A random graph model of cities

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6.268

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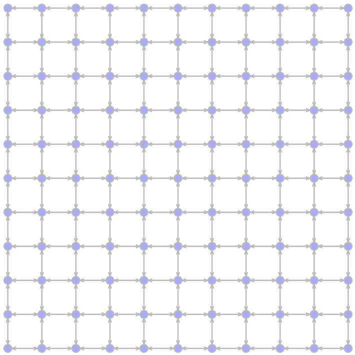
Motivation

Empirical evidence suggests city structure and road coverage powerfully affect population distributions and demand for vehicle miles traveled. We want to see if simple random graph models can capture some of these relationships.

Approach overview

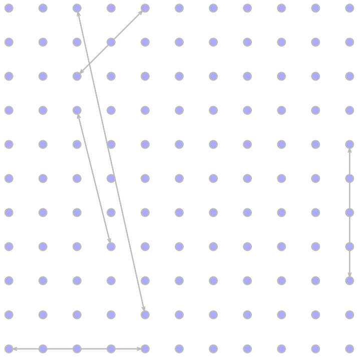
1. Create random graph: we represent a city as a finite two-dimensional lattice with random additional edges. The origin represents downtown.
2. Randomly populate the graph: one-by-one residents arrive at the network and decide which node to live at randomly, combining location and density preferences.
3. Compute optimal population: we solve the MIQP of locating residents on the graph to minimize density and travel times under a simple static model.

Random graph creation



1. Create a lattice on $[-N, N] \times [-N, N]$
2. Add $\lfloor N^{\alpha} \rfloor$ extra edges with one endpoint u chosen uniformly at random and other endpoint v chosen with probability proportional to $d(u, v)^{-\beta}$

Random graph creation



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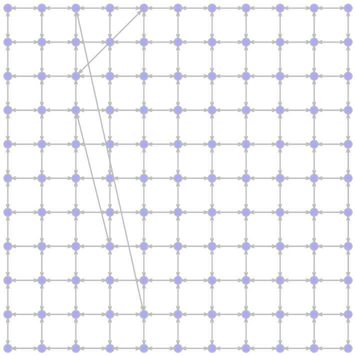

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Random graph population

- $d(v)$: lattice distance from vertex v to the origin
- $p(v, t)$: population of vertex v at time t; $p(\cdot, 1) = 0$
- $\gamma > 0, \delta > 0$ parameters

At times $t = 1, \ldots, M$ a resident arrives at the network and chooses a non-origin vertex to live at with probability proportional to

$$\frac{1}{d(v)^\gamma \cdot p(v,t)^\delta};$$

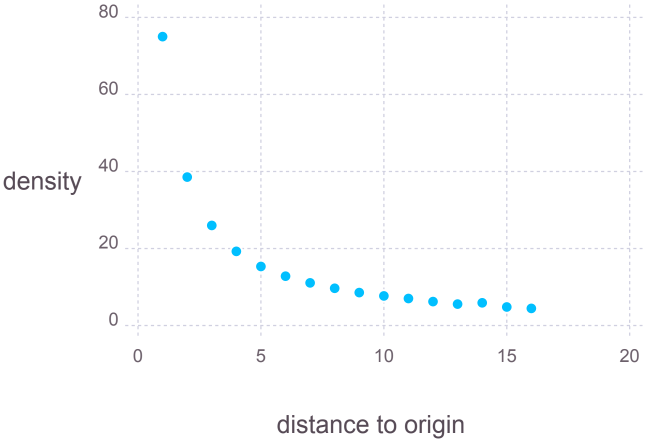
that is, users are likely to select vertices which are close to the origin and have low population.

Optimal population distribution

Can solve a MIQP to optimally place population on network to avoid overcrowding and traffic congestion.
Take $P(k)$ as the set of all edges through node k.

$$
\begin{aligned}
\min \quad & \sum_{k} \left(a x_k + b x_k^2 \right) + \sum_{i,j=-N}^{N} \left(c v_{i,j} + d v_{i,j}^2 \right) \\
\text{s.t.} \quad & x_k = \sum_{(i,j) \in P(k)} v_{i,j} \quad \forall k \\
& \sum_{i,j=-N}^{N} v_{i,j} = M \\
& v_{0,0} = 0 \\
& v_{i,j} \in \mathbb{Z}_+ \quad \forall -N \leq i,j \leq N
\end{aligned}
$$

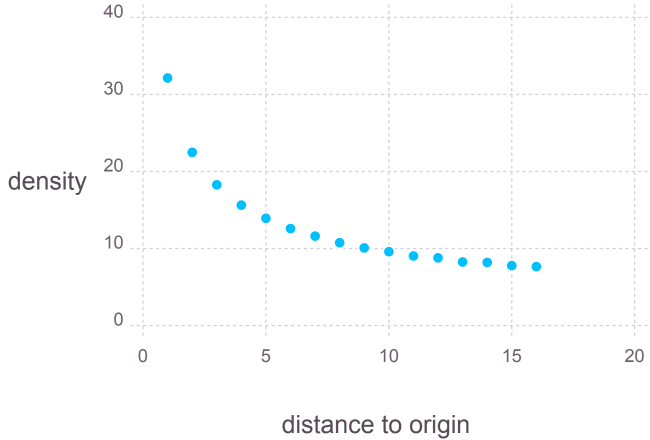
Average density



$\delta = 0$

Averaged over 10 replications for a network with
$N = 10,\ M = 5000,\ \alpha = 1.5,\ \beta = 1,\ \gamma = 1,\ \delta \in \{0, 1, 2, 3\}$

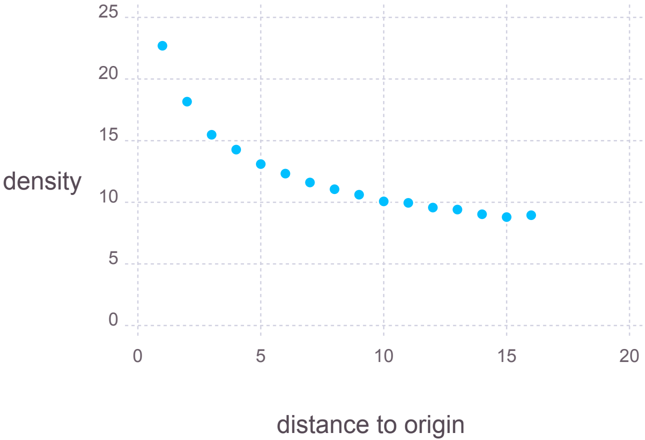
Average density



$\delta = 1$

Averaged over 10 replications for a network with
$N = 10,\ M = 5000,\ \alpha = 1.5,\ \beta = 1,\ \gamma = 1,\ \delta \in \{0, 1, 2, 3\}$

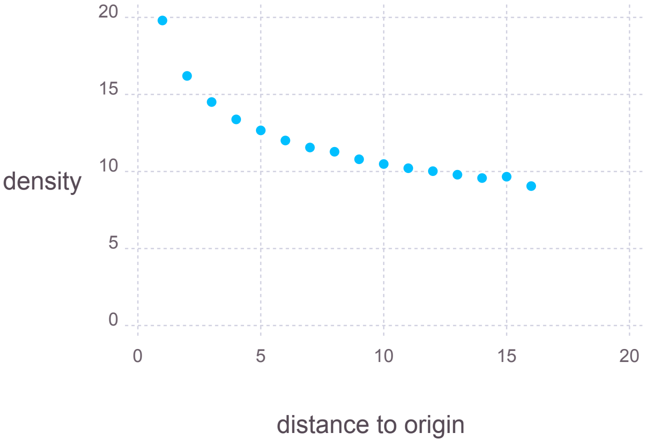
Average density



$$\delta = 2$$

Averaged over 10 replications for a network with
$N = 10,\ M = 5000,\ \alpha = 1.5,\ \beta = 1,\ \gamma = 1,\ \delta \in \{0, 1, 2, 3\}$

Average density

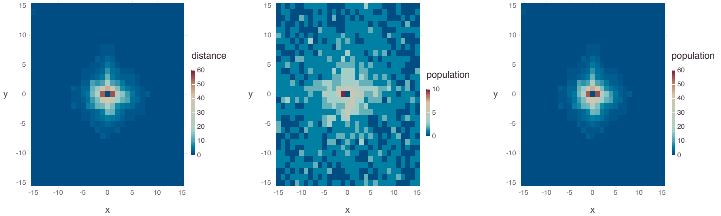


$\delta = 3$

Averaged over 10 replications for a network with
$N = 10,\ M = 5000,\ \alpha = 1.5,\ \beta = 1,\ \gamma = 1,\ \delta \in \{0, 1, 2, 3\}$

Example results

Average 1 person per vertex

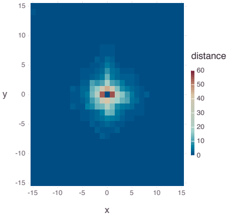


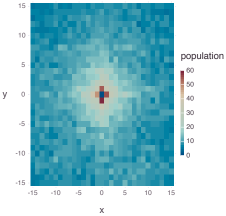
Distance to origin | Random location | Optimal location

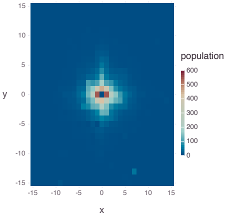
For a network with $N = 15$, $\alpha = 1$, $\beta = 1.5, \gamma = \delta = 2$, $a = 0$, $b = 1$, $c = 0$, and $d = 5$.

Example results

Average 10 people per vertex







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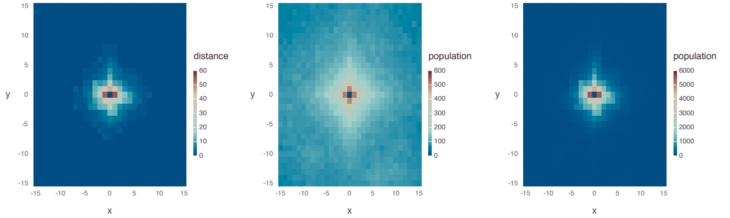
Example results

Average 100 people per vertex

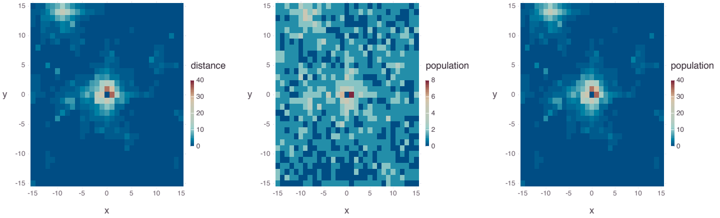


Distance to origin | Random location | Optimal location

For a network with $N = 15$, $\alpha = 1$, $\beta = 1.5, \gamma = \delta = 2$, $a = 0$, $b = 1$, $c = 0$, and $d = 5$.

Example results

Average 1 person per vertex

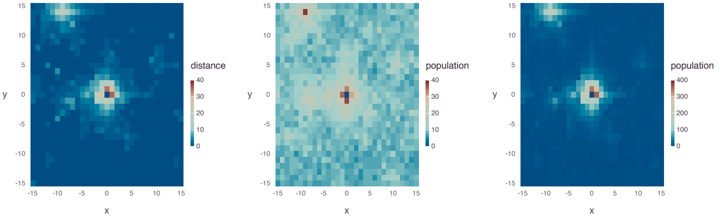


Distance to origin Random location Optimal location

For a network with $N = 15$, $\alpha = 2$, $\beta = 1.5, \gamma = \delta = 2$, $a = 0$, $b = 1$, $c = 0$, and $d = 5$.

Example results

Average 10 people per vertex



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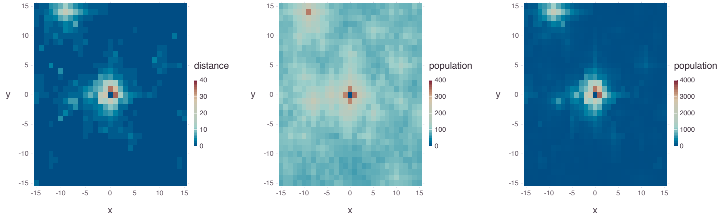

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Conclusion

- Model displays desired decay as distance from origin increases
- Myopic preferential attachment displays early transient behavior...
- ...that settles to expected distribution
- Empirically, appears that larger populations quickly converge to total travel time value strictly greater than minimum possible