Chapter 3

LP Models: Asset/Liability Cash Flow Matching

3.1 Short Term Financing

Corporations routinely face the problem of financing short term cash commitments. Linear programming can help in figuring out an optimal combination of financial instruments to meet these commitments. To illustrate this, consider the following problem. For simplicity of exposition, we keep the example very small.

A company has the following short term financing problem.

Month	Jan	Feb	Mar	Apr	May	Jun
Net Cash Flow	-150	-100	200	-200	50	300

Net cash flow requirements are given in thousands of dollars. The company has the following sources of funds

- A line of credit of up to \$100K at an interest rate of 1% per month,
- In any one of the first three months, it can issue 90-day commercial paper bearing a total interest of 2% for the 3-month period,
- Excess funds can be invested at an interest rate of 0.3% per month.

There are many questions that the company might want to answer. What interest payments will the company need to make between January and June? Is it economical to use the line of credit in some of the months? If so, when? How much? Linear programming gives us a mechanism for answering these questions quickly and easily. It also allows to answer some "what if" questions about changes in the data without having to resolve the problem. What if Net Cash Flow in January were -200 (instead of -150)? What if the limit on the credit line were increased from 100 to 200? What if the negative Net Cash Flow in January is due to the purchase of a machine worth 150 and the vendor allows part or all of the payment on this machine to be made in June at an interest of 3% for the 5-month period? The answers to these questions are readily available when this problem is formulated and solved as a linear program.

There are three steps in applying linear programming: modeling, solving, and interpreting.

3.1.1 Modeling

We begin by modeling the above short term financing problem. That is, we write it in the language of linear programming. There are rules about what one can and cannot do within linear programming. These rules are in place to make certain that the remaining steps of the process (solving and interpreting) can be successful.

Key to a linear program are the *decision variables*, *objective*, and *constraints*.

Decision Variables. The decision variables represent (unknown) decisions to be made. This is in contrast to *problem data*, which are values that are either given or can be simply calculated from what is given. For the short term financing problem, there are several possible choices of decision variables. We will use the following decision variables: the amount x_i drawn from the line of credit in month i, the amount y_i of commercial paper issued in month i, the excess funds z_i in month i and the company's wealth v in June. Note that, alternatively, one could use the decision variables x_i and z_i only, since excess funds and company's wealth can be deduced from these variables.

Objective. Every linear program has an objective. This objective is to be either minimized or maximized. This objective has to be *linear* in the decision variables, which means it must be the sum of constants times decision variables. $3x_1 - 10x_2$ is a linear function. x_1x_2 is not a linear function. In this case, our objective is simply to maximize v.

Constraints. Every linear program also has constraints limiting feasible decisions. Here we have three types of constraints: (i) cash inflow = cash outflow for each month, (ii) upper bounds on x_i and (iii) nonnegativity of the decision variables x_i, y_i and z_i.

For example, in January ($i = 1$), there is a cash requirement of \$150. To meet this requirement, the company can draw an amount x_1 from its line of credit and issue an amount y_1 of commercial paper. Considering the possibility of excess funds z_1 (possibly 0), the cash flow balance equation is as follows.

$$x_1 + y_1 - z_1 = 150$$

Next, in February ($i = 2$), there is a cash requirement of \$100. In addition, principal plus interest of $1.01x_1$ is due on the line of credit and $1.003z_1$ is received on the invested excess funds. To meet the requirement in February, the company can draw an amount x_2 from its line of credit and issue an amount y_2 of commercial paper. So, the cash flow balance equation for February is as follows.

$$x_2 + y_2 - 1.01x_1 + 1.003z_1 - z_2 = 100$$

Similarly, for March we get the following equation:

$$x_3 + y_3 - 1.01x_2 + 1.003z_2 - z_3 = -200$$

For the months of April, May, and June, issuing a commercial paper is no longer an option, so we will not have variables y_4, y_5, and y_6 in the formulation. Furthermore, any commercial paper issued between January and March requires a payment with 2% interest 3 months later. Thus, we have the following additional equations:

$$\begin{array}{rcrcrcrcrcr}
x_4 & - & 1.02y_1 & - & 1.01x_3 & + & 1.003z_3 & - & z_4 & = & 200 \\
x_5 & - & 1.02y_2 & - & 1.01x_4 & + & 1.003z_4 & - & z_5 & = & -50 \\
 & - & 1.02y_3 & - & 1.01x_5 & + & 1.003z_5 & - & v & = & -300
\end{array}$$

Note that x_i is the balance on the credit line in month i, not the incremental borrowing in month i. Similarly, z_i represents the overall excess funds in month i. This choice of variables is quite convenient when it comes to writing down the upper bound and nonnegativity constraints.

$$\begin{array}{rcl}
0 & \leq & x_i \leq 100 \\
y_i & \geq & 0 \\
z_i & \geq & 0.
\end{array}$$

Final Model. This gives us the complete model of this problem:

$$\begin{array}{lrcrcrcrcrcr}
\max & v \\
 & x_1 & + & y_1 & & & & & - & z_1 & = & 150 \\
 & x_2 & + & y_2 & - & 1.01x_1 & + & 1.003z_1 & - & z_2 & = & 100 \\
 & x_3 & + & y_3 & - & 1.01x_2 & + & 1.003z_2 & - & z_3 & = & -200 \\
 & x_4 & - & 1.02y_1 & - & 1.01x_3 & + & 1.003z_3 & - & z_4 & = & 200 \\
 & x_5 & - & 1.02y_2 & - & 1.01x_4 & + & 1.003z_4 & - & z_5 & = & -50 \\
 & & - & 1.02y_3 & - & 1.01x_5 & + & 1.003z_5 & - & v & = & -300 \\
 & x_1 & \leq & 100 \\
 & x_2 & \leq & 100 \\
 & x_3 & \leq & 100 \\
 & x_4 & \leq & 100 \\
 & x_5 & \leq & 100 \\
 & x_i, y_i, z_i & \geq & 0.
\end{array}$$

Formulating a problem as a linear program means going through the above process of clearly defining the decision variables, objective, and constraints.

Exercise 3.1 How would the formulation of the short-term financing problem above change if the commercial papers issued had a 2 month maturity instead of 3?

Exercise 3.2 A company will face the following cash requirements in the next eight quarters (positive entries represent cash needs while negative entries represent cash surpluses).

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8
100	500	100	-600	-500	200	600	-900

The company has three borrowing possibilities.

- a 2-year loan available at the beginning of $Q1$, with a 1% interest per quarter.
- The other two borrowing opportunities are available at the beginning of every quarter: a 6-month loan with a 1.8% interest per quarter, and a quarterly loan with a 2.5% interest for the quarter.

Any surplus can be invested at a 0.5% interest per quarter.

Formulate a linear program that maximizes the wealth of the company at the beginning of Q9.

Exercise 3.3 A home buyer in France can combine several mortgage loans to finance the purchase of a house. Given borrowing needs B and a horizon of T months for paying back the loans, the home buyer would like to minimize his total cost (or equivalently, the monthly payment p made during each of the next T months). Regulations impose limits on the amount that can be borrowed from certain sources. There are n different loan opportunities available. Loan i has a fixed interest rate r_i, a length $T_i \leq T$ and a maximum amount borrowed b_i. The monthly payment on loan i is not required to be the same every month, but a minimum payment m_i is required each month. However the total monthly payment p over all loans is constant. Formulate a linear program that finds a combination of loans that minimizes the home buyer's cost of borrowing. [Hint: In addition to variables x_{ti} for the payment on loan i in month t, it may be useful to introduce a variable for the amount of outstanding principal on loan i in month t.]

3.1.2 Solving the Model with SOLVER

Special computer programs can be used to find solutions to linear programming models. The most widely available program is undoubtedly SOLVER, included in all recent versions of the Excel spreadsheet program. Here are other suggestions:

- MATLAB has a linear programming solver that can be accessed with the command `linprog`. Type `help linprog` to find out details.
- Even if one does not have access to any linear programming software, it is possible to solve linear programs (and other optimization problems) using the website
  `http://www-neos.mcs.anl.gov/neos/`

This is the website for the Network Enabled Optimization Server. Using the JAVA submission tool on this site, one can submit a linear programming problem (in some standard format) and have a remote computer solve his/her problem using one of the several solver options. The solution is then transmitted to the submitting person by e-mail.

- A good open source LP code written in C is CLP available from the following website at the time of this writing:
`http://www.coin-or.org/`

SOLVER, while not a state of the art code (which can cost upwards of $10,000 per copy) is a reasonably robust, easy-to-use tool for linear programming. SOLVER uses standard spreadsheets together with an interface to define variables, objective, and constraints.

Here are a brief outline and some hints and shortcuts on how to create a SOLVER spreadsheet:

- Start with a spreadsheet that has all of the data entered in some reasonably neat way.

  In the short term financing example, the spreadsheet might contain the cash flows, interest rates and credit limit.

- The model will be created in a separate part of the spreadsheet. Identify one cell with each decision variable. SOLVER will eventually put the optimal values in these cells.

  In the short term financing example, we could associate cells `$B$2` to `$B$6` with variables x_1 to x_5 respectively, cells `$C$2` to `$C$4` with the y_i variables, cells `$D$2` to `$D$6` with the z_i variables and, finally, `$E$2` with the variable v.

- A separate cell represents the objective. Enter a formula that represents the objective.

  For the short term financing example, we might assign cell `$B$8` to the objective function. Then, in cell `$B$8`, we enter the function `= $E$2`.

  This formula must be a linear formula, so, in general, it must be of the form: `cell1*cell1' + cell2*cell2' + ...`, where `cell1, cell2` and so on contain constant values and `cell1', cell2'` and so on are the decision variable cells.

- We then have a cell to represent the left hand side of each constraint (again a linear function) and another cell to represent the right hand side (a constant).

  In the short term financing example, cells `$B$10` to `$B$15` might contain the amounts generated through financing, for each month, and cells `$D$10` to `$D$15` the cash requirements for each month. For example, cell `$B$10` would contain the function `= $C$2 + $B$2 -$D$2` and cell `$D$10` the value 150. Similarly, rows 16 to 20 could be used to write the credit limit constraints.

Helpful Hint: Excel has a function `sumproduct()` that is designed for linear programs. `sumproduct(a1..a10,b1..b10)` is identical to `a1*b1+a2*b2+a3*b3+...+a10*b10`. This function can save much time and aggravation. All that is needed is that the length of the first range be the same as the length of the second range (so one can be horizontal and the other vertical).

Helpful Hint: It is possible to assign names to cells and ranges (under the `Insert-Name` menu). Rather than use `a1..a10` as the variables, you can name that range `var` (for example) and then use `var` wherever `a1..a10` would have been used.

- We then select `Solver` under the `Tools` menu. This gives a form to fill out to define the linear program.

- In the `''Set Cell''` box, select the `objective` cell. Choose `Maximize` or `Minimize`.

- In the `''By Changing Cells''`, put in the range containing the variable cells.

- We next add the constraints. Press the `''Add...''` button to add constraints. The dialog box has three parts for the left hand side, the type of constraint, and the right hand side. Put the cell references for a constraint in the form, choose the right type, and press `''Add''`. Continue until all constraints are added. On the final constraint, press `''OK''`.

  Helpful Hint: It is possible to include ranges of constraints, as long as they all have the same type. `c1..e1 <= c3..e3` means `c1 <= c3`, `d1 <= d3`, `e1 <= e3`. `a1..a10 >= 0` means each individual cell must be greater than or equal to 0.

- Push the `options` button and toggle the `''Assume Linear Model''` in the resulting dialog box. This tells Excel to call a linear rather than a nonlinear programming routine so as to solve the problem more efficiently. This also gives you sensitivity ranges, which are not available for nonlinear models.

  Note that, if you want your variables to assume nonnegative values only, you need to specify this in the options box (alternatively, you can add nonnegativity constraints in the previous step, in your constraints).

- Push the `Solve` button. In the resulting dialog box, select `''Answer''` and `''Sensitivity''`. This will put the answer and sensitivity analysis in two new sheets. Ask Excel to `''Keep Solver values''`, and your worksheet will be updated so that the optimal values are in the variable cells.

Exercise 3.4 Solve the linear program formulated in Exercise 3.2 with your favorite software package.

3.1.3 Interpreting the output of SOLVER

If we were to solve the short-term financing problem above using SOLVER, the solution is given in the `''Answer''` report that looks as follows.

Target Cell (Max)

Cell	Name	Original Value	Final Value
$\$B\$8$	Objective	0	92.49694915

Adjustable Cells

Cell	Name	Original Value	Final Value
$\$B\$2$	$x1$	0	0
$\$B\$3$	$x2$	0	50.98039216
$\$B\$4$	$x3$	0	0
$\$B\$5$	$x4$	0	0
$\$B\$6$	$x5$	0	0
$\$C\$2$	$y1$	0	150
$\$C\$3$	$y2$	0	49.01960784
$\$C\$4$	$y3$	0	203.4343636
$\$D\$2$	$z1$	0	0
$\$D\$3$	$z2$	0	0
$\$D\$4$	$z3$	0	351.9441675
$\$D\$5$	$z4$	0	0
$\$D\$6$	$z5$	0	0
$\$E\$2$	v	0	92.49694915

Constraints

Cell	Name	Cell Value	Formula	Slack
$\$B\$10$	January	150	$\$B\$10 = \$D\$10$	0
$\$B\$11$	February	100	$\$B\$11 = \$D\$11$	0
$\$B\$12$	March	-200	$\$B\$12 = \$D\$12$	0
$\$B\$13$	April	200	$\$B\$13 = \$D\$13$	0
$\$B\$14$	May	-50	$\$B\$14 = \$D\$14$	0
$\$B\$15$	June	-300	$\$B\$15 = \$D\$15$	0
$\$B\$16$	$x1$limit	0	$\$B\$16 <= \$D\$16$	100
$\$B\$17$	$x2$limit	50.98039216	$\$B\$17 <= \$D\$17$	49.01960784
$\$B\$18$	$x3$limit	0	$\$B\$18 <= \$D\$18$	100
$\$B\$19$	$x4$limit	0	$\$B\$19 <= \$D\$19$	100
$\$B\$20$	$x5$limit	0	$\$B\$20 <= \$D\$20$	100

This report is fairly easy to read: the company's wealth v in June will be \$92,497. This is reported in `Final Value` of the `Objective` (recall that our units are in \$1000). To achieve this, the company will issue \$150,000 in commercial paper in January, \$49,020 in February and \$203,434 in March. In addition, it will draw \$50,980 from its line of credit in February. Excess cash of \$351,944 in March will be invested for just one month. All this is reported in the `Adjustable Cells` section of the report. For this particular application, the `Constraints` section of the report does not contain anything useful. On the other hand, very useful information can be found in the sensitivity report. This will be discussed in Section 3.3.

Exercise 3.5 Formulate and solve the variation of the short-term financing problem you developed in Exercise 3.1 using SOLVER. Interpret the solution.

Exercise 3.6 Recall Example 2.1. Solve the problem using your favorite linear programming solver. Compare the output provided by the solver to the solution we obtained in Chapter 2.

3.1.4 Modeling Languages

Linear programs can be formulated using modeling languages such as AMPL, GAMS, MOSEL or OPL. The need for these modeling languages arises because the Excel spreadsheet format becomes inadequate when the size of the linear program increases. A modeling language lets people use common notation and familiar concepts to formulate optimization models and examine solutions. Most importantly, large problems can be formulated in a compact way. Once the problem has been formulated using a modeling language, it can be solved using any number of solvers. A user can switch between solvers with a single command and select options that may improve solver performance. The short term financing model would be formulated as follows (all variables are assumed to be nonnegative unless otherwise specified).

```
DATA
LET T=6 be the number of months to plan for
L(t) = Liability in month t=1,...,T
ratex = monthly interest rate on line of credit
ratey = 3-month interest rate on commercial paper
ratez = monthly interest rate on excess funds
VARIABLES
x(t) = Amount drawn from line of credit in month t
y(t) = Amount of commercial paper issued in month t
z(t) = Excess funds in month t
OBJECTIVE (Maximize wealth in June)
Max z(6)
CONSTRAINTS
Month(t=1:T): x(t) - (1+ratex)*x(t-1) + y(t) - (1+ratey)*y(t-3)
-z(t) +(1+ratez)*z(t-1) = L(t)
Month(t=1:T-1):  x(t) < 100
```

```
Boundary conditions on x:  x(0)=x(6) =0
Boundary conditions on y:  y(-2)=y(-1)=y(0)=y(4)=y(5)=y(6) =0
Boundary conditions on z:  z(0) =0
END
```

Exercise 3.7 Formulate the linear program of Exercise 3.3 with one of the modeling languages AMPL, GAMS, MOSEL or OPL.

3.1.5 Features of Linear Programs

Hidden in each linear programming formulation are a number of assumptions. The usefulness of an LP model is directly related to how closely reality matches up with these assumptions.

The first two assumptions are due to the linear form of the objective and constraint functions. The contribution to the objective of any decision variable is proportional to the value of the decision variable. Similarly, the contribution of each variable to the left hand side of each constraint is proportional to the value of the variable. This is the *Proportionality Assumption*.

Furthermore, the contribution of a variable to the objective and constraints is independent of the values of the other variables. This is the *Additivity Assumption*. When additivity or proportionality assumptions are not satisfied, a *nonlinear programming* model may be more appropriate. We discuss such models in Chapters 5 and 6.

The next assumption is the *Divisibility Assumption*: is it possible to take any fraction of any variable? A fractional production quantity may be worrisome if we are producing a small number of battleships or be innocuous if we are producing millions of paperclips. If the Divisibility Assumption is important and does not hold, then a technique called *integer programming* rather than linear programming is required. This technique takes orders of magnitude more time to find solutions but may be necessary to create realistic solutions. We discuss integer programming models and methods in Chapters 11 and 12.

The final assumption is the *Certainty Assumption*: linear programming allows for no uncertainty about the input parameters such as the cash-flow requirements or interest rates we used in the short-term financing model. Problems with uncertain parameters can be addressed using *stochastic programming* or *robust optimization* approaches. We discuss such models in Chapters 16 through 20.

It is very rare that a problem will meet all of the assumptions exactly. That does not negate the usefulness of a model. A model can still give useful managerial insight even if reality differs slightly from the rigorous requirements of the model.

Exercise 3.8 Give an example of an optimization problem where the proportionality assumption is not satisfied.

Exercise 3.9 Give an example of an optimization problem where the additivity assumption is not satisfied.

Exercise 3.10 Consider the LP model we develop for the cash-flow matching problem in Section 3.2. Which of the linear programming assumptions used for this formulation is the least realistic one? Why?

3.2 Dedication

Dedication or *cash flow matching* is a technique used to fund known liabilities in the future. The intent is to form a portfolio of assets whose cash inflows will exactly offset the cash outflows of the liabilities. The liabilities will therefore be paid off, as they come due, without the need to sell or buy assets in the future. The portfolio is formed today and then held until all liabilities are paid off. Dedicated portfolios usually only consist of risk-free non-callable bonds since the portfolio future cash inflows need to be known when the portfolio is constructed. This eliminates interest rate risk completely. It is used by some municipalities and small pension funds. For example, municipalities sometimes want to fund liabilities stemming from bonds they have issued. These pre-refunded municipal bonds can be taken off the books of the municipality. This may allow them to evade restrictive covenants in the bonds that have been pre-refunded and perhaps allow them to issue further debt.

It should be noted however that dedicated portfolios cost typically from 3% to 7% more in dollar terms than do "immunized" portfolios that are constructed based on matching present value, duration and convexity of the assets and of the liabilities. The *present value* of the liability stream L_t for $t = 1, \ldots, T$ is $P = \sum_{t=1}^{T} \frac{L_t}{(1+r_t)^t}$, where r_t denotes the risk-free rate in year t. Its *duration* is $D = \frac{1}{P} \sum_{t=1}^{T} \frac{tL_t}{(1+r_t)^t}$ and its *convexity* is $C = \frac{1}{P} \sum_{t=1}^{T} \frac{t(t+1)L_t}{(1+r_t)^{t+2}}$. Intuitively, duration is the average (discounted) time at which the liabilities occur, whereas convexity, a bit like variance, indicates how concentrated the cash flows are over time. For a portfolio that consists only of risk-free bonds, the present value P^* of the portfolio future cash inflows can be computed using the same risk-free rate r_t (this would not be the case for a portfolio containing risky bonds). Similarly for the duration D^* and convexity C^* of the portfolio future cash inflows. An "immunized" portfolio can be constructed based on matching $P^* = P$, $D^* = D$ and $C^* = C$. Portfolios that are constructed by matching these three factors are immunized against parallel shifts in the yield curve, but there may still be a great deal of exposure and vulnerability to other types of shifts, and they need to be actively managed, which can be costly. By contrast, dedicated portfolios do not need to be managed after they are constructed.

When municipalities use cash flow matching, the standard custom is to call a few investment banks, send them the liability schedule and request bids. The municipality then buys its securities from the bank that offers the lowest price for a successful cash flow match.

Assume that a bank receives the following liability schedule:

Year 1	Year 2	Year 3	Year 4	Year 5	Year 6	Year 7	Year 8
12,000	18,000	20,000	20,000	16,000	15,000	12,000	10,000

The bonds available for purchase today (Year 0) are given in the next table. All bonds have a face value of \$100. The coupon figure is annual. For example, Bond 5 costs \$98 today, and it pays back \$4 in Year 1, \$4 in Year 2, \$4 in Year 3 and \$104 in Year 4. All these bonds are widely available and can be purchased in any quantities at the stated price.

Bond	1	2	3	4	5	6	7	8	9	10
Price	102	99	101	98	98	104	100	101	102	94
Coupon	5	3.5	5	3.5	4	9	6	8	9	7
MaturityYear	1	2	2	3	4	5	5	6	7	8

Formulate and solve a linear program to find the least cost portfolio of bonds to purchase today, to meet the obligations of the municipality over the next eight years. To eliminate the possibility of any reinvestment risk, we assume a 0% reinvestment rate.

Using a modeling language, the formulation might look as follows.

```
DATA
LET T=8 be the number of years to plan for.
LET N=10 be the number of bonds available for purchase today.
L(t) = Liability in year t=1,...,T
P(i) = Price of bond i, i=1,...,N
C(i) = Annual coupon for bond i, i=1,...,N
M(i) = Maturity year of bond i, i=1,...,N
VARIABLES
x(i) = Amount of bond i in the portfolio
z(t) = Surplus at the end of year t, for t=0,...,T
OBJECTIVE (Minimize cost)
Min z(0) + SUM(i=1:N) P(i)*x(i)
CONSTRAINTS Year(t=1:T):
SUM(i=1:N | M(i) > t-1) C(i)*x(i) + SUM(i=1:N | M(i) = t) 100*x(i)
-z(t) + z(t-1) = L(t)
END
```

Exercise 3.11 Solve the dedication linear program above using an LP software package and verify that we can optimally meet the municipality's liabilities for \$93,944 with the following portfolio: 62 Bond1, 125 Bond3, 152 Bond4, 157 Bond5, 123 Bond6, 124 Bond8, 104 Bond9 and 93 Bond10.

Exercise 3.12 A small pension fund has the following liabilities (in million dollars):

Year1	Year2	Year3	Year4	Year5	Year6	Year7	Year8	Year9
24	26	28	28	26	29	32	33	34

It would like to construct a dedicated bond portfolio. The bonds available for purchase are the following:

Bond	1	2	3	4	5	6	7	8
Price	102.44	99.95	100.02	102.66	87.90	85.43	83.42	103.82
Coupon	5.625	4.75	4.25	5.25	0.00	0.00	0.00	5.75
MaturityYear	1	2	2	3	3	4	5	5

Bond	9	10	11	12	13	14	15	16
Price	110.29	108.85	109.95	107.36	104.62	99.07	103.78	64.66
Coupon	6.875	6.5	6.625	6.125	5.625	4.75	5.5	0.00
MaturityYear	6	6	7	7	8	8	9	9

Formulate an LP that minimizes the cost of the dedicated portfolio, assuming a 2% reinvestment rate. Solve the LP using your favorite software package.

3.3 Sensitivity Analysis for Linear Programming

The optimal solution to a linear programming model is the most important output of LP solvers, but it is not the only useful information they generate. Most linear programming packages produce a tremendous amount of *sensitivity information*, or information about what happens when data values are changed.

Recall that in order to formulate a problem as a linear program, we had to invoke a *certainty assumption*: we had to know what value the data took on, and we made decisions based on that data. Often this assumption is somewhat dubious: the data might be unknown, or guessed at, or otherwise inaccurate. How can we determine the effect on the optimal decisions if the values change? Clearly some numbers in the data are more important than others. Can we find the "important" numbers? Can we determine the effect of estimation errors?

Linear programming offers extensive capabilities for addressing these questions. We give examples of how to interpret the SOLVER output. To access the information in SOLVER, one can simply ask for the sensitivity report after optimizing. Rather than simply giving rules for reading the reports, we show how to answer a set of questions from the output.

3.3.1 Short Term Financing

The Solver sensitivity report looks as follows.

Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
B2	$x1$	0	-0.0032	0	0.0032	$1E+30$
B3	$x2$	50.98	0	0	0.0032	0
B4	$x3$	0	-0.0071	0	0.0071	$1E+30$
B5	$x4$	0	-0.0032	0	0.0032	$1E+30$
B6	$x5$	0	0	0	0	$1E+30$
C2	$y1$	150	0	0	0.0040	0.0032
C3	$y2$	49.02	0	0	0	0.0032
C4	$y3$	203.43	0	0	0.0071	0
D2	$z1$	0	-0.0040	0	0.0040	$1E+30$
D3	$z2$	0	-0.0071	0	0.0071	$1E+30$
D4	$z3$	351.94	0	0	0.0039	0.0032
D5	$z4$	0	-0.0039	0	0.0039	$1E+30$
D6	$z5$	0	-0.007	0	0.007	$1E+30$
E2	v	92.50	0	1	$1E+30$	1

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H.Side	Allowable Increase	Allowable Decrease
B10	January	150	-1.0373	150	89.17	150
B11	February	100	-1.030	100	49.020	50.980
B12	March	-200	-1.020	-200	90.683	203.434
B13	April	200	-1.017	200	90.955	204.044
B14	May	-50	-1.010	-50	50	52
B15	June	-300	-1	-300	92.497	$1E+30$
B16	$x1$	0	0	100	$1E+30$	100
B17	$x2$	50.98	0	100	$1E+30$	49.020
B18	$x3$	0	0	100	$1E+30$	100
B19	$x4$	0	0	100	$1E+30$	100
B20	$x5$	0	0	100	$1E+30$	100

The key columns for sensitivity analysis are the `Reduced Cost` and `Shadow Price` columns in SOLVER. The *shadow price* u of a constraint C has the following interpretation:

> If the right hand side of the constraint C changes by an amount Δ, the optimal objective value changes by $u\Delta$, as long as the amount of change Δ is within the allowable range.

For a linear program, the shadow price u is an exact figure, as long as the amount of change Δ is within the allowable range given in the last two columns of the SOLVER output. When the change Δ falls outside this range, the rate of change in the optimal objective value changes and the shadow price u cannot be used. When this occurs, one has to resolve the linear program using the new data.

Next, we consider several examples of sensitivity questions and demonstrate how they can be answered using shadow prices and reduced costs.

- For example, assume that Net Cash Flow in January were -200 (instead of - 150). By how much would the company's wealth decrease at the end of June?

  The answer is in the shadow price of the January constraint, $u = -1.0373$. The RHS of the January constraint would go from 150 to 200, an increase of $\Delta = 50$, which is within the allowable increase (89.17). So the company's wealth in June would decrease by 1.0373 * 50,000 = \$ 51,865.

- Now assume that Net Cash Flow in March were 250 (instead of 200). By how much would the company's wealth increase at the end of June?

  Again, the change $\Delta = -50$ is within the allowable decrease (203.434), so we can use the shadow price $u = -1.02$ to calculate the change in objective value. The increase is (-1.02) * (-50) = \$51,000.

- Assume that the credit limit were increased from 100 to 200. By how much would the company's wealth increase at the end of June?

  In each month, the change $\Delta = 100$ is within the allowable increase ($+\infty$) and the shadow price for the credit limit constraint is $u = 0$. So there is no effect on the company's wealth in June. Note that non-binding constraints–such as the credit limit constraint for months January through May–always have zero shadow price.

- Assume that the negative Net Cash Flow in January is due to the purchase of a machine worth \$150,000. The vendor allows the payment to be made in June at an interest rate of 3% for the 5-month period. Would the company's wealth increase or decrease by using this option? What if the interest rate for the 5-month period were 4%?

  The shadow price of the January constraint is -1.0373. This means that reducing cash requirements in January by \$1 increases the wealth in June by \$1.0373. In other words, the break even interest rate for the 5-month period is 3.73%. So, if the vendor charges 3%, we should accept, but if he charges 4% we should not. Note that the analysis is valid since the amount $\Delta = -150$ is within the allowable decrease.

- Now, let us consider the reduced costs. The basic variables always have a zero reduced cost. The nonbasic variables (which by definition take the value 0) have a nonpositive reduced cost and, frequently their reduced cost is strictly negative. There are two useful interpretations of the reduced cost c, for a nonbasic variable x.

  First, assume that x is set to a positive value Δ instead of its optimal value 0. Then, the objective value is changed by $c\Delta$. For example, what would be the effect of financing part of the January cash needs through the line of credit? The answer is in the reduced cost of variable x_1. Because this reduced cost -0.0032 is strictly negative, the objective function would decrease. Specifically, each dollar financed through the line of credit in January would result in a decrease of \$0.0032 in the company's wealth v in June.

The second interpretation of c is that its magnitude $|c|$ is the minimum amount by which the objective coefficient of x must be increased in order for the variable x to become positive in an optimal solution. For example, consider the variable x_1 again. Its value is zero in the current optimal solution, with objective function v. However, if we changed the objective to $v + 0.0032x_1$, it would now be optimal to use the line of credit in January. In other words, the reduced cost on x_1 can be viewed as the minimum rebate that the bank would have to offer (payable in June) to make it attractive to use the line of credit in January.

Exercise 3.13 Recall Example 2.1. Determine the shadow price and reduced cost information for this problem using an LP software package. How would the solution change if the average maturity of the portfolio is required to be 3.3 instead of 3.6?

Exercise 3.14 Generate the sensitivity report for Exercise 3.2 with your favorite LP solver.

(i) Suppose the cash requirement in Q2 is 300 (instead of 500). How would this affect the wealth in Q9?

(ii) Suppose the cash requirement in Q2 is 100 (instead of 500). Can the sensitivity report be used to determine the wealth in Q9?

(iii) One of the company's suppliers may allow differed payments of \$ 50 from Q3 to Q4. What would be the value of this?

Exercise 3.15 Workforce Planning: Consider a restaurant that is open seven days a week. Based on past experience, the number of workers needed on a particular day is given as follows:

Day	Mon	Tue	Wed	Thu	Fri	Sat	Sun
Number	14	13	15	16	19	18	11

Every worker works five consecutive days, and then takes two days off, repeating this pattern indefinitely. How can we minimize the number of workers that staff the restaurant?

Let the days be numbers 1 through 7 and let x_i be the number of workers who begin their five day shift on day i. The linear programming formulation is as follows.

$$
\begin{array}{llcl}
\text{Minimize} & \sum_i x_i & & \\
\text{Subject to} & & & \\
& x_1+x_4+x_5+x_6+x_7 & \geq & 14 \\
& x_1+x_2+x_5+x_6+x_7 & \geq & 13 \\
& x_1+x_2+x_3+x_6+x_7 & \geq & 15 \\
& x_1+x_2+x_3+x_4+x_7 & \geq & 16 \\
& x_1+x_2+x_3+x_4+x_5 & \geq & 19 \\
& x_2+x_3+x_4+x_5+x_6 & \geq & 18 \\
& x_3+x_4+x_5+x_6+x_7 & \geq & 11 \\
& x_i & \geq & 0 \text{ (for all } i)
\end{array}
$$

Sensitivity Analysis The following table gives the sensitivity report for the solution of the workforce planning problem.

Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
B14	Shift1	4	0	1	0.5	1
B15	Shift2	7	0	1	0	0.333333
B16	Shift3	1	0	1	0.5	0
B17	Shift4	4	0	1	0.5	0
B18	Shift5	3	0	1	0	0.333333
B19	Shift6	3	0	1	0.5	1
B20	Shift7	0	0.333333	1	1E+30	0.333333

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
B24	Monday	14	0.333333	14	1.5	6
B25	Tuesday	17	0	13	4	1E+30
B26	Wednesday	15	0.333333	15	6	3
B27	Thursday	16	0	16	3	4
B28	Friday	19	0.333333	19	4.5	3
B29	Saturday	18	0.333333	18	1.5	6
B30	Sunday	11	0	11	4	1

Answer each of the following questions independently of the others.

1. What is the current total number of workers needed to staff the restaurant?

2. Due to a special offer, demand on Thursdays increases. As a result, 18 workers are needed instead of 16. What is the effect on the total number of workers needed to staff the restaurant?

3. Assume that demand on Mondays decreases: 11 workers are needed instead of 14. What is the effect on the total number of workers needed to staff the restaurant?

4. Every worker in the restaurant is paid $1000 per month. So the objective function in the formulation can be viewed as total wage expenses (in thousand dollars). Workers have complained that Shift 4 is the least desirable shift. Management is considering increasing the wages of workers on Shift 4 to $1100. Would this change the optimal solution? What would be the effect on total wage expenses?

5. Shift 2, on the other hand, is very desirable (Sundays off while on duty Fridays and Saturdays, which are the best days for tips). Management is considering reducing the wages of workers on Shift 1 to $900 per month. Would this change the optimal solution? What would be the effect on total wage expenses?

6. Management is considering introducing a new shift with the days off on Tuesdays and Sundays. Because these days are not consecutive, the wages will be $1200 per month. Will this increase or reduce the total wage expenses?

3.3.2 Dedication

We end this section with the sensitivity report of the dedication problem formulated in Section 3.2.

Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
B5	$x1$	62.13612744	0	102	3	5.590909091
B6	$x2$	0	0.830612245	99	$1E+30$	0.830612245
B7	$x3$	125.2429338	0	101	0.842650104	3.311081442
B8	$x4$	151.5050805	0	98	3.37414966	4.712358277
B9	$x5$	156.8077583	0	98	4.917243419	17.2316607
B10	$x6$	123.0800686	0	104	9.035524153	3.74817022
B11	$x7$	0	8.786840002	100	$1E+30$	8.786840002
B12	$x8$	124.1572748	0	101	3.988878399	8.655456271
B13	$x9$	104.0898568	0	102	9.456887408	0.860545483
B14	$x10$	93.45794393	0	94	0.900020046	$1E+30$
H4	$z0$	0	0.028571429	1	$1E+30$	0.028571429
H5	$z1$	0	0.055782313	0	$1E+30$	0.055782313
H6	$z2$	0	0.03260048	0	$1E+30$	0.03260048
H7	$z3$	0	0.047281187	0	$1E+30$	0.047281187
H8	$z4$	0	0.179369792	0	$1E+30$	0.179369792
H9	$z5$	0	0.036934059	0	$1E+30$	0.036934059
H10	$z6$	0	0.086760435	0	$1E+30$	0.086760435
H11	$z7$	0	0.008411402	0	$1E+30$	0.008411402
H12	$z8$	0	0.524288903	0	$1E+30$	0.524288903

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H.Side	Allowable Increase	Allowable Decrease
B19	$year1$	12000	0.971428571	12000	$1E+30$	6524.293381
B20	$year2$	18000	0.915646259	18000	137010.161	13150.50805
B21	$year3$	20000	0.883045779	20000	202579.3095	15680.77583
B22	$year4$	20000	0.835764592	20000	184347.1716	16308.00686
B23	$year5$	16000	0.6563948	16000	89305.96314	13415.72748
B24	$year6$	15000	0.619460741	15000	108506.7452	13408.98568
B25	$year7$	12000	0.532700306	12000	105130.9798	11345.79439
B26	$year8$	10000	0.524288903	10000	144630.1908	10000

Exercise 3.16 Analyze the solution tables above and

- Interpret the shadow price in year t $(t = 1, \dots, 8)$
- Interpret the reduced cost of bond i $(i = 1, \dots, 10)$
- Interpret the reduced cost of each surplus variable z_t $(t = 0, \dots, 7)$

Answers:

- The shadow price in Year t is the cost of the bond portfolio that can be attributed to a dollar of liability in Year t. For example, each dollar of liability in Year 3 is responsible for \$ 0.883 in the cost of the bond portfolio. Note that, by setting the shadow price in Year t equal to $\frac{1}{(1+r_t)^t}$, we get a term structure of interest rates. Here $r_3 = 0.0423$. In Figure 3.3.2 we plot the term structure of interest rates we compute from this solution. How does this compare with the term structure of Treasury rates?

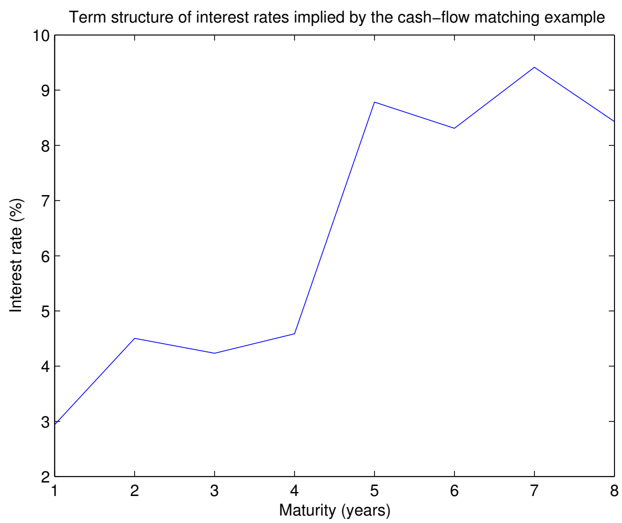

Figure 3.1: Interest rates implied by shadow prices

- The reduced cost of bond i indicates by how much bond i is overpriced for inclusion in the optimal portfolio. For example, bond 2 would have to be \$ 0.83 lower, at \$ 98.17, for inclusion in the optimal portfolio.

  Exercise 3.17 Note that the optimal solution has no holdings in Bond 7 which matures in Year 5, despite the \$16,000 liability in Year 5. This is likely due to a mispricing of this bond at \$100. What would be a more realistic price for this bond?

  Answer: Row 7 of the "Adjustable Cells" table indicates that variable x_7, corresponding to Bond 7 holdings, will become positive only if the price is reduced by 8.786 or more. So, a more realistic price for this bond would be just above \$91. By checking the reduced costs, one may sometimes spot errors in the data!

- The reduced cost of the surplus variable z_t indicates what the interest rate on cash reinvested in Year t would have to be in order to keep excess cash in Year t.

Exercise 3.18 Generate the sensitivity report for Exercise 3.12.

(i) Suppose that the liability in Year 3 is 29 (instead of 28). What would be the increase in cost of the dedicated portfolio?

(ii) Draw a graph of the term structure of interest rates implied by the shadow prices.

(iii) Bond 4 is not included in the optimal portfolio. By how much would the price of Bond 4 have to decrease for Bond 4 to become part of the optimal portfolio?

(iv) The fund manager would like to have 10000 units of Bond 3 in the portfolio. By how much would this increase the cost of the portfolio?

(v) Is there any bond that looks badly mispriced?

(vi) What interest rate on cash would make it optimal to include cash as part of the optimal portfolio?

3.4 Case Study

We are currently in year i. A municipality sends you the following liability stream (in million dollars) in years $i+1$ to $i+8$:

$6/15/i+1$	$12/15/i+1$	$6/15/i+2$	$12/15/i+2$	$6/15/i+3$	$12/15/i+3$
6	6	9	9	10	10

$6/15/i+4$	$12/15/i+4$	$6/15/i+5$	$12/15/i+5$	$6/15/i+6$	$12/15/i+6$
10	10	8	8	8	8

$6/15/i+7$	$12/15/i+7$	$6/15/i+8$	$12/15/i+8$
6	6	5	5

Your job:

- Value the liability using the Treasury curve.

- Identify between 30 and 50 assets that are suitable for a dedicated portfolio (non-callable bonds, treasury bills or notes). Explain why they are suitable. You can find current data on numerous web sites such as `www.bondsonline.com`

- Set up a linear program to identify a lowest cost dedicated portfolio of assets (so no short selling) and solve with Excel's solver (or any other linear programming software that you prefer). What is the cost of your portfolio? Discuss the composition of your portfolio. Discuss the assets and the liabilities in light of the Sensitivity Report. What is the term structure of interest rates implied by the shadow prices? Compare with the term structure of Treasury rates. (Hint: refer to Section 3.3.2.)

- Set up a linear program to identify a lowest cost portfolio of assets (no short selling) that matches present value, duration and convexity (or a related measure) between the liability stream and the bond portfolio. Solve the linear program with your favorite software. Discuss the solution. How much would you save by using this immunization strategy instead of dedication? Can you immunize the portfolio against nonparallel shifts of the yield curve? Explain.

- Set up a linear program to identify a lowest cost portfolio of assets (no short selling) that combines a cash matching strategy for the liabilities in the first 3 years and an immunization strategy based on present value, duration and convexity for the liabilities in the last 5 years. Compare the cost of this portfolio with the cost of the two previous portfolios.

- The municipality would like you to make a second bid: what is your lowest cost dedicated portfolio of riskfree bonds if short sales are allowed? Discuss the feasibility of your solution.

Chapter 4

LP Models: Asset Pricing and Arbitrage

4.1 Derivative Securities and The Fundamental Theorem of Asset Pricing

One of the most widely studied problems in financial mathematics is the pricing of *derivative securities*, also known as *contingent claims*. These are securities whose price depends on the value of another *underlying security*. Financial *options* are the most common examples of derivative securities. For example, a European call option gives the holder the right to purchase an underlying security for a prescribed amount (called the *strike price*) at a prescribed time in the future, known as the *expiration* or *exercise date*. The exercise date is also known as the *maturity date* of the derivative security. Recall the similar definitions of European put options as well as American call and put options from Section 1.3.2.

Options are used mainly for two purposes: speculation and hedging. By speculating on the direction of the future price movements of the underlying security, investors can take (bare) positions in options on this security. Since options are often much cheaper than their underlying security, this bet results in much larger earnings in relative terms if the price movements happen in the expected direction compared to what one might earn by taking a similar position in the underlying. Of course, if one guesses the direction of the price movements incorrectly, the losses are also much more severe.

The more common and sensible use of options is for hedging. Hedging refers to the reduction of risk in an investor's overall position by forming a suitable portfolio of assets that are expected to have opposing risks. For example, if an investor holds a share of XYZ and is concerned that the price of this security may fall significantly, she can purchase a put option on XYZ and protect herself against price levels below a certain threshold–the strike price of the put option.

Recall the option example in the simple one-period binomial model of Section 1.3.2. Below, we summarize some of the information from that example:

We consider the share price of XYZ stock which is currently valued at \$40. A month from today, we expect the share price of XYZ to either double or halve, with equal probabilities. We also consider a European call option on XYZ with a strike price of \$50 which will expire a month from today. The payoff function for the call is shown in Figure 4.1.

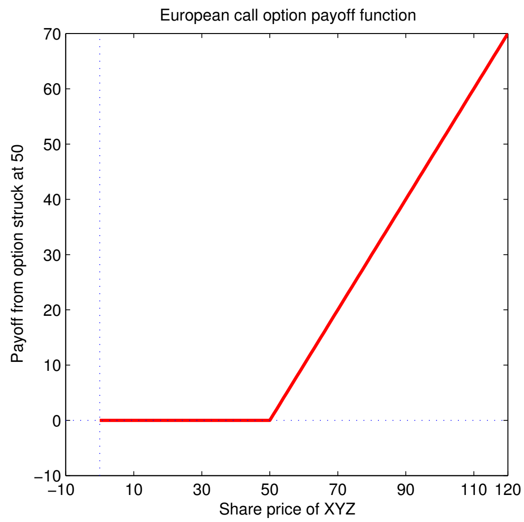

Figure 4.1: Piecewise linear payoff function for a call option

We assume that interest rates for cash borrowing or lending are zero and that any amount of XYZ shares can be bought or sold with no commission.

$$S_0{=}\$40 \begin{cases} \nearrow 80{=}S_1(u) \\ \searrow 20{=}S_1(d) \end{cases} \qquad \text{and} \quad C_0{=}? \begin{cases} \nearrow (80-50)^+ = 30 \\ \searrow (20-50)^+ = 0 \end{cases}$$

In Section 1.3.2 we obtained a fair price of \$10 for the option using a replication strategy and the no-arbitrage principle. Two portfolios of securities that have identical future payoffs under all possible realizations of the random states must have the same value today. In the example, the first portfolio is the option while the second one is the portfolio of $\frac{1}{2}$ share of XYZ and -\$10 in cash. Since we know the current value of the second portfolio, we can deduce the fair price of the option. To formalize this approach, we first give a definition of arbitrage:

Definition 4.1 *An* arbitrage *is a trading strategy*

- *that has a positive initial cash flow and has no risk of a loss later (type A), or*

- *that requires no initial cash input, has no risk of a loss, and a positive probability of making profits in the future (type B).*

In the example, any price other than \$10 for the call option would lead to a type A arbitrage–guaranteed profits at the initial time point and no future obligations. We do not need to have a guarantee of profits for type B arbitrage–all we need is a guarantee of no loss, and a positive probability of a gain. Prices adjust quickly so that arbitrage opportunities cannot persist in the markets. Therefore, in pricing arguments it is often assumed that arbitrage opportunities do not exist.

4.1.1 Replication

In the above example, we formulated and solved the following question to determine the fair price of an option: Can we form a portfolio of the underlying security (long or short) and cash (borrowed or lent) today, such that the payoff of the portfolio at the expiration date of the option will match the payoff of the option? In other words, can we replicate the option using a portfolio of the underlying security and cash?

Let us work in a slightly more general setting. Let S_0 be the current price of the underlying security and assume that there are two possible outcomes at the end of the period: $S_1^u = S_0 \cdot u$ and $S_1^d = S_0 \cdot d$. Assume $u > d$. We also assume that there is a fixed interest rate of r on cash positions for the given period. Let $R = 1 + r$.

Now we consider a derivative security which has payoffs of C_1^u and C_1^d in the up and down states respectively:

$$\begin{tikzcd}[row sep=small] & S_1^u = S_0 \cdot u \\ S_0 \arrow[ur] \arrow[dr] & \\ & S_1^d = S_0 \cdot d \end{tikzcd} \qquad \begin{tikzcd}[row sep=small] & C_1^u \\ C_0 =? \arrow[ur] \arrow[dr] & \\ & C_1^d \end{tikzcd}$$

To price the derivative security, we will replicate its payoff. For replication consider a portfolio of Δ shares of the underlying and \$$B$ cash. For what values of Δ and B does this portfolio have the same payoffs at the expiration date as the derivative security?

In the "up" state, the replicating portfolio will have value $\Delta S_0 \cdot u + BR$ and in the "down" state it will be worth $\Delta S_0 \cdot d + BR$. Therefore, for perfect replication, we need to solve the following simple system of equations:

$$\begin{aligned} \Delta S_0 \cdot u + BR &= C_1^u \\ \Delta S_0 \cdot d + BR &= C_1^d. \end{aligned}$$

We obtain:

$$\begin{aligned} \Delta &= \frac{C_1^u - C_1^d}{S_0(u-d)} \\ B &= \frac{uC_1^d - dC_1^u}{R(u-d)}. \end{aligned}$$

Since this portfolio is worth $S_0\Delta + B$ today, that should be the price of the derivative security as well:

$$C_0 = \frac{C_1^u - C_1^d}{u-d} + \frac{uC_1^d - dC_1^u}{R(u-d)}$$

$$= \frac{1}{R}\left[\frac{R-d}{u-d}C_1^u + \frac{u-R}{u-d}C_1^d\right].$$

4.1.2 Risk-Neutral Probabilities

Let

$$p_u = \frac{R-d}{u-d} \quad \text{and} \quad p_d = \frac{u-R}{u-d}.$$

Note that we must have $d < R < u$ to avoid arbitrage opportunities as indicated in the following simple exercise.

Exercise 4.1 Let S_0 be the current price of a security and assume that there are two possible prices for this security at the end of the current period: $S_1^u = S_0 \cdot u$ and $S_1^d = S_0 \cdot d$. (Assume $u > d$.) Also assume that there is a fixed interest rate of r on cash positions for the given period. Let $R = 1 + r$. Show that there is an arbitrage opportunity if $u > R > d$ is not satisfied.

An immediate consequence of this observation is that both $p_u > 0$ and $p_d > 0$. Noting also that $p_u + p_d = 1$ one can interpret p_u and p_d as probabilities. In fact, these are the so-called *risk-neutral probabilities* (RNPs) of up and down states, respectively. Note that they are completely independent from the physical probabilities of these states.

The price of any derivative security can now be calculated as the present value of the expected value of its future payoffs where the expected value is taken using the risk-neutral probabilities.

In our example above $u = 2$, $d = \frac{1}{2}$ and $r = 0$ so that $R = 1$. Therefore:

$$p_u = \frac{1-1/2}{2-1/2} = \frac{1}{3} \quad \text{and} \quad p_d = \frac{2-1}{2-1/2} = \frac{2}{3}.$$

As a result, we have

$$\begin{aligned} S_0 = 40 &= \frac{1}{R}(p_u S_1^u + p_d S_1^d) = \frac{1}{3}80 + \frac{2}{3}20, \\ C_0 = 10 &= \frac{1}{R}(p_u C_1^u + p_d C_1^d) = \frac{1}{3}30 + \frac{2}{3}0, \end{aligned}$$

as expected. Using risk neutral probabilities we can also price other derivative securities on the XYZ stock. For example, consider a European put option on the XYZ stock struck at \$60 and with the same expiration date as the call of the example.

$$P_0 = ? \begin{cases} \nearrow P_1^u = \max\{0, 60 - 80\} = 0 \\ \searrow P_1^d = \max\{0, 60 - 20\} = 40 \end{cases}$$

We can easily compute:

$$P_0 = \frac{1}{R}(p_u P_1^u + p_d P_1^d) = \frac{1}{3}0 + \frac{2}{3}40 = \frac{80}{3},$$

without needing to replicate the option again.

Exercise 4.2 Compute the price of a binary (digital) call option on the XYZ stock that pays \$1 if the XYZ price is above the strike price of \$50.

Exercise 4.3 Assume that the XYZ stock is currently priced at \$40. At the end of the next period, the XYZ price is expected to be in one of the following two states: $S_0 \cdot u$ or $S_0 \cdot d$. We know that $d < 1 < u$ but do not know d or u. The interest rate is zero. If a European call option with strike price \$50 is priced at \$10 while a European call option with strike price \$40 is priced at \$13. If we assume that these prices do not contain any arbitrage opportunities, what is the fair price of a European put option with a strike price of \$40?
Hint: First note that $u > \frac{5}{4}$–otherwise the first call would be worthless. Then we must have $10 = p_u(S_0 \cdot u - 50)$ and $13 = p_u(S_0 \cdot u - 40)$. From these equations determine p_u and then u and d.

Exercise 4.4 Assume that the XYZ stock is currently priced at \$40. At the end of the next period, the XYZ price is expected to be in one of the following two states: $S_0 \cdot u$ or $S_0 \cdot d$. We know that $d < 1 < u$ but do not know d or u. The interest rate is zero. European call options on XYZ with strike prices of \$30, \$40, \$50, and \$60 are priced at \$10, \$7, $\$\frac{10}{3}$, and \$0. Which one of these options is mispriced? Why?

Remark 4.1 *Exercises 4.3 and 4.4 are much simplified and idealized examples of the pricing problems encountered by practitioners. Instead of a set of possible future states for prices which may be difficult to predict, they must work with a set of market prices for related securities. Then, they must extrapolate prices for an unpriced security using no-arbitrage arguments.*

Next we move from our binomial setting to a more general setting and let

$$\Omega = \{\omega_1, \omega_2, \dots, \omega_m\} \tag{4.1}$$

be the (finite) set of possible future "states". For example, these could be prices for a security at a future date.

For securities S^i, $i = 0 \dots n$, let $S^i_1(\omega_j)$ denote the price of this security in state ω_j at time 1. Also let S^i_0 denote the current (time 0) price of security S^i. We use $i = 0$ for the "riskless" security that pays the interest rate $r \geq 0$ between time 0 and time 1. It is convenient to assume that $S^0_0 = 1$ and that $S^0_1(\omega_j) = R = 1 + r, \forall j$.

Definition 4.2 *A risk-neutral probability measure on the set* $\Omega = \{\omega_1, \omega_2, \dots, \omega_m\}$ *is a vector of positive numbers* $(p_1, p_2, \dots, p_m)$ *such that*

$$\sum_{j=1}^{m} p_j = 1$$

and for every security S^i, $i = 0, \dots, n$,

$$S^i_0 = \frac{1}{R}\left(\sum_{j=1}^{m} p_j S^i_1(\omega_j)\right) = \frac{1}{R}\hat{\mathbf{E}}[S^i_1].$$

Above, $\hat{\mathbf{E}}[S]$ denotes the expected value of the random variable S under the probability distribution $(p_1, p_2, \ldots, p_m)$.

4.1.3 The Fundamental Theorem of Asset Pricing

In this section we state the first fundamental theorem of asset pricing and prove it for finite Ω. This proof is a simple exercise in linear programming duality that also utilizes the following well-known result of Goldman and Tucker on the existence of strictly complementary optimal solutions of LPs:

Theorem 4.1 (Goldman and Tucker [30]) *When both the primal and dual linear programming problems*

$$\begin{array}{rrcl} \min_x & c^T x & & \\ & Ax & = & b \\ & x & \geq & 0 \end{array} \tag{4.2}$$

and

$$\begin{array}{rrcl} \max_y & b^T y & & \\ & A^T y & \leq & c, \end{array} \tag{4.3}$$

have feasible solutions, they have optimal solutions satisfying strict complementarity, i.e., there exist x^* and y^* optimal for the respective problems such that

$$x^* + (c - A^T y^*) > 0.$$

Now, we are ready to prove the following theorem:

Theorem 4.2 (The First Fundamental Theorem of Asset Pricing) *A risk-neutral probability measure exists if and only if there is no arbitrage.*

Proof:
We provide the proof for the case when the state space Ω is finite and is given by (4.1). We assume without loss of generality that every state has a positive probability of occuring (since states that have no probability of occuring can be removed from Ω.) Given the current prices S_0^i and the future prices $S_1^i(\omega_j)$ in each state ω_j, for securities 0 to n, consider the following linear program with variables x_i, for $i = 0, \ldots, n$:

$$\begin{array}{rrcl} \min_x & \sum_{i=0}^n S_0^i x_i & & \\ & \sum_{i=0}^n S_1^i(\omega_j) x_i & \geq & 0, \quad j = 1, \ldots, m. \end{array} \tag{4.4}$$

Note that type-A arbitrage corresponds to a feasible solution to this LP with a negative objective value. Since $x = (x_1, \ldots, x_n)$ with $x_i = 0, \forall i$ is a feasible solution, the optimal objective value is always non-positive. Furthermore, since all the constraints are homogeneous, if there exists a feasible solution such that $\sum S_0^i x_i < 0$ (this corresponds to type-A arbitrage), the problem is unbounded. In other words, there is no type-A arbitrage if and only if the optimal objective value of this LP is 0.

Suppose that there is no type-A arbitrage. Then, there is no type-B arbitrage if and only if all constraints are tight for all optimal solutions of

(4.4) since every state has a positive probability of occuring. Note that these solutions must have objective value 0.

Consider the dual of (4.4):

$$\begin{array}{llcl} \max_p & \sum_{j=1}^m 0 p_j & & \\ & \sum_{j=1}^m S_1^i(\omega_j) p_j & = & S_0^i, \quad i = 0, \ldots, n, \\ & p_j & \geq & 0, \quad j = 1, \ldots, m. \end{array} \tag{4.5}$$

Since the dual objective function is constant at zero for all dual feasible solutions, any dual feasible solution is also dual optimal.

When there is no type-A arbitrage, (4.4) has an optimal solution. Now, Theorem 2.2–Strong Duality Theorem–indicates that the dual must have a feasible solution. If there is no type-B arbitrage either, Goldman and Tucker's theorem indicates that there exists a feasible and therefore optimal dual solution p^* such that $p^* > 0$. This follows from strict complementarity with primal constraints $\sum_{i=1}^n S_1^i(\omega_j) x_i \geq 0$ which are tight. From the dual-nconstraint corresponding to $i = 0$, we have that $\sum_{j=1}^m p_j^* = \frac{1}{R}$. Multiplying p^* by R one obtains a risk-neutral probability distribution. Therefore, the "no arbitrage" assumption implies the existence of RNPs.

The converse direction is proved in an identical manner. The existence of a RNP measure implies that (4.5) is feasible, and therefore its dual, (4.4) must be bounded, which implies that there is no type-A arbitrage. Furthermore, since we have a strictly feasible (and optimal) dual solution, any optimal solution of the primal must have tight constraints, indicating that there is no type-B arbitrage. □

4.2 Arbitrage Detection Using Linear Programming

The linear programming problems (4.4) and (4.5) formulated in the proof of Theorem 4.2 can naturally be used for detection of arbitrage opportunities. However, as we discussed above, this argument works only for finite state spaces. In this section, we discuss how LP formulations can be used to detect arbitrage opportunities without limiting consideration to finite state spaces. The price we pay for this flexibility is the restriction on the selection of the securities: we only consider the prices of a set of derivative securities written on the same underlying with same maturity. This discussion is based on Herzel [38].

Consider an underlying security with a current, time 0, price of S_0 and a random price S_1 at time 1. Consider n derivative securities written on this security that mature at time 1, and have piecewise linear payoff functions $\Psi_i(S_1)$, each with a single breakpoint K_i, for $i = 1, \ldots, n$. The obvious motivation is the collection of calls and puts with different strike prices written on this security. If, for example, the i-th derivative security were a European call with strike price K_i, we would have $\Psi_i(S_1) = (S_1 - K_i)^+$. We assume that the K_is are in increasing order, without loss of generality. Finally, let S_0^i denote the current price of the i-th derivative security.

Consider a portfolio $x = (x_1, \ldots, x_n)$ of the derivative securities 1 to n and let $\Psi^x(S_1)$ denote the payoff function of the portfolio:

$$\Psi^x(S_1) \quad := \quad \sum_{i=1}^{n} \Psi_i(S_1) x_i. \tag{4.6}$$

The cost of forming the portfolio x at time 0 is given by

$$\sum_{i=1}^{n} S_0^i x_i. \tag{4.7}$$

To determine whether a static arbitrage opportunity exists in the current prices S_0^i, we consider the following problem: What is the cheapest portfolio of the derivative securities 1 to n whose payoff function $\Psi^x(S_1)$ is nonnegative for all $S_1 \in [0, \infty)$? Non-negativity of $\Psi^x(S_1)$ corresponds to "no future obligations" part of the arbitrage definition. If the minimum initial cost of such a portfolio is negative, then we have a type-A arbitrage.

Since all $\Psi_i(S_1)$s are piecewise linear, so is $\Psi^x(S_1)$. It will have up to n breakpoints at points K_1 through K_n. Observe that a piecewise linear function is nonnegative over $[0, \infty)$ if and only if it is nonnegative at 0 and at all the breakpoints, and if the slope of the function is nonnegative to the right of the largest breakpoint. From our notation, $\Psi^x(S_1)$ is nonnegative for all non-negative values of S_1 if and only if

1. $\Psi^x(0) \geq 0$,
2. $\Psi^x(K_j) \geq 0, \ \forall j$,
3. and $[(\Psi^x)'_+(K_n)] \geq 0$.

Now consider the following linear programming problem:

$$\begin{array}{lrcl} \min_x & \sum_{i=1}^n S_0^i x_i & & \\ & \sum_{i=1}^n \Psi_i(0) x_i & \geq & 0 \\ & \sum_{i=1}^n \Psi_i(K_j) x_i & \geq & 0, \quad j = 1, \ldots, n \\ & \sum_{i=1}^n (\Psi_i(K_n + 1) - \Psi_i(K_n)) x_i & \geq & 0 \end{array} \tag{4.8}$$

Since all $\Psi_i(S_1)$'s are piecewise linear, the quantity $\Psi_i(K_n + 1) - \Psi_i(K_n)$ gives the right-derivative of $\Psi_i(S_1)$ at K_n. Thus, the expression in the last constraint is the right derivative of $\Psi^x(S_1)$ at K_n. The following observation follows from our arguments above:

Proposition 4.1 *There is no type-A arbitrage in prices S_0^i if and only if the optimal objective value of (4.8) is zero.*

Similar to the previous section, we have the following result:

Proposition 4.2 *Suppose that there are no type-A arbitrage opportunities in prices S_0^i. Then, there are no type-B arbitrage opportunities if and only if the dual of the problem (4.8) has a strictly feasible solution.*

Exercise 4.5 Prove Proposition 4.2.

Next, we focus on the case where the derivative securities under consideration are European call options with strikes at K_i for $i = 1, \ldots, n$, so that $\Psi_i(S_1) = (S_1 - K_i)^+$. Thus

$$\Psi_i(K_j) = (K_j - K_i)^+.$$

In this case, (4.8) reduces to the following problem:

$$\begin{array}{rrcl} \min_x & c^T x & & \\ & Ax & \geq & 0, \end{array} \tag{4.9}$$

where $c^T = [S_0^1, \ \ldots, S_0^n]$ and

$$A = \begin{bmatrix} K_2 - K_1 & 0 & 0 & \cdots & 0 \\ K_3 - K_1 & K_3 - K_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ K_n - K_1 & K_n - K_2 & K_n - K_3 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix}. \tag{4.10}$$

This formulation is obtained by removing the first two constraints of (4.8) which are redundant in this particular case.

Using this formulation and our earlier results, one can prove a theorem giving necessary and sufficient conditions for a set of call option prices to contain arbitrage opportunities:

Theorem 4.3 *Let $K_1 < K_2 < \cdots < K_n$ denote the strike prices of European call options written on the same underlying security with the same maturity. There are no arbitrage opportunities if and only if the prices S_0^i satisfy the following conditions:*

1. $S_0^i > 0, \ i = 1, \ldots, n$
2. $S_0^i > S_0^{i+1}, \ i = 1, \ldots, n-1$
3. *The function $C(K_i) := S_0^i$ defined on the set $\{K_1, K_2, \ldots, K_n\}$ is a strictly convex function.*

Exercise 4.6 Use Proposition 4.2 to show that there are no arbitrage opportunities for the option prices in Theorem 4.3 if and only if there exists strictly positive scalars $y_1, \ldots, y_n$ satisfying $y_n = S_0^n$, $y_{n-1} = (S_0^{n-1} - S_0^n)/(K^n - K^{n-1})$, and

$$y_i = \frac{S_0^i - S_0^{i+1}}{K^{i+1} - K^i} - \frac{S_0^{i+1} - S_0^{i+2}}{K^{i+2} - K^{i+1}}, i = 1, \ldots, n-2.$$

Use this observation to prove Theorem 4.3

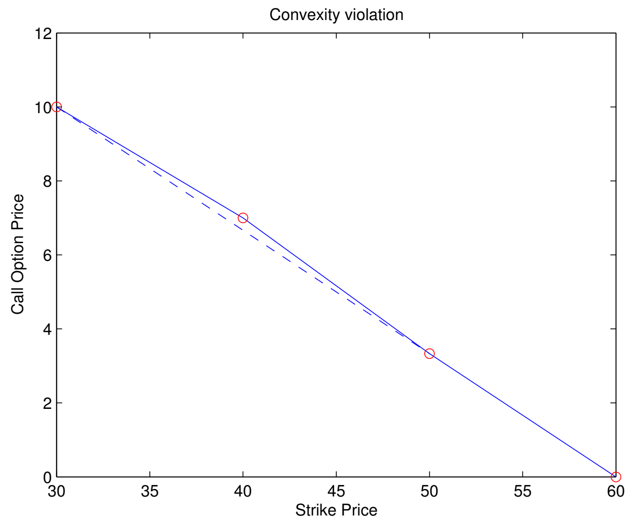

Figure 4.2: Nonconvexity in the call price function indicates arbitrage

As an illustration of Theorem 4.3, consider the scenario in Exercise 4.4: XYZ stock is currently priced at \$40. European call options on XYZ with strike prices of \$30, \$40, \$50, and \$60 are priced at \$10, \$7, $\$\frac{10}{3}$, and \$0. Do these prices exhibit an arbitrage opportunity? As we see in Figure 4.2, the option prices violate the third condition of the Theorem and therefore must carry an arbitrage opportunity.

Exercise 4.7 Construct a portfolio of the options in the example above that provides a type-A arbitrage opportunity.

4.3 Additional Exercises

Exercise 4.8 Consider the linear programming problem (4.9) that we developed to detect arbitrage opportunities in the prices of European call options with a common underlying security and common maturity (but different strike prices). This formulation implicitly assumes that the i^{th} call can be bought or sold at the same current price of S_0^i. In real markets, there is always a gap between the price a buyer pays for a security and the amount the seller collects called the *bid-ask spread*.

Assume that the ask price of the i^{th} call is given by S_a^i while its bid price is denoted by S_b^i with $S_a^i > S_b^i$. Develop an analogue of the LP (4.9) in the case where we can purchase the calls at their ask prices or sell them at their bid prices. Consider using two variables for each call option in your new LP.

Exercise 4.9 Consider all the call options on the S&P 500 index that expire on the same day, about three months from the current date. Their current prices can be downloaded from the website of the Chicago Board of Options Exchange at `www.cboe.com` or several other market quote websites. Formulate the linear programming problem (4.9) (or, rather the version you developed for Exercise 4.8 since market quotes will include bid and ask prices) to determine whether these prices contain any arbitrage opportunities. Solve this linear programming problem using an LP software.

Sometimes, illiquid securities can have misleading prices since the reported price corresponds to the last transaction in that security which may have happened several days ago, and if there were to be a new transaction, this value would change dramatically. As a result, it is quite possible that you will discover false "arbitrage opportunities" because of these misleading prices. Repeat the LP formulation and solve it again, this time only using prices of the call options that have had a trading volume of at least 100 on the day you downloaded the prices.

Exercise 4.10 (i) You have \$20,000 to invest. Stock XYZ sells at \$20 per share today. A European call option to buy 100 shares of stock XYZ at \$15 exactly six months from today sells for \$1000. You can also raise additional funds which can be immediately invested, if desired, by selling call options with the above characteristics. In addition, a 6-month riskless zero-coupon bond with \$100 face value sells for \$90. You have decided to limit the number of call options that you buy or sell to at most 50.

You consider three scenarios for the price of stock XYZ six months from today: the price will be the same as today, the price will go up to \$40, or drop to \$12. Your best estimate is that each of these scenarios is equally likely. Formulate and solve a linear program to determine the portfolio of stocks, bonds, and options that maximizes expected profit.

Answer: First, we define the decision variables.
B = number of bonds purchased,
S = number of shares of stock XYZ purchased,
C = number of call options purchased (if > 0) or sold (if < 0).

The expected profits (per unit of investment) are computed as follows.

Bonds: 10
Stock XYZ: $\frac{1}{3}(20 + 0 - 8) = 4$
Call Option: $\frac{1}{3}(1500 - 500 - 1000) = 0$

Therefore, we get the following linear programming formulation.

$$
\begin{array}{llllllll}
\max & 10B & + & 4S & & & & \\
 & 90B & + & 20S & + & 1000C & \leq & 20000 & \text{(budget constraint)} \\
 & & & & & C & \leq & 50 & \text{(limit on number of call options purchased)} \\
 & & & & & C & \geq & -50 & \text{(limit on number of call options sold)} \\
 & & B \geq 0, & S \geq 0 & & & & & \text{(nonnegativity).}
\end{array}
$$

Solving (using SOLVER, say), we get the optimal solution $B = 0$, $S = 3500$, C = -50 with an expected profit of \$14,000.

Note that, with this portfolio, the profit is not positive under all scenarios. In particular, if the price of stock XYZ goes to \$40, a loss of \$5000 will be incurred.

(ii) Suppose that the investor wants a profit of at least \$2000 in any of the three scenarios. Write a linear program that will maximize the investor's expected profit under this additional constraint.

Answer: This can be done by introducing three additional variables.
P_i = profit in scenario i
The formulation is now the following.

$$
\begin{array}{lrcrcrrcl}
\max & \frac{1}{3}P_1 & + & \frac{1}{3}P_2 & + & \frac{1}{3}P_3 & & & \\
 & 90B & + & 20S & + & 1000C & & \leq & 20000 \\
 & 10B & + & 20S & + & 1500C & & = & P_1 \\
 & 10B & & & - & 500C & & = & P_2 \\
 & 10B & - & 8S & - & 1000C & & = & P_3 \\
 & & & & & & P_1 & \geq & 2000 \\
 & & & & & & P_2 & \geq & 2000 \\
 & & & & & & P_3 & \geq & 2000 \\
 & & & & & C & & \leq & 50 \\
 & & & & & C & & \geq & -50 \\
 & \multicolumn{8}{l}{B \geq 0, \quad S \geq 0.}
\end{array}
$$

(iii) Solve this linear program with SOLVER to find out the expected profit.

How does it compare with the earlier figure of \$14,000?

Answer: The optimum solution is to buy 2,800 shares of XYZ and sell 36 call options. The resulting expected worth in six months will be \$31,200. Therefore, the expected profit is \$11,200 (=\$31,200 - 20,000).

(iv) *Riskless profit* is defined as the largest possible profit that a portfolio is guaranteed to earn, no matter which scenario occurs. What is the portfolio that maximizes riskless profit for the above three scenarios?

Answer: To solve this question, we can use a slight modification of the previous model, by introducing one more variable.
Z = riskless profit.
Here is the formulation.

$$
\begin{array}{lrcrcrrcl}
\max & Z & & & & & & & \\
 & 90B & + & 20S & + & 1000C & & \leq & 20000 \\
 & 10B & + & 20S & + & 1500C & & = & P_1 \\
 & 10B & & & - & 500C & & = & P_2 \\
 & 10B & - & 8S & - & 1000C & & = & P_3 \\
 & & & & & & P_1 & \geq & Z \\
 & & & & & & P_2 & \geq & Z \\
 & & & & & & P_3 & \geq & Z \\
 & & & & & C & & \leq & 50 \\
 & & & & & C & & \geq & -50 \\
 & \multicolumn{8}{l}{B \geq 0, \quad S \geq 0.}
\end{array}
$$

The result is (obtained using SOLVER) a riskless profit of \$7272. This is obtained by buying 2,273 shares of XYZ and selling 25.45 call options. The resulting expected profit is \$9,091 in this case.

Exercise 4.11 *Arbitrage in the Currency Market*

Consider the global currency market. Given two currencies, say the Yen and the USDollar, there is an exchange rate between them (about 118 Yens to the Dollar in February 2006). It is axiomatic of arbitrage-free markets that there is no method of converting, say, a Dollar to Yens then to Euros, then Pounds, and to Dollars so that you end up with more than a dollar. How would you recognize when there is an arbitrage opportunity?

The following are actual trades made on February 14, 2002.

from		Dollar	Euro	Pound	Yen
into	Dollar		.8706	1.4279	.00750
	Euro	1.1486		1.6401	.00861
	Pound	.7003	.6097		.00525
	Yen	133.38	116.12	190.45	

For example, one dollar converted into euros yielded 1.1486 euros. It is not obvious from the chart above, but in the absence of any conversion costs, the Dollar-Pound-Yen-Dollar conversion actually makes \$0.0003 per dollar converted while changing the order to Dollar-Yen-Euro-Dollar loses about \$0.0002 per dollar converted. How can one formulate a linear program to identify such arbitrage possibilities?

Answer:

```
   VARIABLES
DE = quantity of dollars changed into euros
DP = quantity of dollars changed into pounds
DY = quantity of dollars changed into yens
ED = quantity of euros changed into dollars
EP = quantity of euros changed into pounds
EY = quantity of euros changed into yens
PD = quantity of pounds changed into dollars
PE = quantity of pounds changed into euros
PY = quantity of pounds changed into yens
YD = quantity of yens changed into dollars
YE = quantity of yens changed into euros
YP = quantity of yens changed into pounds
D = quantity of dollars generated through arbitrage
OBJECTIVE
Max D
CONSTRAINTS
Dollar:  D + DE + DP + DY - 0.8706*ED - 1.4279*PD - 0.00750*YD =
1
Euro:  ED + EP + EY - 1.1486*DE - 1.6401*PE - .00861*YE = 0
```

```
Pound:  PD + PE + PY - 0.7003*DP - 0.6097*EP - 0.00525*YP = 0
Yen:  YD + YE + YP - 133.38*DY - 116.12*EY - 190.45*PY = 0
BOUNDS
D < 10000
END
```

Solving this linear program, we find that, in order to gain \$10,000 in arbitrage, we have to change about \$34 million dollars into euros, then convert these euros into yens and finally change the yens into dollars. There are other solutions as well. The arbitrage opportunity is so tiny (\$0.0003 to the dollar) that, depending on the numerical precision used, some LP solvers do not find it, thus concluding that there is no arbitrage here. An interesting example illustrating the role of numerical precision in optimization solvers!

4.4 Case Study: Tax Clientele Effects in Bond Portfolio Management

The goal is to construct an optimal tax-specific bond portfolio, for a given tax bracket, by exploiting the price differential of an after-tax stream of cash flows. This objective is accomplished by purchasing at the ask price "underpriced" bonds (for the specific tax bracket), while simultaneously selling at the bid price "overpriced" bonds. The following model was proposed by E.I. Ronn [62]. See also S.M. Schaefer [65].

Let

$J = \{1, \ldots, j, \ldots, N\}$ = set of riskless bonds.

P_j^a = asked price of bond j

P_j^b = bid price of bond j

X_j^a = amount of bond j bought

X_j^b = amount of bond j sold short, and

We make the natural assumption that $P_j^a > P_j^b$. The objective function of the program is

$$Z = \max \sum_{j=1}^{N} P_j^b X_j^b - \sum_{j=1}^{N} P_j^a X_j^a \tag{4.11}$$

since the long side of an arbitrage position must be established at ask prices while the short side of the position must be established at bid prices. Now consider the future cash-flows of the portfolio.

$$C_1 = \sum_{j=1}^{N} a_j^1 X_j^a - \sum_{j=1}^{N} a_j^1 X_j^b \tag{4.12}$$

$$\text{For } t = 2, \ldots, T, \qquad C_t = (1+\rho)C_{t-1} + \sum_{j=1}^{N} a_j^t X_j^a - \sum_{j=1}^{N} a_j^t X_j^b, \tag{4.13}$$

where ρ = Exogenous riskless reinvestment rate

a_j^t = coupon and/or principal payment on bond j at time t.

For the portfolio to be riskless, we require

$$C_t \geq 0 \qquad t = 1, \ldots, T. \tag{4.14}$$

Since the bid-ask spread has been explicitly modeled, it is clear that $X_j^a \geq 0$ and $X_j^b \geq 0$ are required. Now the resulting linear program admits two possible solutions. Either all bonds are priced to within the bid-ask spread, i.e. $Z = 0$, or infinite arbitrage profits may be attained, i.e. $Z = \infty$. Clearly any attempt to exploit price differentials by taking extremely large positions in these bonds would cause price movements: the bonds being bought would appreciate in price; the bonds being sold short would decline in value. In order to provide a finite solution, the constraints $X_j^a \leq 1$ and $X_j^b \leq 1$ are imposed. Thus, with

$$0 \leq X_j^a,\ X_j^b \leq 1 \qquad j = 1, \ldots, N, \tag{4.15}$$

the complete problem is now specified as (4.11)-(4.15).

Taxes

The proposed model explicitly accounts for the taxation of income and capital gains for specific investor classes. This means that the cash flows need to be adjusted for the presence of taxes.

For a discount bond (i.e. when $P_j^a < 100$), the after-tax cash-flow of bond j in period t is given by

$$a_j^t = c_j^t(1 - \tau),$$

where c_j^t is the semiannual coupon payment

and τ is the ordinary income tax rate.

At maturity, the j^{th} bond yields

$$a_j^t = (100 - P_j^a)(1 - g) + P_j^a,$$

where g is the capital gains tax rate.

For premium bond (i.e. when $P_j^a > 100$), the premium is amortized against ordinary income over the life of the bond, giving rise to an after-tax coupon payment of

$$a_j^t = \left[c_j^t - \frac{P_j^a - 100}{n_j}\right](1 - \tau) + \frac{P_j^a - 100}{n_j}$$

where n_j is the number of coupon payments remaining to maturity.

A premium bond also makes a nontaxable repayment of

$$a_j^t = 100$$

at maturity.

Data

The model requires that the data contain bonds with perfectly forcastable cash flows. All callable bonds are excluded from the sample. For the same reason, flower bonds of all types are excluded. Thus, all noncallable bonds and notes are deemed appropriate for inclusion in the sample.

Major categories of taxable investors are Domestic Banks, Insurance Companies, Individuals, Nonfinancial Corporations, Foreigners. In each case, one needs to distinguish the tax rates on capital gains versus ordinary income.

The fundamental question to arise from this study is: does the data reflect tax clientele effects or arbitrage opportunities?

Consider first the class of tax-exempt investors. Using current data, form the optimal "purchased" and "sold" bond portfolios. Do you observe the same tax clientele effect as documented by Schaefer for British government securities; namely, the "purchased" portfolio contains high coupon bonds and the "sold" portfolio is dominated by low coupon bonds. This can be explained as follows: The preferential taxation of capital gains for (most) taxable investors causes them to gravitate towards low coupon bonds. Consequently, for tax-exempt investors, low coupon bonds are "overpriced" and not desirable as investment vehicles.

Repeat the same analysis with the different types of taxable investors. Do you observe:

1. a clientele effect in the pricing of US Government investments, with tax-exempt investors, or those without preferential treatment of capital gains, gravitating towards high coupon bonds?

2. that not all high coupon bonds are desirable to investors without preferential treatment of capital gains? Nor are all low coupon bonds attractive to those with preferential treatment of capital gains. Can you find reasons why this may be the case?

The dual price, say u_t, associated with constraint (4.13) represents the present value of an additional dollar at time t. Explain why. It follows that u_t may be used to compute the term structure of spot interest rates R_t, given by the relation

$$R_t = \left(\frac{1}{u_t}\right)^{\frac{1}{t}} - 1.$$

Compute this week's term structure of spot interest rates for tax-exempt investors.