - \sum_{j \in N} A_B^{-1} A_j x_j \geq 0, x_N \geq 0 \Big\} \end{aligned}$$

Since $\bar{x}_B > 0$, all small enough values of x_N are feasible. Thus,

$$c^\top x = c_B^\top \bar{x}_B + \sum_{j \in N} \underbrace{\left(c_j - c_B^\top A_B^{-1} A_j\right)}_{\bar{c}_j \equiv \text{reduced cost}} x_j$$

- B optimal for $\max$ problem $\Leftrightarrow c_j - c_B^\top A_B^{-1} A_j \leq 0$ for all $j \in N$

Move to adjacent corners

Suppose we are at the corner (B, N). Suppose $\bar{x}_B = A_B^{-1} b > 0$.

$$\begin{aligned}\mathcal{F} &= \left\{x : x_B = A_B^{-1} b - A_B^{-1} A_N x_N \geq 0, x_N \geq 0\right\} \\ &= \left\{x : x_B = \bar{x}_B - \sum_{j \in N} A_B^{-1} A_j x_j \geq 0, x_N \geq 0\right\}\end{aligned}$$

Since $\bar{x}_B > 0$, all small enough values of x_N are feasible. Thus,

$$c^\top x = c_B^\top \bar{x}_B + \sum_{j \in N} \underbrace{\left(c_j - c_B^\top A_B^{-1} A_j\right)}_{\bar{c}_j \equiv \text{reduced cost}} x_j$$

- B optimal for $\max$ problem $\Leftrightarrow c_j - c_B^\top A_B^{-1} A_j \leq 0$ for all $j \in N$
- B optimal for $\min$ problem $\Leftrightarrow c_j - c_B^\top A_B^{-1} A_j \geq 0$ for all $j \in N$

Move to adjacent corners

Suppose we are at the corner (B, N). Suppose $\bar{x}_B = A_B^{-1} b > 0$.

$$
\begin{aligned}
\mathcal{F} &= \left\{x : x_B = A_B^{-1} b - A_B^{-1} A_N x_N \geq 0, x_N \geq 0\right\} \\
&= \left\{x : x_B = \bar{x}_B - \sum_{j \in N} A_B^{-1} A_j x_j \geq 0, x_N \geq 0\right\}
\end{aligned}
$$

Since $\bar{x}_B > 0$, all small enough values of x_N are feasible. Thus,

$$
c^\top x = c_B^\top \bar{x}_B + \sum_{j \in N} \underbrace{(c_j - c_B^\top A_B^{-1} A_j)}_{\bar{c}_j \equiv \text{reduced cost}} x_j
$$

- B optimal for $\max$ problem $\Leftrightarrow c_j - c_B^\top A_B^{-1} A_j \leq 0$ for all $j \in N$
- B optimal for $\min$ problem $\Leftrightarrow c_j - c_B^\top A_B^{-1} A_j \geq 0$ for all $j \in N$
- steepest ascent: increase x_{j^*} where $j^* = \operatorname{argmax}\{c_j - c_B^\top A_B^{-1} A_j\}$

Move to adjacent corners

Suppose we are at the corner (B, N). Suppose $\bar{x}_B = A_B^{-1} b > 0$.

$$\begin{aligned} \mathcal{F} &= \left\{ x : x_B = A_B^{-1} b - A_B^{-1} A_N x_N \geq 0, x_N \geq 0 \right\} \\ &= \left\{ x : x_B = \bar{x}_B - \sum_{j \in N} A_B^{-1} A_j x_j \geq 0, x_N \geq 0 \right\} \end{aligned}$$

Since $\bar{x}_B > 0$, all small enough values of x_N are feasible. Thus,

$$c^\top x = c_B^\top \bar{x}_B + \sum_{j \in N} \underbrace{(c_j - c_B^\top A_B^{-1} A_j)}_{\bar{c}_j \equiv \text{reduced cost}} x_j$$

- B optimal for $\max$ problem $\Leftrightarrow$ $c_j - c_B^\top A_B^{-1} A_j \leq 0$ for all $j \in N$
- B optimal for $\min$ problem $\Leftrightarrow$ $c_j - c_B^\top A_B^{-1} A_j \geq 0$ for all $j \in N$
- steepest ascent: increase x_{j^*} where $j^* = \operatorname{argmax}\{c_j - c_B^\top A_B^{-1} A_j\}$
- steepest descent: increase x_{j^*} where $j^* = \operatorname{argmin}\{c_j - c_B^\top A_B^{-1} A_j\}$

Move to an adjacent corner

Suppose we are at the corner (B, N). Suppose $\bar{x}_B = A_B^{-1} b > 0$.

An adjacent corner is one where one basic variable in B is replaced by one non-basic variable from N.

Suppose we increase x_j for some $j \in N$. Then we must have

$$A_B x_B + A_j x_j = b \quad \Rightarrow \quad x_B = \underbrace{A_B^{-1} x_B}_{\bar{x}_B} + \underbrace{(-A_B^{-1} A_j)}_{d} x_j$$

The objective

$$c^\top x = c_B^\top x_B + c_j x_j = c_B^\top \bar{x}_B + (\underbrace{c_j - c_B^\top A_B^{-1} A_j}_{\bar{c}_j}) x_j$$

Move in direction x_j if reduced cost $\bar{c}_j$ has an appropriate sign.

How far can one move in the direction x_j? We need $x_B \geq 0$, therefore

$$x_j \leq \min \left\{ \frac{-\bar{x}_B(k)}{d_k} : d_k < 0 \right\}$$

Move to adjacent corner

In the example, $B = \{3, 4\}$, $\bar{x}_B = \begin{bmatrix} 480 \\ 160 \end{bmatrix}$ and $N = \{1, 2\}$

- $\bar{c}_1 = c_1 - c_B^\top A_B^{-1} A_1 = c_1 = 13$
- $\bar{c}_2 = c_2 - c_B^\top A_B^{-1} A_2 = c_2 = 23$

Steepest ascent direction: x_2

Move in direction x_2: How far can one go?

- $x_1 = 0$
- $x_B = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \bar{x}_B - A_B^{-1} A_2 x_2 = \begin{bmatrix} 480 \\ 160 \end{bmatrix} - \begin{bmatrix} 15 \\ 4 \end{bmatrix} x_2 \geq 0$
- So, $x_2 \leq \min\left\{\frac{480}{15}, \frac{160}{4}\right\} = \min\{32, 40\} = 32$
- Thus, new basis $B = \{2, 4\}$ and $N = \{1, 3\}$.
- What is this corner (geometrically)?

Move to adjacent corner

In the example, $B = \{3, 4\}$, $\bar{x}_B = \begin{bmatrix} 480 \\ 160 \end{bmatrix}$ and $N = \{1, 2\}$

- $\bar{c}_1 = c_1 - c_B^\top A_B^{-1} A_1 = c_1 = 13$
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Move in direction x_2: How far can one go?

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- $x_B = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \bar{x}_B - A_B^{-1} A_2 x_2 = \begin{bmatrix} 480 \\ 160 \end{bmatrix} - \begin{bmatrix} 15 \\ 4 \end{bmatrix} x_2 \geq 0$
- So, $x_2 \leq \min \left\{ \frac{480}{15}, \frac{160}{4} \right\} = \min \{32, 40\} = 32$
- Thus, new basis $B = \{2, 4\}$ and $N = \{1, 3\}$.
- What is this corner (geometrically)? Corner C on page 19

Move to adjacent corner

$B = \{2, 4\}$, $\bar{x}_B = \begin{bmatrix} x_2 \\ x_4 \end{bmatrix} = \begin{bmatrix} 32 \\ 32 \end{bmatrix}$ and $N = \{1, 3\}$

- $\bar{c}_1 = c_1 - c_B^\top A_B^{-1} A_1 = \frac{16}{3}$
- $\bar{c}_3 = c_3 - c_B^\top A_B^{-1} A_3 = -\frac{23}{15}$

Steepest ascent direction: x_1

Move in direction x_2: How far can one go?

- $x_3 = 0$
- $x_B = \begin{bmatrix} x_2 \\ x_4 \end{bmatrix} = \bar{x}_B - A_B^{-1} A_1 x_1 = \begin{bmatrix} 32 \\ 32 \end{bmatrix} - \begin{bmatrix} \frac{1}{3} \\ \frac{8}{3} \end{bmatrix} x_1 \geq 0$
- So, $x_1 \leq \min\{96, 12\} = 12$
- Thus, new basis $B = \{1, 2\}$ and $N = \{3, 4\}$.
- What is this corner (geometrically)?

Move to adjacent corner

$B = \{2,4\}$, $\bar{x}_B = \begin{bmatrix} x_2 \\ x_4 \end{bmatrix} = \begin{bmatrix} 32 \\ 32 \end{bmatrix}$ and $N = \{1,3\}$

- $\bar{c}_1 = c_1 - c_B^\top A_B^{-1} A_1 = \frac{16}{3}$
- $\bar{c}_3 = c_3 - c_B^\top A_B^{-1} A_3 = -\frac{23}{15}$

Steepest ascent direction: x_1

Move in direction x_2: How far can one go?

- $x_3 = 0$
- $x_B = \begin{bmatrix} x_2 \\ x_4 \end{bmatrix} = \bar{x}_B - A_B^{-1} A_1 x_1 = \begin{bmatrix} 32 \\ 32 \end{bmatrix} - \begin{bmatrix} \frac{1}{3} \\ \frac{8}{3} \end{bmatrix} x_1 \geq 0$
- So, $x_1 \leq \min\{96, 12\} = 12$
- Thus, new basis $B = \{1,2\}$ and $N = \{3,4\}$.
- What is this corner (geometrically)? Corner A on page 19

Move to adjacent corner

$B = \{1, 2\}$, $\bar{x}_B = \begin{bmatrix} x_2 \\ x_4 \end{bmatrix} = \begin{bmatrix} 12 \\ 28 \end{bmatrix}$ and $N = \{3, 4\}$

- $\bar{c}_3 = c_3 - c_B^\top A_B^{-1} A_3 = -1$
- $\bar{c}_4 = c_4 - c_B^\top A_B^{-1} A_4 = -2$

Current basis is optimal! Algorithm terminates at Corner A.

Degenerate basic solution

Recall that all we need is that $x_B = A_B^{-1} b \geq 0$.

We call a basic solution x **degenerate** if $x_i = 0$ for some $i \in B$.

At any basic solution

$$c^\top x = c_B^\top x_B + c_j x_j = c_B^\top \bar{x}_B + \sum_{j \in N} \underbrace{(c_j - c_B^\top A_B^{-1} A_j)}_{\bar{c}_j} x_j$$

At degenerate basic solution, there are directions d^j for which the step $x_j = 0$. The solution can be optimal even if some reduced costs do not have the correct sign.

Simplex can **cycle** if one is not careful. Bland's rule ensures that simplex terminates.

Initial feasible basis

Standard form LP

$$\begin{aligned} \max \quad & c^\top x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

Can assume that $b \geq 0$. Why?

Initial feasible basis

Standard form LP

$$\begin{aligned} \text{max} \quad & c^\top x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

Can assume that $b \geq 0$. Why?

Form a new LP (called the Phase I LP)

$$\begin{aligned} v = \quad \text{min} \quad & \mathbf{1}^\top y \\ \text{s.t.} \quad & Au + y = b \\ & u, y \geq 0 \end{aligned}$$

- This LP always has an initial feasible basis: $B = \{y\}$
- Optimal value $v > 0$: Original LP is infeasible! Why?
- Optimal value $v = 0$: Feasible basis for original LP $\equiv$ non-zero components of optimal u^*

Interior point methods

The simplex method goes from corner to corner.

- Worst case running time is exponential in the problem size. (Klee-Minty example)
- Average case running time is polynomial (Smoothed analysis by Spielman and Teng)

Interior point methods: move along the interior of the polytope.

$$\begin{array}{rl} \max & c^\top x, \\ \text{s.t.} & Ax = b, \\ & x \geq 0 \end{array} \qquad \approx \qquad \begin{array}{rl} \max & c^\top x + \mu \sum_{i=1}^n \ln(x_i) \\ \text{s.t.} & Ax = b \end{array}$$

- $\mu \ln(x)$ is a barrier for $x \geq 0$ for all $\mu > 0$
- Interior point methods solve LP by computing $x^*(\mu)$ for $\mu \searrow 0$.
- One never gets to the corner … need "rounding"

Sensitivity Analysis

Two quantities of interest

- solution of the LP $x^*(A, b, c)$
- value of the LP $f(A, b, c) = c^\top x^*(A, b, c)$

How does x^* and f change as a function of small changes b and c?

Sensitivity w.r.t. c: $(\bar{A}, \bar{b}, \bar{c}) \equiv$ current values and $\bar{x}^*$ current optimal

- Recall that for small changes in c
  - x^* remains constant, i.e. $x^* = \bar{x}^*$.
  - $f(c) = c^\top \bar{x}^*$ is a linear function of c and $\boldsymbol{\nabla} f(c) = \bar{x}^*$
- What are the perturbations for which $x^* = \bar{x}^*$?

Sensitivity Analysis

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- solution of the LP $x^*(A, b, c)$
- value of the LP $f(A, b, c) = c^\top x^*(A, b, c)$

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- Recall that for small changes in c
  - x^* remains constant, i.e. $x^* = \bar{x}^*$.
  - $f(c) = c^\top \bar{x}^*$ is a linear function of c and $\nabla f(c) = \bar{x}^*$
- What are the perturbations for which $x^* = \bar{x}^*$?
  - max problem:$c_j - c_B^\top A_B^{-1} A_j \leq 0$ for all $j \in \bar{N}$
  - min problem:$c_j - c_B^\top A_B^{-1} A_j \geq 0$ for all $j \in \bar{N}$

Sensitivity w.r.t. b

$f(b)$ = value of linear program as a function of b

- The optimal solution $x^*(b)$ can never remain constant
- However, the basis B^* can remain invariant as b is perturbed.

The optimal basis B^* at $b = \bar{b}$ remains optimal provided

$$x(b) = A_{B^*}^{-1} b = \bar{x}_{B^*} + A_{B^*}^{-1}(b - \bar{b}) \geq 0$$

For all b such that B^* is optimal, the optimal value

$$f(b) = c_{B^*}^\top x(b) = c_{B^*}^\top A_{B^*}^{-1} b = (A_{B^*}^{-T} c_{B^*})^\top b$$

is a linear function of b with the partial derivative (gradient)

$$v^* = \nabla f(b) = A_{B^*}^{-\top} c_{B^*}$$

Sensitivity analysis for simple example

Range for the c vector for which the current solution is optimal

- c_1: $\bar{c}_2 = 23$

$$c_N - (c_{B^*}^\top A_{B^*}^{-1} A_N)^\top = \begin{bmatrix} 0 \\ 0 \end{bmatrix} - \begin{bmatrix} -0.1 & 0.1 \\ 0.375 & -0.125 \end{bmatrix} \begin{bmatrix} c_1 \\ 23 \end{bmatrix} \leq 0$$

Thus, $7.667 \leq c_1 \leq 23$

- c_2: $\bar{c}_1 = 13$

$$c_N - (c_{B^*}^\top A_{B^*}^{-1} A_N)^\top = \begin{bmatrix} 0 \\ 0 \end{bmatrix} - \begin{bmatrix} -0.1 & 0.1 \\ 0.375 & -0.125 \end{bmatrix} \begin{bmatrix} 13 \\ c_2 \end{bmatrix} \leq 0$$

Thus, $13 \leq c_2 \leq 39$

Sensitivity analysis for simple example

Range of the right hand side vector for which the current basis is optimal

- b_1: $\bar{b}_2 = 160$

$$x = A_{B^*}^{-1} \begin{bmatrix} b_1 \\ 160 \end{bmatrix} = \begin{bmatrix} -0.1 & 0.375 \\ 0.1 & -0.125 \end{bmatrix} \begin{bmatrix} b_1 \\ 160 \end{bmatrix} = \begin{bmatrix} -0.1b_1 + 60 \\ 0.1b_1 - 20 \end{bmatrix} \geq 0$$

  Thus, $200 \leq b_1 \leq 600$

- b_2: $b_1 = 480$

$$x = A_{B^*}^{-1} \begin{bmatrix} 480 \\ b_2 \end{bmatrix} = \begin{bmatrix} -0.1 & 0.375 \\ 0.1 & -0.125 \end{bmatrix} \begin{bmatrix} 480 \\ b_2 \end{bmatrix} = \begin{bmatrix} -48 + 0.375b_2 \\ 48 - 0.125b_2 \end{bmatrix} \geq 0$$

  Thus, $128 \leq b_2 \leq 384$

- Shadow price

$$v^* = (c_{B^*}^\top A_{B^*}^{-1})^\top = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Sensitivity analysis

Question: What is the value of the LP if $b_1 = 500$? Is this exact or approximate?

New $b_1 = 500 \in [200, 600]$ in the RHS range for b_1. Therefore,

$$f(b_1) = f(\bar{b}_1) + v_1^*(b_1 - \bar{b}_1) = 800 + (500 - 480) = 820$$

Question: What is we had to pay 0.5/unit to purchase the extra 20 units of b_1. Is it worth it?

Yes. Since $v_1 = 1 > 0.5$.

Question: What is the value of the LP if $b_2 = 120$? Is this exact or approximate?

New $b_2 = 120 \notin [128, 384]$ in the RHS range for b_1. Therefore, the partial derivatives are not valid. Will return to this topic later in the course.

Sensitivity analysis

Question: The company introduces a new product that uses 4 units of resource 1 and 3 units of resource 2. What is the minimum price p at which it is optimal to produce this product?

Approach 1: Use reduced costs

$$\bar{c}_3 = p - c_B^\top A_B^{-1} A_3 = p - \begin{bmatrix} 13 & 23 \end{bmatrix} \begin{bmatrix} 5 & 15 \\ 4 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ 3 \end{bmatrix} = p - 10.$$

Optimal to produce new product when $p \geq 10$

Approach 2: Use shadow prices. Produce an infinitesimal amount δ of the new product. Two different impacts:

- Get revenue from the product: $p\delta$
- Change in resources $-A_3\delta$: change in revenue $= (A_3^\top v^*)\delta$
- Net change: $(p - A_3^\top v^*)\delta = (p - 10)\delta$

Sensitivity for the dedication example

Question: What is the impact on the cost of the portfolio if the liability in year 4 goes up to 22K?

2K is in the allowable increase. Therefore, the δcost $= 2v_4^* = 1.672$K

Question: How will the portfolio and cost change if the price p_6 of bond 6 increases by 20¢?

0.2 is in the allowable range. Therefore, optimal portfolio will not change. Therefore, δcost $= \delta p_6 x_6^* = (0.2)(0.123) = 0.00246$K

Question: What is the implied yield curve faced by the portfolio manager?

$$
\begin{aligned}
v_t^* &= \frac{\partial \mathsf{obj}}{\partial \ell_t} = \text{ cost of financing 1 unit of liability at time } t \\
&= \frac{1}{(1+r_t)^t} \quad \Rightarrow \quad r_t = e^{-\frac{1}{t}\ln(v_t^*)} - 1.
\end{aligned}
$$

Sensitivity for the dedication example

Question: At what interest rate would the portfolio manager hold cash?

Suppose interest rate $= r$ and the investor holds δ units of cash at time t.

- At time t: bonds have to fund a "net" liability $= \ell_t + \delta$
- A time $t+1$: the net liability $= \ell_{t+1} - (1+r)\delta$
- Change in cost of portfolio $= \delta(v_t^* - (1+r)v_{t+1}^*)$
- Will hold cash at time t if $r \geq \frac{v_t^*}{v_{t+1}^*} - 1$
- Will hold cash at some point if $r \geq \min_{0 \leq t \leq 7} \left\{ \frac{v_t^*}{v_{t+1}^*} - 1 \right\}$, $v_0^* = 1$.

Sensitivity analysis for short-term financing

Question: What is the impact if the liability in January goes *up* by 50K?

Since $\delta b_{\mathsf{Jan}} = 50 \leq 135$, it follows that $\delta \mathsf{obj} = v^*_{\mathsf{Jan}} \delta b_{\mathsf{Jan}} = -51.86$

Question: Would a higher credit limit help?

No. The credit limit constraints are slack, therefore impact is zero.

Question: Suppose the vendor could move all/part of the liability in Jan to Jun by paying a total interest r. When is this profitable?

Two liabilities change simultaneously

- $\delta b_1 = -\delta$
- $\delta b_6 = (1 + r)\delta$

Therefore, the net change in objective $\delta \mathsf{obj} = (-v_1^* + (1 + r)v_6^*)\delta \geq 0$ only if $r \leq \frac{v_1^*}{v_6^*} - 1 = 3.72\%$

Sensititivity analysis for short term financing

Question: How much of the liability should be deferred? Are your calculations exact?

Since two liabilities are changing simultaneously, the RHS ranges cannot ordinarily be used to compute the amount moved.

However, June is the terminal month so it not "really" a constraint. Only the allowable decrease in Jan matters, and since allowable decrease is 150, all the liability can be deferred.

Interpreting sensitivity tables

General linear program

$$\begin{array}{rll} \text{min} & c^\top x & \\ \text{s.t.} & Ax <> b & : u \geq 0 \\ & x <> \{0, \text{free}\} & \end{array}$$

x^* = optimal primal solution $\qquad u^*$ = optimal dual solution

Sensitivity with respect to objective vector: current obj $= \bar{c}$

Variable	Red. Cost	Allow Dec.	Cur. Val.	Allow Inc.

- Reduced cost: impact on objective (meaningful when LP in std form)
- Allow Inc/Dec:
  - Range for coefficient c_j for which x^* is constant
  - All other coefficients $c_k = \bar{c}_k$ for $k \neq j$

Sensitivity tables (contd)

Objective sensitivity ranges are **not** valid when ≥ 2 components change

x^* optimal for $\bar{c} + \delta c$ if dual feasible (u, v) that satisfies CS with x^*

Sensitivity tables (contd)

Objective sensitivity ranges are **not** valid when ≥ 2 components change

x^* optimal for $\bar{c} + \delta c$ if dual feasible (u, v) that satisfies CS with x^*

Sensitivity with respect to RHS vector: current rhs $= \bar{b}$

Constraint	Dual	Allow Dec.	Cur. Val.	Allow Inc.

- Dual: change in objective per unit change in rhs component
- Allow Inc/Dec:
  - Range for rhs component b_i for which the dual u^* is constant
  - All other components $b_k = \bar{b}_k$ for $k \neq i$

RHS sensitivity ranges are **not** valid when ≥ 2 components change

u^* optimal for $\bar{b} + \delta b$ if primal feasible x that satisfies CS with u^*

Duality

Will show that $v_i = \left.\frac{\partial f}{\partial b_i}\right|_{b_i = \bar{b}_i}$ are the optimal solution to another LP!

Dual linear program of standard form LP: v_i for each constraint in (P).

$$
\begin{aligned}
(P) = \quad \max \quad & c^\top x \\
\text{s.t.} \quad & Ax = b \\
& x \geq 0
\end{aligned}
\qquad \Rightarrow \qquad
\begin{aligned}
(D) = \quad \min \quad & b^\top v \\
\text{s.t.} \quad & A_j^\top v \geq c_j
\end{aligned}
$$

Weak and strong duality

Suppose x is feasible for primal LP and v is feasible for the dual LP.

$$c^\top x \leq (A^\top v)^\top x \leq v^\top (Ax) = v^\top b$$

Thus, $P \leq D$. This is called weak duality.

Suppose $P < \infty$, and B is an optimal basis.

- $x_B^* = A_B^{-1} b$ and $P = c_B^\top A_B^{-1} b$
- B optimal basis $\Rightarrow c^\top - c_B^\top A_B^{-1} A \leq 0 \Rightarrow A^\top (A_B^{-\top} c_B) \geq c$
- $v = A_B^{-T} c_B$ feasible for D, and

$$D \leq b^\top v = b^\top A_B^{-\top} c_B = c_B^\top A_B^{-1} b = P \quad \Rightarrow \quad v \text{ is optimal}$$

- Thus, $D = P$. This is called strong duality.

Complementary slackness (CS)

Suppose x^* and v^* are primal-dual optimal. Then

$$\begin{aligned} c^\top x^* &= b^\top v^* = (Ax^*)^\top v^* \\ &= \sum_{j=1}^n x_j^* A_j^\top v^* \quad \Rightarrow \quad \sum_{j=1}^n (A_j^\top v^* - c_j) x_j^* = 0. \end{aligned}$$

Since $A_j^* v^* \geq c_j$ and $x_j^* \geq 0$, it follows that $(A_j^\top v^* - c_j) x_j^* = 0$ for all j. This is called complementary slackness.

Suppose x and v are primal-dual feasible, and satisfy complementary slackness. Then

$$0 = \sum_{j=1}^n x_j (A_j^\top v - c_j) = \Big(\sum_{j=1}^n x_j A_j\Big)^\top v - c^\top x = b^\top v - c^\top x$$

Thus, x and v are primal-dual optimal.

Duals of general linear programs

Suppose primal LP has

- m main constraints
- d variables

Rules for constructing duals

- A dual variable for each primal main constraint
- Objective: sense max $\leftrightarrow$ min and function $\sum_{i=1}^{m} b_i v_i = b^\top v$
- A dual main constraint for each primal variable x_j: sense undetermined

$$\sum_{i=1}^{m} a_{ij} v_i = A_j^\top v \quad <> \quad c_j$$

- A dual variable constraint for each primal main constraint

$$v_i \quad <> \quad \{0, \text{free}\}$$

Duals (contd)

- Sensible constraint $\Leftrightarrow$ sensible variables, etc.

	Constraint		Variable
	max	min	
Sensible	$\leq$	$\geq$	≥ 0
Odd	$=$	$=$	free
Bizarre	$\geq$	$\leq$	≤ 0

- Complementary slackness (CS) conditions:

(primal main constraint)*(dual variable) $= 0$, and vice versa

Duals (contd)

- Sensible constraint $\Leftrightarrow$ sensible variables, etc.

	Constraint		Variable
	max	min	
Sensible	$\leq$	$\geq$	$\geq\ 0$
Odd	$=$	$=$	free
Bizarre	$\geq$	$\leq$	$\leq\ 0$

- Complementary slackness (CS) conditions:

  (primal main constraint)*(dual variable) $=$ 0, and vice versa

$$\begin{array}{ll} \text{max} & 13x_1 + 23x_2 + 20x_3 \\ \text{s.t.} & 5x_1 + 15x_2 + 12x_3 \leq 480 \\ & 4x_1 + 4x_2 + 5x_3 = 160 \\ & x_1, x_2 \geq 0, x_3 \text{ free} \end{array}$$

$$\begin{array}{ll} \text{min} & 480v_1 + 160v_2 \\ \text{s.t.} & 5v_1 + 4v_2 <> 13 \\ & 15v_1 + 4v_2 <> 23 \\ & 12v_1 + 5v_2 <> 20 \\ & v_1 <> 0, v_2 <> 0 \end{array}$$

Duals (contd)

- Sensible constraint $\Leftrightarrow$ sensible variables, etc.

	Constraint		Variable
	max	min	
Sensible	$\leq$	$\geq$	$\geq\ 0$
Odd	$=$	$=$	free
Bizarre	$\geq$	$\leq$	$\leq\ 0$

- Complementary slackness (CS) conditions:

(primal main constraint)*(dual variable) = 0, and vice versa

$$\begin{array}{ll} \text{max} & 13x_1 + 23x_2 + 20x_3 \\ \text{s.t.} & 5x_1 + 15x_2 + 12x_3 \leq 480 \\ & 4x_1 + 4x_2 + 5x_3 = 160 \\ & x_1, x_2 \ \geq \ 0, x_3 \text{ free} \end{array}$$

$$\begin{array}{ll} \text{min} & 480v_1 + 160v_2 \\ \text{s.t.} & 5v_1 + 4v_2 \ \geq \ 13 \\ & 15v_1 + 4v_2 \ \geq \ 23 \\ & 12v_1 + 5v_2 = 20 \\ & v_1 \geq 0, v_2 \text{ free} \end{array}$$

Strong duality yields sensitivity analysis

$$
\begin{array}{rclllcll}
f(b) & = & \max & c^\top x & & = & \min & b^\top v \\
 & & \text{s.t.} & Ax = b & & & \text{s.t.} & A^\top v \geq c \\
 & & & x \geq 0 & & & &
\end{array}
$$

$\bar{v}^*$ will remain optimal for small perturbations of the b vector, i.e.

$$f(b) = b^\top \bar{v}^* = \bar{b}^\top \bar{v}^* + (\bar{v}^*)^\top (b - \bar{b}) = f(\bar{b}) + (\bar{v}^*)^\top (b - \bar{b})$$

Thus, $\bar{v}^*$ is the partial derivative at $\bar{b}$.

What are the perturbations for which $\bar{v}^*$ remains optimal?

- Find a primal feasible x that satisfies CS with $\bar{v}^*$

Computing duals from complementary slackness

2 variable LP and its dual

$$\begin{array}{ll} \text{max} & 13x_1 + 23x_2 \\ \text{s. t.} & 5x_1 + 15x_2 \le 480 \\ & 4x_1 + 4x_2 \le 160 \\ & x_1, x_2 \ge 0 \end{array} \qquad \begin{array}{ll} \text{min} & 480v_1 + 160v_2 \\ \text{s.t.} & 5v_1 + 4v_2 \ge 13 \\ & 15v_1 + 4v_2 \ge 23 \\ & v_1, v_2 \ge 0 \end{array}$$

$x^* = (12, 28)$ is optimal if only if there exists v satisfying CS

$$\left.\begin{array}{l} x_1 > 0 \Rightarrow 5v_1 + 4v_2 = 13 \\ x_2 > 0 \Rightarrow 15v_1 + 4v_2 = 23 \end{array}\right\} \quad \Rightarrow \quad v = (1, 2)$$

Feasible for the variable constraints, therefore optimal.

For what values of b is v^* optimal?

$$\left.\begin{array}{l} v_1^* > 0 \Rightarrow 5x_1 + 15x_2 = b_1 \\ v_2^* > 0 \Rightarrow 4x_1 + 4x_2 = b_2 \end{array}\right\} \Rightarrow x = \left(\frac{15b_2 - 4b_1}{40}, \frac{4b_1 - 5b_2}{40}\right)$$

Need to ensure that $x \ge 0$

Ranges for the RHS

Range for b_1

$$x_1 \geq 0 \Rightarrow 15\bar{b}_2 - 4b_1 \geq 0 \Rightarrow b_1 \leq \frac{15\bar{b}_2}{4} = 600$$

$$x_2 \geq 0 \Rightarrow 4b_1 - 5\bar{b}_2 \geq 0 \Rightarrow b_1 \geq \frac{5\bar{b}_2}{4} = 200$$

Note that b_2 is constant when we change b_1

Ranges for the RHS

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Note that b_2 is constant when we change b_1

Range for b_2

$$x_1 \geq 0 \Rightarrow 15b_2 - 4\bar{b}_1 \geq 0 \Rightarrow b_2 \geq \frac{4\bar{b}_1}{15} = 128$$

$$x_2 \geq 0 \Rightarrow 4\bar{b}_1 - 5b_2 \geq 0 \Rightarrow b_2 \leq \frac{4\bar{b}_1}{5} = 384$$

Note that b_1 is constant when we change b_2

Ranges for the RHS

Range for b_1

$$x_1 \geq 0 \Rightarrow 15\bar{b}_2 - 4b_1 \geq 0 \Rightarrow b_1 \leq \frac{15\bar{b}_2}{4} = 600$$

$$x_2 \geq 0 \Rightarrow 4b_1 - 5\bar{b}_2 \geq 0 \Rightarrow b_1 \geq \frac{5\bar{b}_2}{4} = 200$$

Note that b_2 is constant when we change b_1

Range for b_2

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$$x_2 \geq 0 \Rightarrow 4\bar{b}_1 - 5b_2 \geq 0 \Rightarrow b_2 \leq \frac{4\bar{b}_1}{5} = 384$$

Note that b_1 is constant when we change b_2

What if $b = \bar{b} + \theta \begin{bmatrix} 1 & -1 \end{bmatrix}^\top$

$$x = \bar{x} + \theta \begin{bmatrix} 5 & 15 \\ 4 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \bar{x} + \begin{bmatrix} -0.475 \\ 0.225 \end{bmatrix} \theta \geq 0 \Rightarrow -\frac{\bar{x}_2}{0.225} \leq \theta \leq \frac{\bar{x}_1}{0.475}$$

Another example of duality

Primal linear program

$$
\begin{array}{rrcrcrl}
\max & c_1x_1 & + & c_2x_2 & & & \\
\text{s.t.} & -3x_1 & + & 4x_2 & \leq & 4 & u_1 \\
 & 3x_1 & + & 4x_2 & \leq & 16 & u_2 \\
 & -x_1 & + & x_2 & \geq & -3 & u_3 \\
 & x_1 & & x_2 & \geq & 0 &
\end{array}
$$

Dual linear program

$$
\begin{array}{rrcrcrcrl}
\max & 4u_1 & + & 16u_2 & - & 3u_3 & & & \\
\text{s.t.} & -3u_1 & + & 3u_2 & - & u_3 & \geq & c_1 & x_1 \\
 & 4u_1 & + & 4u_2 & + & u_3 & \geq & c_2 & x_2 \\
 & u_1 & & & & & \geq & 0 & \text{1st main constraint} \\
 & & & u_2 & & & \geq & 0 & \text{2nd main constraint} \\
 & & & & & u_3 & \leq & 0 & \text{3rd main constraint}
\end{array}
$$

Another example of duality (contd)

Complementary slackness conditions:

$$\begin{array}{lrcl}
(1) & (-3x_1 + 4x_2 - 4)u_1 & = & 0 \\
(2) & (3x_1 + 4x_2 - 16)u_2 & = & 0 \\
(3) & (-x_1 + x_2 + 3)u_3 & = & 0
\end{array}$$

$$\begin{array}{lrcl}
(4) & (-3u_1 + 3u_2 - u_3 - c_1)x_1 & = & 0 \\
(5) & (4u_1 + 4u_2 + u_3 - c_1)x_2 & = & 0
\end{array}$$

Show that $x = (4, 1)$ is optimal for $c = (1, 1)$

$$(1) \Rightarrow u_1 = 0$$

$$(4) \text{ and } (5) \Rightarrow \left.\begin{array}{rcl} 3u_2 - u_3 & = & 1 \\ 4u_2 + u_3 & = & 1 \end{array}\right\} \Rightarrow \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Another example of duality (contd)

Set of c vectors for which $x = (4, 1)$ is optimal?

$$(4) \text{ and } (5) \Rightarrow \left.\begin{array}{rcl} 3u_2 - u_3 & = & c_1 \\ 4u_1 + u_3 & = & c_2 \end{array}\right\} \Rightarrow \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} c_1 + c_2 \\ -4c_1 + 3c_2 \end{bmatrix}$$

For this u to be feasible, we must have

$$\begin{array}{rcl} u_2 \geq 0 & \Rightarrow & c_1 + c_2 \geq 0 \\ u_3 \leq 0 & \Rightarrow & 3c_2 \leq 4c_1 \end{array}$$

Another example of duality (contd)

Set of b vectors for which $u = \frac{1}{7}(0, 2, -1)$ is optimal?

$$(2) \text{ and } (3) \Rightarrow \left.\begin{array}{rcl} 3x_1 + 4x_2 & = & b_2 \\ -x_1 + x_2 & = & b_3 \end{array}\right\} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} b_2 - 4b_3 \\ b_2 + 3b_3 \end{bmatrix}$$

For this x to be feasible, we must have

$$\begin{array}{rcl} x_1 \geq 0 & \Rightarrow & b_2 \geq 4b_3 \\ x_2 \geq 0 & \Rightarrow & b_2 \geq -3b_3 \\ -3x_1 + 4x_2 \leq b_3 & \Rightarrow & b_2 + 12b_3 \leq 7b_1 \end{array}$$

Martingale pricing via LPs

2 period market with d assets

- 1 state at time 0: prices $p = (p_1, \ldots, p_d)^\top$
- m states at time 1: payoffs

$$S = \begin{bmatrix} S_1(\omega_1) & S_2(\omega_1) & \ldots & S_d(\omega_1) \\ S_1(\omega_2) & S_2(\omega_2) & \ldots & S_d(\omega_2) \\ \vdots & \vdots & \ddots & \vdots \\ S_1(\omega_m) & S_2(\omega_m) & \ldots & S_d(\omega_m) \end{bmatrix}$$

Positions $\theta = (\theta_1, \ldots, \theta_d)^\top$

- Price at time 0: $p^\top \theta$
- Payoff at time 1: $S\theta \in \mathbb{R}^m$ (one for each possible state)

Types of arbitrage

Type A arbitrage: A position θ with negative cost (i.e. positive payoff at time 0) that has non-negative payoff (i.e. no obligation) in all states of the world, i.e.

$$p^\top \theta < 0 \qquad S\theta \geq 0$$

Type B arbitrage: A position θ with non-positive cost (i.e. non-negative payoff at time 0) that has non-negative payoff (i.e. no obligation) in all states of the world, and a strictly positive payoff in some state of the world.

$$p^\top \theta \leq 0 \qquad S\theta \geq 0 \qquad S\theta \neq 0$$

Assumption: No Type A or B arbitrage in the market. Reasonable?

No arbitrage and state prices

Theorem: A 2-period market has no arbitrage if, and only if, it has a state price deflator, i.e. a vector $\pi \in \mathbb{R}^m$ such that

$$\pi > 0 \qquad p = S^\top \pi \quad \Leftrightarrow \quad p_j = \sum_{i=1}^{m} S_j(\omega_i)\pi_i$$

Suppose there exists a state price deflator π.

- θ such that $S\theta \geq 0$

$$0 \leq \pi^\top (S\theta) = (S^\top \pi)^\top \theta = p^\top \theta \quad \Rightarrow \quad \text{No Type A arb}$$

- θ such that $S\theta \geq 0$ and $S\theta \neq 0$

$$0 < \pi^\top (S\theta) = (S^\top \pi)^\top \theta = p^\top \theta \quad \Rightarrow \quad \text{No Type B arb}$$

Need to show the other direction.

No arbitrage and LPs

Define $M = \begin{bmatrix} -p^\top \\ S \end{bmatrix}$.

$$\begin{aligned} \text{No arbitrage} \quad &\Leftrightarrow \quad \nexists\ \theta \text{ such that } \; M\theta \geq 0, \quad M\theta \neq 0 \\ &\Leftrightarrow \quad \text{For all } i = 1, \ldots, m+1,\ \nexists\ \theta \text{ such that } \; M\theta \geq e^{(i)} \end{aligned}$$

where $e^{(i)} \in \mathbb{R}^{m+1}$ with the i-th component equal to 1 and the rest 0.

Primal-dual pair of LPs: Weak Duality $P \geq D$

$$\begin{aligned} P_i = \; & \min \quad 0^\top \theta \\ & \text{s.t.} \quad M\theta \geq e^{(i)} \end{aligned} \qquad\qquad \begin{aligned} D_i = \; & \max \quad (e^{(i)})^\top y = y_i \\ & \text{s.t.} \quad M^\top y = 0 \\ & \qquad\;\; y \geq 0 \end{aligned}$$

No arbitrage $\Rightarrow P_i = \infty$. Therefore, $D_i = +\infty$ or $-\infty$. But $y = 0$ feasible, therefore $D_i = +\infty$.

No arbitrage and LPs

$D_i = \infty$ implies that there exists $y^{(i)} \in \mathbb{R}^{m+1}$ such that

$$y_i^{(i)} > 0 \qquad M^\top y^{(i)} = 0 \qquad y^{(i)} \geq 0$$

Let $\bar{y} = \sum_{i=1}^{m+1} y^{(i)}$. Then $\bar{y}_i > 0$, for all i, and $M^\top \bar{y} = 0$

Define $\pi = \frac{1}{\bar{y}_1} \begin{bmatrix} \bar{y}_2 \\ \vdots \\ \bar{y}_{m+1} \end{bmatrix}$. Then $\pi > 0$, and

$$0 = M^\top \bar{y} = \begin{bmatrix} -p & S^\top \end{bmatrix} \bar{y} = -\bar{y}_1 p + S^\top \begin{bmatrix} \bar{y}_2 \\ \vdots \\ \bar{y}_{m+1} \end{bmatrix} \quad \Rightarrow \quad p = S^\top \pi$$

π is a state price deflator.

State prices and Martingale measures

Suppose $S_1(\omega) > 0$ for all $\omega \in \{1, \ldots, m\}$

- S_1 is called a numeraire security
- For example, S_1 could be risk-free security. Then $S_1(\omega) = (1 + r)$.

Since $p_j = \sum_{i=1}^{m} S_j(\omega_i)\pi_i$, we have

$$\frac{p_i}{p_1} = \frac{\sum_{j=1}^{m} S_j(\omega_i)\pi_i}{p_1} = \sum_{i=1}^{m} \frac{S_j(\omega_i)}{S_1(\omega_i)} \cdot \underbrace{\frac{S_1(\omega_i)\pi_i}{p_1}}_{\equiv q_i}$$

State prices and Martingale measures

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Claim: q is a probability mass function

$$q > 0 \qquad \sum_{i=1}^m q_i = \frac{1}{p_1} \sum_{i=1}^m S_1(\omega_i)\pi_i = \frac{1}{\cancel{p_1}} \cdot \cancel{p_1} = 1.$$

State prices and Martingale measures

Suppose $S_1(\omega) > 0$ for all $\omega \in \{1, \ldots, m\}$

- S_1 is called a numeraire security
- For example, S_1 could be risk-free security. Then $S_1(\omega) = (1 + r)$.

Since $p_j = \sum_{i=1}^m S_j(\omega_i)\pi_i$, we have

$$\frac{p_i}{p_1} = \frac{\sum_{j=1}^m S_j(\omega_i)\pi_i}{p_1} = \sum_{i=1}^m \frac{S_j(\omega_i)}{S_1(\omega_i)} \cdot \underbrace{\frac{S_1(\omega_i)\pi_i}{p_1}}_{\equiv q_i} = \mathbb{E}_q\left[\frac{S_j}{S_1}\right]$$

Claim: q is a probability mass function

$$q > 0 \qquad \sum_{i=1}^m q_i = \frac{1}{p_1}\sum_{i=1}^m S_1(\omega_i)\pi_i = \frac{1}{\cancel{p_1}} \cdot \cancel{p_1} = 1.$$

Large scale LPs: many constraints

Consider the linear program

$$\begin{aligned} z^* = \ \min_x \ & c^\top x \\ \text{s.t.} \ & Hx = g \\ & a_i^\top x \geq b_i, \quad i = 1, \dots, m \end{aligned}$$

where $x \in \mathbb{R}^d$, $d \approx 100$, and $m \approx 10^6$. We would like so solve this iteratively.

Form a smaller LP where we only take a subset of the variables:

$$\begin{aligned} z_S^* = \ \min_x \ & c^\top x \\ \text{s.t.} \ & Hx = g \\ & a_i^\top x \geq b_i, \quad i \in S. \end{aligned}$$

Let x_S^* denote the optimal solution of this LP.

- Can we determine whether this solution is optimal for the full LP?
- If not, is there a way to update the LP?

Large scale LPs: many constraints

Suppose $a_i^\top x_S^* \geq b_i$ **for all** $i = 1, \ldots, m$

- x_S^* is feasible for the full LP
- Smaller LP has fewer constraints; $c^\top x_S^* = z_S^* \leq z^*$
- x_S^* is optimal

If not, choose a constraint j with $a_j^\top x_S^* < b_j$

- Update $S \leftarrow S \cup \{j\}$
- Recompute x_S^*

Large scale LPs: many variables

Consider the linear program

$$
\begin{aligned}
z^* = \quad \min_x \quad & c^\top x \\
\text{s.t.} \quad & Ax = b, \\
& x \geq 0
\end{aligned}
$$

where $x \in \mathbb{R}^d$, $d \approx 10^6$, and $b \in \mathbb{R}^m$ with $m \approx 100$. We would like so solve this iteratively.

Form a smaller LP where we only take a subset of the variables:

$$
\begin{aligned}
z_S^* = \quad \min_x \quad & \textstyle\sum_{i \in S} c_i x_i^* \\
\text{s.t.} \quad & \textstyle\sum_{i \in S} A_i x_i = b \quad & : v^* \\
& x_i \geq 0, & i \in S.
\end{aligned}
$$

Let x_S^* denote the optimal solution of this LP. And v^* the optimal dual.

- Can we determine whether this solution is optimal for the full LP?
- If not, is there a way to update the LP?

Large scale LPs: many variables

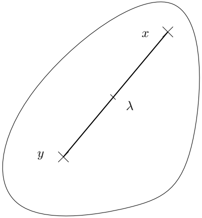
Suppose the reduced costs of the variables $\ell \notin S$ are all positive.

- Recall the reduced cost $\bar{c}_l = c_l - A_l^\top v^*$
- x_S^* is optimal since all other directions are suboptimal

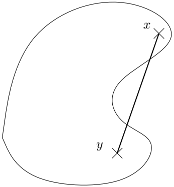
If not, choose a variable ℓ with $\bar{c}_\ell = c_\ell - A_\ell^\top v^* < 0$.

- Update $S \leftarrow S \cup \{\ell\}$
- Recompute x_S^*

Convex sets and functions



Convex set



Non-convex set

C is convex set, if and only if, for all $x, y \in C$, $\lambda \in [0,1]$

$$\lambda x + (1-\lambda)y \in C$$

Set contains line segment $[x, y]$

Intersection property: C_1, C_2 convex $\Rightarrow C_1 \cap C_2$ convex.

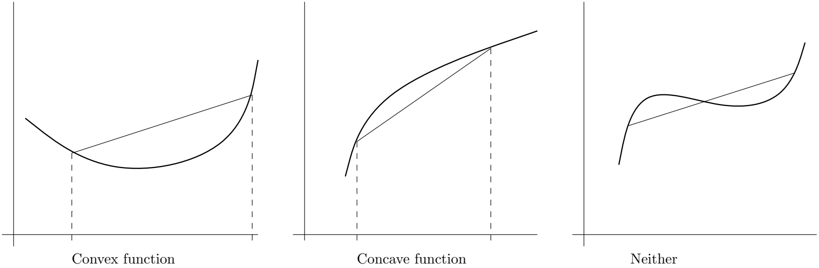
$f : \mathbb{R}^d \mapsto \mathbb{R}$ is a convex function, if and only if, for all x, y, and $\lambda \in [0,1]$

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$$

The line segment $[f(x), f(y)]$ lies above the function.

$f : \mathbb{R}^d \mapsto \mathbb{R}$ is a concave function iff $-f(x)$ is convex.

Convex functions



Properties:

- f_1, f_2 convex (concave), $\alpha_1, \alpha_2 \geq 0 \Rightarrow \alpha_1 f_1 + \alpha_2 f_2$ convex (concave)
- f_1, f_2 convex (concave) $\Rightarrow \max\{f_1, f_2\}$ convex ($\min\{f_1, f_2\}$ concave)
- f convex (concave) $\Rightarrow \{x : f(x) \leq \beta\}$ convex ($\{x : f(x) \geq \beta\}$ convex)

Simple examples of convex sets and functions

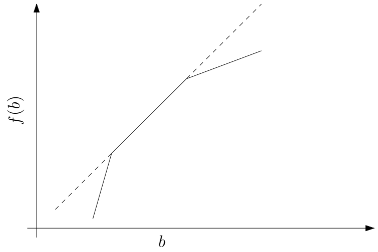
Convex functions

- $f(x) = c^\top x$ is both convex and concave
- $f(x) = \max\{c_1^\top x, c_2^\top x\}$ is convex
- $f(x) = \min\{c_1^\top x, c_2^\top x\}$ is concave
- $f(x) = x^\top Q x$ is convex is Q has non-negative eigenvalues

Convex sets

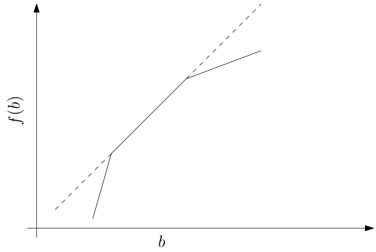
- halfspace $\{x : a^\top x \leq b\}$ is convex
- hyperplane $\{x : a^\top x = b\}$ is convex
- polytope $\{x : a_i^\top x \leq b_i, i = 1, \ldots, m\}$ = intersection of halfspaces

Behavior of LP outside the range



$$
\begin{aligned}
f(b) &= \max \; c^\top x \\
&\quad \text{s.t.} \; Ax <> b \\
&= \min \; b^\top v \\
&\quad \text{s.t.} \; A^\top v = c \\
&\qquad\quad v <> 0 \\
&= \min_{\{v: A^\top v = c, v <> 0\}} \{v^\top b\} \\
&= \text{piece-wise linear concave}
\end{aligned}
$$

Behavior of LP outside the range



$$
\begin{aligned}
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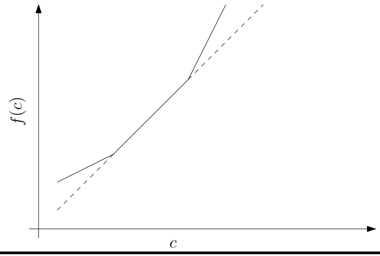
$$



$$
\begin{aligned}
f(c) &= \quad \max \quad c^\top x \\
&\qquad\ \ \text{s.t.} \quad Ax <> b \\
&= \text{piece-wise linear convex}
\end{aligned}
$$

Convex optimization problems

Come in two varieties

$$\begin{array}{rl} \max & f(x) \text{ concave} \\ \text{s.t.} & x \in C \text{ convex} \end{array} \qquad \qquad \begin{array}{rl} \min & f(x) \text{ convex} \\ \text{s.t.} & x \in C \text{ convex} \end{array}$$

Theorem: When the function f is piece-wise linear and the convex set is defined by piece-wise linear functions, a convex optimization problem can be reformulated into an LP.

Scenario based portfolio selection (contd.)

Extend to $d \approx 100$ and number of scenarios $m \approx 10,000$

$$\begin{array}{ll} \text{min} & c^\top x \\ \text{s. t.} & Sx \geq r \\ & x \geq 0 \end{array}$$

Problem with this formulation?

Scenario based portfolio selection (contd.)

Extend to $d \approx 100$ and number of scenarios $m \approx 10,000$

$$\begin{array}{ll} \text{min} & c^\top x \\ \text{s. t.} & Sx \geq r \\ & x \geq 0 \end{array}$$

Problem with this formulation? May be infeasible or very expensive

Scenario based portfolio selection (contd.)

Extend to $d \approx 100$ and number of scenarios $m \approx 10,000$

$$\begin{array}{rl} \text{min} & c^\top x \\ \text{s. t.} & Sx \geq r \\ & x \geq 0 \end{array}$$

Problem with this formulation? May be infeasible or very expensive

Relaxed formulation

- Shortfall in scenario i: $\max\left\{r_i - \sum_{j=1}^{d} S_{ij} x_j, 0\right\}$
- Shortfall vector: $\max\left\{r - Sx, 0\right\}$
- Expected Shortfall is 1% of expected requirement

$$\mathbf{1}^\top \max\left\{r - Sx, 0\right\} \leq 0.01(\mathbf{1}^\top r)$$

- Optimization problem: not a linear program!

$$\begin{array}{rl} \text{min} & c^\top x \\ \text{s. t.} & \mathbf{1}^\top \max\left\{r - Sx, 0\right\} \leq 0.01(\mathbf{1}^\top r) \\ & x \geq 0 \end{array}$$

LP reformulation

$\mathbf{1}^\top \max\{r - Sx, 0\} =$ sum of piece-wise linear convex functions.

Introduce a new variable

$$z \geq \max\{r - Sx, 0\}$$

Equivalent to two sets of linear constraints.

LP reformulation

$\mathbf{1}^\top \max\{r - Sx, 0\}$ = sum of piece-wise linear convex functions.

Introduce a new variable

$$z \geq \max\{r - Sx, 0\} \quad \Leftrightarrow \quad z \geq r - Sx, \quad z \geq 0$$

Equivalent to two sets of linear constraints.

LP reformulation

$\mathbf{1}^\top \max\{r - Sx, 0\}$ = sum of piece-wise linear convex functions.

Introduce a new variable

$$z \geq \max\{r - Sx, 0\} \quad \Leftrightarrow \quad z \geq r - Sx, \quad z \geq 0$$

Equivalent to two sets of linear constraints.

Linear programming formulation

$$\begin{array}{ll} \text{min} & c^\top x \\ \text{s. t.} & \mathbf{1}^\top z \leq 0.01(\mathbf{1}^\top r) \\ & z + Sx \geq r, \\ & x, z \geq 0 \end{array}$$

Very close to the conditional Value-at-Risk portfolio selection

Computing state price vectors

Two period market with d assets

- price at time t_0: $p \in \mathbb{R}^d$
- payoff matrix for m states at time t_1: $S \in \mathbb{R}^{m \times d}$

No arbitrage $\Leftrightarrow$ there exists $\pi \in \mathbb{R}^m$ such that $\pi > 0$ and $S^\top \pi = p$

Perhaps solve the LP: What is the problem here?

$$
\begin{aligned}
\max \quad & 0^\top \pi \\
\text{s.t.} \quad & S^\top \pi = p \\
& \pi \geq 0
\end{aligned}
$$

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- price at time t_0: $p \in \mathbb{R}^d$
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No arbitrage $\Leftrightarrow$ there exists $\pi \in \mathbb{R}^m$ such that $\pi > 0$ and $S^\top \pi = p$

Perhaps solve the LP: What is the problem here?

$$\begin{aligned} \max \quad & 0^\top \pi \\ \text{s.t.} \quad & S^\top \pi = p \\ & \pi \geq 0 \end{aligned}$$

Want to maximize the minimum of π: objective should be

$$\max \quad \min_{1 \leq i \leq m} \{\pi_i\}$$

Computing state price vectors

Optimization problem: Convex problem?

$$\begin{array}{rl} \max & \min_{1\leq i\leq m}\{\pi_i\} \\ \text{s.t.} & S^\top \pi = p \\ & \pi \geq 0 \end{array}$$

Computing state price vectors

Optimization problem: Convex problem?

$$\begin{array}{rl} \max & \min_{1 \le i \le m}\{\pi_i\} \\ \text{s.t.} & S^\top \pi = p \\ & \pi \ge 0 \end{array}$$

Introduce a new variable: $\beta \le \min_{1 \le i \le m}\{\pi_i\}$ or equivalently

$$\beta \le \pi_i \quad i = 1, \ldots, m$$

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Equivalent linear program

$$\begin{array}{rl} \max & \beta \\ \text{s.t.} & \pi - \beta \mathbf{1} \ge 0 \\ & S^\top \pi = p \\ & \pi \ge 0 \end{array}$$

Bounds on option prices in incomplete markets

Upper bound on a security with payoff a

$$
\begin{array}{rlcrl}
\min & p^\top \theta & = & \max & a^\top \pi, \\
\text{s.t.} & S\theta \geq a & & \text{s.t.} & S^\top \pi = p, \\
 & & & & \pi \geq 0.
\end{array}
$$

Let π^* denote the optimal solution and let $I = \{i : \pi_i^* > 0\}$. Then CS implies that $\sum_{j=1}^{d} S(\omega_i)\theta_j = a_i$ for all $i \in I$

Sensitivity analysis will give correct bound for a new payoff b provided π^* is optimal for the corresponding dual.

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Sensitivity analysis will give correct bound for a new payoff b provided π^* is optimal for the corresponding dual.

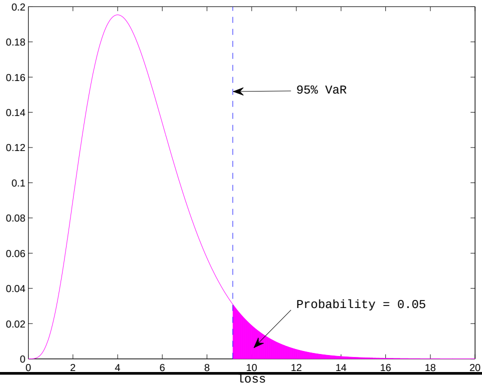
- Compute a new portfolio ϕ such that $\sum_{j=1}^d S(\omega_i)\phi_j = b_i$ for all $i \in I$
- Check if ϕ is primal feasible, i.e. $S\phi \geq b$

Value at Risk

Definition. The value at risk $\mathrm{VaR}_p(L)$ of random loss L at the confidence level $p \in (0,1)$ is defined as

$$\mathrm{VaR}_p(L) := p^{\text{th}}\text{-quantile of } L \quad \approx \quad F_L^{-1}(p)$$

where F_L is the CDF of the random loss L.



VaR is a "tail" risk measure

VaR_p is increasing in p

$\mathrm{VaR}_{0.99}(L) \geq \mathrm{VaR}_{0.95}(L)$

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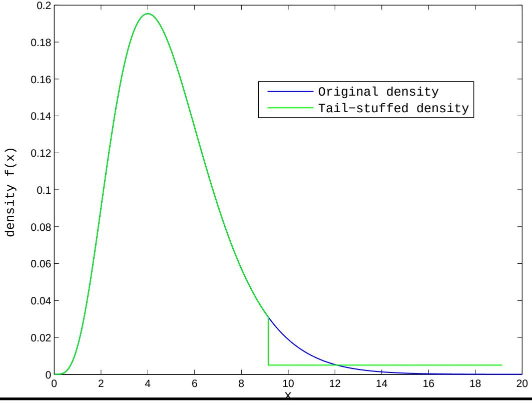

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$\mathrm{VaR}_{0.99}(L) \geq \mathrm{VaR}_{0.95}(L)$

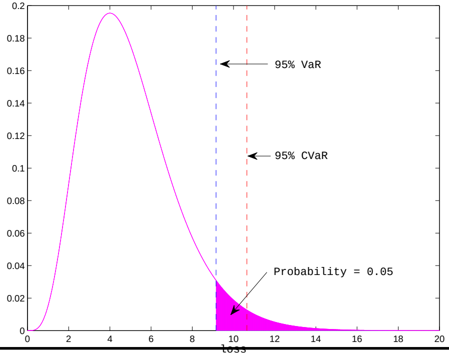
VaR creates incentives for tail stuffing.

Conditional Value at Risk

Definition. The conditional value at risk $\mathsf{CVaR}_p(L)$ of random variable L at the confidence level $p \in (0,1)$ is defined as

$$\mathsf{CVaR}_p(L) = \mathbb{E}\left[L \mid L \geq \mathsf{VaR}_p(L)\right] = \frac{\int_{\mathsf{VaR}_p(L)}^{\infty} x f_L(x)dx}{\mathbb{P}(L \geq \mathsf{VaR}_p(L))}$$

where f_L is the density of the random loss L.



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Other names: Tail conditional expectation and Expected Shortfall

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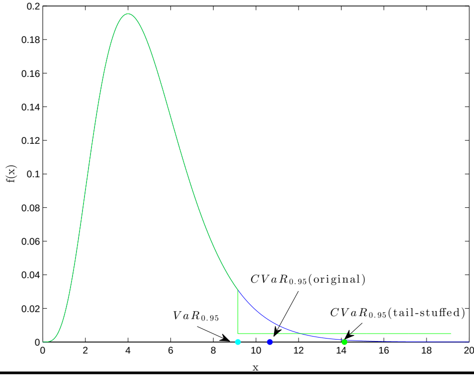

CVaR is also a “tail” risk measure

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Penalizes tail stuffing

Sample approximation of CVaR

Let $L_1, \ldots, L_N$ denote N IID samples of a random loss $\tilde{L}$

Let $L_{(1)}, \ldots, L_{(N)}$ denote the order statistics of the random samples.

$$\begin{aligned}
\mathsf{VaR}_p &= L_{(\lceil pN \rceil)} = (pN)\text{-th largest loss} \\
\mathsf{CVaR}_p &= \frac{\sum_{k=(\lceil pN \rceil)+1}^{N} L_{(k)}}{(1-p)N} \\
&\approx \text{Average of largest } (1-p)N \text{ losses}
\end{aligned}$$

LP formulation for CVaR

$$\begin{array}{rl}
\max & \sum_{k=1}^{N} q_k L_k \\
\text{s.t.} & \sum_{k=1}^{N} q_k = 1 \\
& q_k \leq \frac{1}{(1-p)N}, \; k = 1, \ldots, N, \\
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\end{array}
\quad = \quad
\begin{array}{rl}
\min & \beta + \frac{1}{(1-p)N} \sum_{k=1}^{N} v_k \\
\text{s.t.} & \beta + v_k \geq L_k,\ k = 1, \ldots, N, \\
& v \geq 0, \beta \text{ free}
\end{array}
$$

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& q_k \ge 0
\end{array}
\quad = \quad \min \quad \beta + \frac{1}{(1-p)N} \sum_{k=1}^{N} (L_k - \beta)^+
$$

Justification of LP formulation for CVaR

Suppose $p = \frac{\sigma}{N}$. Then

$$\begin{array}{rcl} \mathsf{Var}_p(L) & = & L_{(\sigma)} \equiv \ \sigma\text{-th largest loss} \\ \mathsf{CVaR}_p(L) & = & \frac{1}{(1-p)N} \sum_{k=\sigma+1}^{N} L_{(k)} \end{array}$$

Guess primal feasible solution: $q^*_{(k)} = \frac{1}{(1-p)N}$ for $k = \sigma, \ldots, N$. Then

$$\sum_{k=1}^{N} q^*_{(k)} L_{(k)} = \frac{1}{(1-p)N} \sum_{k=\sigma+1}^{N} L_{(k)} = \mathsf{CVaR}_p(L)$$

Guess dual feasible solution: $\beta^* = L_{(\sigma)}$ and $v^*_k = (L_k - L_{(\sigma)})^+$. Then

$$\begin{array}{rl} \multicolumn{2}{l}{\beta^* + \dfrac{1}{(1-p)N} \displaystyle\sum_{k=1}^{N} v^*_k} \\ = & L_{(\sigma)} + \frac{1}{(1-p)N} \sum_{k=\sigma+1}^{N} (L_{(k)} - L_{(\sigma)}) \\ = & \frac{1}{(1-p)N} \sum_{k=\sigma+1}^{N} L_{(k)} = \mathsf{CVaR}_p(L) \end{array}$$

Thus, q^* and (β^*, v^*) are primal-dual optimal

Mean-CVaR portfolio selection

$x = (x_1, \ldots, x_d)$ portfolio of d assets

$\ell_{ij} = i$-th sample of the rate of loss on asset j

Samples of losses on portfolio x: $\{\ell_i(x) = \sum_{j=1}^{d} \ell_{ij} x_j : i = 1, \ldots, N\}$

Mean-CVaR portfolio selection

$$\begin{array}{rl} \text{max} & \mu^\top x \\ \text{s.t.} & \text{CVaR}_p(\ell(x)) \leq \gamma \\ & \mathbf{1}^\top x = 1 \end{array}$$

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Mean-CVaR portfolio selection: replace $\mathrm{CVaR}_p(\ell(x))$ by its dual

$$\begin{array}{rl} \text{max} & \mu^\top x \\ \text{s.t.} & \min_\beta \left\{ \beta + \frac{1}{(1-p)N} \sum_{i=1}^N (\ell_i(x) - \beta)^+ \right\} \leq \gamma \\ & \mathbf{1}^\top x = 1 \end{array}$$

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Analysis of the constraint

$$f_i(x, \beta) = (\ell_i(x) - \beta)^+ = \max\Big\{ \sum_{j=1}^{d} \ell_{ij} x_j - \beta, 0 \Big\}$$

piece-wise linear convex function. Mean-CVaR piecewise linear convex optimization problem.

LP formulation of the mean-CVaR problem

Introduce a new variable $z_i = (\ell_i(x) - \beta)^+$. Then

$$z_i \geq \ell_i(x) - \beta \qquad z_i \geq 0$$

LP formulation

$$\begin{array}{ll}
\text{max} & \mu^\top x \\
\text{s.t.} & \beta + \frac{1}{(1-p)N} \sum_{i=1}^{N} z_i \leq \gamma \\
& z_i - \ell_i(x) + \beta \geq 0, \qquad i = 1, \ldots, N \\
& \mathbf{1}^\top x = 1 \\
& z \geq 0.
\end{array}$$

Minimum absolute deviation risk measure

$x = (x_1, \ldots, x_d)$ portfolio of d assets

$r_{ij} = i$-th sample of the rate of return on asset j

Mean-absolute deviation $\mathsf{MAD}(x) = \frac{1}{N}\sum_{i=1}^{N}\left|\sum_{j=1}^{d}(r_{ij} - \mu_j)x_j\right|$

Mean-MAD portfolio selection problem

$$\begin{array}{rl} \min & \mathsf{MAD}(x) \\ \text{s.t.} & \mu^\top x \geq R \\ & \mathbf{1}^\top x = 1 \end{array}$$

Analysis of objective function

$$f_i(x) = \left|\sum_{j=1}^{d}(r_{ij} - \mu_j)x_j\right| = \max\left\{\sum_{j=1}^{d}(r_{ij} - \mu_j)x_j, -\sum_{j=1}^{d}(r_{ij} - \mu_j)x_j\right\}$$

piece-wise linear convex function. Mean-MAD problem is a piece-wise linear convex problem.

LP formulation of the mean-MAD problem

Introduce a new variable $z_i = \left| \sum_{j=1}^{d} (r_{ij} - \mu_j) x_j \right|$. Then

$$z_i \geq \sum_{j=1}^{d} (r_{ij} - \mu_j) x_j \qquad z_i \geq -\sum_{j=1}^{d} (r_{ij} - \mu_j) x_j$$

LP formulation of the mean-MAD problem

Introduce a new variable $z_i = \left| \sum_{j=1}^{d} (r_{ij} - \mu_j) x_j \right|$. Then

$$z \geq Rx - \mathbf{1}\mu^\top x \qquad z \geq -Rx + \mathbf{1}\mu^\top x$$

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LP formulation

$$\begin{array}{rl} \text{min} & \mathbf{1}^\top z \\ \text{s.t.} & z - (R - \mathbf{1}\mu^\top)x \geq 0 \\ & z + (R - \mathbf{1}\mu^\top)x \geq 0 \\ & \mu^\top x \geq R \\ & \mathbf{1}^\top x = 1 \end{array}$$

Sparsity

Unknown sparse signal: x Measurements: $Ax = b$

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Sparse recovery problem: $\mathbf{1}(a) = 1$ is a is true and 0 otherwise

$$\begin{array}{rl} \text{min} & \sum_{i=1}^{d} \mathbf{1}(|x_i| > 0) \\ \text{s. t.} & Ax = b \end{array}$$

Non-convex problems ... does **not** scale well with d

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Non-convex problems ... does **not** scale well with d

ℓ_1 recovery problem: ℓ_p-norm $(p \geq 1)$ $\|x\|_p = \left(\sum_{i=1}^{d} |x_i|^p \right)^{\frac{1}{p}}$

$$\begin{array}{rl} \text{min} & \|x\|_1 = \sum_{i=1}^{d} |x_i| \\ \text{s. t.} & Ax = b \end{array}$$

Convex piecewise-linear problem ... **does** scale well with d

ℓ_1-minimization is an LP

Introduce $z_i = |x_i| = \max\{x_i, -x_i\}$ and relax to $z_i \geq \max\{x_i, -x_i\}$

$$z - x \geq 0, \qquad z + x \geq 0$$

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$$z - x \geq 0, \qquad z + x \geq 0$$

Reformulation of the ℓ_1-minimization problem

$$\begin{array}{ll} \text{min} & \mathbf{1}^\top z \\ \text{s. t.} & Ax = b, \\ & z + x \geq 0, \\ & z - x \geq 0. \end{array}$$