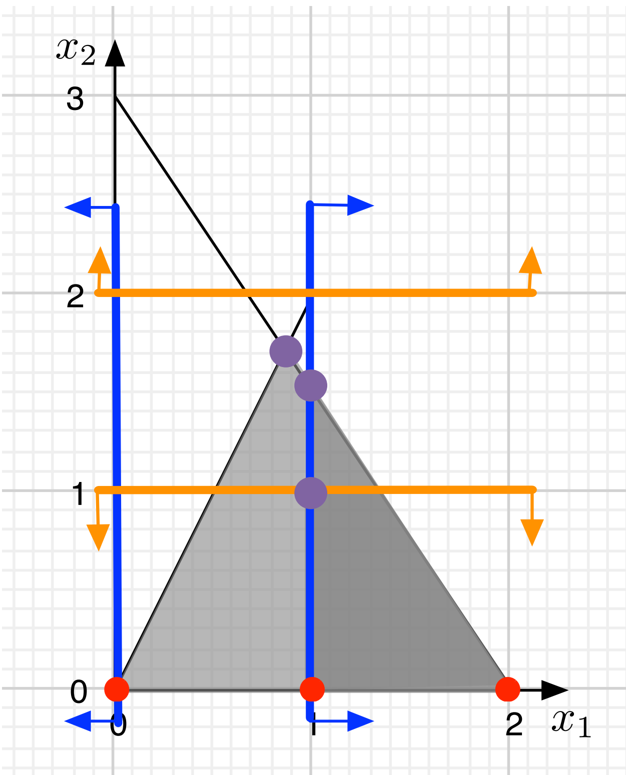
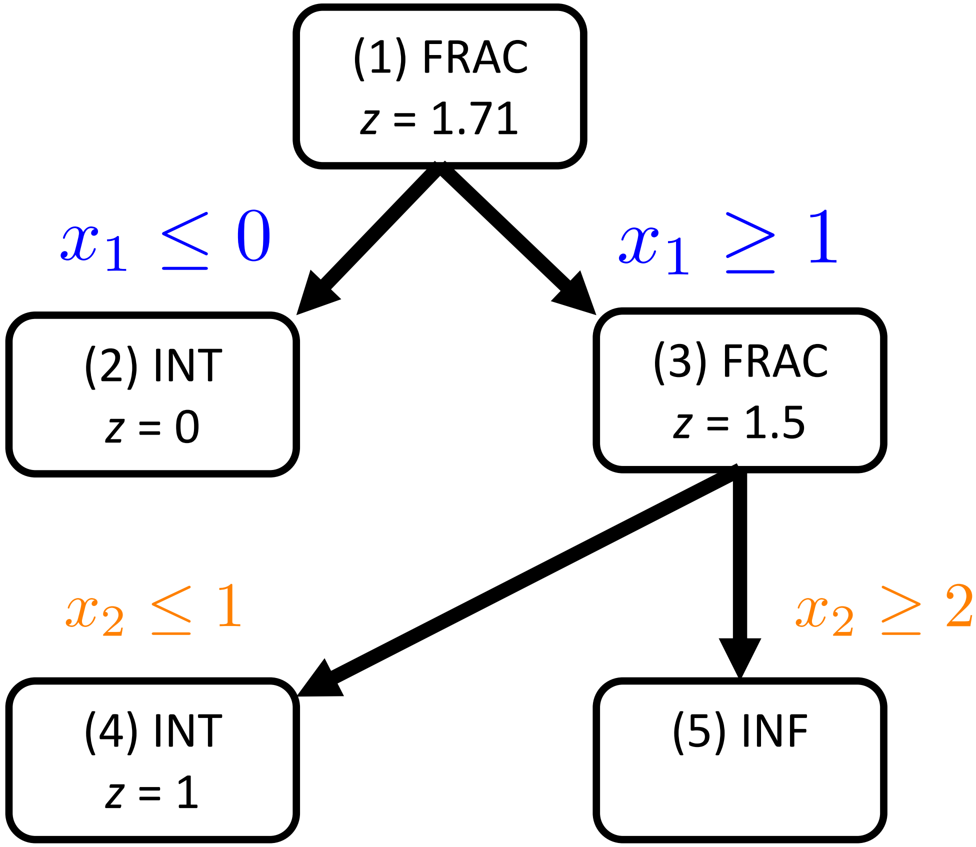
Polyhedra and MIP formulations

Basic Branch-and-Bound





$$
\begin{aligned}
\max z := x_2 & \\
3x_1 + 2x_2 &\leq 6 \\
-2x_1 + x_2 &\leq 0 \\
x_1,\ x_2 &\geq 0
\end{aligned}
$$

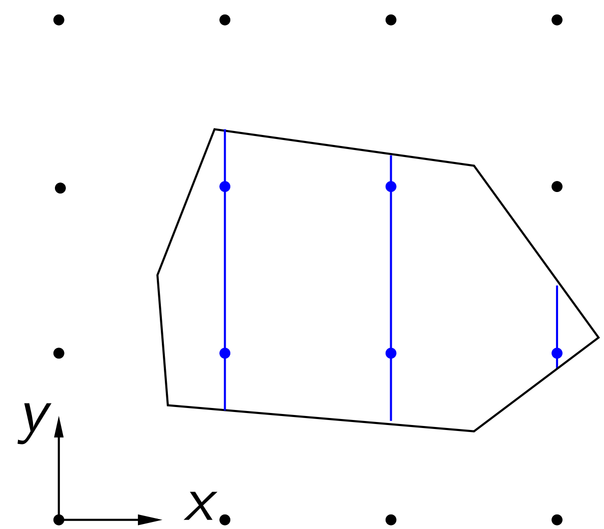
← Linear Programming (LP) Relaxation

$$x_1,\ x_2 \in \mathbb{Z}$$

Key : LP is (usually) much easier than MIP

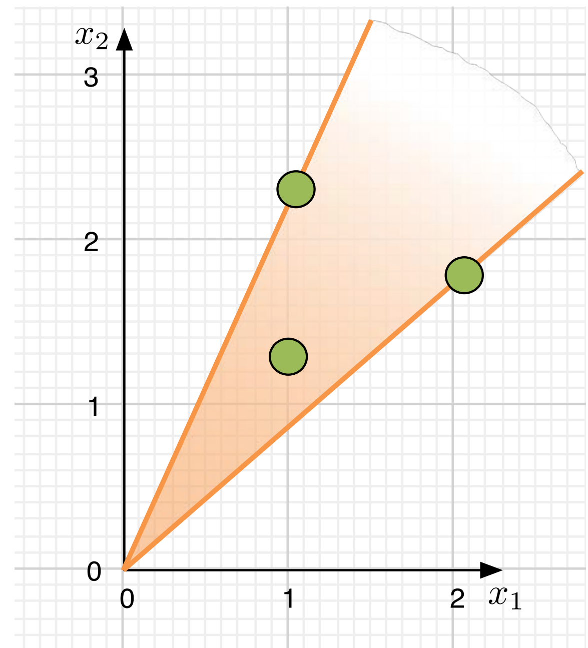
Why Polyhedral Theory? LP Relaxation

$$
\begin{aligned}
\max \quad & a \cdot x + h \cdot y \\
s.t. \quad & \\
& \boxed{Ax + By \leq b} \\
& x \in \mathbb{Z}^n \\
& y \in \mathbb{R}^m
\end{aligned}
$$



Conic and Convex Combinations

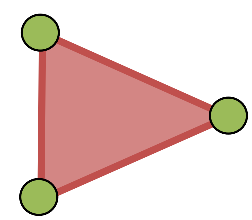
- Finite set V
- Conic Hull:



$$\mathrm{cone}(V) := \left\{ x \in \mathbb{R}^n \ : \ \exists \lambda \in \mathbb{R}_+^V \text{ s.t. } x = \sum_{v \in V} v \lambda_v \right\}$$

- Convex Hull:

$$\Delta^V := \left\{ \lambda \in \mathbb{R}_+^V \ : \ \sum_{v \in V} \lambda_v = 1 \right\}$$

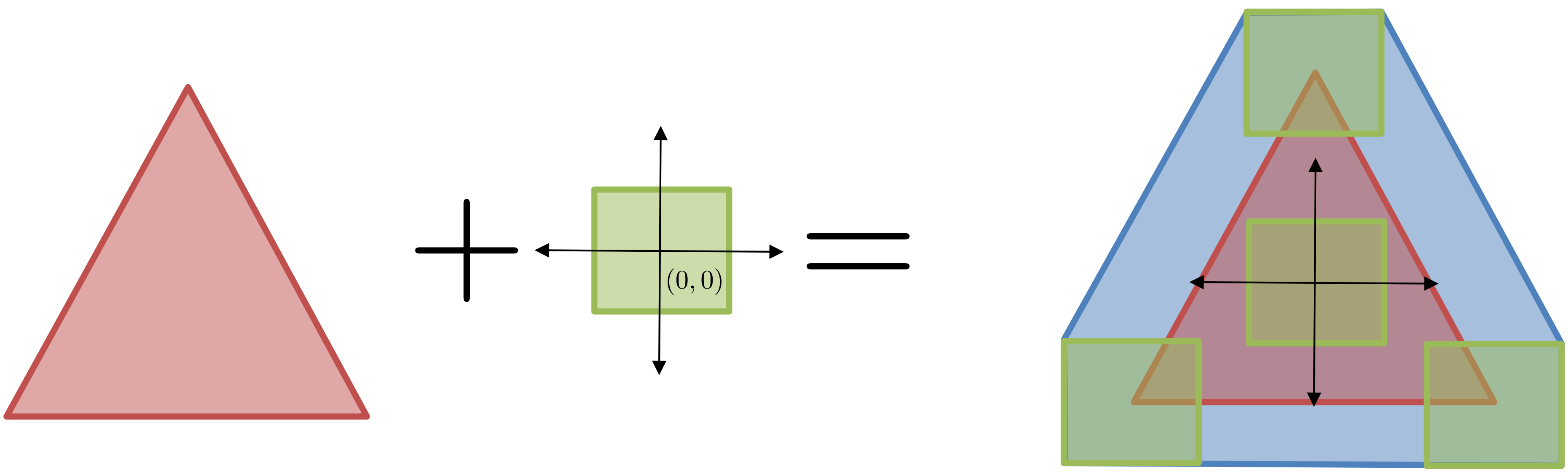


$$\mathrm{conv}(V) := \left\{ x \in \mathbb{R}^n \ : \ \exists \lambda \in \Delta^V \text{ s.t. } x = \sum_{v \in V} v \lambda_v \right\}$$

Minkowski Sum of Two Sets

- $A,\ B \subseteq \mathbb{R}^n$

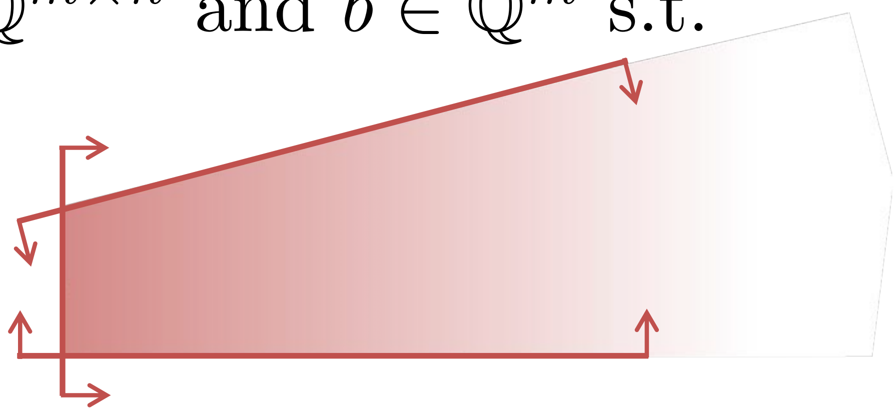
$$A + B := \{a + b \ : \ a \in A,\ b \in B\}$$



Two Types of Polyhedra

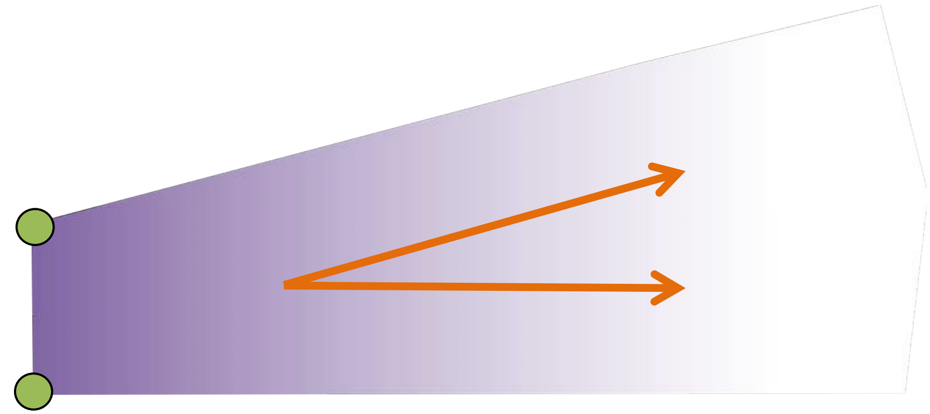
- $\mathcal{H}$-*polyhedron* iff $\exists\ A \in \mathbb{Q}^{m \times n}$ and $b \in \mathbb{Q}^m$ s.t.

$$P = \{x \in \mathbb{Q}^n \ :\ Ax \leq b\}$$



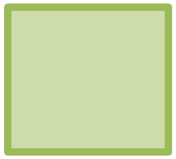
- $\mathcal{V}$-*polyhedron* iff $\exists$ finite sets $V \subseteq \mathbb{Q}^n$ and $R \subseteq \mathbb{Q}^n$ s.t.

$$P = \text{conv}\,(V) + \text{cone}\,(R)$$



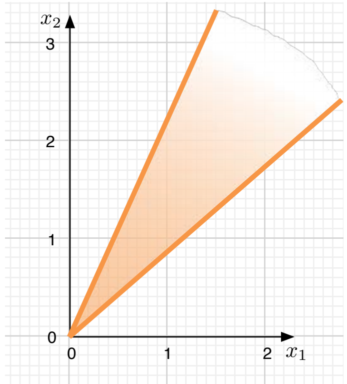
Bounded and Unbounded Polyhedra

- Let P be a $\mathcal{H}$ or $\mathcal{V}$-polyhedron
  - *P* is a polytope iff it is bounded



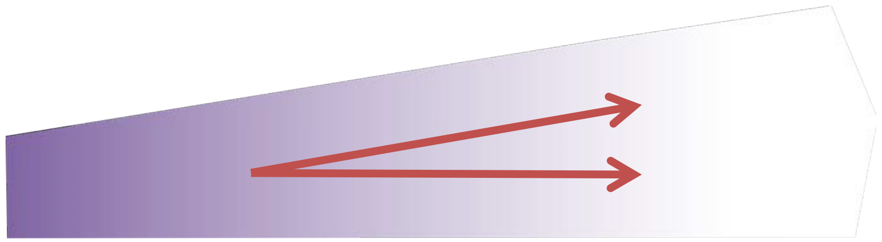
  - *P* is a cone iff:

$$\lambda x \in P \quad \forall \lambda \geq 0,\ x \in P$$



  - The *recession cone* of *P* is:

$$P_\infty := \{d \in \mathbb{Q}^n \ : \ x + \lambda d \in P \quad \forall x \in P, \quad \lambda \geq 0\}$$



Properties of Recession Cones

- If a non-empty polyhedron *P* is bounded iff

$$P_\infty = \{0\}$$

- The recession cone is always a cone
- If $P = \{x \in \mathbb{Q}^n \ : \ Ax \leq b\}$ then

$$P_\infty = \{x \in \mathbb{Q}^n \ : \ Ax \leq 0\}$$

- If $P = \operatorname{conv}(V) + \operatorname{cone}(R)$ then

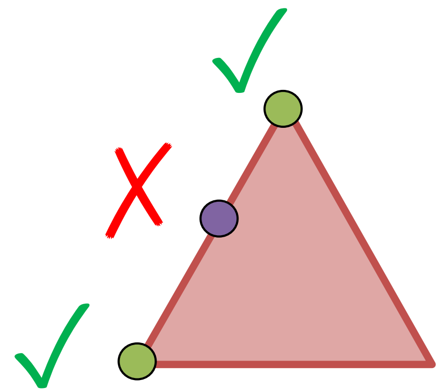
$$P_\infty = \operatorname{cone}(R)$$

Extreme Points and Extreme Rays

- $x \in P$ is an *extreme point* of P iff

$$\nexists x^1, x^2 \in P,\ x^1 \neq x^2$$

$$x \in \operatorname{conv}\left(\{x^1, x^2\}\right)$$



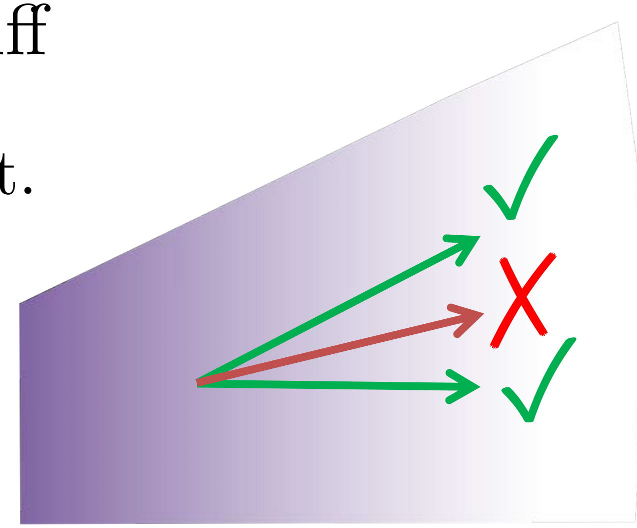
  - $\operatorname{ext}(P) :=$ set of all extreme points

- $r^1, r^2 \in P_\infty$ are equivalent iff $\exists \lambda > 0$ s.t. $r^1 = \lambda r^2$

  $r \in P_\infty \setminus \{0\}$ is an *extreme ray* of P iff

$$\nexists \text{ non-equivalent } r^1, r^2 \in P_\infty \setminus \{0\} \text{ s.t.}$$

$$r = r^1 + r^2$$



  - $\operatorname{ray}(P) :=$ set of all extreme rays (only one per equivalence class)

Basic Feasible Solutions and Extreme Points/Rays

LEMMA *Let $P \subseteq \mathbb{Q}^n$ be an $\mathcal{H}$-polyhedron. Then*

- *A point $x \in P$ is an extreme point of P if and only it is a basic feasible solution of P.*
- *A direction $r \in P_\infty \setminus \{0\}$ is an extreme ray of P if and only if it satisfies $n-1$ of the linear inequalities of P_∞ at equality and the left hand sides of these inequalities are linearly independent.*

$\mathcal{H}$-polyhedron = $\mathcal{V}$-polyhedron

- *P* is *pointed* iff it has at least one extreme point
- Assumption: *P* is non-empty and pointed
- Minkowski-Weyl Theorem:
  - P is a $\mathcal{V}$-polyhedron iff it is a $\mathcal{H}$-polyhedron
  - If *P* is a polyhedron then
    - $|\mathrm{ext}(P)| < \infty, \ |\mathrm{ray}(P)| < \infty$
    - $P = \mathrm{conv}\left(\mathrm{ext}(P)\right) + \mathrm{cone}\left(\mathrm{ray}(P)\right)$

Minkowski-Weyl: From cones to polyhedra

- Given:

Theorem 3.11 (Minkowski–Weyl Theorem for Cones). *A subset of $\mathbb{R}^n$ is a finitely generated cone if and only if it is a polyhedral cone.*

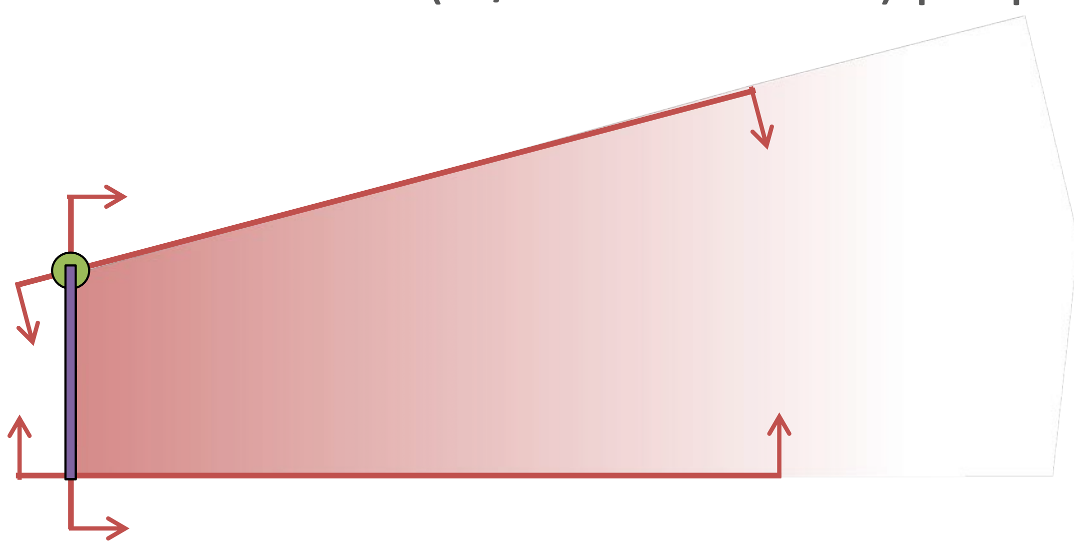
- Prove that:

Theorem 3.13 (Minkowski–Weyl Theorem [279, 348]). *A subset P of $\mathbb{R}^n$ is a polyhedron if and only if $P = Q + C$ for some polytope $Q \subset \mathbb{R}^n$ and finitely generated cone $C \subseteq \mathbb{R}^n$.*

Faces and Facets

$P := \{x \in \mathbb{Q}^n \, : \, Ax \leq b\}$, a^i is i-th row of $A \in \mathbb{Q}^{m \times n}$

- F is a
  - Face of P iff $\exists L \subseteq [m] := \{1, \ldots, m\}$ s.t.
    $$F = \{x \in P \, : \, a^l \cdot x = b_l \quad \forall l \in L\}$$
  - Proper Face if additionally $\emptyset \neq F \neq P$
  - Facet if it is a maximal (w/r to inclusion) proper face



Inequalities, Equalities and Redundancy

$P := \{x \in \mathbb{Q}^n \,:\, Ax \leq b\}$, a^i is i-th row of $A \in \mathbb{Q}^{m \times n}$

- $a^i \cdot x \leq b_i$ is a
  - Implied equality of *P* iff

$$a^i \cdot x = b_i \;\forall\, x \in P$$

  - Facet defining for *P* iff

$$F := \left\{x \in P \,:\, a^i \cdot x = b_i\right\} \text{ is a facet}$$

  - Redundant for *P* and $L \subseteq [m]$ (with $i \in L$) iff

$$P = \left\{x \in \mathbb{Q}^n \,:\, a^l \cdot x \leq b_l \quad \forall l \in L \setminus \{i\}\right\}$$

- $L \subseteq [m]$ is a minimal representation of *P* iff

$$P = \left\{x \in \mathbb{Q}^n \,:\, a^l \cdot x \leq b_l \quad \forall l \in L\right\}$$

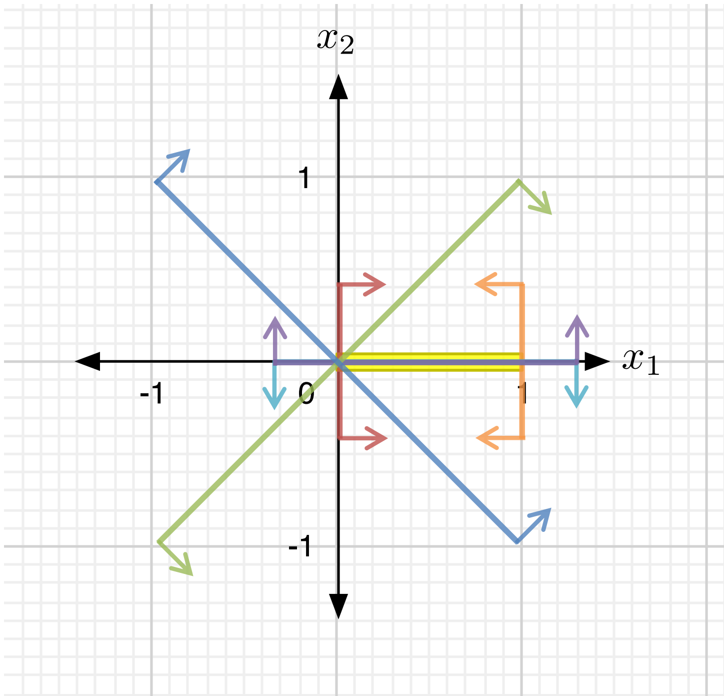
  - And no $i \in L$ is redundant for *P* and *L*

Example

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -1 \\ 1 & 0 \\ -1 & -1 \\ -1 & 1 \\ -1 & 0 \end{bmatrix}, \quad b = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{aligned} +x_2 &\leq 0 \quad (1) \\ -x_2 &\leq 0 \quad (2) \\ x_1 \qquad &\leq 1 \quad (3) \\ -x_1 - x_2 &\leq 0 \quad (4) \\ -x_1 + x_2 &\leq 0 \quad (5) \\ -x_1 \qquad &\leq 0 \quad (6) \end{aligned}$$

- Implied equations?
- Facet defining inequalities?
- Redundant inequalities?
- Minimal representation?



Facets and Minimal Representations

$P := \{x \in \mathbb{Q}^n \;:\; Ax \leq b\}$, a^i is i-th row of $A \in \mathbb{Q}^{m\times n}$

- Any facet has a facet defining inequality
  - # of facets is finite
- Let
  - F ⊆[m] set of facet defining inequalities and f = # of facets,
  - E ⊆[m] set of implied equalities and $r = \operatorname{rank}\left([A_l]_{l\in E}\right)$

    $\exists F' \subseteq F, \quad |F'| = f$
  - Then

$$\exists E' \subseteq E, \quad |E'| = r$$

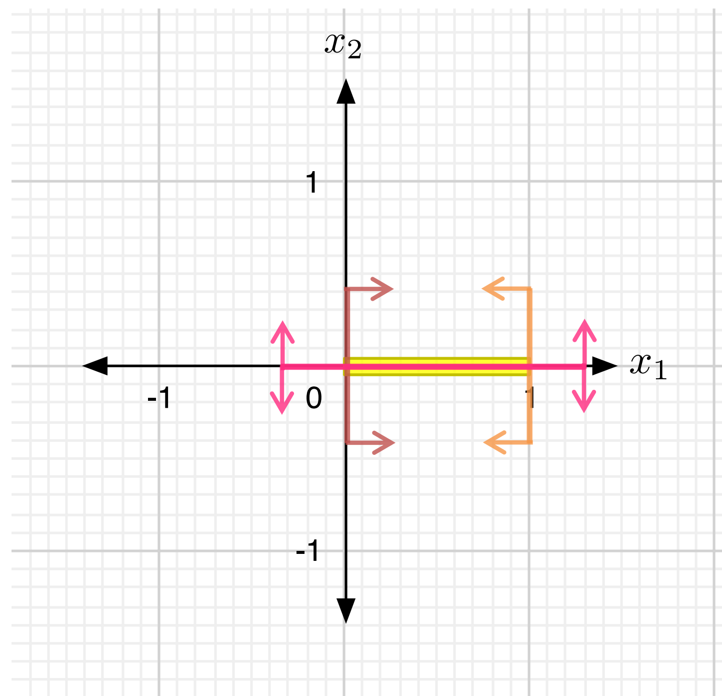
$$P = \left\{ x \in \mathbb{Q}^n \;:\; \begin{array}{ll} a^l \cdot x \leq b_l & \forall l \in F' \\ a^l \cdot x = b_l & \forall l \in E' \end{array} \right\}$$

Example: Minimal Representation

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -1 \\ 1 & 0 \\ -1 & -1 \\ -1 & 1 \\ -1 & 0 \end{bmatrix}, \quad b = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{aligned}
&\cancel{+x_2 \le 0} && (1) \\
&\cancel{-x_2 \le 0} && (2) \\
&x_1 \le 1 && (3) \\
&\cancel{-x_1 - x_2 \le 0} && (4) \\
&\cancel{-x_1 + x_2 \le 0} && (5) \\
&-x_1 \le 0 && (6)
\end{aligned}$$

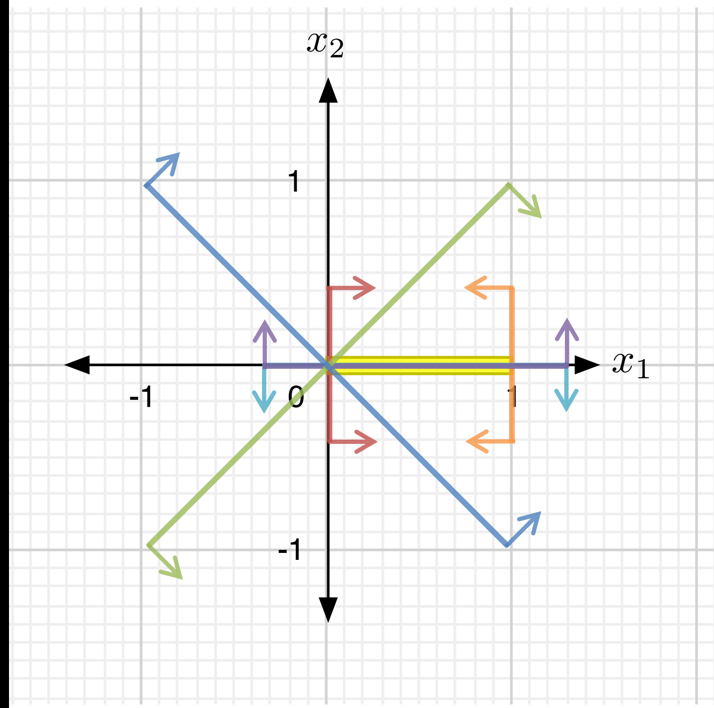
$$x_2 = 0$$



Equalities, Inequalities and Polyhedral.jl

```
julia> @variable(model, x[1:2]);
julia> @constraint(model, x[2]<=0);
julia> @constraint(model, -x[2]<=0);
julia> @constraint(model, x[1]<=1);
julia> @constraint(model, -x[1]-x[2]<=0);
julia> @constraint(model, -x[1]+x[2]<=0);
julia> @constraint(model, -x[1]<=0);
julia> poly = polyhedron(model, CDDLibrary(:exact));
julia> simpleh = SimpleHRepresentation(poly);
julia> @variable(model2,x[1:2]);
julia> @constraint(model2,convert.(Int64,simpleh.A)*x
.<= convert.(Int64,simpleh.b))…
 x[2] ≤ 0
 -x[2] ≤ 0
 x[1] ≤ 1
 -x[1] - x[2] ≤ 0
 -x[1] + x[2] ≤ 0
 -x[1] ≤ 0
julia> for eq in eqs(poly)
       println(dot(convert.(Int64,eq.a),x), " == ",
convert.(Int64,eq.β))
       end
julia> for ineq in ineqs(poly)
       println(dot(convert.(Int64,ineq.a),x), " <= ",
convert.(Int64,ineq.β))
       end
x[2] <= 0
-x[2] <= 0
x[1] <= 1
-x[1] - x[2] <= 0
-x[1] + x[2] <= 0
-x[1] <= 0
```

$$
\begin{aligned}
+x_2 &\le 0 \quad (1)\\
-x_2 &\le 0 \quad (2)\\
x_1 \qquad &\le 1 \quad (3)\\
-x_1 - x_2 &\le 0 \quad (4)\\
-x_1 + x_2 &\le 0 \quad (5)\\
-x_1 \qquad &\le 0 \quad (6)
\end{aligned}
$$



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julia> @constraint(model, -x[1]<=0);
julia> poly = polyhedron(model, CDDLibrary(:exact));
julia> removehredundancy!(poly);
julia> simpleh = SimpleHRepresentation(poly);
julia> @variable(model2,x[1:2]);
julia> @constraint(model2,convert.(Int64,simpleh.A)*x
.<= convert.(Int64,simpleh.b))…
 x[2] ≤ 0
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julia> for eq in eqs(poly)
       println(dot(convert.(Int64,eq.a),x), " == ",
convert.(Int64,eq.β))
       end
x[2] == 0
julia> for ineq in ineqs(poly)
       println(dot(convert.(Int64,ineq.a),x), " <= ",
convert.(Int64,ineq.β))
       end
x[1] <= 1
-x[1] <= 0
```

$$
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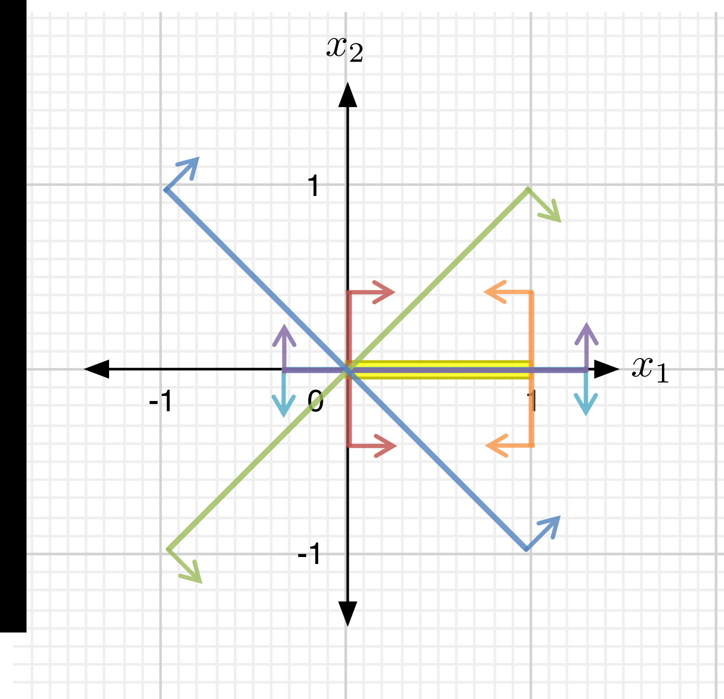
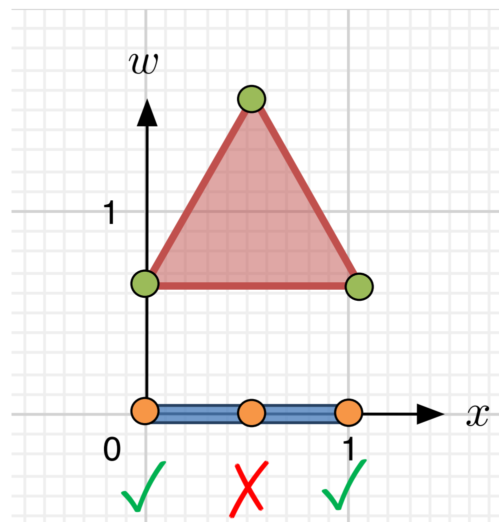
$$



Linear Transformations and Projections

- Let $L : \mathbb{R}^n \to \mathbb{R}^p$ be linear
  (i.e. $L(\alpha x + \beta y) = \alpha L(x) + \beta L(y)$ for all $\alpha, \beta \in \mathbb{R}$ and $x, y \in \mathbb{R}^n$)
- Let $L(S) := \{L(x) \,:\, x \in S\}$ for any set $S \subseteq \mathbb{Q}^n$
- Then $\operatorname{ext}(L(P)) \subseteq L(\operatorname{ext}(P))$
  - In particular let
    - $Q = \{(x, w) \in \mathbb{Q}^{n+p} \,:\, Ax + Dw \leq b\}$
    - $\operatorname{Proj}(Q) := \{x \in \mathbb{Q}^n \,:\, \exists w \in \mathbb{Q}^p \text{ s.t. } (x, w) \in Q\}$
  - Then

$$\operatorname{ext}(\operatorname{Proj}_x(Q)) \subseteq \operatorname{Proj}_x(\operatorname{ext}(Q))$$



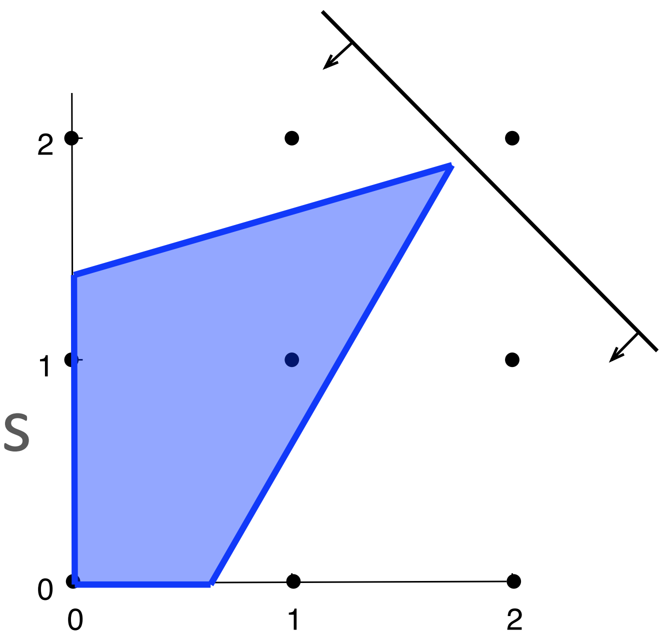
Farkas Lemma

- $P := \{x \in \mathbb{Q}^n \ : \ Ax \leq b\}$
- $A \in \mathbb{Q}^{m \times n}, b \in \mathbb{Q}^m$
- Version 1: $P \subseteq \{x \in \mathbb{Q}^n \ : \ c \cdot x \leq d\}$ iff

$$\exists \mu \in \mathbb{Q}^m_+ \text{ s.t.}$$

$$c = A^{\mathsf{T}} \mu, \quad b \cdot \mu \leq d$$

(P nonempty)



- Version 2: P is empty iff there exists $\mu \geq 0$ such that $\mu^T b < \mu^T A = 0$

Constructing the Projection of a Polyhedron

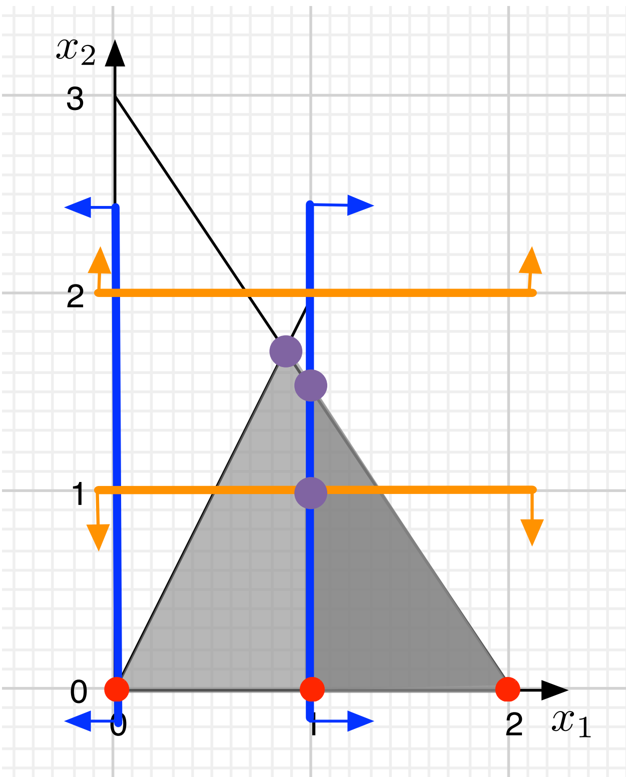
- $Q = \left\{(x, w) \in \mathbb{Q}^{n+p} \ : Ax + Dw \leq b\right\}$
- $\operatorname{Proj}(Q) := \{x \in \mathbb{Q}^{n} \ : \ \exists w \in \mathbb{Q}^{p} \text{ s.t. } \ (x, w) \in Q\}$
- $C = \left\{\mu \in \mathbb{Q}_{+}^{m} \ : D^{\intercal}\mu = 0\right\}$
- Then

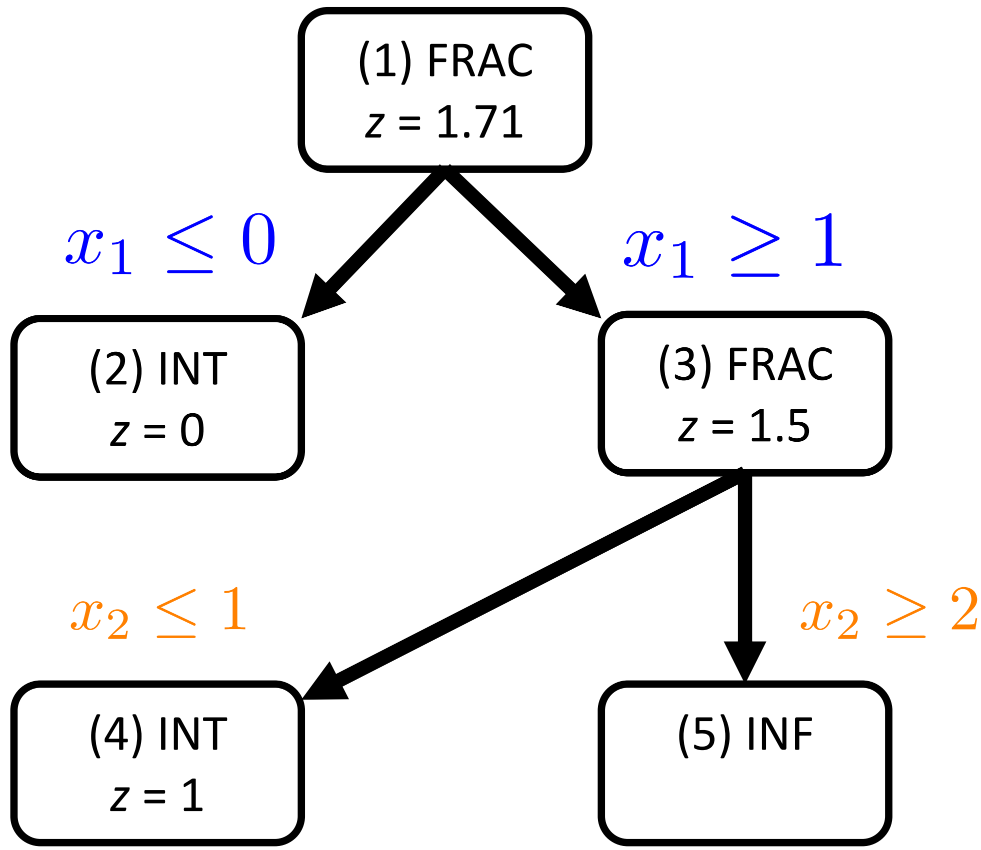
$$\operatorname{Proj}(Q) = \left\{x \in \mathbb{Q}^{n} \ : \mu^{\intercal} Ax \leq \mu^{\intercal} b \quad \forall \mu \in \operatorname{ray}(C)\right\}$$
$$\operatorname{Proj}(Q)_{\infty} = \operatorname{Proj}(Q_{\infty})$$

- Exercise: Prove it!
- Sanity check: When $C = \{0\}$, what is $\operatorname{Proj}_{x}(Q)$?
- Can also explicitly compute projections using Fourier-Motzkin elimination

Good MIP formulations

Basic Branch-and-Bound





$$
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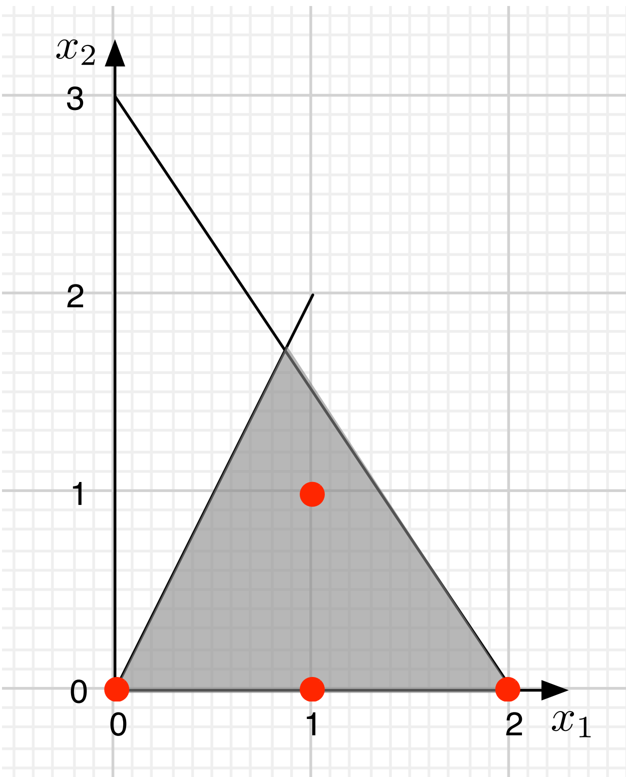
3x_1 + 2x_2 &\leq 6 \\
-2x_1 + x_2 &\leq 0 \\
x_1,\ x_2 &\geq 0
\end{aligned}
$$

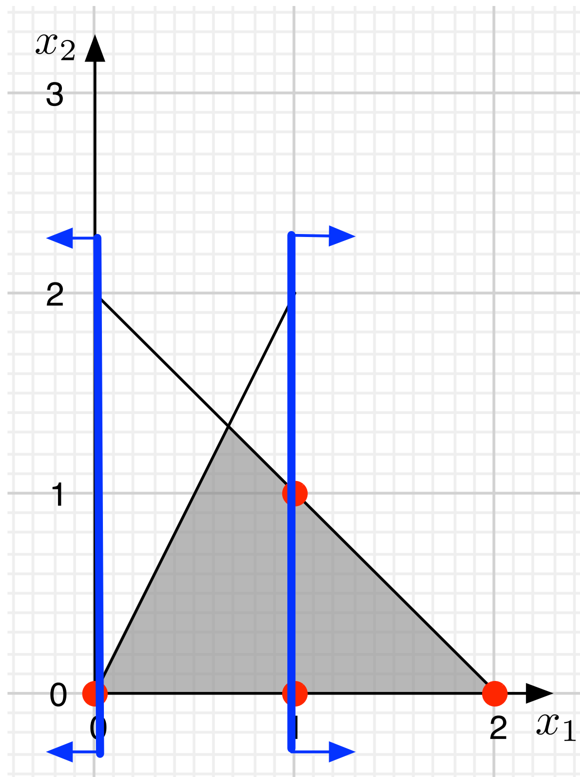
← Linear Programming (LP) Relaxation

$$x_1,\ x_2 \in \mathbb{Z}$$

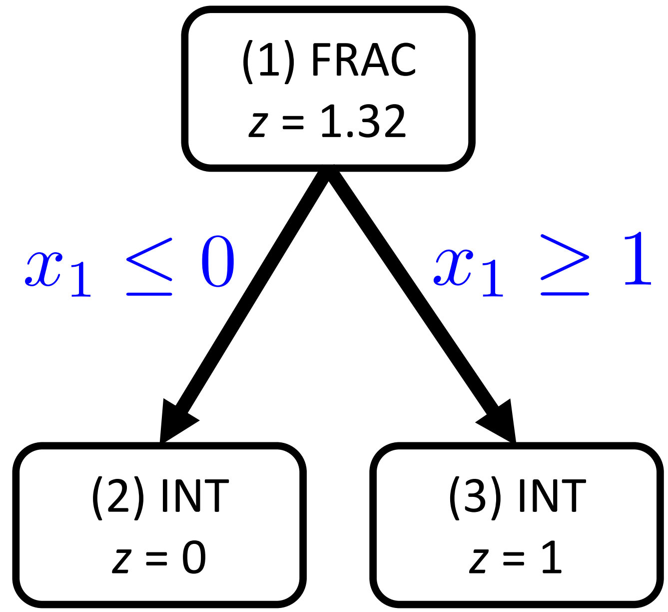
“Stronger Formulations” = Faster Solves ?



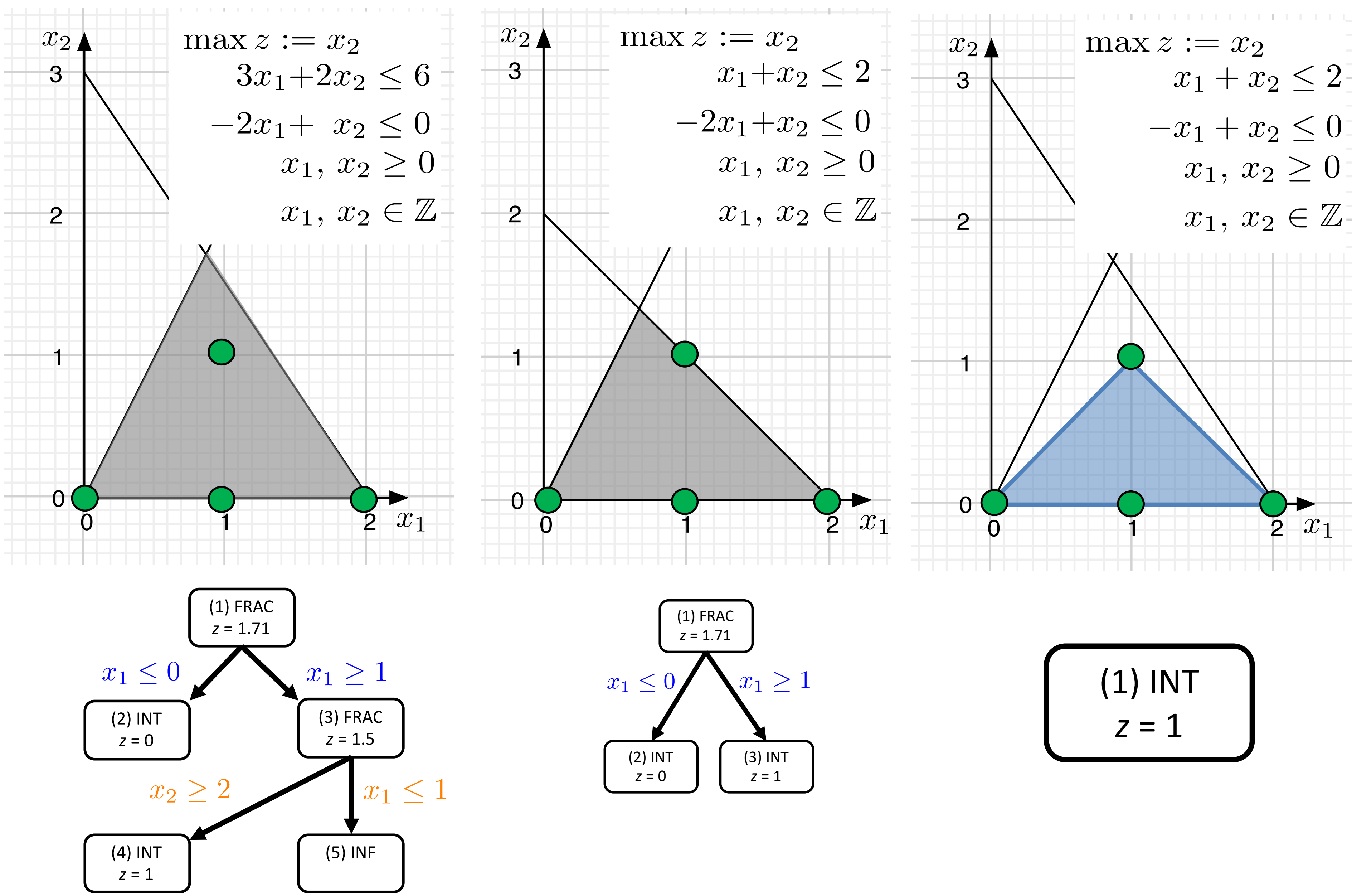
$$\begin{aligned} \max z &:= x_2 \\ 3x_1 + 2x_2 &\leq 6 \\ -2x_1 + x_2 &\leq 0 \\ x_1,\ x_2 &\geq 0 \\ x_1,\ x_2 &\in \mathbb{Z} \end{aligned}$$



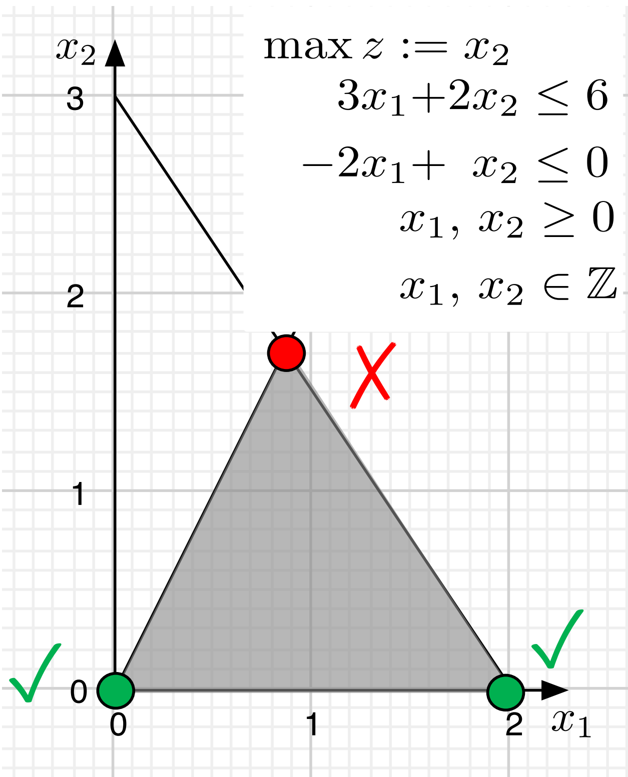
$$\begin{aligned} \max z &:= x_2 \\ x_1 + x_2 &\leq 2 \\ -2x_1 + x_2 &\leq 0 \\ x_1,\ x_2 &\geq 0 \\ x_1,\ x_2 &\in \mathbb{Z} \end{aligned}$$

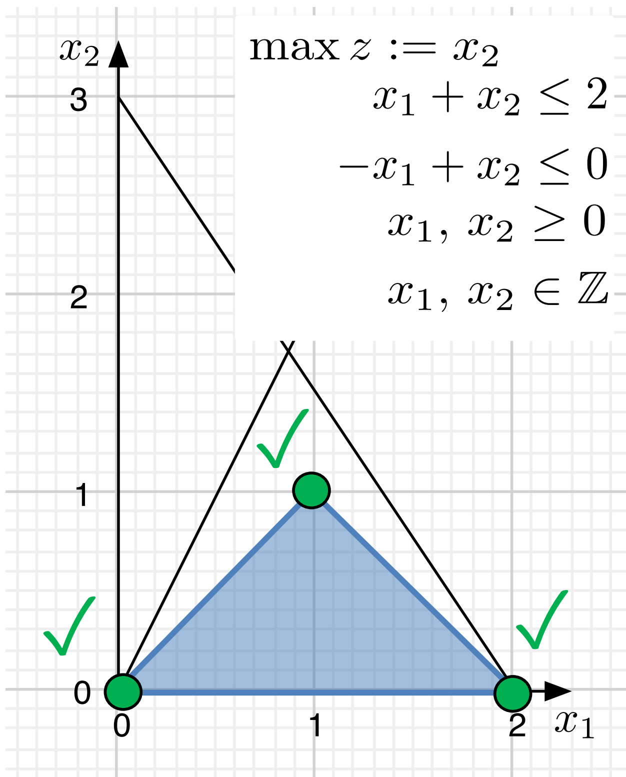


"Stronger Formulations" = Faster Solves ?



Strongest Formulation = “Integral” or “Ideal”





- “Integral” or “Ideal” formulation:
  - All extreme points or basic feasible solutions of LP relaxation are “integral” (satisfy integer constraint)
  - Optimizing a linear function = Solve LP

(Linear) Mixed Integer Programming Formulation

- Let
  - $S \subseteq \mathbb{R}^n$,
  - $n_1 + n_2 = n,\ p_1 + p_2 = p,\ A \in \mathbb{Q}^{m \times n},\ D \in \mathbb{Q}^{m \times p},\ b \in \mathbb{Q}^m$
  - $P := \{(x, w) \in \mathbb{R}^n \times \mathbb{R}^p \ :\ Ax + Dw \leq b\}$
  - $P_I := P \cap (\mathbb{R}^{n_1} \times \mathbb{Z}^{n_2} \times \mathbb{R}^{p_1} \times \mathbb{Z}^{p_2})$
- P_I is a MIP formulation of S iff

$$S = \operatorname{Proj}_x (P_I)$$

- x = original integer and continuous variables
- w = auxiliary integer and continuous variables
- A formulation is *integral* or *ideal* iff

$$\operatorname{ext}(P) \subseteq (\mathbb{R}^{n_1} \times \mathbb{Z}^{n_2} \times \mathbb{R}^{p_1} \times \mathbb{Z}^{p_2})$$

Advantage of Integral Formulations

- If P_I is a formulation of S then:

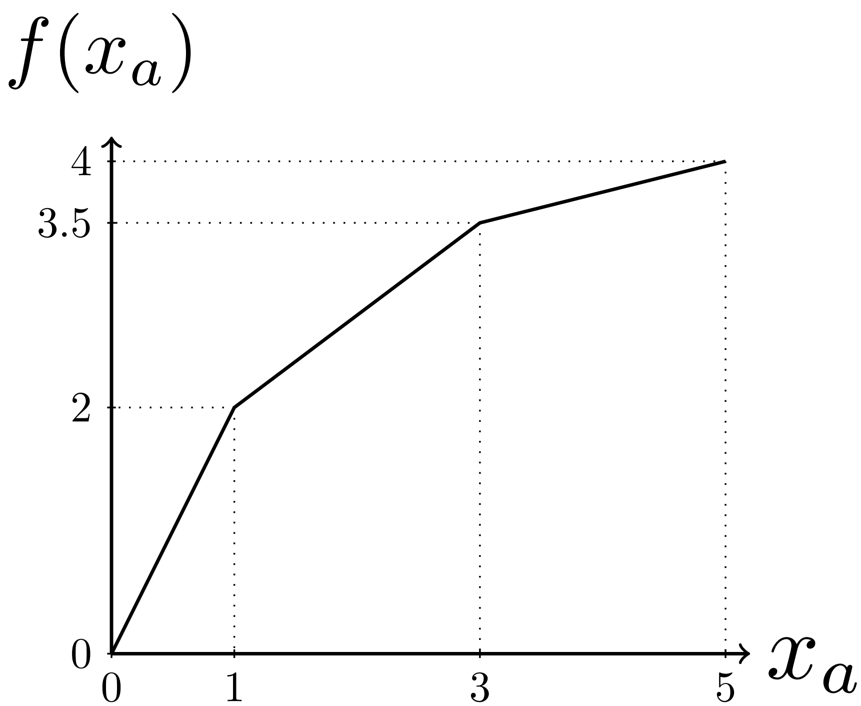
$$\max_{x} \left(c \cdot x \ : \ x \in S\right) = \max_{x,w} \left(c \cdot x \ : \ (x, w) \in P_I\right)$$

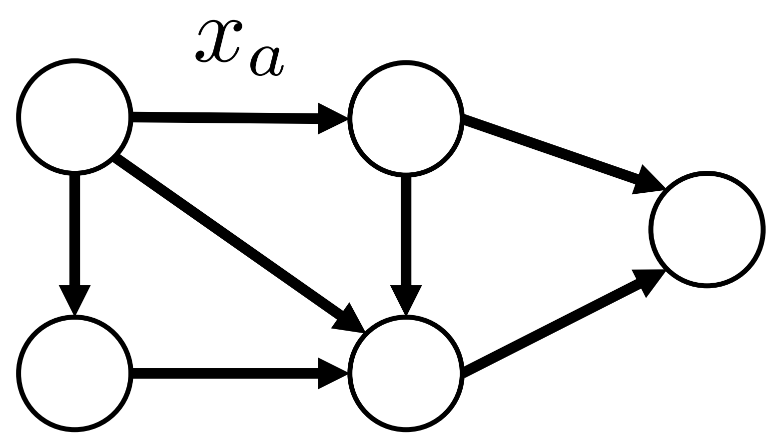
- If P_I is an ideal formulation of S then:

$$\max_{x,w} \left(c \cdot x \ : \ (x, w) \in P_I\right) = \max_{x,w} \left(c \cdot x \ : \ (x, w) \in P\right)$$

- In practice, S is one of many constraints:
  - Ideal (or strong) formulations tend to be more effective

Example: Piecewise Linear Network Flow





$$gr(f_i) := \{(x_i, z_i) \; : \; f_i(x_i) = z_i\}$$

- Network flow or transportation problem
- Economies of scale for transportation costs

Constructing a MIP Formulation

$$\min \quad \sum_{i=1}^{n} f_i(x_i)$$

$$s.t.$$

$$Ex \leq h$$

$$0 \leq x_i \leq u \quad \forall i \in [n]$$

$$\min \quad \sum_{i=1}^{n} z_i$$

$$s.t.$$

$$Ex \leq h$$

$$0 \leq x_i \leq u \qquad \forall i \in [n]$$

$$(x_i, z_i) \in gr(f_i) \quad \forall i \in [n].$$

$$\min \qquad \sum_{i=1}^{n} z_i$$

$$s.t.$$

$$Ex \leq h$$

$$0 \leq x_i \leq u \qquad \forall i \in [n]$$

$$A^i \binom{x_i}{z_i} + B^i \lambda^i + D^i y^i \leq b^i \qquad \forall i \in [n]$$

$$y^i \in \mathbb{Z}^k \quad \forall i \in [n]$$

MIP Formulation

$$gr(f_i) := \{(x_i, z_i) \; : \; f_i(x_i) = z_i\}$$

Strong, but not Necessarily Ideal

$$\min \quad \sum_{i=1}^{n} z_i$$

$$s.t.$$

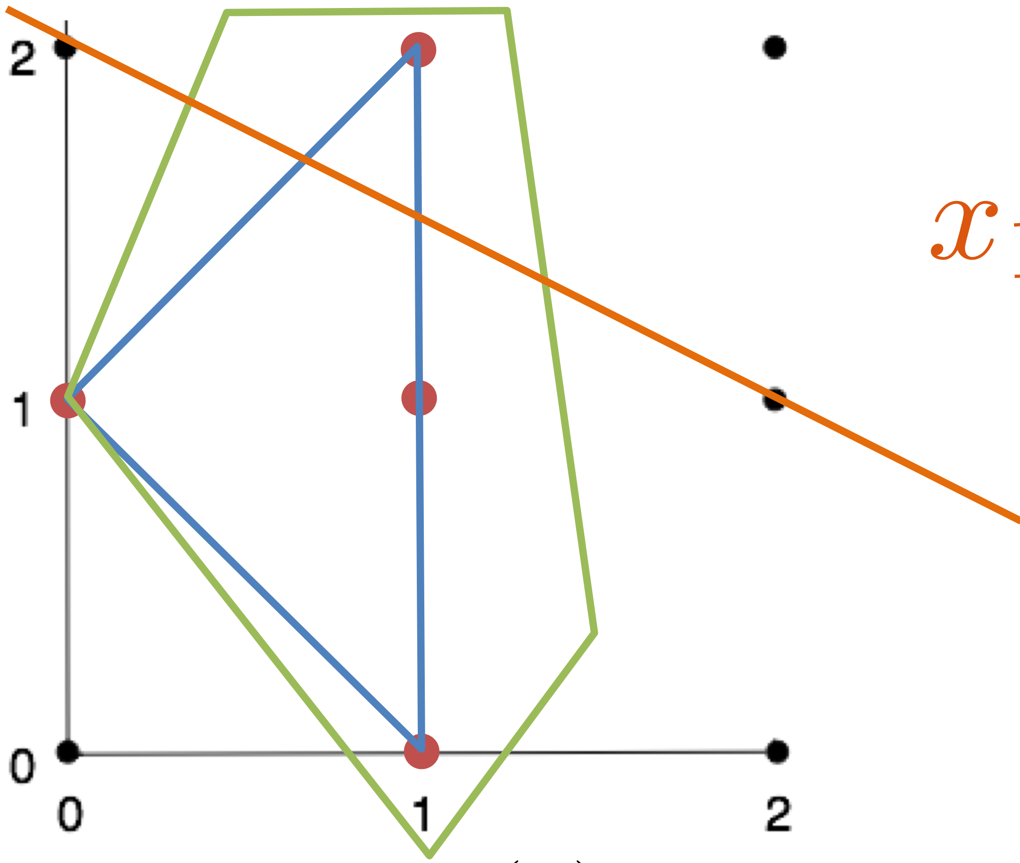
$$\boxed{\begin{aligned} Ex &\leq h \\ 0 \leq x_i &\leq u \\ A^i \begin{pmatrix} x_i \\ z_i \end{pmatrix} + B^i \lambda^i + D^i y^i &\leq b^i \\ y^i &\in \mathbb{Z}^k \end{aligned}} \quad \begin{aligned} & \\ &\forall i \in [n] \\ &\left.\begin{aligned}&\forall i \in [n] \\ &\forall i \in [n]\end{aligned}\right\} \text{Ideal for each i} \end{aligned}$$

Not necessarily ideal for complete problem

Example

- Pure Integer $S \subseteq \mathbb{Z}^n$

$$P \cap \mathbb{Z}^n = S \ (P \subseteq \mathbb{R}^n)$$



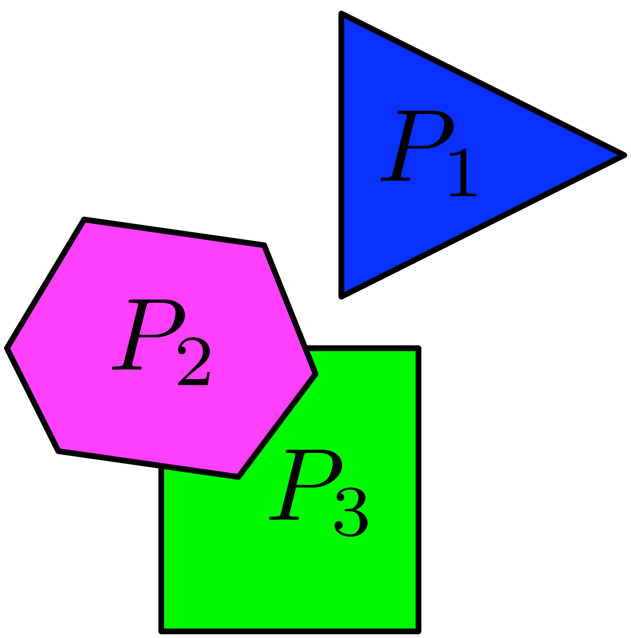
$$x_1 + 2x_2 \leq 4$$

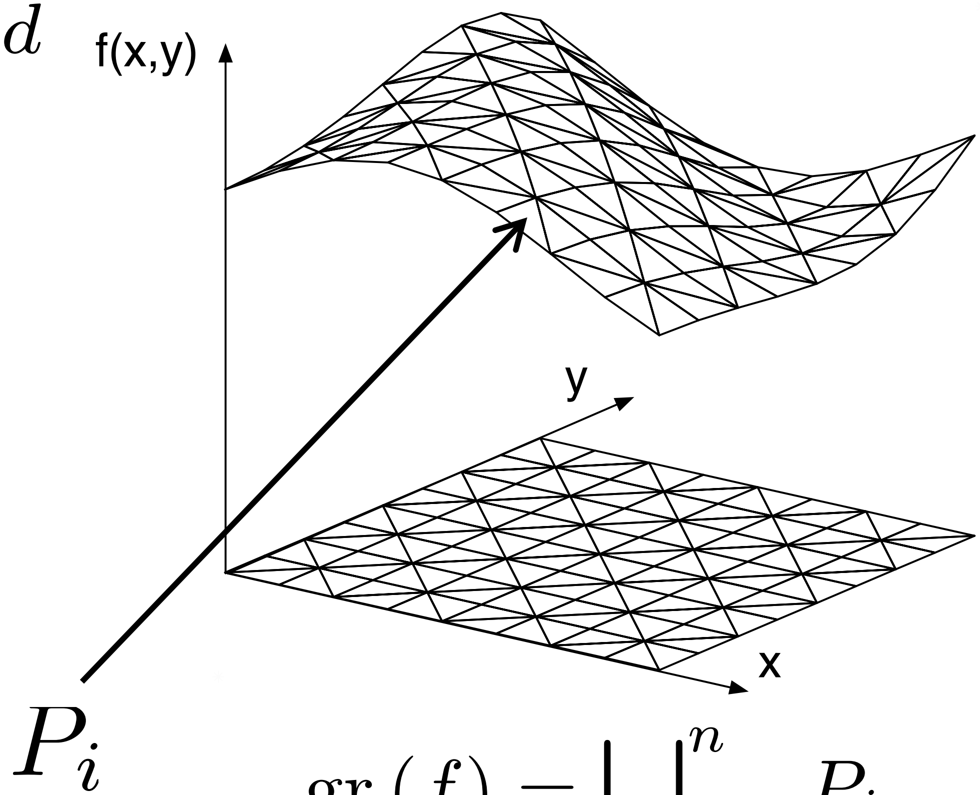
$$Q := \operatorname{conv}(S)$$

Mixed 0-1 Formulations

- Modeling Finite Alternatives = Unions of Polyhedra
  - Bounded or unbounded polyhedra, but bounded for now

$$x \in \bigcup_{i=1}^{n} P_i \subseteq \mathbb{R}^d$$

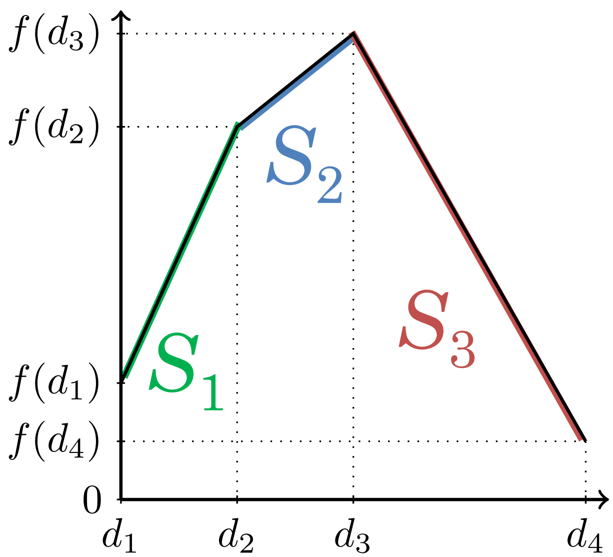




$$\mathrm{gr}\,(f) = \bigcup_{i=1}^{n} P_i$$

Graph of a Piecewise Linear Function

$$f(x) = \begin{cases} m_1 x + c_1 & x \in [d_1, d_2] \\ & \vdots \\ m_k x + c_k & x \in [d_k, d_{k+1}] \end{cases}$$

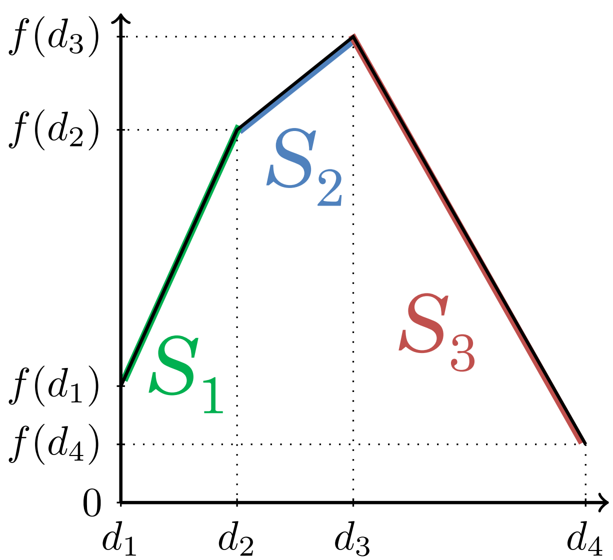


$$S = \mathrm{gr}\,(f) = \bigcup_{i=1}^{k} \underbrace{\left\{ (x, z) \in \mathbb{R}^2 \; : \; \begin{matrix} d_i \leq x \leq d_{i+1} \\ m_i x + c_i = z \end{matrix} \right\}}_{S_i}$$

Very Bad Formulation for PWL Function : Big-M

$$S = \mathrm{gr}(f) = \bigcup_{i=1}^{k} \left\{ (x,z) \in \mathbb{R}^2 \;:\; \begin{array}{r} d_i \le x \le d_{i+1} \\ m_i x + c_i = z \end{array} \right\}$$

Not Ideal!



$$\begin{aligned} d_i - (d_i - d_1)(1 - y_i) &\le x && \forall i \in [k] \\ d_{i+1} + (d_{k+1} - d_{i+1})(1 - y_i) &\ge x && \forall i \in [k] \\ m_i x + c_i - \underline{M}_i (1 - y_i) &\le z && \forall i \in [k] \\ m_i x + c_i + \overline{M}_i (1 - y_i) &\ge z && \forall i \in [k] \end{aligned}$$

$$\sum_{i=1}^{k} y_i = 1$$

$$y \in \{0,1\}^k$$

$$f(x) = \begin{cases} m_1 x + c_1 & x \in [d_1, d_2] \\ & \vdots \\ m_k x + c_k & x \in [d_k, d_{k+1}] \end{cases}$$

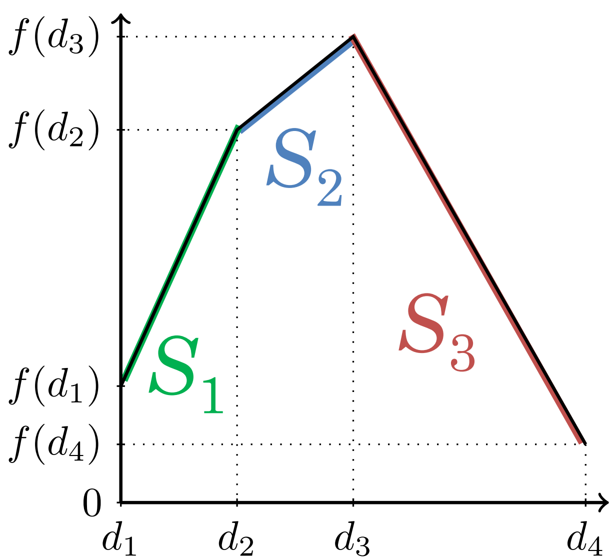
$$\underline{M}_i := \max_{j=1}^{k+1} \{ m_i d_j + c_i - f(d_j) \}$$

$$\overline{M}_i := \max_{j=1}^{k+1} \{ f(d_j) - m_i d_j - c_i \}$$

"Better" Formulation : Convex Combination (CC)

$$S = \mathrm{gr}(f) = \bigcup_{i=1}^{k} \left\{ (x,z) \in \mathbb{R}^2 : \begin{array}{r} d_i \le x \le d_{i+1} \\ m_i x + c_i = z \end{array} \right\}$$

"Stronger" than Big-M, but still not ideal



$$\sum_{i=1}^{k+1} \lambda_i d_i = x$$

$$\sum_{i=1}^{k+1} \lambda_i f(d_i) = z$$

$$\sum_{i=1}^{k+1} \lambda_i = 1$$

$$\lambda_1 \le y_1$$

$$\lambda_i \le y_{i-1} + y_i \quad \forall 2 \le i \le k$$

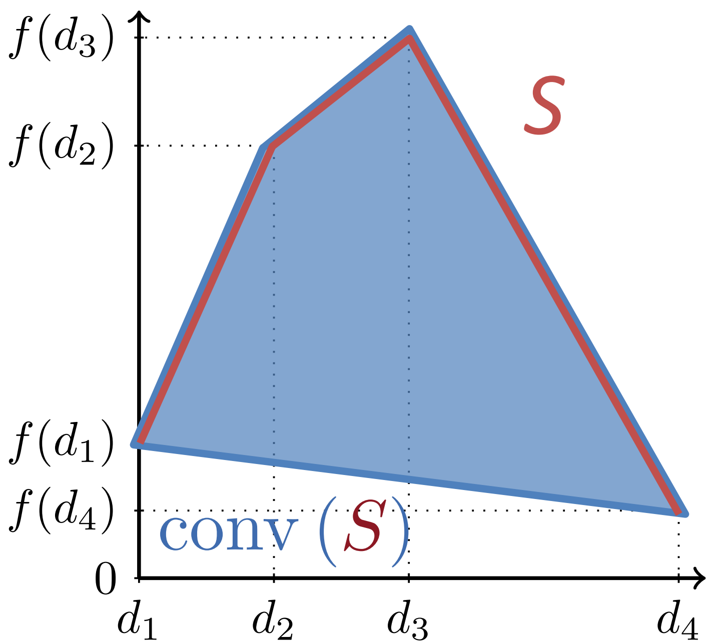
$$\lambda_{k+1} \le y_k$$

$$\sum_{i=1}^{k} y_i = 1$$

$$y \in \{0,1\}^k$$

$$f(x) = \begin{cases} m_1 x + c_1 & x \in [d_1, d_2] \\ & \vdots \\ m_k x + c_k & x \in [d_k, d_{k+1}] \end{cases}$$

“Relatively” strong = “Convex Hull” or “Sharp”



$$\sum_{i=1}^{4} d_i \lambda_i = x, \qquad \sum_{i=1}^{4} f(d_i)\lambda_i = z$$

$$\sum_{i=1}^{4} \lambda_i = 1, \qquad \lambda_i \geq 0$$

$$\sum_{i=1}^{3} y_i = 1, \qquad y_i \in \{0,1\}$$

$$\lambda_1 \leq y_1, \qquad \lambda_2 \leq y_1 + y_2$$

$$\lambda_3 \leq y_2 + y_3, \qquad \lambda_4 \leq y_3$$

$$f(x) = \begin{cases} m_1 x + c_1 & x \in [d_1, d_2] \\ & \vdots \\ m_k x + c_k & x \in [d_k, d_{k+1}] \end{cases}$$

Not integral, but LP relaxation gives convex hull

Sharp Formulations

- A MIP formulation P_I of S is *sharp or convex hull* iff

$$\operatorname{conv}(S) = \operatorname{Proj}_x(P)$$

- If P_I is a *sharp* formulation of S then:

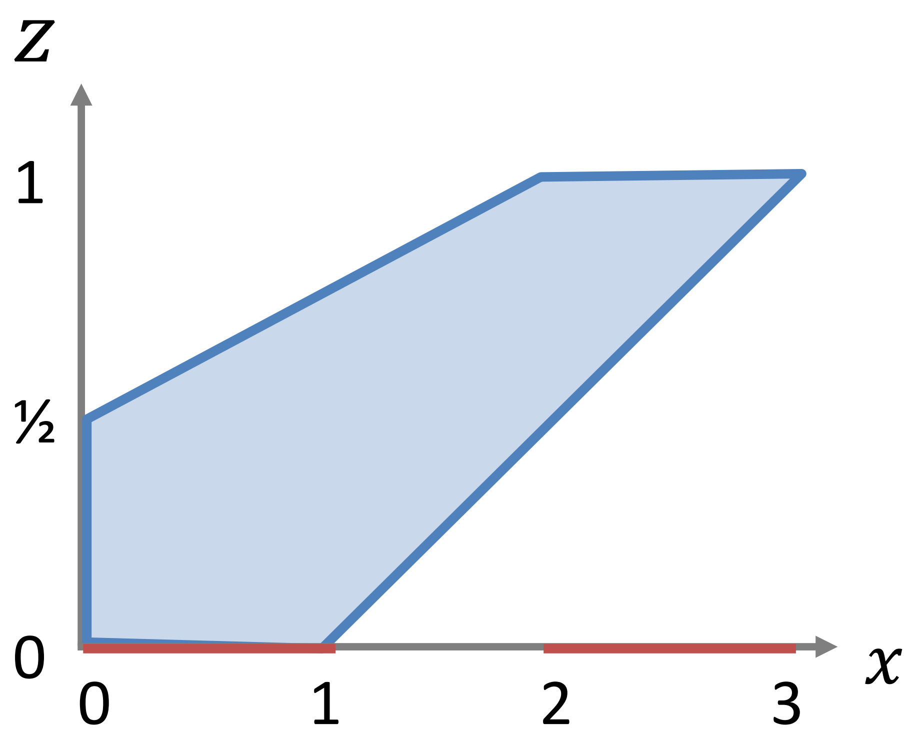
$$\max_{x,w}\left(c \cdot x \;:\; (x,w) \in P_I\right) = \max_{x,w}\left(c \cdot x \;:\; (x,w) \in P\right)$$

- CC is a sharp formulation for piecewise linear functions, but Big-M is not sharp.
  – Exercise: Show for *x=2* and

$$f(x) = \begin{cases} 1 - x & x \in [0,1] \\ 2x - 2 & x \in [1,2] \\ 6 - 2x & x \in [2,3] \\ x - 3 & x \in [3,4] \end{cases}$$

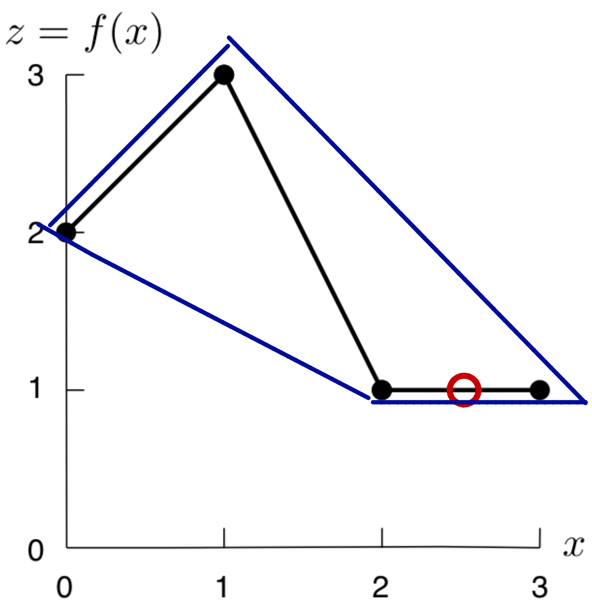
Ideal and Sharp Formulations

- Ideal formulations are sharp
- Example of non-ideal sharp formulation:
  - For $S = [0,1] \cup [2,3]$
  - $4z - x \leq 2,$
  - $x \geq 0,$
  - $x - 2z \leq 1,$
  - $0 \leq z \leq 1,$
  - $z \in \mathbb{Z}$



Remember: CC is not Ideal

Example: $S = \{(x,z) : f(x) = z\}$



$$
\begin{aligned}
0\lambda_0 + 1\lambda_1 + 2\lambda_2 + 3\lambda_3 &= x \\
2\lambda_0 + 3\lambda_1 + 1\lambda_2 + 1\lambda_3 &= z \\
\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3 &= 1 \\
\lambda_0 &\le y_1 \\
\lambda_1 &\le y_1 + y_2 \\
\lambda_2 &\le y_2 + y_3 \\
\lambda_3 &\le y_3 \\
y_1 + y_2 + y_3 &= 1 \\
\lambda_i &\ge 0 && \forall i \in \{0, \ldots, 3\} \\
0 \le y_i &\le 1 && \forall i \in \{1, 2, 3\} \\
y_i &\in \mathbb{Z} && \forall i \in \{1, 2, 3\}\,.
\end{aligned}
$$

Sharp

Extreme point: $\lambda_2 = \lambda_3 = 1/2$, $\lambda_0 = \lambda_1 = 0$

$x = 2.5$, $z = 1$ $\quad$ $y_1 = y_3 = 1/2$, $y_2 = 0$

The power of auxiliary variables

Needed for Small, Strong Formulations?

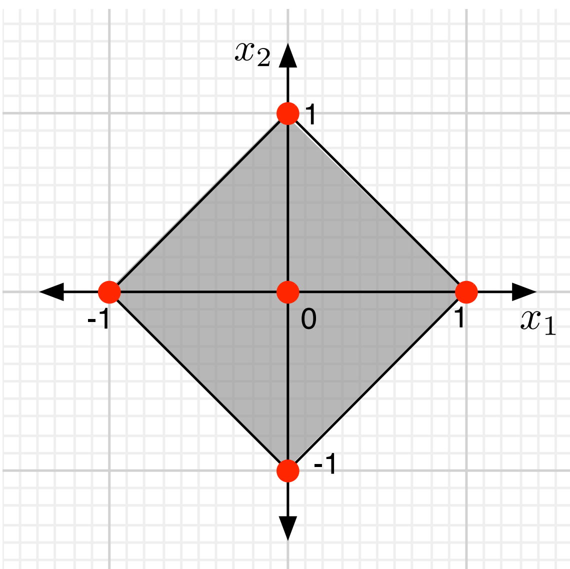
$$S := \left\{ x \in \mathbb{Z}^n \; : \; \sum_{i=1}^{n} |x_i| \leq 1 \right\}$$

Exponential sized ideal formulation:

$$\sum_{i=1}^{n} s_i x_i \leq 1 \quad \forall s \in \{-1, 1\}^n$$

$$x \in \mathbb{Z}^n$$

Poly-sized non-extended formulation?



- Formulations with auxiliary variables are usually called **extended formulations** (Usually for ideal formulations)

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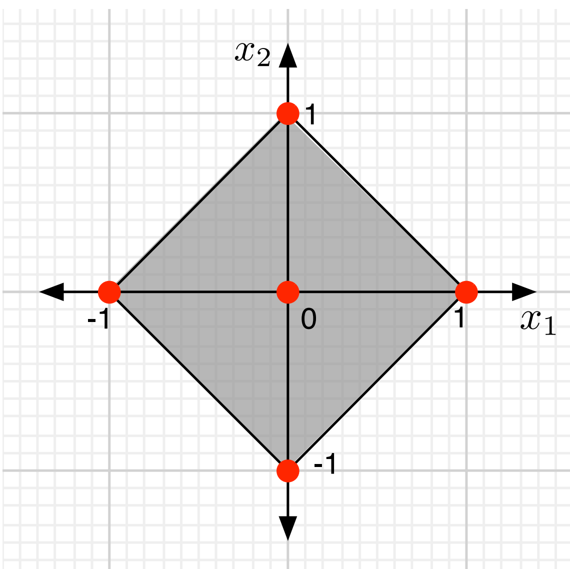

Exponential sized ideal formulation:

$$\sum_{i=1}^n s_i x_i \leq 1 \quad \forall s \in \{-1, 1\}^n$$
$$x \in \mathbb{Z}^n$$

Polynomial sized non-ideal formulation:

$$s_1 x_i + s_2 x_j \leq 1 \quad \forall s \in \{-1, 1\}^2, \quad i \neq j$$
$$x \in \mathbb{Z}^n$$

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Needed for Small, Strong Formulations?

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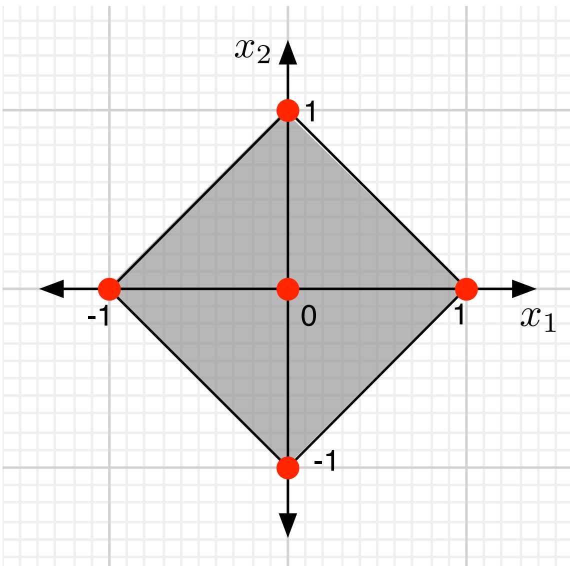

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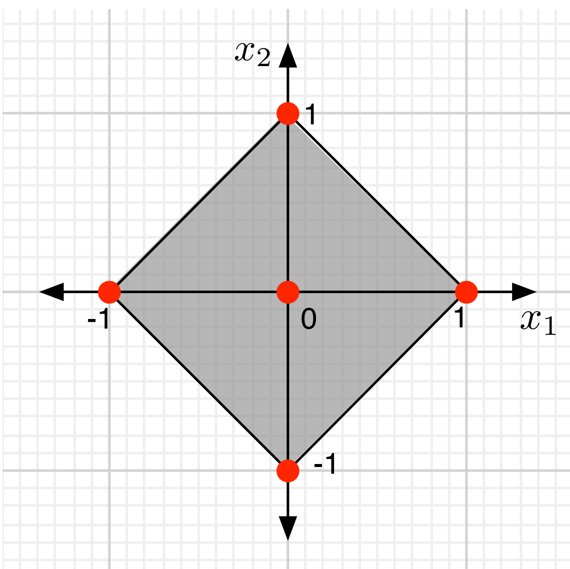
Needed for Small, Strong Formulations?

$$S := \left\{ x \in \mathbb{Z}^n \ : \ \sum_{i=1}^n |x_i| \leq 1 \right\}$$



Exponential sized ideal formulation:

$$\sum_{i=1}^n s_i x_i \leq 1 \quad \forall s \in \{-1,1\}^n$$
$$x \in \mathbb{Z}^n$$

Polynomial sized non-ideal formulation:

$$s_1 x_i + s_2 x_j \leq 1 \quad \forall s \in \{-1,1\}^2, \quad i \neq j$$
$$x \in \mathbb{Z}^n$$

Polynomial sized ideal extended formulation:

$$\sum_{i=1}^n y_i \leq 1 \qquad x \in \mathbb{Z}^n$$
$$x_i \leq y_i \quad \forall i \in [n]$$
$$-x_i \leq y_i \quad \forall i \in [n]$$

- Formulations with auxiliary variables are usually called **extended formulations** (Usually for ideal formulations)

Sometimes non-ideal formulations are large

- Write a MIP formulation for "Parity":
  - All vectors in $\{0,1\}^n$ with an odd number of ones.

Sometimes non-ideal formulations are large

- Write a MIP formulation for "Parity":
  - All vectors in $\{0,1\}^n$ with an odd number of ones.
- Modelling Trick 1: How to exclude a point $x^* \in \{0,1\}^n$
  - $\displaystyle\sum_{i:x_i^*=1} x_i + \sum_{i:x_i^*=0} (1-x_i) \leq n-1$
- Ideal Formulation for "Parity":

$$\sum_{i\in A} x_i + \sum_{i\notin A} (1-x_i) \leq n-1 \quad \forall A \subseteq [n],\ |A| \text{ even}$$

$$0 \leq x_i \leq 1 \qquad \forall i \in [n]$$

$$x \in \mathbb{Z}^n$$

Exponential number of inequalities!

- Proposition: "Every" non-extended formulation (even non-ideal) needs exponential many inequalities

Smaller Formulation with "Auxiliary" Variables

- Alternative formulation:

$$\sum_{i=1}^{n} x_i = 2w + 1$$
$$0 \leq x_i \leq 1 \qquad \forall i \in [n]$$
$$w \geq 0$$
$$x \in \mathbb{Z}^n$$
$$w \in \mathbb{Z}$$

- However formulation is not ideal:
  - Let: $x^* \in \{0,1\}^n$ s.t. $\sum_{i=1}^{n} x_i^* = k$ is even
  - Then: $(x^*, (k-1)/2)$ is an extreme point of LP relaxation
- Small ideal formulation for Parity: In a few slides

Key to Size Advantage of Auxiliary Variables

- Projecting a polyhedron can exponentially increase the number of facets.

$$\begin{aligned} \sum_{i=1}^{n} y_i &\leq 1 \\ x_i &\leq y_i \quad \forall i \in [n] \\ -x_i &\leq y_i \quad \forall i \in [n] \end{aligned} \xrightarrow{\mathrm{Proj}_x} \sum_{i=1}^{n} s_i x_i \leq 1 \quad \forall s \in \{-1, 1\}^n$$

Limits of Projection

- $Q = \{(x,w) \in \mathbb{Q}^{n+p} \ : Ax + Dw \leq b\}$
- $\mathrm{Proj}_x(Q) := \{x \in \mathbb{Q}^n \ : \ \exists w \in \mathbb{Q}^p \text{ s.t. } \ (x,w) \in Q\}$
- $C = \{\mu \in \mathbb{Q}^m_+ \ : D^\intercal \mu = 0\}$
- Then
  - $\mathrm{Proj}_x(Q) = \{x \in \mathbb{Q}^n \ : \mu^\intercal A x \leq \mu^\intercal b \quad \forall \mu \in \mathrm{ray}(C)\}$
- If Q has m inequalities
  - what is the largest # of facets of $\mathrm{Proj}_x(Q)$?
  - = # extreme rays of C

$$\binom{m}{m-p-1} = \binom{m}{p+1}$$

- Only polynomial increase if p is fixed

What About Matching?

- General maximum weight matching:

$$\max \quad \sum_{e \in E} t_e x_e$$

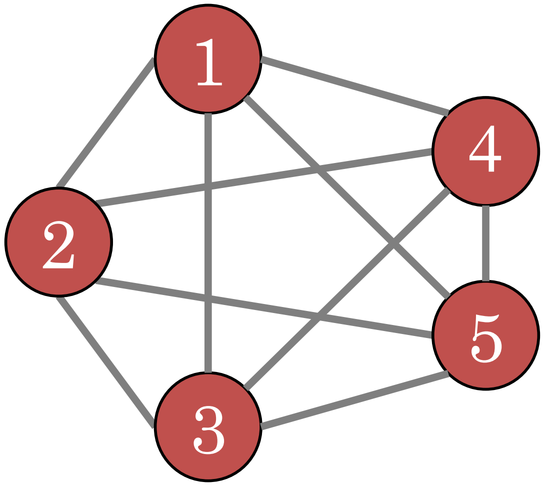
$$s.t.$$

$$\sum_{e \in \delta(v)} x_e \leq 1 \quad \forall v \in V$$

$$0 \leq x_e \leq 1 \quad \forall e \in E$$

$$x_e \in \mathbb{Z} \quad \forall e \in E$$

$$\sum_{e \in E(S)} x_e \leq \left\lfloor \frac{1}{2} |S| \right\rfloor \quad \forall S \subseteq V$$

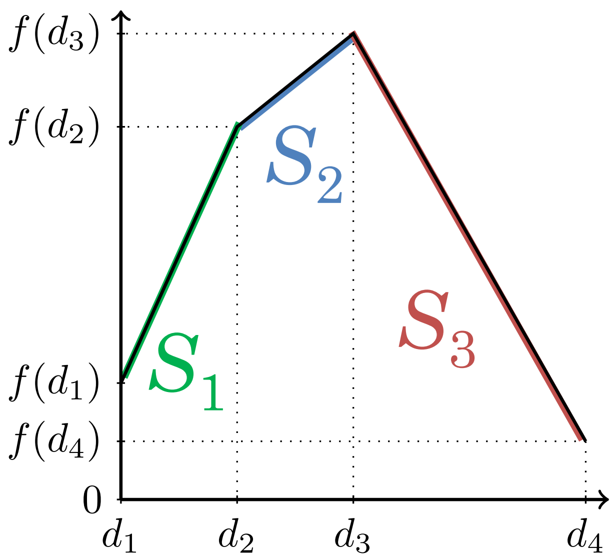


- There is no poly-size ideal extended formulation for general matching (Rothvoss '13)

Combinatorial disjunctive constraints

Graph of a Piecewise Linear Function

$$f(x) = \begin{cases} m_1 x + c_1 & x \in [d_1, d_2] \\ & \vdots \\ m_k x + c_k & x \in [d_k, d_{k+1}] \end{cases}$$

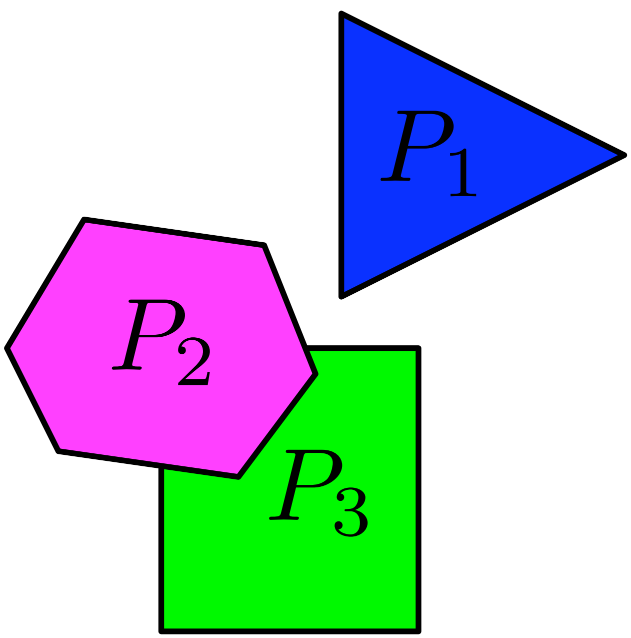


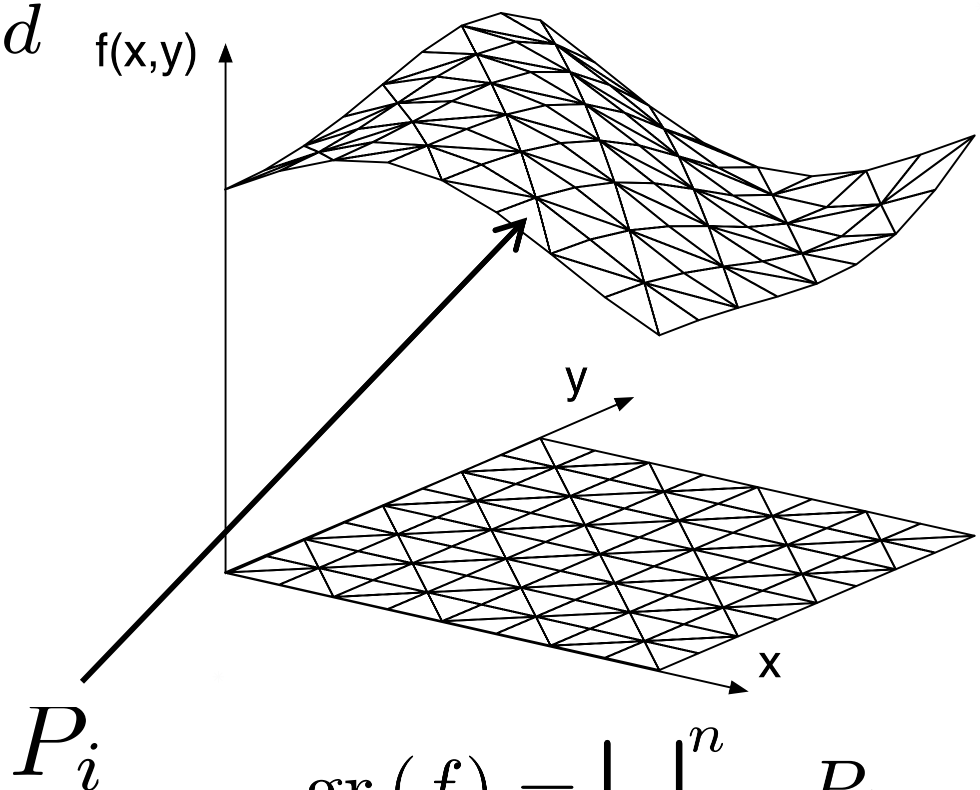
$$S = \mathrm{gr}\,(f) = \bigcup_{i=1}^{k} \underbrace{\left\{ (x, z) \in \mathbb{R}^2 \; : \; \begin{array}{r} d_i \leq x \leq d_{i+1} \\ m_i x + c_i = z \end{array} \right\}}_{S_i}$$

Mixed 0-1 Formulations

- Modeling Finite Alternatives = Unions of Polyhedra
  - Bounded or unbounded polyhedra, but bounded for now

$$x \in \bigcup_{i=1}^{n} P_i \subseteq \mathbb{R}^d$$



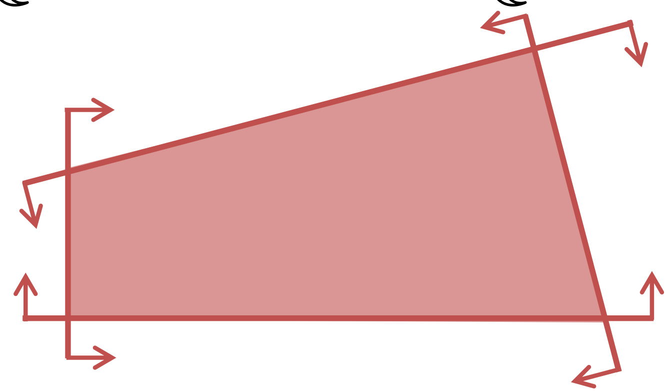


$$\mathrm{gr}\,(f) = \bigcup_{i=1}^{n} P_i$$

Two Types of Bounded Polyhedra (Polytopes)

- $\mathcal{H}$*-polyhedron* iff $\exists\ A \in \mathbb{Q}^{m \times n}$ and $b \in \mathbb{Q}^m$ s.t.

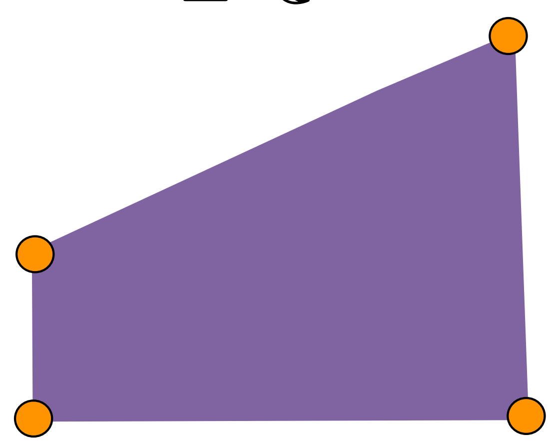
$$P = \{x \in \mathbb{Q}^n \ : \ Ax \leq b\}$$



- $\mathcal{V}$*-polyhedron* iff $\exists$ finite sets $V \subseteq \mathbb{Q}^n$ s.t.

$$P = \operatorname{conv}(V)$$

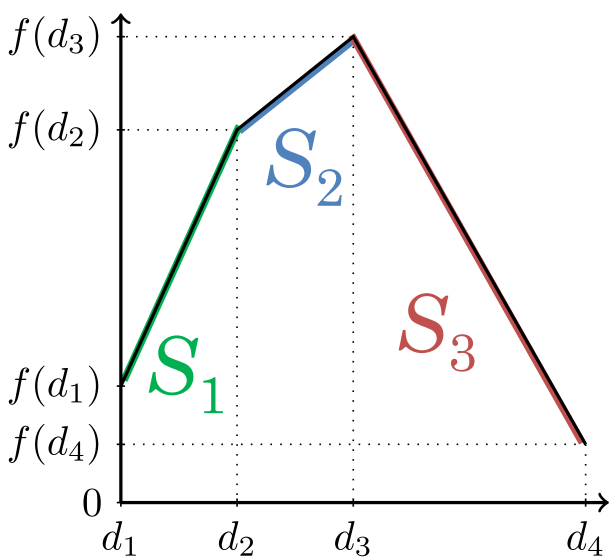
can take $V = \operatorname{ext}(P)$



Example of an $\mathcal{H}$-based Formulation

$$S = \operatorname{gr}(f) = \bigcup_{i=1}^{k} \left\{ (x,z) \in \mathbb{R}^2 : \boxed{\begin{aligned} d_i \le x \le d_{i+1} \\ m_i x + c_i = z \end{aligned}} \right\}$$

Data = Linear inequalities
mixed with formulation



$$\boxed{\begin{aligned} d_i - (d_i - d_1) \\ d_{i+1} + (d_{k+1} - d_{i+1}) \\ m_i x + c_i - \underline{M}_i \\ m_i x + c_i + \overline{M}_i \end{aligned}} \begin{aligned} (1 - y_i) &\le x && \forall i \in [k] \\ (1 - y_i) &\ge x && \forall i \in [k] \\ (1 - y_i) &\le z && \forall i \in [k] \\ (1 - y_i) &\ge z && \forall i \in [k] \end{aligned}$$

$$\sum_{i=1}^{k} y_i = 1$$

$$y \in \{0,1\}^k$$

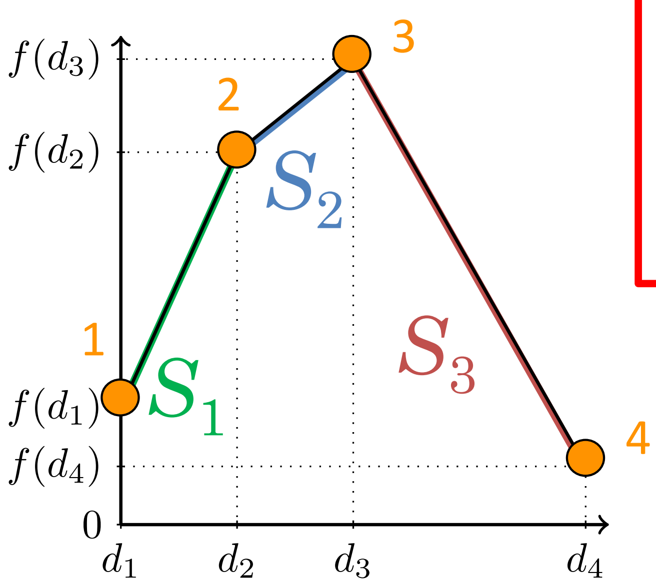
$$f(x) = \begin{cases} m_1 x + c_1 & x \in [d_1, d_2] \\ \vdots & \\ m_k x + c_k & x \in [d_k, d_{k+1}] \end{cases}$$

$$\underline{M}_i := \max_{j=1}^{k+1} \{ m_i d_j + c_i - f(d_j) \}$$

$$\overline{M}_i := \max_{j=1}^{k+1} \{ f(d_j) - m_i d_j - c_i \}$$

Example of an $\mathcal{V}$-based Formulation

$$S = \mathrm{gr}(f) = \bigcup_{i=1}^{k} \left\{ (x,z) \in \mathbb{R}^2 \; : \; \begin{array}{c} d_i \leq x \leq d_{i+1} \\ m_i x + c_i = z \end{array} \right\}$$

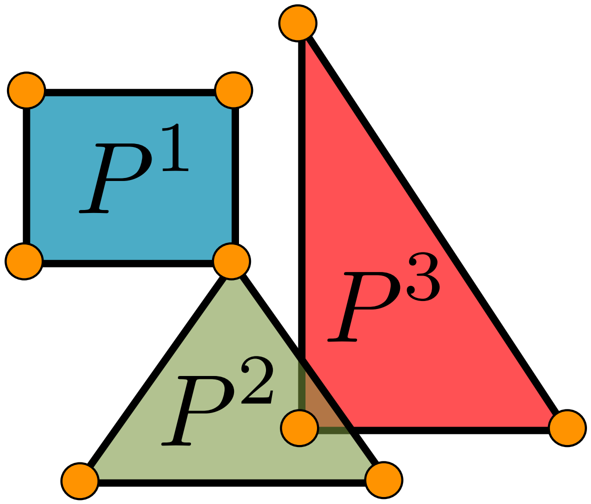


$$\sum_{i=1}^{k+1} \lambda_i d_i = x$$
$$\sum_{i=1}^{k+1} \lambda_i f(d_i) = z$$

$$\sum_{i=1}^{k+1} \lambda_i = 1$$
$$\lambda_1 \leq y_1$$
$$\lambda_i \leq y_{i-1} + y_i \quad \forall 2 \leq i \leq k$$
$$\lambda_{k+1} \leq y_k$$
$$\sum_{i=1}^{k} y_i = 1$$
$$y \in \{0,1\}^k$$

Data = vertices
in linear transformation
separate from formulation

Linear Transformation for $\mathcal{V}$-Formulation



$$V := \bigcup_{i=1}^{k} \operatorname{ext}\left(P^i\right)$$

$$\Delta^V := \left\{\lambda \in \mathbb{R}_+^V \ : \ \sum_{v \in V} \lambda_v = 1\right\}$$

$$Q^i := \left\{\lambda \in \Delta^V \ : \ \lambda_v \leq 0 \quad \forall v \notin \operatorname{ext}\left(P^i\right)\right\}$$

$$\boxed{x \in \bigcup_{i=1}^{k} P^i} \quad \longleftrightarrow \quad \boxed{x = \sum_{v \in V} v \lambda_v} \quad \boxed{\lambda \in \bigcup_{i=1}^{k} Q^i}$$

Dependent on specific data from polytopes

Purely Combinatorial

Constructing Ideal Formulations

- Add inequalities to fix non-ideal formulation

$\mathcal{H}$-based Formulation

$$Ax + Dy \leq d \quad \longrightarrow \quad (x, y) \in \operatorname{conv}\left(\left\{(x, y) \in \mathbb{R}^n \times \mathbb{Z}^k \,:\, Ax + Dy \leq b\right\}\right)$$

$$y \in \mathbb{Z}^k \qquad\qquad y \in \mathbb{Z}^k$$

$\mathcal{V}$-based Formulation

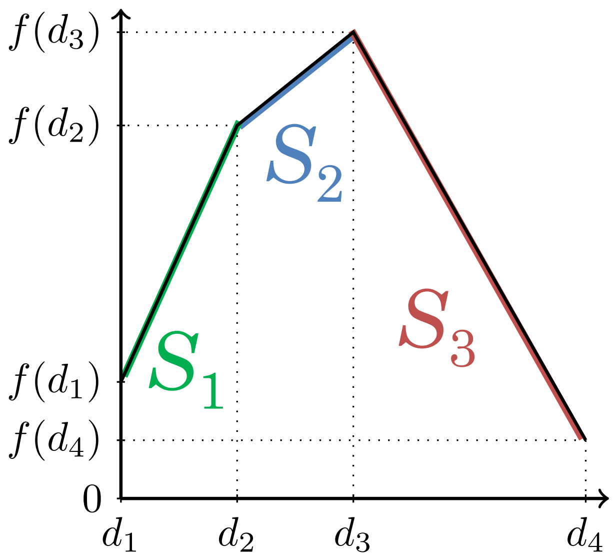
$$x = L\lambda \qquad\qquad x = L\lambda$$

$$\boxed{A\lambda + Dy \leq d \quad \longrightarrow \quad (\lambda, y) \in \operatorname{conv}\left(\left\{(\lambda, y) \in \mathbb{R}^V \times \mathbb{Z}^k \,:\, A\lambda + Dy \leq b\right\}\right)}$$

$$y \in \mathbb{Z}^k \qquad\qquad y \in \mathbb{Z}^k$$

Purely Combinatorial / Data Independent = Often Simpler

Fixing Big-M Formulation



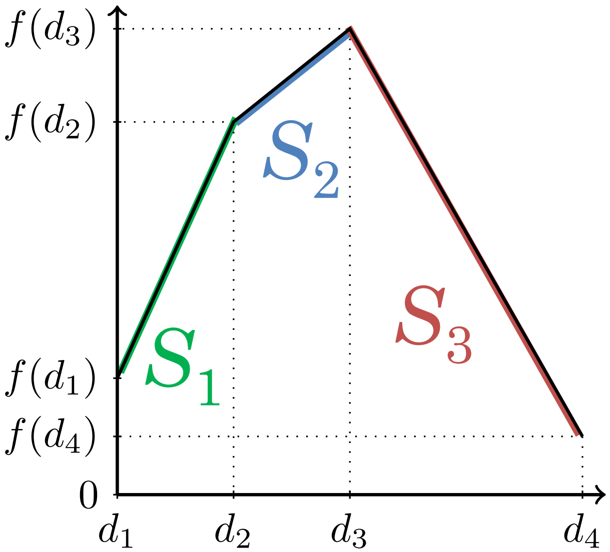
$$
\begin{aligned}
d_i - (d_i - d_1)(1 - y_i) &\leq x && \forall i \in [k] \\
d_{i+1} + (d_{k+1} - d_{i+1})(1 - y_i) &\geq x && \forall i \in [k] \\
m_i x + c_i - \underline{M}_i (1 - y_i) &\leq z && \forall i \in [k] \\
m_i x + c_i + \overline{M}_i (1 - y_i) &\geq z && \forall i \in [k] \\
\sum_{i=1}^{k} y_i &= 1 \\
y &\in \{0,1\}^k
\end{aligned}
$$

- How to construct convex hull of infinite # of feasible solutions to Big-M:
  - Take convex hull of

$$
\boldsymbol{e}^j = \begin{cases} 1 & \text{if } i = j \\ 0 & o.w. \end{cases}
$$

$$
\bigcup_{i=1}^{k} \left\{ (x, z, y) \in \mathbb{R}^{k+2} \; : \; y = \boldsymbol{e}^i, \quad (x, z) \in \{(d_i, f(d_i)), (d_{i+1}, f(d_{i+1}))\} \right\}
$$

Fixing CC Formulation



$$\sum_{i=1}^{k+1} \lambda_i d_i = x \qquad \sum_{i=1}^{k} y_i = 1$$

$$\sum_{i=1}^{k+1} \lambda_i f(d_i) = z \qquad y \in \{0,1\}^k$$

$$\sum_{i=1}^{k+1} \lambda_i = 1$$

$$\lambda_1 \leq y_1$$

$$\lambda_i \leq y_{i-1} + y_i \quad \forall 2 \leq i \leq k$$

$$\lambda_{k+1} \leq y_k$$

- Take convex hull of

$$\bigcup_{i=1}^{k} \left\{ (x,z,y,\lambda) \in \mathbb{R}^{2k+3} \;:\; \begin{array}{c} y = \boldsymbol{e}^i, \\ (\lambda, x, z) \in \left\{ \left(\boldsymbol{e}^i, d_i, f(d_i)\right), \left(\boldsymbol{e}^{i+1}, d_{i+1}, f(d_{i+1})\right) \right\} \end{array} \right\}$$

Extended Formulation for Unions of Polytopes

- $P^i := \{x \in \mathbb{R}^n \,:\, A^i x \leq b^i\}$ rational polytopes and

  $S = \bigcup_{i=1}^k P^i$

- then an ideal formulation for S is

$$A^i x^i \leq b^i y_i \quad \forall i \in [k] \qquad \sum_{i=1}^k y_i = 1$$

$$\sum_{i=1}^k x^i = x \qquad\qquad y_i \geq 0 \quad \forall i \in [k]$$

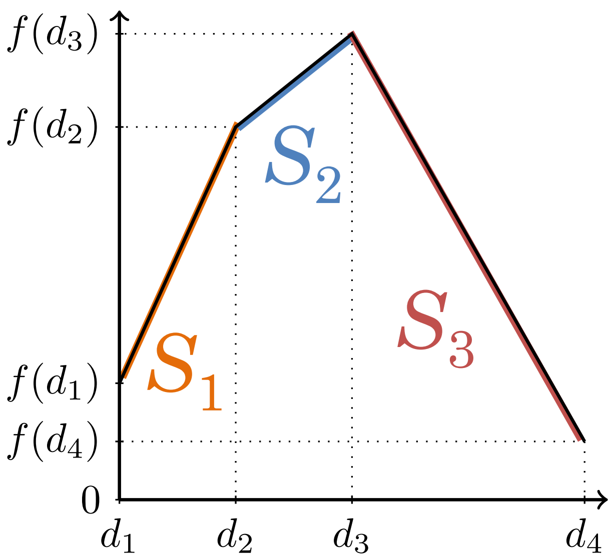
$$y \in \mathbb{Z}^k$$

$$x^i \in \mathbb{R}^n \quad \forall i \in [k]$$

- Exercise: Why is this formulation valid?

Ideal Formulation for PWL Functions

$$S = \mathrm{gr}(f) = \bigcup_{i=1}^{k} \left\{ (x,z) \in \mathbb{R}^2 \;:\; \begin{array}{r} d_i \le x \le d_{i+1} \\ m_i x + c_i = z \end{array} \right\}$$



MC Formulation:

$$d_i y_i \le x^i \le d_{i+1} y_i \qquad \forall i \in [k]$$

$$m_i x^i + c_i y_i = z^i \qquad \forall i \in [k]$$

$$\sum_{i=1}^{k} x^i = x$$

$$\sum_{i=1}^{k} z^i = z$$

$$\sum_{i=1}^{k} y_i = 1$$

$$y \in \{0,1\}^k$$

Extended Formulations for $x \in \bigcup_{i=1}^{k} P^i$

$$P^i = \left\{ x \in \mathbb{R}^n \ : \ A^i x \leq b^i \right\}$$

$$\begin{aligned} A^i x^i &\leq b^i y_i \quad \forall i \in [k] \\ \sum_{i=1}^{k} x^i &= x \\ \sum_{i=1}^{k} y_i &= 1 \\ y &\in \{0,1\}^k \end{aligned}$$

$\mathcal{H}$-formulation

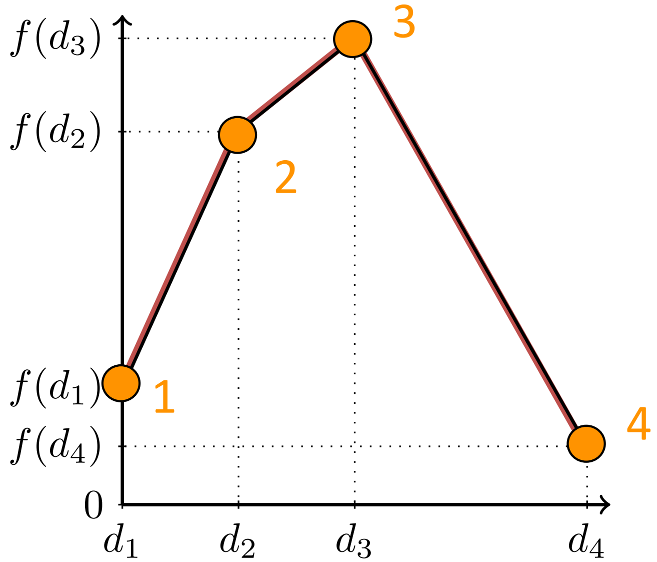
$$\begin{aligned} \sum_{i=1}^{k} \sum_{v \in \operatorname{ext}(P^i)} v \lambda_v^i &= x \\ \sum_{v \in \operatorname{ext}(P^i)} \lambda_v^i &= y_i \quad \forall i \in [k] \\ \sum_{i=1}^{k} y_i &= 1 \\ \lambda^i &\in \mathbb{R}_+^{\operatorname{ext}(P_i)} \\ y &\in \{0,1\}^k \end{aligned}$$

$\mathcal{V}$-formulation

- Both formulations are ideal and use copies of variables

Extended $\mathcal{V}$-formulation for PWL Functions

$$S = \mathrm{gr}(f) = \bigcup_{i=1}^{k} \left\{ (x,z) \in \mathbb{R}^2 \; : \; \begin{array}{r} d_i \leq x \leq d_{i+1} \\ m_i x + c_i = z \end{array} \right\}$$



DCC Formulation:

$$\sum_{i=1}^{k} \lambda_i^i d_i + \lambda_{i+1}^i d_{i+1} = x$$

$$\sum_{i=1}^{k} \lambda_i^i f(d_i) + \lambda_{i+1}^i f(d_{i+1}) = z$$

$$\lambda_i^i + \lambda_{i+1}^i = y_i \qquad \forall i \in [k]$$

$$\sum_{i=1}^{k} y_i = 1$$

$$y \in \{0,1\}^k$$

Much more to come…

Exercise: Small ideal formulation for "Parity"

- Write a polynomial-sized ideal MIP formulation for "Parity":
  - All vectors in $\{0,1\}^n$ with an even number of ones.

Exercise: Small ideal formulation for "Parity"

- Write a polynomial-sized ideal MIP formulation for "Parity":
  - All vectors in $\{0,1\}^n$ with an even number of ones.

$$
\begin{aligned}
x_i = \sum_{k \in N^{even}} x_i^k \qquad & \forall i \in N \\
\sum_{i \in N} x_i^k = k\lambda_k \qquad & \forall k \in N^{even} \\
\sum_{k \in N^{even}} \lambda_k = 1 \qquad & \\
x_i^k \le \lambda_k \qquad & \forall i \in N, k \in N^{even} \\
x_i^k \ge 0 \qquad & \forall i \in N, k \in N^{even} \\
\lambda_k \ge 0 \qquad & \forall k \in N^{even}
\end{aligned}
$$

$$
\begin{aligned}
N &= \{1, \ldots, n\} \\
N^{even} &= \text{even elements of } N
\end{aligned}
$$

More On Formulations

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Mixed Integer Linear Programming Formulation Techniques*

Juan Pablo Vielma†

Better Multi-Big-M $\mathcal{H}$-based Formulation

$$P^i = \{x \in \mathbb{R}^n \,:\, A^i x \le b^i\} \to P^i = \{x \in \mathbb{R}^n \,:\, Dx \le d^i\}$$

$$D = \begin{bmatrix} A^1 \\ \vdots \\ A^k \end{bmatrix}, \quad d^i \text{ appropriately constructed (Big-Ms)}$$

$$Dx \le \sum_{i=1}^k d^i y_i$$

$$\sum_{i=1}^k y_i = 1$$

$$y \in \{0,1\}^k$$

- Usually not ideal, but often stronger than standard Big-M

General $\mathcal{V}$-based CC-Like Formulation

$$V := \bigcup_{i=1}^{k} \operatorname{ext}\left(P^i\right)$$

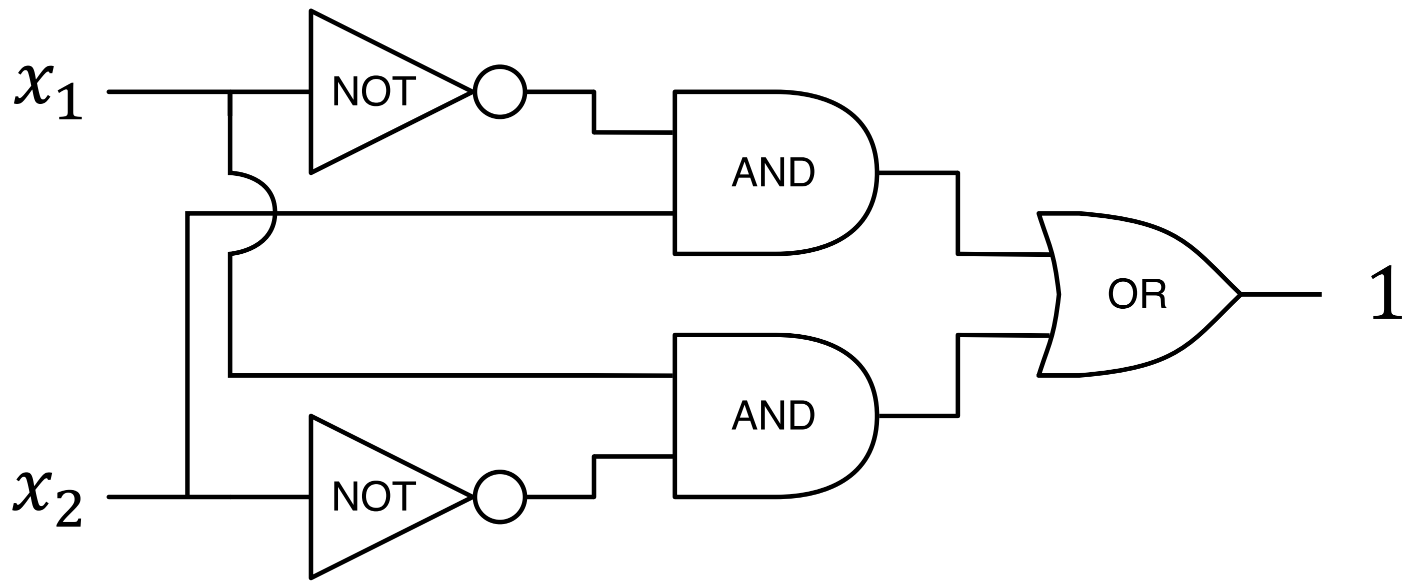
$$\begin{aligned} \sum_{v \in V} v\lambda_v &= x \\ \sum_{v \in V} \lambda_v &= 1 \\ \lambda_v &\leq \sum_{i : v \in \operatorname{ext}(P_i)} y_i \\ \sum_{i=1}^{k} y_i &= 1 \end{aligned}$$

$$y \in \{0,1\}^k, \quad \lambda \in \mathbb{R}_+^V$$

- Usually not ideal, but “relatively” strong

Universal Polynomial-Size Extended Formulation

- Combinatorial optimization problem in NP:
  - Poly-time algorithm to recognize feasible $x \in \{0,1\}^n$
  - Fixed n: poly-size circuit to recognize feasible $x \in \{0,1\}^n$

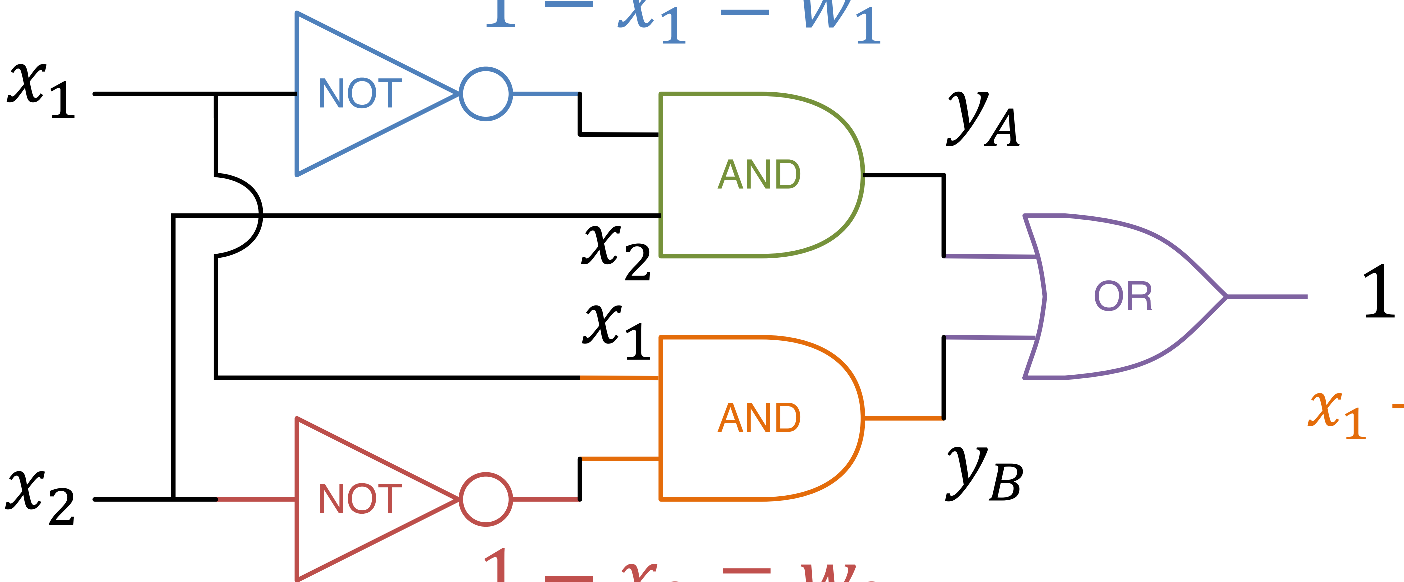


Circuit to recognize parity for $n = 2$

Universal Polynomial-Size Extended Formulation

- From circuit to MIP:

$$1 - x_1 = w_1$$



$$1 - x_2 = w_2$$

$$x_1, x_2, w_1, w_2, y_A, y_B \in \{0,1\}$$

$$w_1 + x_2 \leq 1 + y_A$$
$$y_A \leq w_1$$
$$y_A \leq x_2$$
$$0 \leq y_A$$

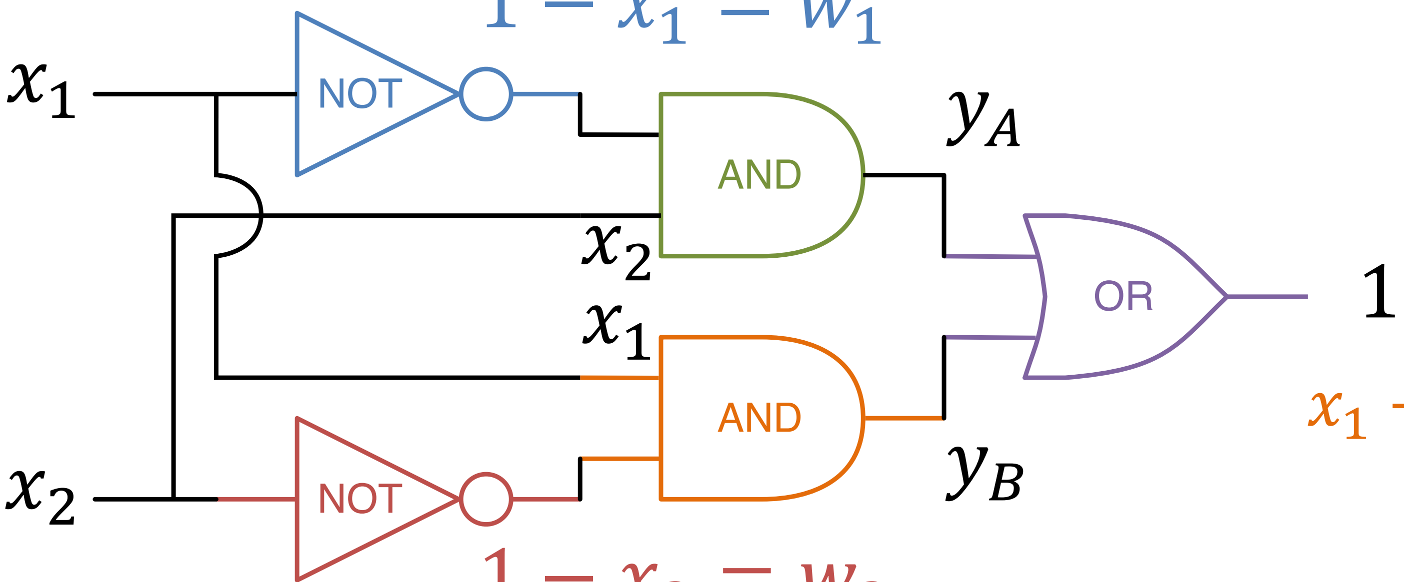
$$1 \leq y_A + y_B$$

$$x_1 + w_2 \leq 1 + y_B$$
$$y_B \leq x_1$$
$$y_B \leq w_2$$
$$0 \leq y_B$$

Is Universal Formulation Ideal?

- From circuit to MIP:

$$1 - x_1 = w_1$$



$$1 - x_2 = w_2$$

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$$w_1 + x_2 \leq 1 + y_A$$
$$y_A \leq w_1$$
$$y_A \leq x_2$$
$$0 \leq y_A$$

$$1 \leq y_A + y_B$$

$$x_1 + w_2 \leq 1 + y_B$$
$$y_B \leq x_1$$
$$y_B \leq w_2$$
$$0 \leq y_B$$

- All 5 pieces of the formulation are ideal!

Universal 2-Parity Formulation is Not Ideal

```
julia> model = Model();  @variable(model,x[1:2],Bin);
@variable(model,w[1:2],Bin); @variable(model,y[["A","B"]],Bin);
julia> @constraint(model,[i=1:2], 1-x[i]==w[i]);
julia> @constraint(model,w[1]+x[2]<=1+y["A"]);
julia> @constraint(model,x[1]+w[2]<=1+y["B"]);
julia> @constraint(model,y["A"]<=w[1]);
julia> @constraint(model,y["A"]<=x[2]);
julia> @constraint(model,y["B"]<=w[2]);
julia> @constraint(model,y["B"]<=x[1]);
julia> poly = polyhedron(model, CDDLibrary(:exact));
julia> SimpleVRepresentation(poly); removevredundancy!(poly);
SimpleVRepresentation(poly);
V-representation
begin
 6 7 rational
 1 1//1 0//1 0//1 1//1 0//1 1//1
 1 1//2 1//2 1//2 1//2 1//2 0//1
 1 1//2 1//2 1//2 1//2 0//1 1//2
 1 0//1 0//1 1//1 1//1 0//1 0//1
 1 0//1 1//1 1//1 0//1 1//1 0//1
 1 1//1 1//1 0//1 0//1 0//1 0//1
end
```

Intersecting formulations does not necessarily preserve their strength