Advanced Mixed Integer Programming Formulation Techniques

Constructing MIP Formulations using Convex Hulls

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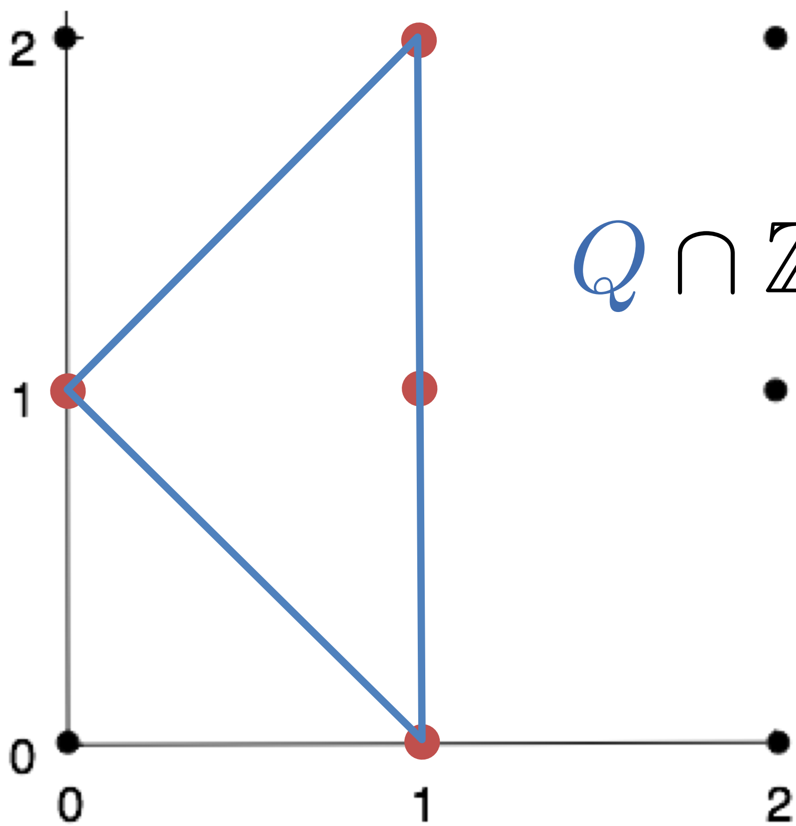
Course Material

- Code:
  - https://github.com/joehuchette/ISCO-spring-school/tree/master/code
- Slides:
  - https://github.com/joehuchette/ISCO-spring-school/tree/master/slides

Constructing Ideal Formulations For Pure Integer

- Pure Integer :

$$Q := \operatorname{conv}\left(\{p^i\}_{i=1}^{n}\right)$$



$Q \cap \mathbb{Z}^2$ ← Ideal

Integral Formulations for Small Sets

- $S = \{x \in \{0,1\}^3 : \ x_3 = x_1 \times x_2\}$
- Constraints = $\operatorname{conv}(S)$
  - $\operatorname{conv}(W) := \left\{\sum_{w \in W} w\lambda_w \ : \ \lambda \in \Delta^W\right\}$
  - $\Delta^W := \left\{\lambda \in \mathbb{R}^W_+ \ : \ \sum_{w \in W} \lambda_w = 1\right\}$
- $S = \{(0,0,0), (1,0,0), (0,1,0), (1,1,1)\}$
- $\operatorname{conv}(S)$:

Integral Formulations for Small Sets

```
julia> using Polyhedra, CDDLib, JuMP
julia> points = SimpleVRepresentation([0 0 0; 1 0 0; 0 1 0; 1 1 1]);
julia> poly = polyhedron(points, CDDLibrary(:exact))
julia> ineq = SimpleHRepresentation(poly)
H-representation
begin
 4 4 rational
 1//1 -1//1 -1//1 1//1
 0//1 1//1 0//1 -1//1
 0//1 0//1 1//1 -1//1
 0//1 0//1 0//1 1//1
end
julia> model = Model(); @variable(model,x[1:3]);
julia> @constraint(model,convert.(Int64,ineq.A)*x .<= convert.(Int64,ineq.b))
4-element
Array{JuMP.ConstraintRef{JuMP.Model,JuMP.GenericRangeConstraint{JuMP.GenericAf
fExpr{Float64,JuMP.Variable}}},1}:
 x[1] + x[2] - x[3] ≤ 1
 -x[1] + x[3] ≤ 0
 -x[2] + x[3] ≤ 0
 -x[3] ≤ 0
```

Integral Formulations for Small Sets

- $S = \{x \in \{0,1\}^3 \ : \ x_3 = x_1 \times x_2\}$
- Constraints = $\operatorname{conv}(S)$
  - $\operatorname{conv}(W) := \left\{ \sum_{w \in W} w \lambda_w \ : \ \lambda \in \Delta^W \right\}$
  - $\Delta^W := \left\{ \lambda \in \mathbb{R}^W_+ \ : \ \sum_{w \in W} \lambda_w = 1 \right\}$
- $S = \{(0,0,0), (1,0,0), (0,1,0), (1,1,1)\}$
- $\operatorname{conv}(S)$:
  - $x_1 + x_2 \leq 1 + x_3$
  - $x_3 \leq x_1$
  - $x_3 \leq x_2$
  - $0 \leq x_3$
- Always works for 0-1 problems

Careful With General Integer Problems

- All different : $S = \{x \in [n]^n : \; x_i \neq x_j \;\; \forall\, i \neq j\}$
- Permutahedron:
  - $\operatorname{conv}(S) = \left\{ x \in \mathbb{R}^n : \begin{array}{c} \sum_{i=1}^n x_i = \sum_{i=1}^n i \\ \sum_{i \in I} x_i \geq \binom{|I|+1}{2} \;\; \forall I\colon \emptyset \neq I \subseteq [n] \end{array} \right\}$
- If n is odd then:
  - $\left(\frac{n+1}{2}, \dots, \frac{n+1}{2}\right) \in \operatorname{conv}(S) \cap \mathbb{Z}^n$
  - $\left(\frac{n+1}{2}, \dots, \frac{n+1}{2}\right) \notin S$
- "Hole-Free": $S \subseteq \mathbb{R}^n$ such that $\operatorname{conv}(S) \cap \mathbb{Z}^n = S$

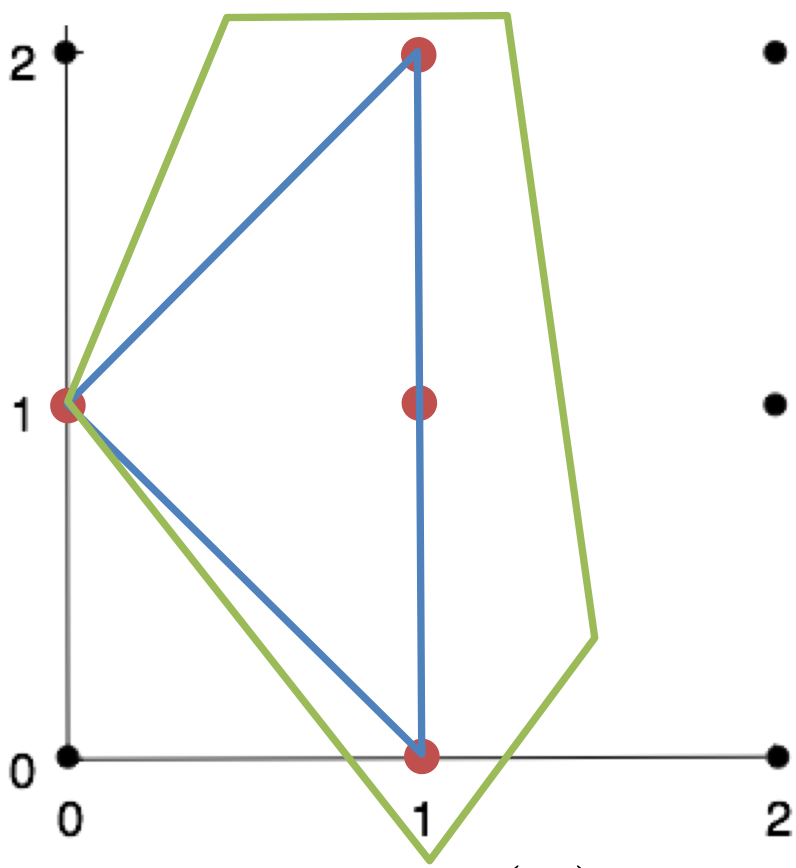
Solution: Extended Formulations

- (Finite?) Non-hole free subset of integers is always projection of hole-free subset of integers
- All different : $S = \{x \in [n]^n : \ x_i \neq x_j \ \ \forall\, i \neq j\}$
- $$T := \left\{ (x, Y) \in \mathbb{R}^n \times \{0,1\}^{n\times n} : \begin{array}{r} \sum_{i=1}^n Y_{i,j} = 1 \ \forall\, j \in [n] \\ \sum_{j=1}^n Y_{i,j} = 1 \ \forall\, i \in [n] \\ \sum_{j=1}^n j\, Y_{i,j} = x_i \ \forall i \in [n] \end{array} \right\}$$
- $S = \mathrm{proj}_x(T)$

What About Mixed-Integer Sets?

- Pure Integer $S \subseteq \mathbb{Z}^n$

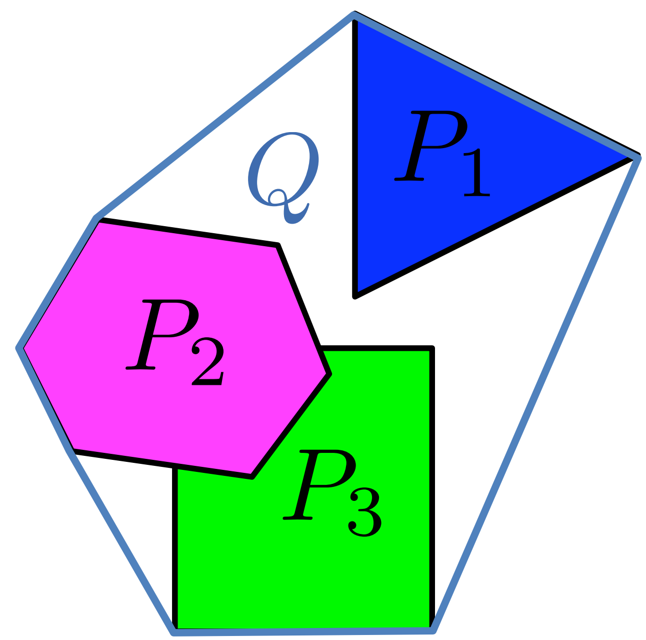
$$P \cap \mathbb{Z}^n = S \ (P \subseteq \mathbb{R}^n)$$



$$Q := \mathrm{conv}\,(S)$$

- Mixed-Integer $S = \bigcup_{i=1}^{n} P^i \subseteq \mathbb{R}^n$

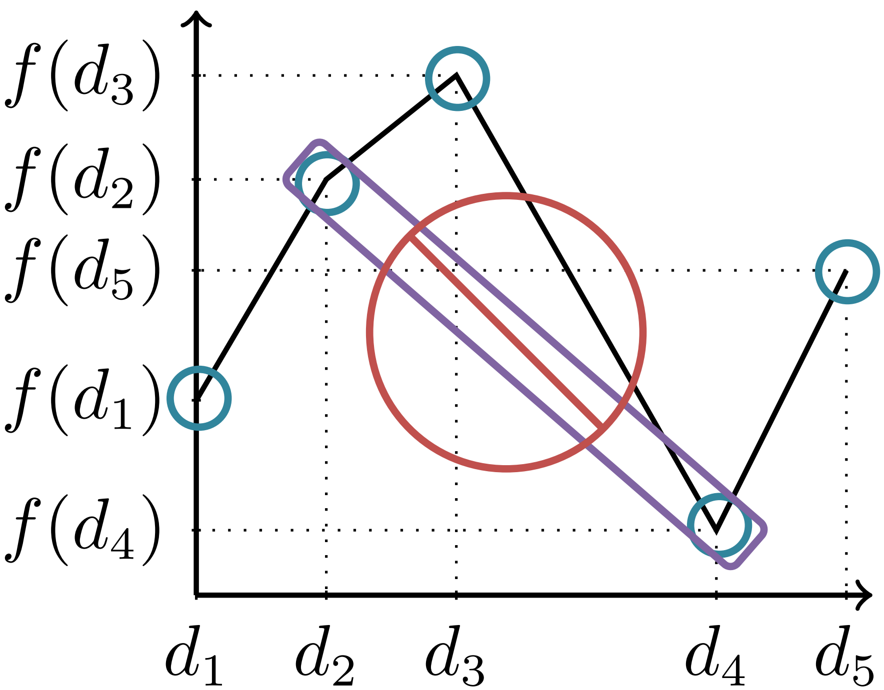
$$Q := \mathrm{conv}\,(S)?$$



What is a MIP Formulation?

Simple Formulation for Univariate Functions

$$z = f(x)$$



$$\text{Size} = O(\#\text{ of segments})$$

Non-Ideal: Fractional Extreme Points

$$\begin{pmatrix} x \\ z \end{pmatrix} = \sum_{j=1}^{5} \begin{pmatrix} d_j \\ f(d_j) \end{pmatrix} \lambda_j$$

$$1 = \sum_{j=1}^{5} \lambda_j, \quad \lambda_j \geq 0$$

$$y \in \{0,1\}^4, \quad \sum_{i=1}^{4} y_i = 1$$

$$0 \leq \lambda_1 \leq y_1$$

$$0 \leq \lambda_2 \leq y_1 + y_2$$

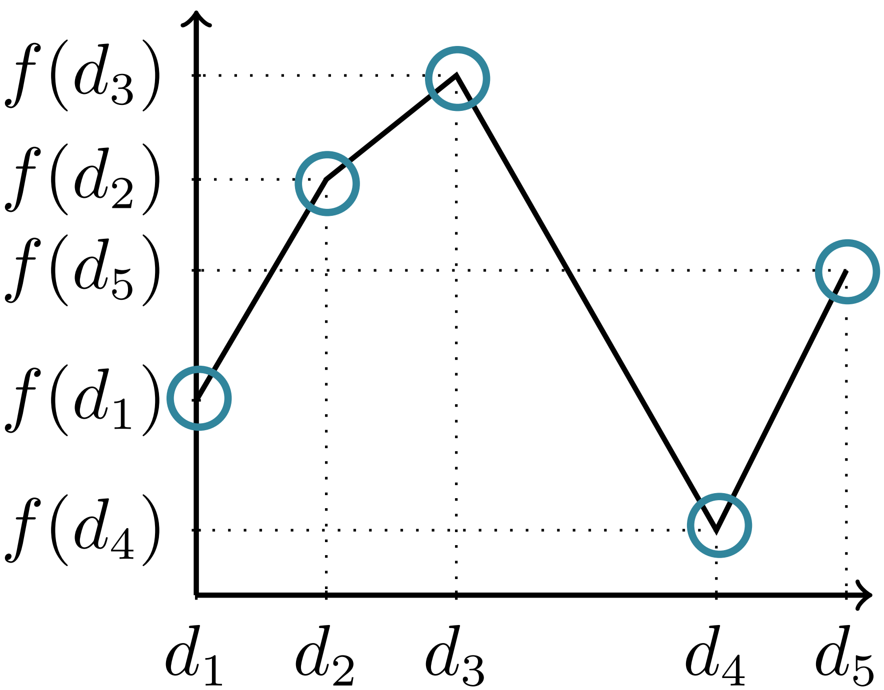
$$0 \leq \lambda_3 \leq y_2 + y_3$$

$$0 \leq \lambda_4 \leq y_3 + y_4$$

$$0 \leq \lambda_5 \leq y_4$$

Advanced Formulation for Univariate Functions

$$z = f(x)$$



$$\text{Size} = O\left(\log_2 \#\text{ of segments}\right)$$

Ideal: Integral Extreme Points

$$\begin{pmatrix} x \\ z \end{pmatrix} = \sum_{j=1}^{5} \begin{pmatrix} d_j \\ f(d_j) \end{pmatrix} \lambda_j$$

$$1 = \sum_{j=1}^{5} \lambda_j, \quad \lambda_j \geq 0$$

$$y \in \{0,1\}^2$$

$$0 \leq \lambda_1 + \lambda_5 \leq 1 - y_1$$

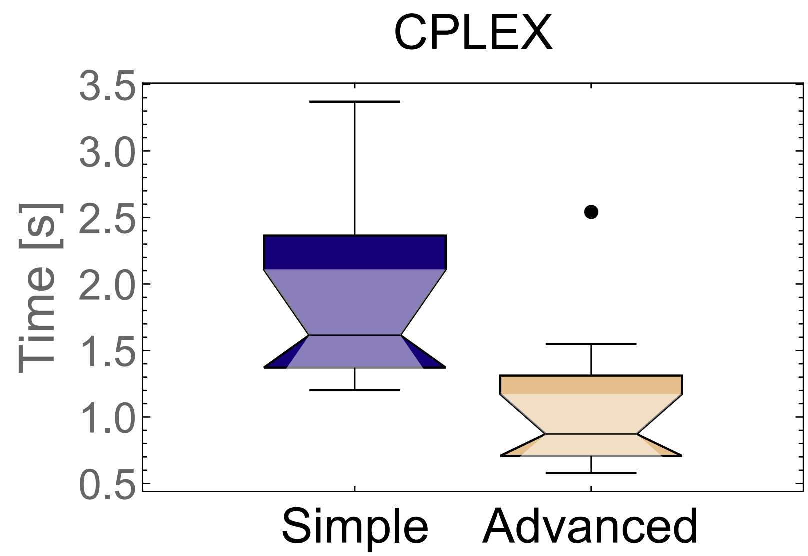
$$0 \leq \lambda_3 \leq y_1$$

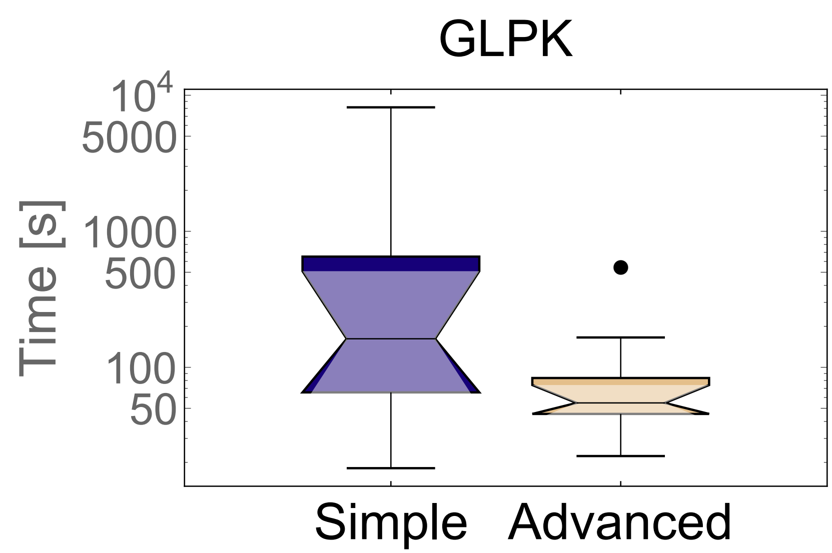
$$0 \leq \lambda_4 + \lambda_5 \leq 1 - y_2$$

$$0 \leq \lambda_1 + \lambda_2 \leq y_2$$

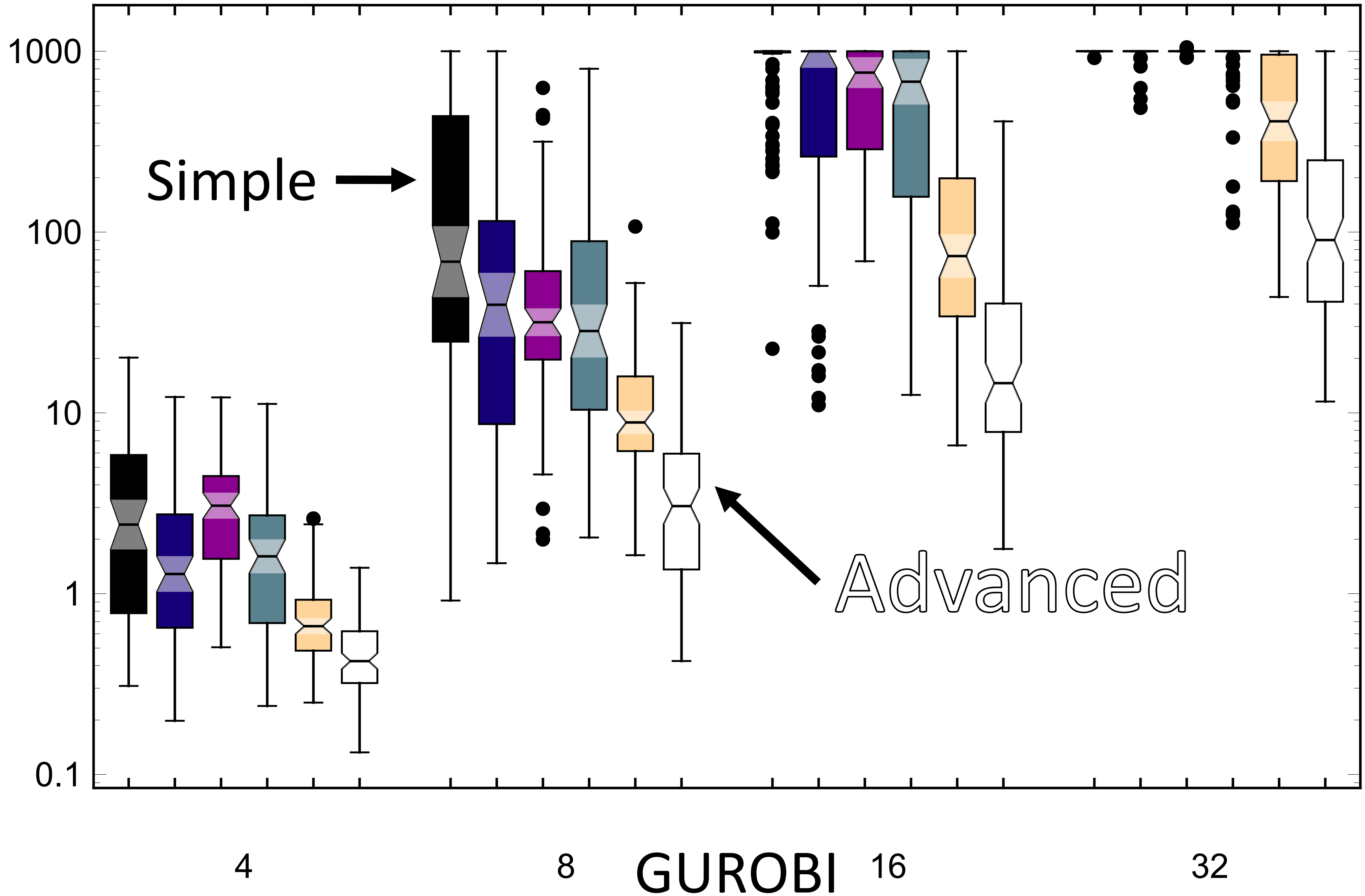
Computational Performance

- Advanced formulations provide an computational advantage
- Advantage is significantly more important for free solvers
- State of the art commercial solvers can be significantly better that free solvers
- Still, free is free!





Formulation Improvements can be Significant



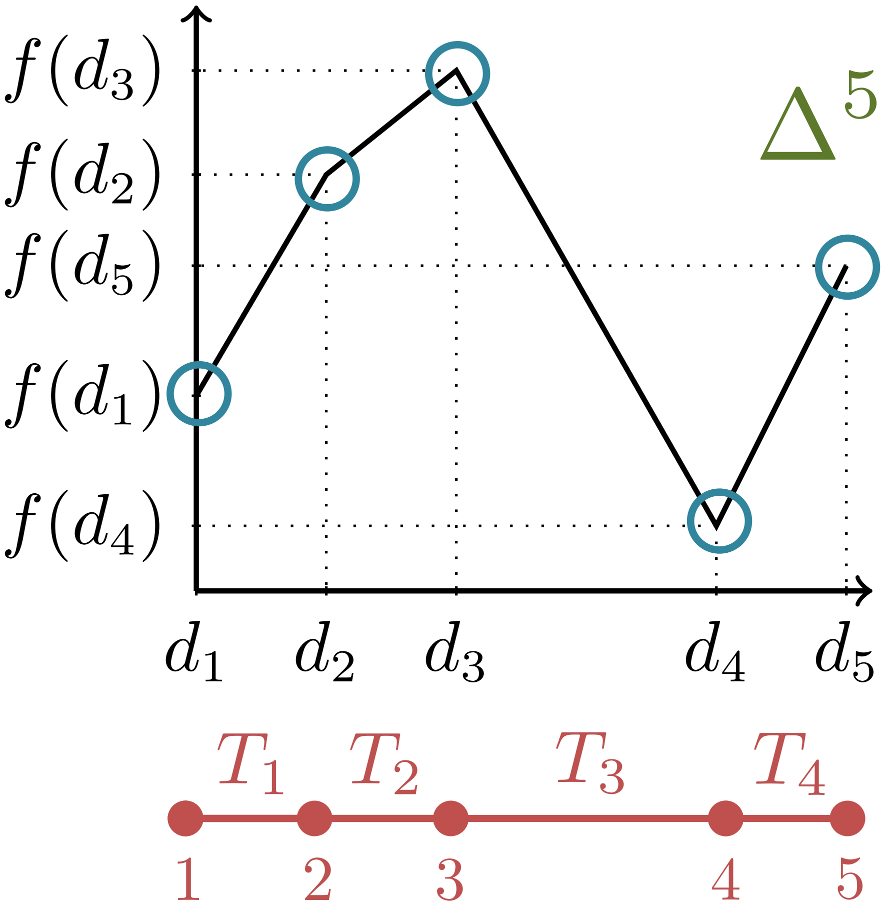
Abstracting Univariate Functions

$$P_i := \{\lambda \in \Delta^5 : \lambda_j = 0 \quad \forall j \notin T_i\}$$

$$z = f(x)$$

$$\cancel{\begin{pmatrix} x \\ z \end{pmatrix} = \sum_{j=1}^{5} \begin{pmatrix} d_j \\ f(d_j) \end{pmatrix} \lambda_j}$$

$$T_i := \{i, i+1\} \qquad i \in \{1, \ldots, 4\}$$



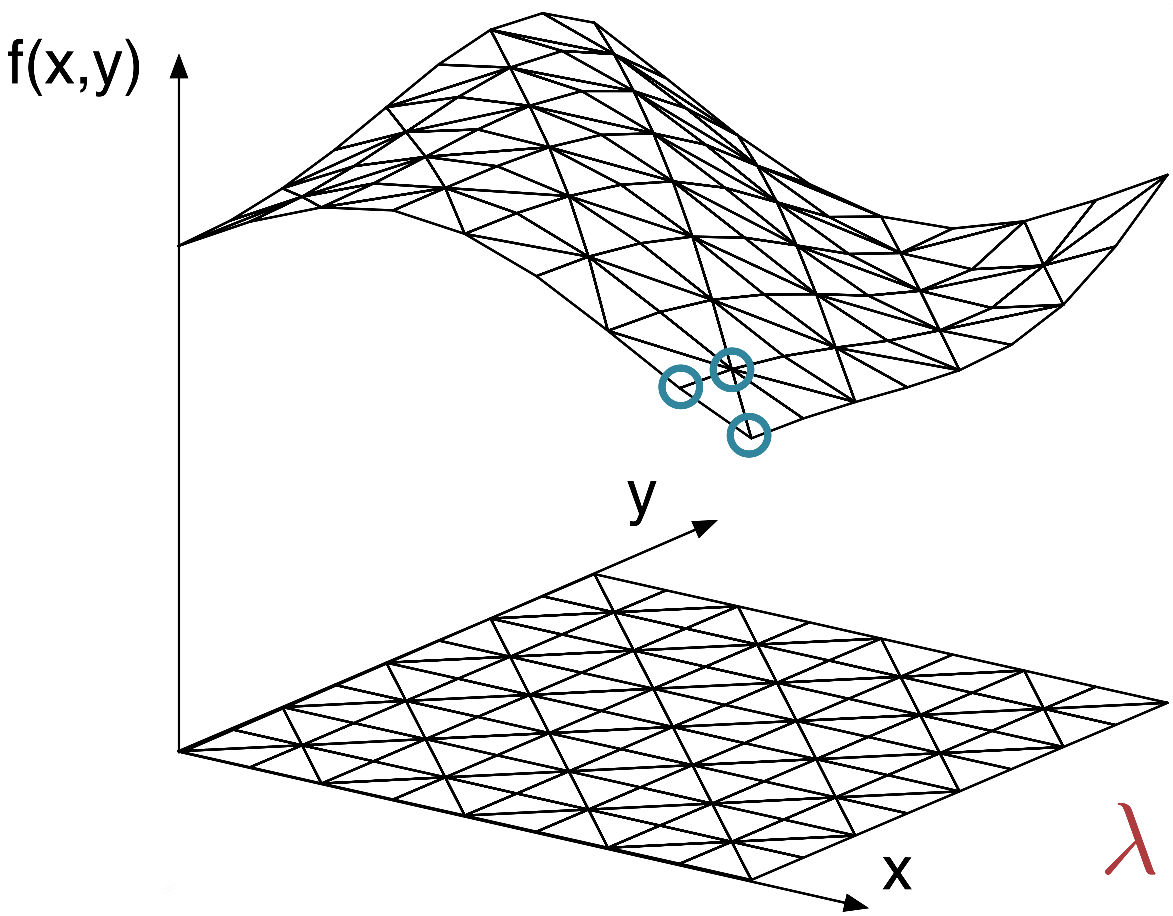
$$\Delta^5 = \boxed{1 = \sum_{j=1}^{5} \lambda_j, \quad \lambda_j \geq 0}$$

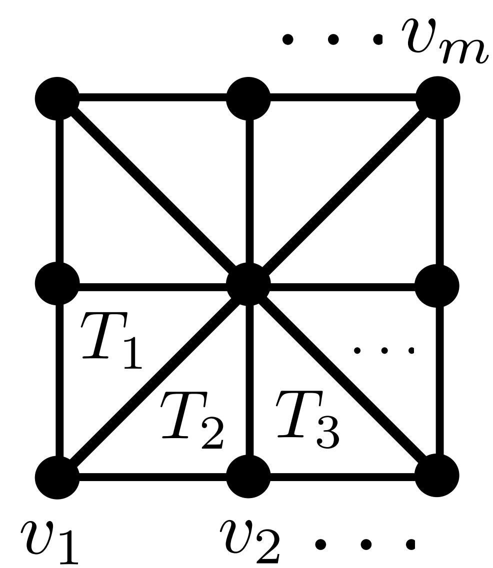
Special Ordered Sets of Type 2 (SOS2)

$$\lambda \in \bigcup_{i=1}^{4} P_i \subseteq \Delta^5$$

Abstraction Works for Multivariate Functions

$$P_i := \{\lambda \in \Delta^m : \lambda_j = 0 \quad \forall v_j \notin T_i\}$$





$$\lambda \in \bigcup_{i=1}^{n} P_i \subseteq \Delta^m$$

Standard Formulation for SOS2 = Unary Encoding

Q = LP relaxation

$$\sum_{i=1}^{5} \lambda_i = 1$$

$$y \in \{0,1\}^4, \quad \sum_{i=1}^{4} y_i = 1$$

$$0 \le \lambda_1 \le y_1$$
$$0 \le \lambda_2 \le y_1 + y_2$$
$$0 \le \lambda_3 \le y_2 + y_3$$
$$0 \le \lambda_4 \le y_3 + y_4$$
$$0 \le \lambda_5 \le y_4$$

$$(\lambda, y) \in Q \cap \left(\mathbb{R}^5 \times \mathbb{Z}^4\right)$$
$$\Updownarrow$$
$$y = e^i \ \wedge \ \lambda \in P_i$$

Unary Encoding

$$P_i := \{\lambda \in \Lambda^5 \ : \ \lambda_j = 0 \quad j \notin \{i, i+1\}\}$$

Advanced = Binary Encoded Formulation

Q = LP relaxation → $\boxed{\sum_{i=1}^{5} \lambda_i = 1}$

$$h^1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ h^2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix},\ h^3 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ h^4 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- V. and Nemhauser '08.

$$\begin{aligned}
0 &\le \lambda_1 + \lambda_5 \le 1 - y_1 \\
0 &\le \lambda_3 \qquad\quad\ \le y_1 \\
0 &\le \lambda_4 + \lambda_5 \le 1 - y_2 \\
0 &\le \lambda_1 + \lambda_2 \le y_2
\end{aligned}$$

$$(\lambda, y) \in Q \cap \left(\mathbb{R}^5 \times \mathbb{Z}^2\right)$$

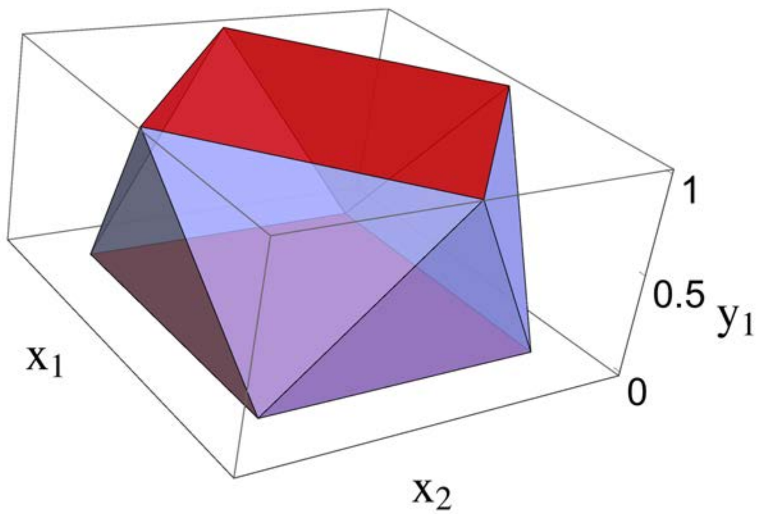
$$\Updownarrow$$

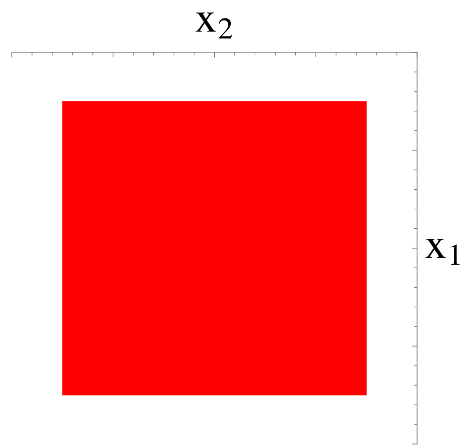
$$y = h^i \ \wedge\ \lambda \in P_i$$

Binary Encoding

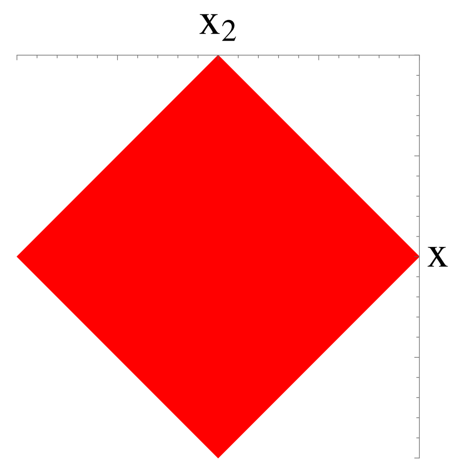
$$P_i := \{\lambda \in \Lambda^5 : \lambda_j = 0 \quad j \notin \{i, i+1\}\}$$

Embedding Formulation = Ideal non-Extended





P_1



P_2

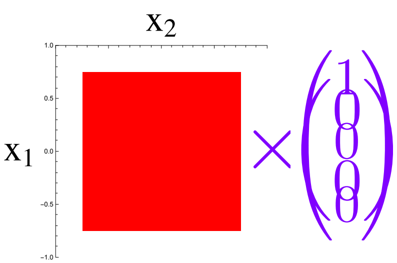
$$Q\left(H\right) := \operatorname{conv}\left(\bigcup_{i=1}^{n} P_i \times \left\{h^i\right\}\right)$$

$$(x, y) \in Q \cap \left(\mathbb{R}^d \times \mathbb{Z}^k\right) \quad \Leftrightarrow \quad y = h^i \;\wedge\; x \in P_i$$

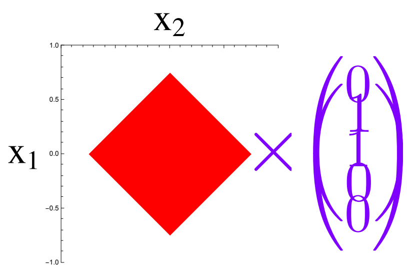
$$\operatorname{ext}(Q) \subseteq \mathbb{R}^d \times \mathbb{Z}^k \qquad H := \left\{h^i\right\}_{i=1}^{n} \subseteq \{0,1\}^k, \quad h^i \neq h^j$$

Alternative Encodings

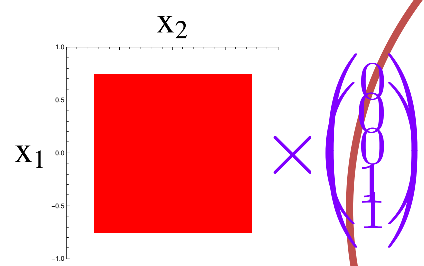
- Careful with general integers:



P_1

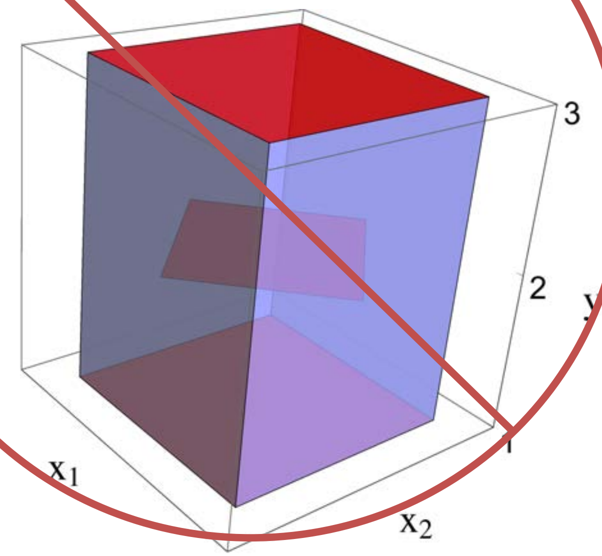


P_2



P_3

$$\bigcup_{i=1}^{n} P_i \times \{i\}$$



- Options for 0-1 encodings:
  - Traditional or **Unary** encoding

$$H = \left\{ y \in \{0,1\}^n \ : \ \sum_{i=1}^{n} y_i = 1 \right\}$$

$$= \{\mathbf{e}^i\}_{i=1}^{n}$$

$$\mathbf{e}_j^i = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

  - **Binary** encodings: $H \equiv \{0,1\}^{\log_2 n}$
  - Others (e.g. **incremental** encoding $\equiv$ unary)

Embedding Formulations and Complexity

- **Embedding formulation** of $\lambda \in \bigcup_{i=1}^{n} P_i \subseteq \mathbb{R}^V$:
  - Encoding $H := \{h^i\}_{i=1}^{n} \subseteq \{0,1\}^k, \quad h^i \neq h^j$
  - $Q(H) := \operatorname{conv}\left(\bigcup_{i=1}^{n} P_i \times \{h^i\}\right)$
- **Embedding complexity** = size smallest formulation
  - $$\operatorname{mc}(\mathcal{P}) := \min_H \{\operatorname{size}(Q(H))\},$$

    $$\operatorname{size}(Q(H)) := \#\ \text{facets}$$

Special Ordered Sets = Simplex Faces = $\mathcal{P} := \{P_i\}_{i=1}^n$

- $$\Delta^{d+1} := \left\{x \in \mathbb{R}_+^{d+1} \;:\; \sum_{i=1}^{d+1} x_i = 1\right\} = \operatorname{conv}\left(\left\{e^i\right\}_{i=1}^{d+1}\right)$$

  $$P_i := \operatorname{conv}\left(\left\{e^j\right\}_{j \in T_i}\right) = \left\{x \in \Delta^{d+1} \;:\; \sum_{j \notin T_i} x_i \leq 0\right\}$$

  $$T_i \subseteq \{1, \ldots, d+1\}$$

- $$\operatorname{mc}(\mathcal{P}) := \min_H \{\operatorname{size}(Q(H))\},$$

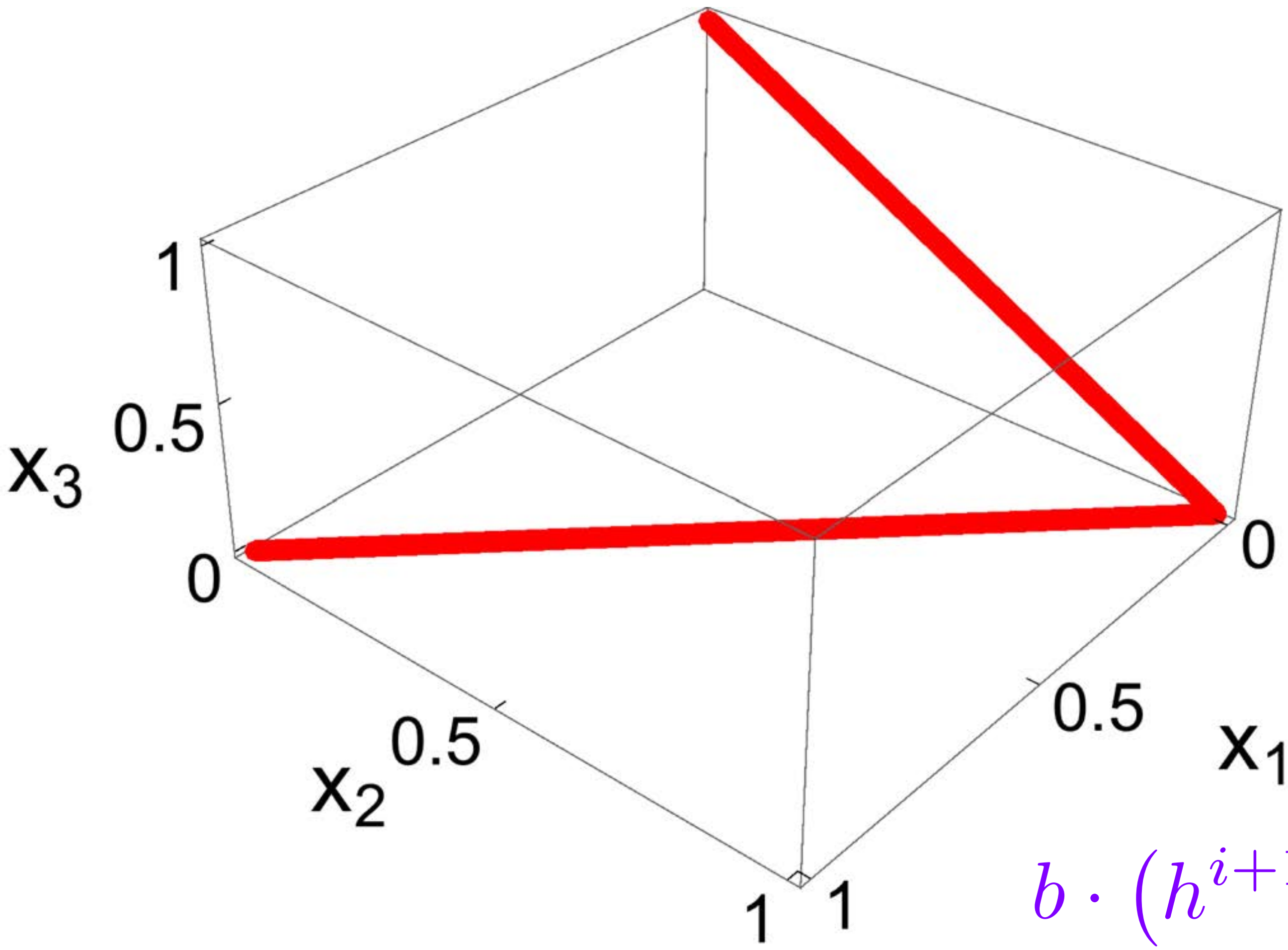
  $$\operatorname{size}(Q(H)) := \#\ \text{facets}$$

- $$\operatorname{mc}_G(\mathcal{P}) := \min_H \{\operatorname{size}_G(Q(H))\},$$

  $$\operatorname{size}_G(Q(H)) := \#\ \text{non-bound facets}$$

Special Ordered Sets of Type 2 (SOS2) = $\mathcal{P} := \{P_i\}_{i=1}^{n}$

- $P_i := \operatorname{conv}\left(\{e^i, e^{i+1}\}\right) \subseteq \Delta^{n+1}, \quad i \in [n]$



$],$

$2 \lceil \log_2 n \rceil$

- (

$^{i}\})$

- (

$, b \cdot h^{i+1}\}$

$b \in$

$$\underbrace{b \cdot \left(h^{i+1} - h^{i}\right)}_{} = 0$$

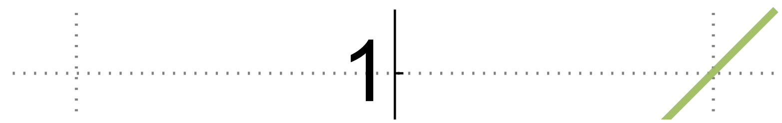
Embedding Formulation for SOS2: Part 1

- From encodings (H) to hyperplanes:

$$\{h^i\}_{i=1}^{n}$$

$$h^1 = \begin{pmatrix}1\\1\end{pmatrix},\ h^2 = \begin{pmatrix}0\\0\end{pmatrix},\ h^3 = \begin{pmatrix}1\\0\end{pmatrix}$$

$$c^i = h^{i+1} - h^i$$

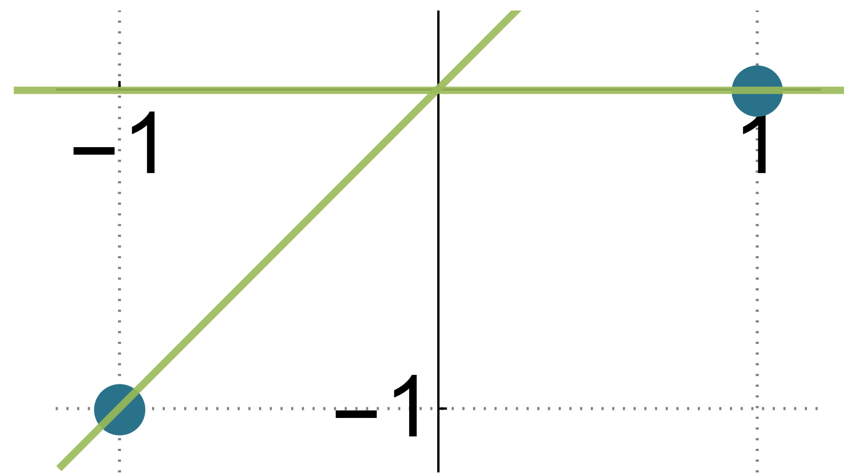


non-bound facets = 2 × # of hyperplanes

$$\{c^i\}_{i=1}$$

Hyperplanes spanned by

$$\{b^i \cdot y = 0\}_{j=1}^{L}$$



Embedding Formulation for SOS2: Part 2

$$\left\{b^i \cdot y = 0\right\}_{j=1}^{L}$$

$$Q(H) = \qquad L(H) := \operatorname{aff}(H) - h^1$$

$$\left(b^j \cdot h^1\right) x_1 + \sum_{i=2}^{n} \min\left\{b^j \cdot h^i, b^j \cdot h^{i-1}\right\} x_i + \left(b^j \cdot h^n\right) x_{n+1} \leq \quad b^j \cdot y \quad \forall j$$

$$-\left(b^j \cdot h^1\right) x_1 - \sum_{i=2}^{n} \max\left\{b^j \cdot h^i, b^j \cdot h^{i-1}\right\} x_i - \left(b^j \cdot h^n\right) x_{n+1} \leq -b^j \cdot y \quad \forall j$$

$$\sum_{i=1}^{n+1} x_i = 1, \qquad x \in \mathbb{R}_+^{n+1}$$

$$y \in L(H)$$

- # non-bound facets = 2 × # of hyperplanes

Embedding Complexity for SOS2

- Lower Bound: $L(H) := \operatorname{aff}(H) - h^1$

$$\operatorname{mc}_G(\mathcal{P}) \geq 2 \times \min \#\text{ of hyperplanes}$$

$$\min \#\text{ of hyperplanes} \geq \dim(L(H))$$

$$\dim(L(H)) \geq \lceil \log_2 n \rceil$$

- Upper Bound: $H = \{0,1\}^{\lceil \log_2 n \rceil}$
  - Gray code: $\{h^i - h^{i+1}\}_{i=1}^{n-1} \equiv \{e^i\}_{i=1}^{\lceil \log_2 n \rceil}$

$$\operatorname{size}_G(Q(H)) = 2\lceil \log_2 n \rceil$$

$$n + 1 \leq \operatorname{mc}(\mathcal{P}) \leq n + 1 + 2\lceil \log_2 n \rceil$$

Embedding Complexity for SOS2

- Unary encoding (Padberg / Lee and Wilson, early 00's):

$$\mathrm{size}_G\left(Q\left(H\right)\right) = 2(n-1), \quad \mathrm{size}\left(Q\left(H\right)\right) = 2n$$

- **Smallest Binary** encoding (V. and Nemhauser '08, Muldoon '12):

$$\mathrm{size}_G\left(Q\left(H\right)\right) = 2\left\lceil \log_2 n \right\rceil,$$

$$2 + 2\left\lceil \log_2 n \right\rceil \leq \mathrm{size}\left(Q\left(H\right)\right) \leq n + 1 + 2\left\lceil \log_2 n \right\rceil$$

- Adding lower bounds (# hyperplanes ≥ dimension):

$$\mathrm{mc}_G\left(\mathcal{P}\right) = 2\left\lceil \log_2 n \right\rceil,$$

$$n + 1 \leq \mathrm{xc}\left(\mathcal{P}\right) \leq \mathrm{mc}\left(\mathcal{P}\right) \leq n + 1 + 2\left\lceil \log_2 n \right\rceil$$

Validity of Formulation May Not Be Evident

$$\left\{b^i \cdot y = 0\right\}_{j=1}^{L}$$

$$Q\left(H\right) =$$

$$L\left(H\right) := \operatorname{aff}\left(H\right) - h^1$$

$$\left(b^j \cdot h^1\right) x_1 + \sum_{i=2}^{n} \min\left\{b^j \cdot h^i, b^j \cdot h^{i-1}\right\} x_i + \left(b^j \cdot h^n\right) x_{n+1} \leq \quad b^j \cdot y \quad \forall j$$

$$-\left(b^j \cdot h^1\right) x_1 - \sum_{i=2}^{n} \max\left\{b^j \cdot h^i, b^j \cdot h^{i-1}\right\} x_i - \left(b^j \cdot h^n\right) x_{n+1} \leq -b^j \cdot y \quad \forall j$$

$$\sum_{i=1}^{n+1} x_i = 1, \qquad x \in \mathbb{R}_{+}^{n+1}$$

$$y \in L(H)$$

- # non-bound facets = 2 × # of hyperplanes

Validity of Formulation May Not Be Evident

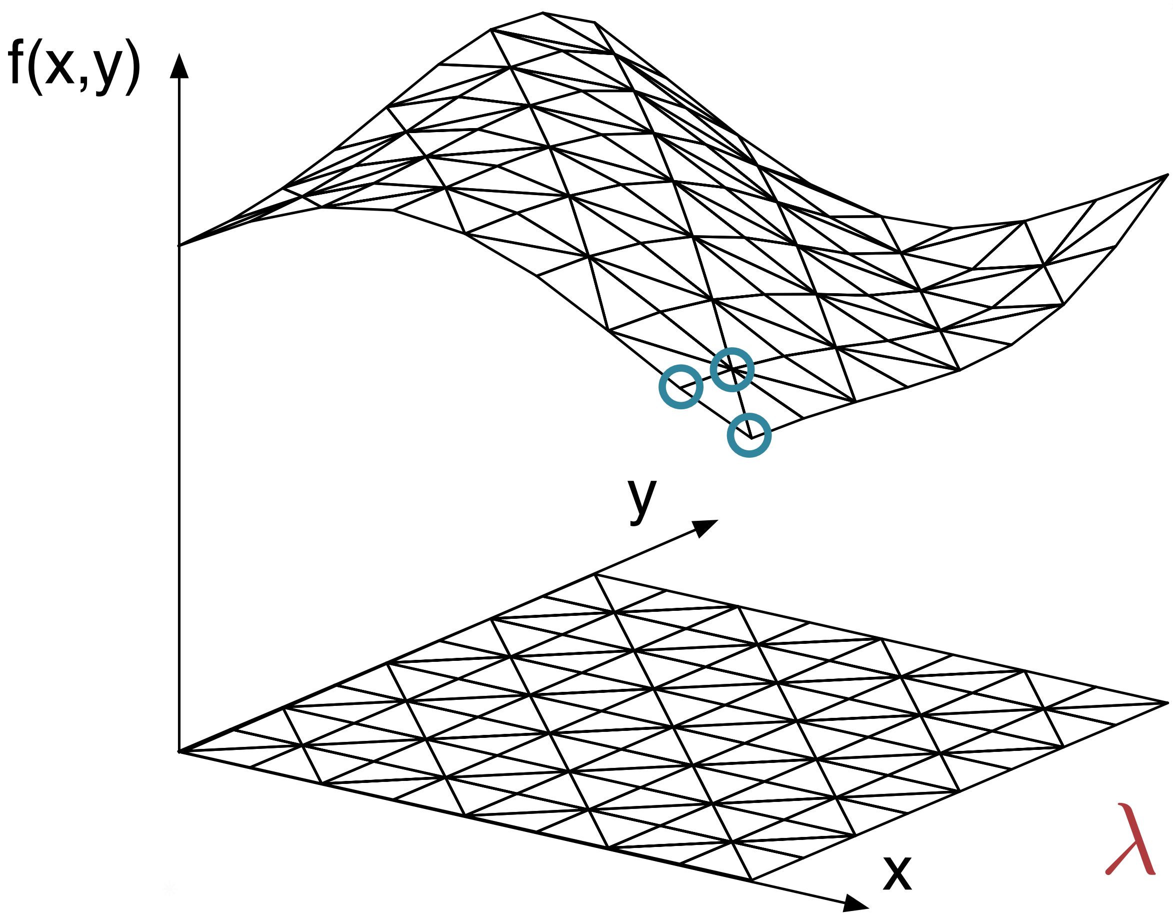
- $H = (0,1,1,1)^T, (0,1,0,0)^T, (0,0,0,0)^T, (0,1,0,1)^T, (0,0,0,1)^T,$
  $(1,0,0,0)^T, (1,1,0,1)^T, (1,0,1,1)^T, (1,1,1,1)^T$
- $(c^i)_{i=1}^8 = (0,0,-1,-1)^T, (0,-1,0,0)^T, (0,1,0,1)^T, (0,-1,0,0)^T,$
  $(1,0,0,-1)^T, (0,1,0,1)^T, (0,-1,1,0)^T, (0,1,0,0)^T$
- $b^1 = (1,0,0,-1,1)^T$, $b^2 = (1,0,0,1)^T$, $b^3 = (1,-1,-1,1)^T$, $b^4 = (1,0,0,0)^T$ and $b^5 = (0,0,1,0)^T$

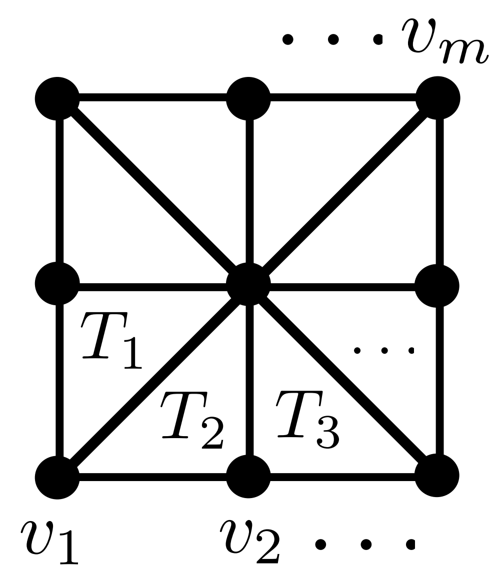
Validity of Formulation May Not Be Evident

$$
\begin{aligned}
\sum_{j=1}^{10} \lambda_j &= 1, \\
\lambda_5 + \lambda_6 + \lambda_7 + \lambda_8 + \lambda_9 + \lambda_{10} &\leq y_1 - y_3 + y_4 \\
\lambda_4 + \lambda_5 + \lambda_6 + 2\lambda_7 + 2\lambda_8 + \lambda_9 + \lambda_{10} &\geq y_1 - y_3 + y_4 \\
\lambda_1 + \lambda_5 + \lambda_6 + \lambda_7 + 2\lambda_8 + 2\lambda_9 + 2\lambda_{10} &\leq y_1 + y_4 \\
\lambda_1 + \lambda_2 + \lambda_4 + \lambda_5 + \lambda_6 + 2\lambda_7 + 2\lambda_8 + 2\lambda_9 + 2\lambda_{10} &\geq y_1 + y_4 \\
-\lambda_1 - \lambda_2 - \lambda_3 + \lambda_6 + \lambda_7 + \lambda_8 &\leq y_1 - y_2 - y_3 + y_4 \\
-\lambda_1 - \lambda_2 + \lambda_5 + \lambda_6 + \lambda_7 + \lambda_8 + \lambda_9 &\geq y_1 - y_2 - y_3 + y_4 \\
\lambda_7 + \lambda_8 + \lambda_9 + \lambda_{10} &\leq y_1 \\
\lambda_6 + \lambda_7 + \lambda_8 + \lambda_9 + \lambda_{10} &\geq y_1 \\
\lambda_1 + \lambda_9 + \lambda_{10} &\leq y_3 \\
\lambda_1 + \lambda_2 + \lambda_8 + \lambda_9 + \lambda_{10} &\geq y_3 \\
\lambda_j &\geq 0
\end{aligned}
$$

Abstraction Works for Multivariate Functions

$$P_i := \{\lambda \in \Delta^m \;:\; \lambda_j = 0 \quad \forall v_j \notin T_i\}$$





$$\lambda \in \bigcup_{i=1}^{n} P_i \subseteq \Delta^m$$

Formulations and Complexity for Triangulations

- Lower bound:

$$\left(\sqrt{n/2}+1\right)^2 \leq \mathrm{mc}\left(\mathcal{P}\right)$$

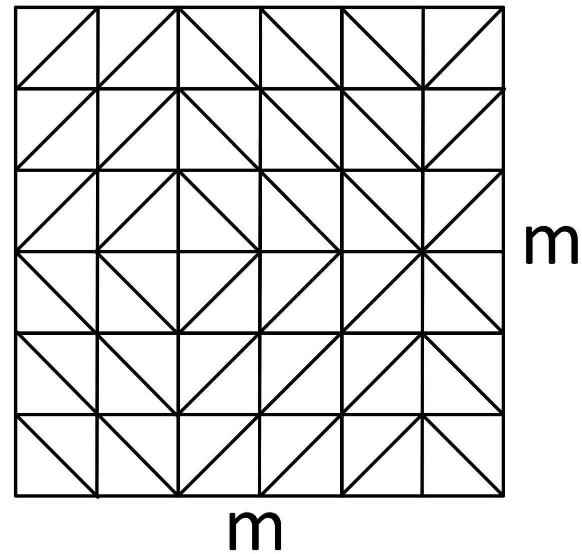
- Size of unary formulation is:
  (Lee and Wilson '01)

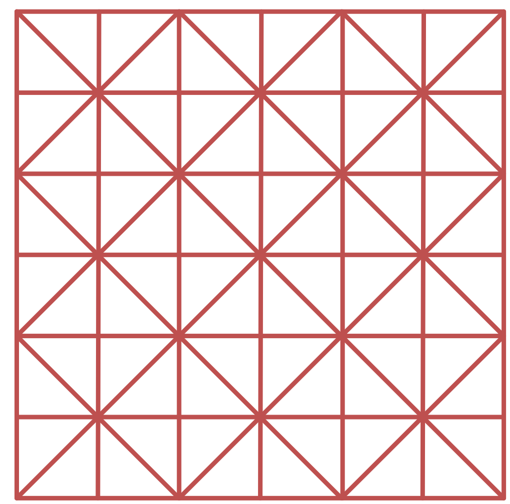
$$\mathrm{mc}\left(\mathcal{P}\right) \leq \binom{2\sqrt{n/2}}{\sqrt{n/2}} + \left(\sqrt{n/2}+1\right)^2$$

- Small binary formulation for union jack triangulation of size:
  (V. and Nemhauser '08)

$$\mathrm{mc}\left(\mathcal{P}\right) \leq 4\log_2\sqrt{n/2}+2+\left(\sqrt{n/2}+1\right)^2$$

$n = 2m^2$





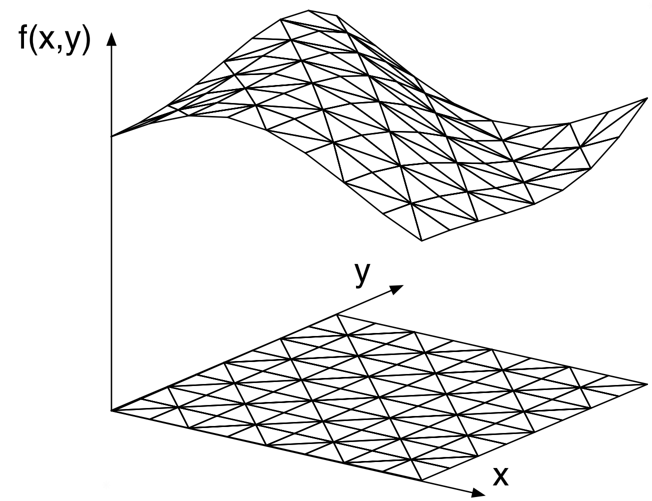
Encoding Selection Matters

- Size of unary formulation is:
  (Lee and Wilson ’01)

$$\binom{2\sqrt{n/2}}{\sqrt{n/2}} + \left(\sqrt{n/2}+1\right)^2$$

General Inequalities

Variable Bounds

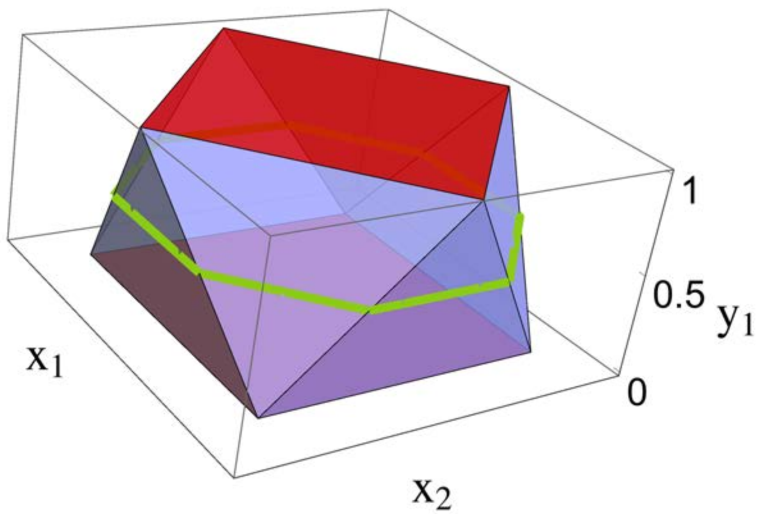


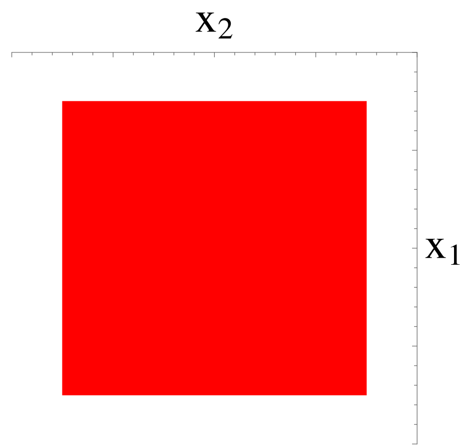
- Size of one binary formulation:
  (V. and Nemhauser ’08)

$$4\log_2\sqrt{n/2}+2+\left(\sqrt{n/2}+1\right)^2$$

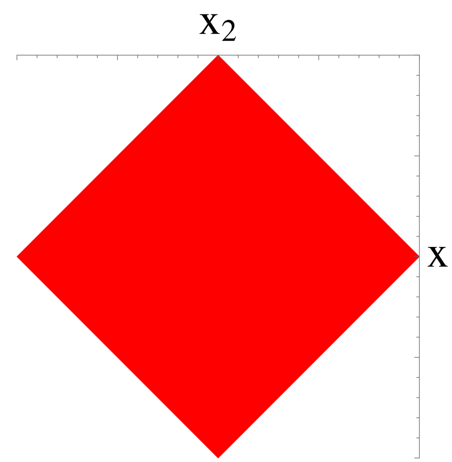
- Right embedding = significant computational advantage over alternatives (Extended, Big-M, etc.)

Unary Encoding, Minkowski Sum and Cayley Trick

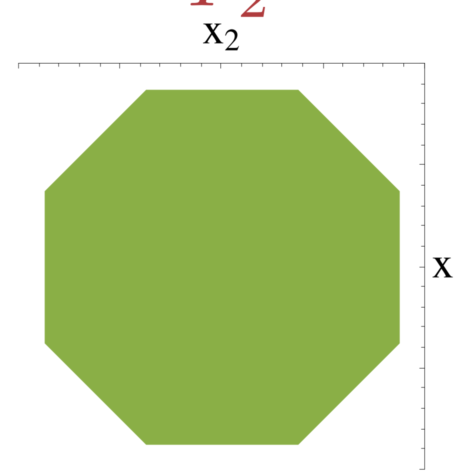




P_1



P_2



$$Q \cap \left(\mathbb{R}^2 \times \{0.5\}\right) \equiv P_1 + P_2 =$$

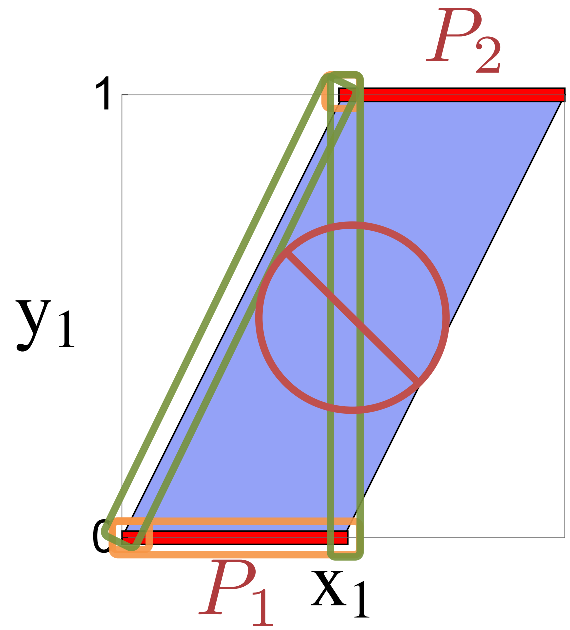
$$H = \{e^i\}_{i=1}^n$$

$$Q(H) \cap \left(\mathbb{R}^d \times \left\{\tfrac{1}{n} \textstyle\sum_{i=1}^n \mathbf{e}^i\right\}\right) \equiv \textstyle\sum_{i=1}^n P_i$$

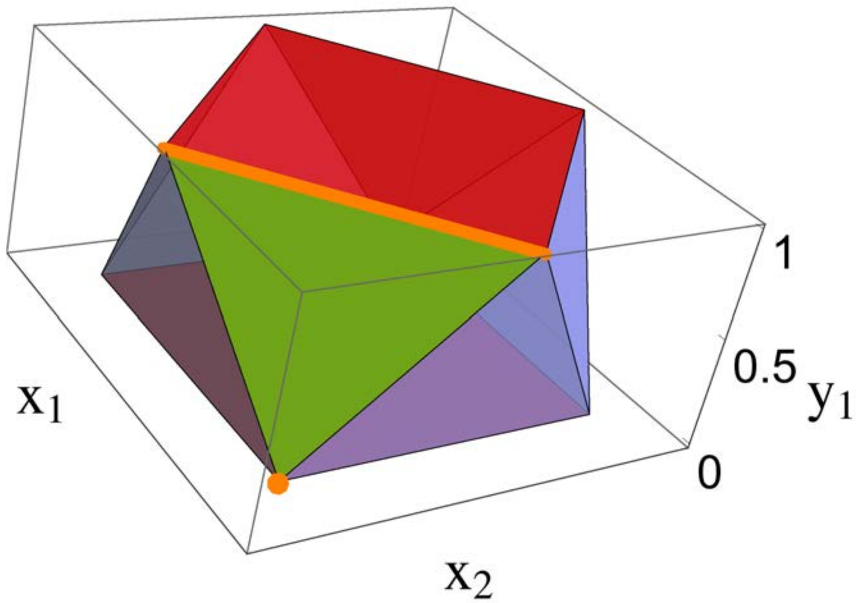
Faces of Cayley Embedding

- Two types of facets (or faces):
  - $P_1 \times \{0\} \equiv y_i \geq 0$
  - $\mathrm{conv}\left((F_1 \times 0) \cup (F_2 \times 1)\right)$

    F_i proper face of P_i
  - Not all combinations of faces
  - Which ones are valid?



Valid Combinations = Common Normals

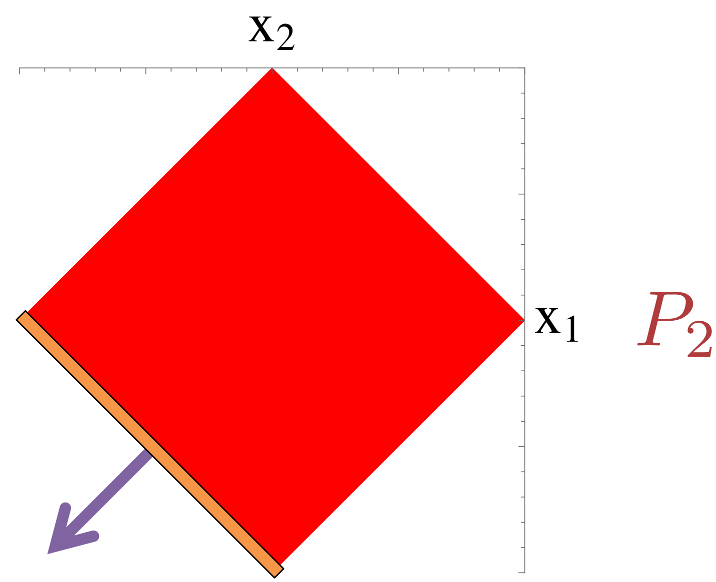


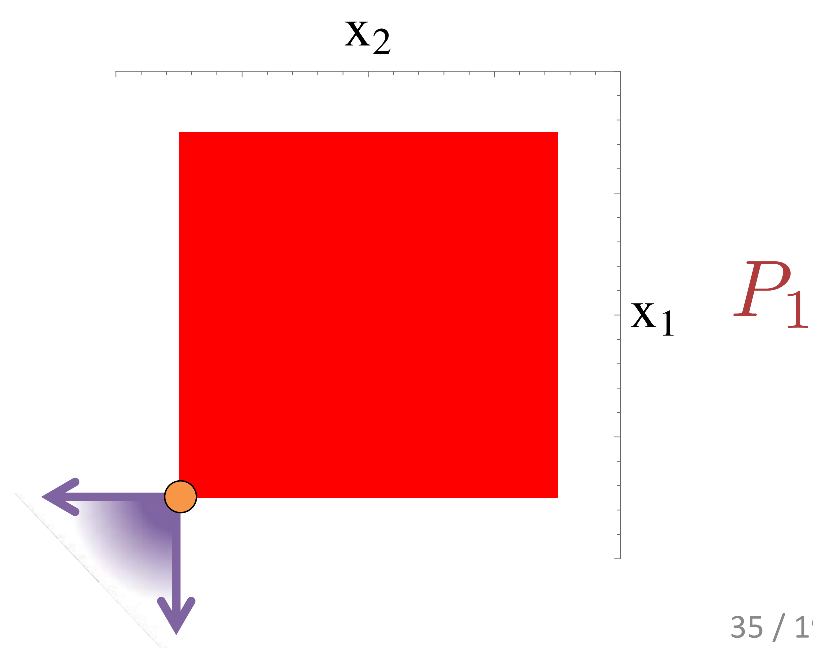
$$N(F_1) \cap N(F_2) \neq \emptyset$$

$$\Updownarrow$$

$$\text{conv}\left((F_1 \times 0) \cup (F_2 \times 1)\right)$$

is face of $Q(H)$





Redundancy in Embedding Formulations

Formulations and Complexity for Triangulations

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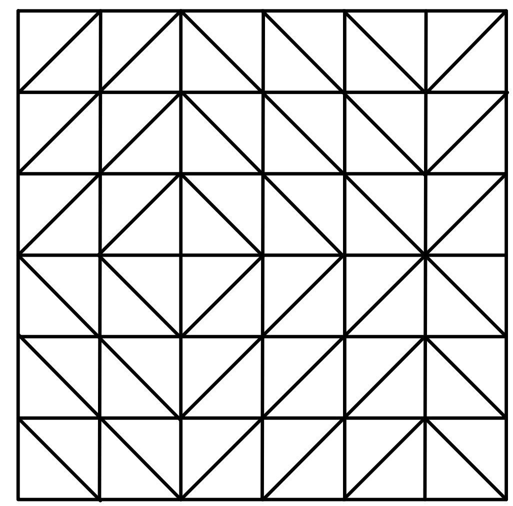
$$\left(\sqrt{n/2}+1\right)^2 \leq \mathrm{mc}\left(\mathcal{P}\right)$$

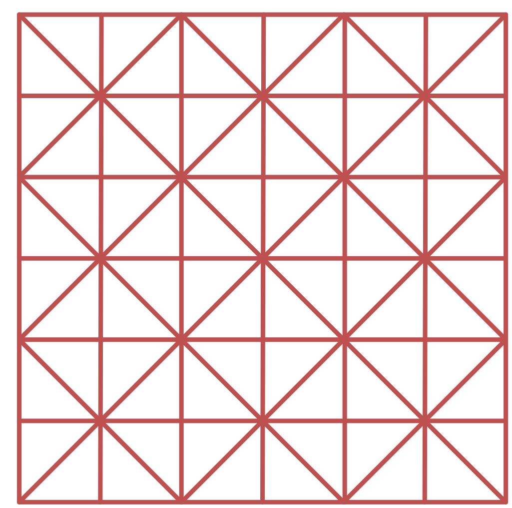


- Size of unary formulation is:
  (Lee and Wilson '01)

$$\mathrm{mc}\left(\mathcal{P}\right) \leq \binom{2\sqrt{n/2}}{\sqrt{n/2}} + \left(\sqrt{n/2}+1\right)^2$$

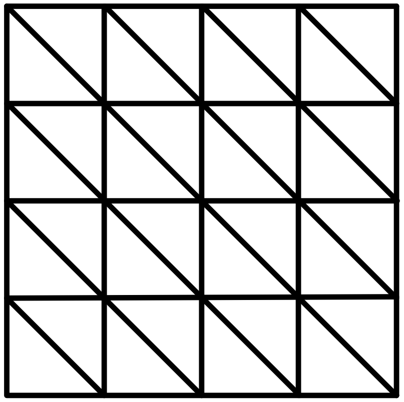
- Small binary formulation for union jack triangulation of size:
  (V. and Nemhauser '08)

$$\mathrm{mc}\left(\mathcal{P}\right) \leq 4\log_2\sqrt{n/2}+2+\left(\sqrt{n/2}+1\right)^2$$



What About Other Triangulations

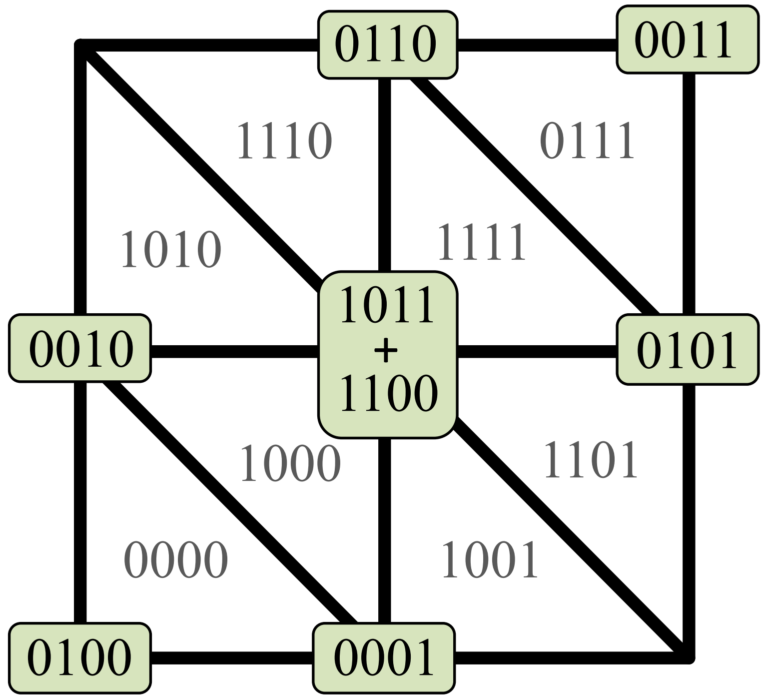
- $$\min_{H \in \mathcal{H}_3(8)} \operatorname{size}_M \left(Q\left(\mathcal{T}, H\right)\right) \geq 9.$$



$$\begin{aligned}
\lambda_{(1,1)} + \lambda_{(3,3)} &\leq 1 - y_1, & \lambda_{(1,3)} + \lambda_{(2,2)} + \lambda_{(3,1)} &\leq y_1 \\
\lambda_{(1,2)} + \lambda_{(2,1)} &\leq 1 - y_2, & \lambda_{(2,3)} + \lambda_{(3,2)} &\leq y_2 \\
\lambda_{(1,1)} + \lambda_{(2,1)} + \lambda_{(3,1)} &\leq 1 - y_3, & \lambda_{(1,3)} + \lambda_{(2,3)} + \lambda_{(3,3)} &\leq y_3 \\
\lambda_{(1,1)} + \lambda_{(1,2)} + \lambda_{(1,3)} &\leq 1 - y_4, & \lambda_{(3,1)} + \lambda_{(3,2)} + \lambda_{(3,3)} &\leq y_4
\end{aligned}$$

$$\sum\nolimits_{v \in V} \lambda_v = 1, \quad \lambda \in \mathbb{R}^V_+, \quad y \in \{0,1\}^4 .$$

Redundant Embedding Formulation



- $\mathrm{size}_M\left(Q\left(\mathcal{T},\{h^i\}_{i=1}^{8}\right)\right) = 19$

- v/s 8 inequalities