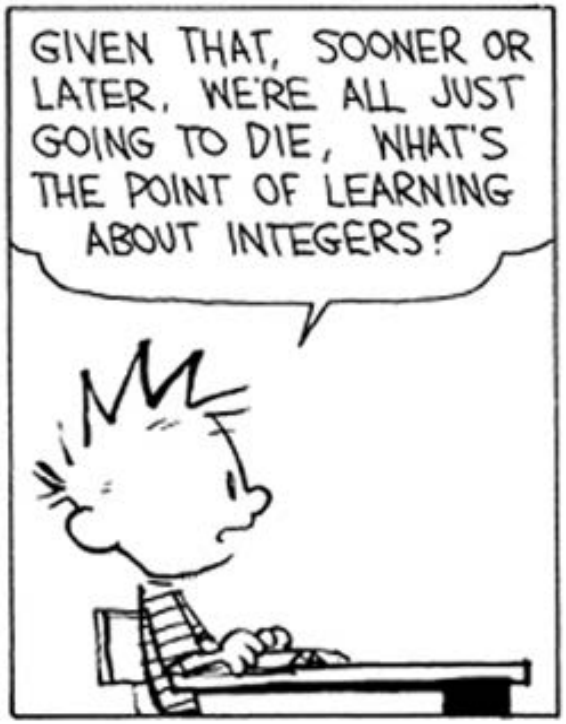
Advanced Mixed Integer Programming Formulation Techniques

MIP Representability



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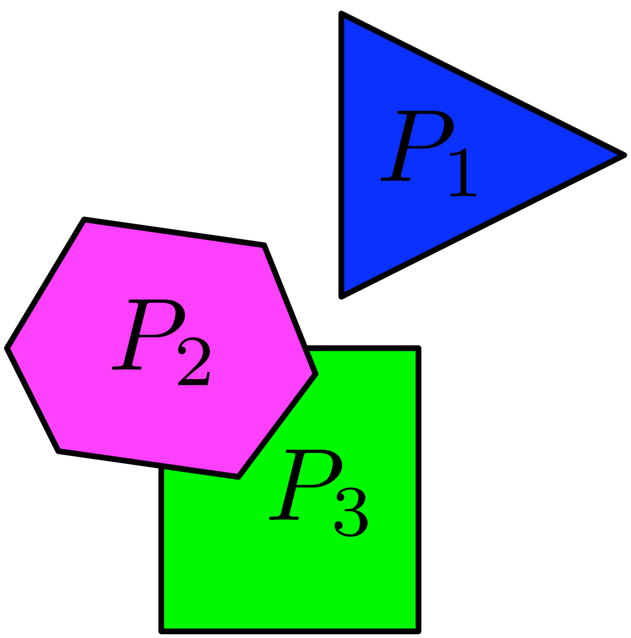
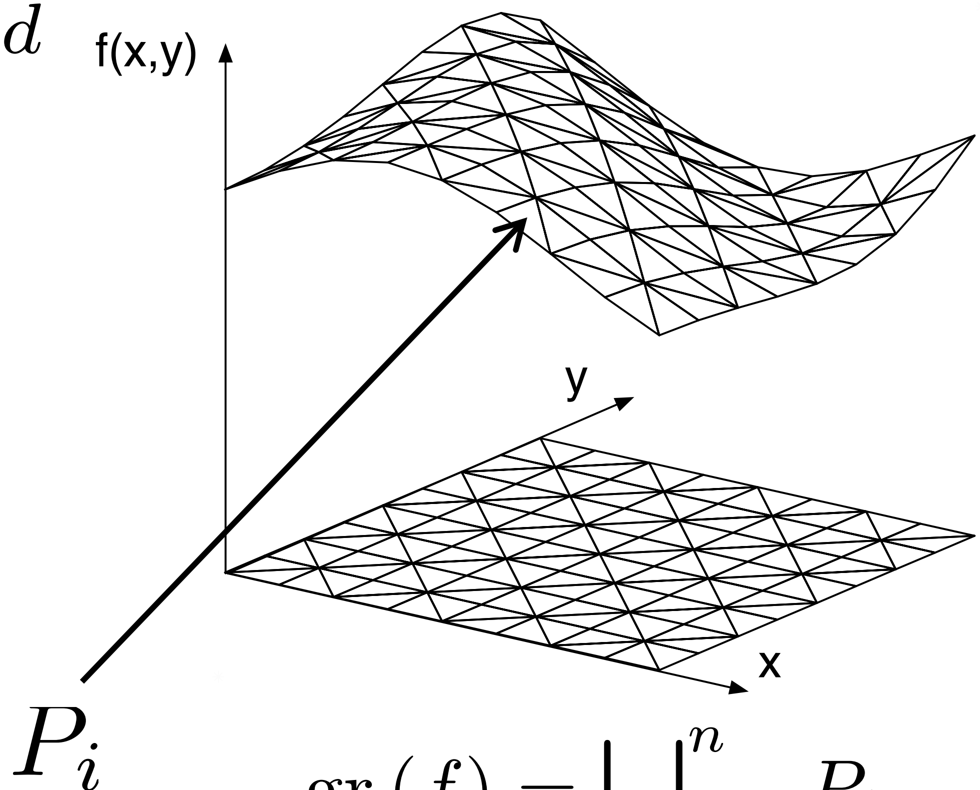
Mixed 0-1 Formulations

- Modeling Finite Alternatives = Unions of Polyhedra
  - Bounded or unbounded polyhedra, but bounded for now

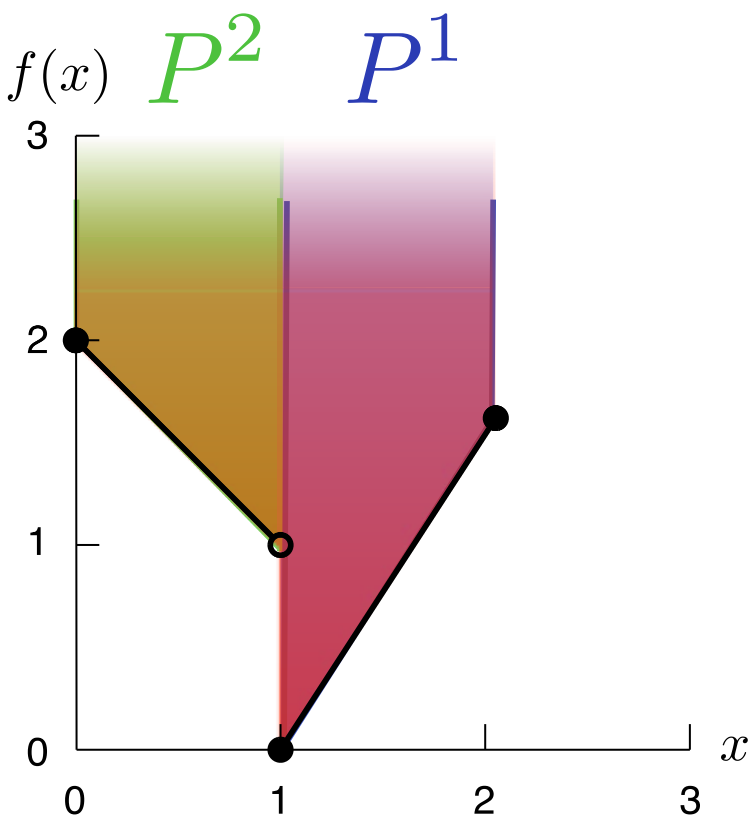
$$x \in \bigcup_{i=1}^{n} P_i \subseteq \mathbb{R}^d$$





$$\operatorname{gr}(f) = \bigcup_{i=1}^{n} P_i$$

What About Discontinuous Piecewise Linear



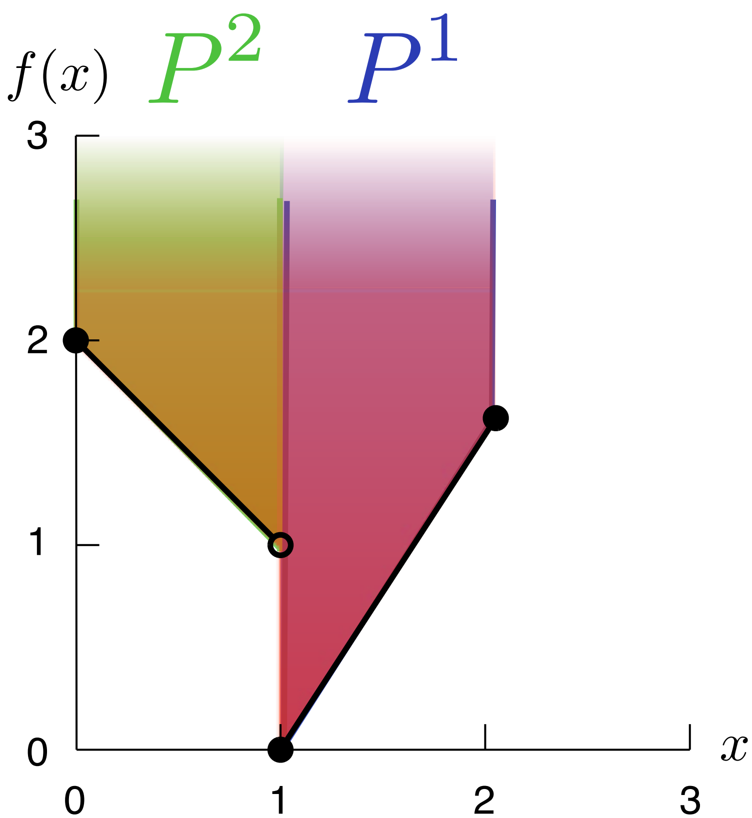
$$f(x) = \begin{cases} -x+2 & x \in [0,1) \\ x-1 & x \in [1,2] \end{cases}$$

$$\text{epi}\,(f) = \{(x,z) \,:\, f(x) \leq z\}$$

$$\text{epi}\,(f) = P^1 \cup P^2$$

- Works for piecewise linear functions:
  - Bounded domain
  - Lower semicontinuous = closed epigraph
  - Epigraph is union of **unbounded** polyhedra

What About Discontinuous Piecewise Linear



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$$P^1_\infty = P^2_\infty = \{(x,z) \in \mathbb{R}^2 \,:\, x = 0, \quad z \geq 0\}$$

- Works for piecewise linear functions:
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Extended Formulation for Unions of Polyhedra

- $P^i := \{x \in \mathbb{R}^n \ : \ A^i x \leq b^i\}$ rational polyhedra and

  $S = \bigcup_{i=1}^k P^i$

- If $P^i_\infty = P^j_\infty \quad \forall i, j \in [k]$
- then an ideal formulation for S is

$$A^i x^i \leq b^i y_i$$

$$\sum_{i=1}^k x^i = x$$

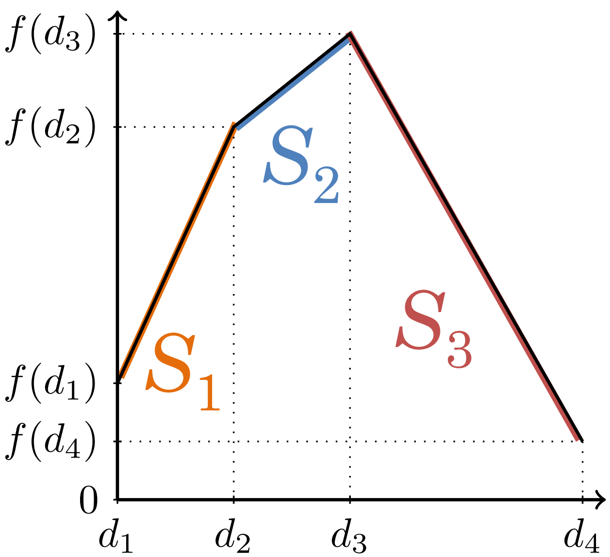
$$x^i \in \mathbb{R}^n \qquad \forall i \in [n]$$

$$\sum_{i=1}^k y_i = 1$$

$$y \in \mathbb{Z}^n$$

Extended Formulation for PWL Functions

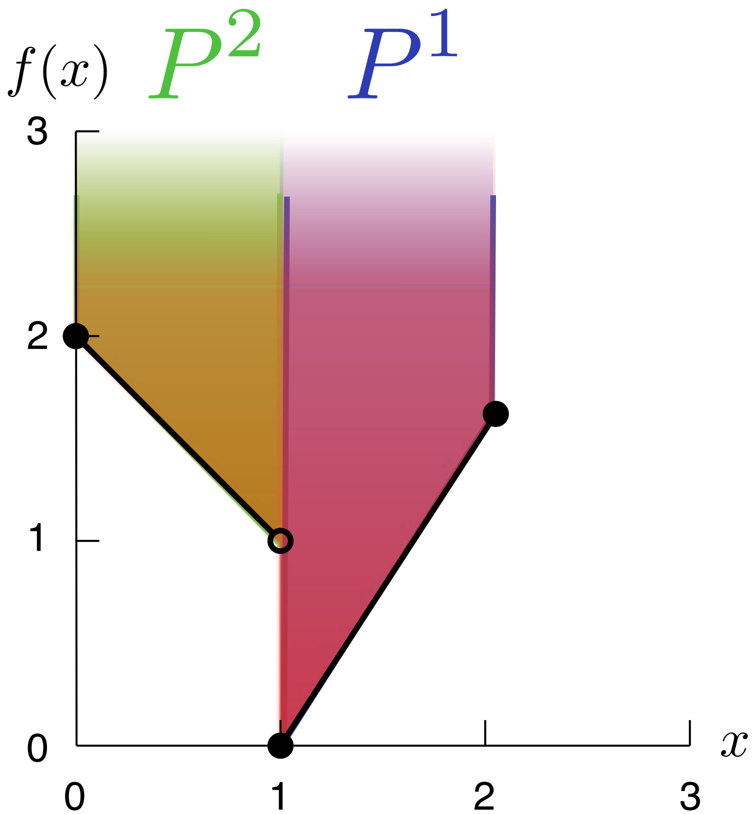
$$S = \mathrm{gr}\,(f) = \bigcup_{i=1}^{k} \left\{ (x, z) \in \mathbb{R}^2 \;:\; \begin{array}{r} d_i \leq x \leq d_{i+1} \\ m_i x + c_i = z \end{array} \right\}$$



MC Formulation:

$$\begin{aligned} d_i y_i \leq x^i &\leq d_{i+1} y_i \quad && \forall i \in [k] \\ m_i x^i + c_i y_i &= z^i && \forall i \in [k] \\ \sum_{i=1}^{k} x^i &= x \\ \sum_{i=1}^{k} z^i &= z \\ \sum_{i=1}^{k} y_i &= 1 \\ y &\in \{0, 1\}^k \end{aligned}$$

Example: Discontinuous Piecewise Linear



$$f(x) = \begin{cases} -x+2 & x \in [0,1) \\ x-1 & x \in [1,2] \end{cases}$$

$$d_i y_i \leq x^i \leq d_{i+1} y_i \qquad \forall i \in [k]$$

$$m_i x^i + c_i y_i \leq z^i \qquad \forall i \in [k]$$

$$\sum\nolimits_{i=1}^{k} x^i = x$$

$$\sum\nolimits_{i=1}^{k} z^i = z$$

$$\sum\nolimits_{i=1}^{k} y_i = 1$$

$$y \in \{0,1\}^k$$

$$\operatorname{epi}(f) = \{(x,z) \,:\, f(x) \leq z\}$$

$$\operatorname{epi}(f) = P^1 \cup P^2$$

$$P^1_\infty = P^2_\infty = \{(x,z) \in \mathbb{R}^2 \,:\, x = 0, \quad z \geq 0\}$$

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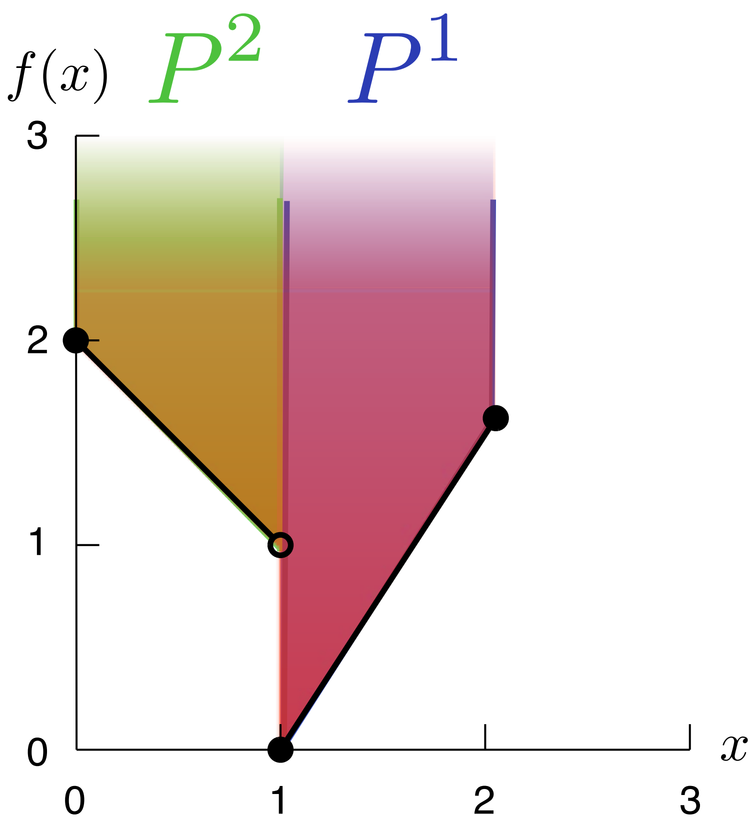

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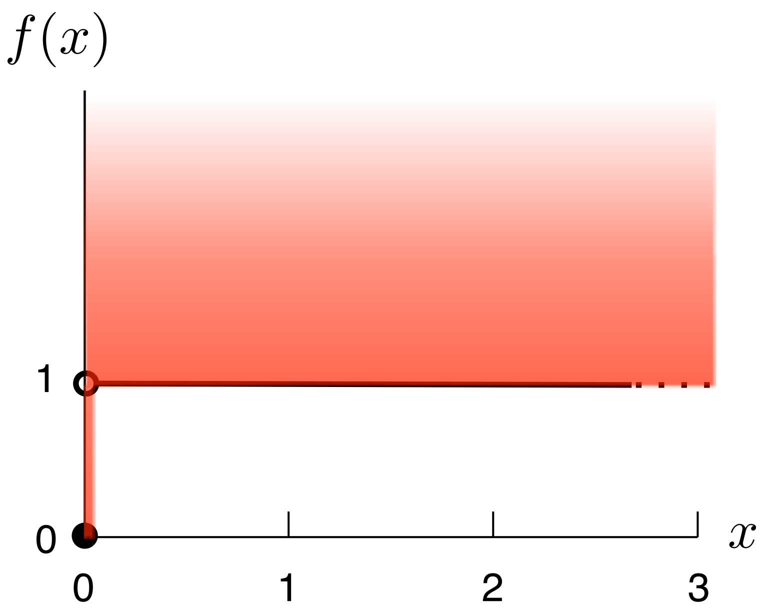
- Works for piecewise linear functions:
  - Bounded domain
  - Lower semicontinuous = closed epigraph

What if Recession Cones are Different?



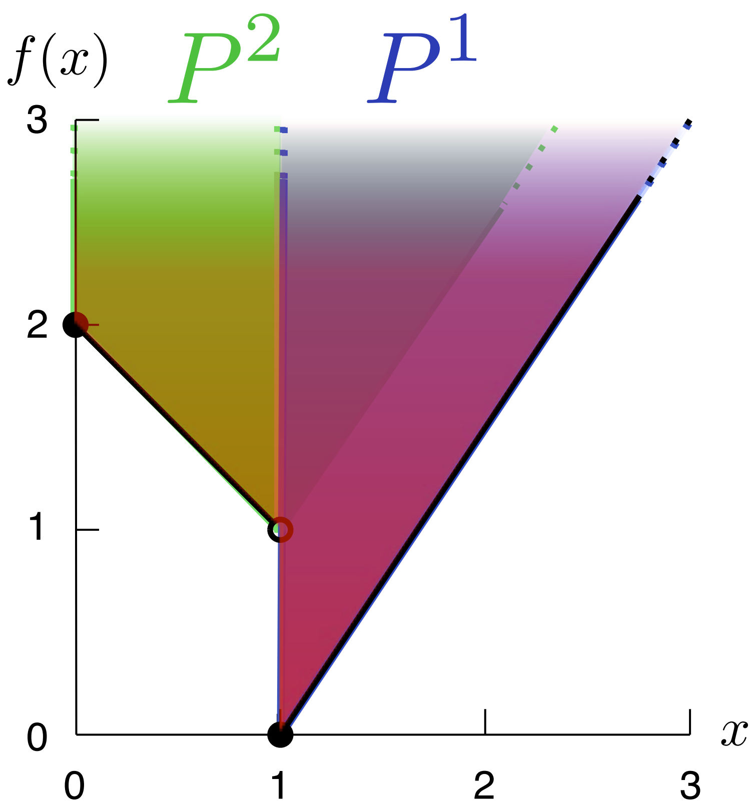
$$f : [0, \infty) \to \mathbb{R}$$

$$f(x) = \begin{cases} 0 & x = 0 \\ 1 & x > 0 \end{cases}$$

$$\operatorname{epi}(f) = \{(x, z) \,:\, f(x) \leq z\}$$

- No linear MIP formulation:
  - 0-1 or general integer
  - More later

Bounded MIP Representability May be Hidden



$$f(x) = \begin{cases} -x+2 & x \in [0,1) \\ x-1 & x \in [1,\infty) \end{cases}$$

$$\mathrm{epi}(f) = \{(x,z) \,:\, f(x) \le z\}$$

$$\mathrm{epi}(f) = P^1 \cup P^2$$

$$P^2_\infty = \cancel{\{(x,z) \in \mathbb{R}^2 \,:\, x = 0, \quad z \ge 0\}}$$

$$\not\Vdash \quad \{(x,z) \in \mathbb{R}^2 \,:\, x \ge 0, \quad z \ge x\}$$

$$P^1_\infty = \{(x,z) \in \mathbb{R}^2 \,:\, x \ge 0, \quad z \ge x\}$$

$$P^2 \to P^2 \cup P^1_\infty$$

- Obtaining a MIP formulation may require redundancy

Finite, Binary or 0-1 Linear MIP Representability

- Let
  - $S \subseteq \mathbb{R}^n$,
  - $n_1 + n_2 = n,\ p_1 + p_2 = p,\ A \in \mathbb{Q}^{m \times n},\ D \in \mathbb{Q}^{m \times p},\ b \in \mathbb{Q}^m$
  - $P := \{(x, w) \in \mathbb{R}^n \times \mathbb{R}^p \ :\ Ax + Dw \leq b\}$
  - $P_I := P \cap (\mathbb{R}^{n_1} \times \mathbb{Z}^{n_2} \times \mathbb{R}^{p_1} \times \mathbb{Z}^{p_2})$
  - P_I is a MIP formulation of S iff $S = \mathrm{Proj}_x (P_I)$
- If $\exists M > 0$ s.t. $P \subseteq \mathbb{R}^{n_1} \times [-M, M]^{n_2} \times \mathbb{R}^{p_1} \times [-M, M]^{p_2}$
  - Then S is a finite union of rational polyhedra with the same recession cone
- S is finite/binary linear MIP representable if it has a linear MIP formulation
- Sometimes denoted “Bounded” MIP representable

Mixed-Integer Formulations: General Integer

Beyond Unions of Polyhedra

- $P^i := \left\{x \in \mathbb{R}^n \ : \ A^i x \leq b^i\right\}$ rational polyhedra and $S = \bigcup_{i=1}^k P^i \cap \boxed{\left(\mathbb{R}^{n_1} \times \mathbb{Z}^{n_2}\right)}$
- If $P^i_\infty = P^j_\infty \quad \forall i, j \in [k]$
- then a formulation for S is

$$A^i x^i \leq b^i y_i \qquad \sum_{i=1}^k y_i = 1$$

$$\sum_{i=1}^k x^i = x \qquad y \in \mathbb{Z}^n$$

$$x^i \in \mathbb{R}^n \quad \forall i \in [n] \qquad x \in \mathbb{R}^{n_1} \times \mathbb{Z}^{n_2}$$

- Formulation is ideal if

$$\operatorname{ext}\left(P^i\right) \subseteq \mathbb{R}^{n_1} \times \mathbb{Z}^{n_2}$$

General Linear MIP Representability

- $P^i := \{x \in \mathbb{R}^n \ : \ A^i x \leq b^i\}$ rational polyhedra and

  $S = \bigcup_{i=1}^{k} P^i \cap (\mathbb{R}^{n_1} \times \mathbb{Z}^{n_2})$

- If $P^i_\infty = P^j_\infty \quad \forall i, j \in [k]$
- What about?
  - $S = \{0\} \cup ([2, \infty) \cap \mathbb{Z})$

    $= \{0, 2, 3, 4, \ldots\}$

  - MIP formulation:

$$x = 2y_1 + 3y_2, \quad y_1, y_2 \geq 0, \quad y \in \mathbb{Z}^2$$

(Linear) Mixed Integer Programming Formulation

- Let
  - $S \subseteq \mathbb{R}^n$,
  - $n_1 + n_2 = n,\ p_1 + p_2 = p,\ A \in \mathbb{Q}^{m \times n},\ D \in \mathbb{Q}^{m \times p},\ b \in \mathbb{Q}^m$
  - $P := \{(x, w) \in \mathbb{R}^n \times \mathbb{R}^p \ :\ Ax + Dw \leq b\}$
  - $P_I := P \cap \left(\mathbb{R}^{n_1} \times \mathbb{Z}^{n_2} \times \mathbb{R}^{p_1} \times \mathbb{Z}^{p_2}\right)$
- P_I is a MIP formulation of S iff

$$S = \operatorname{Proj}_x \left(P_I\right)$$

- S is linear MIP representable if it has a linear MIP formulation

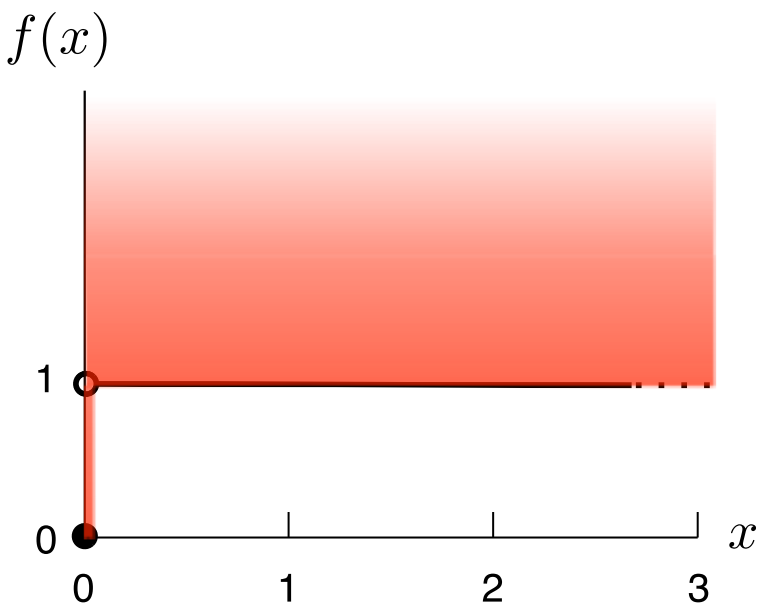
General Linear MIP Representability

- $S \subseteq \mathbb{R}^n$ is MIP representable if and only if there exist rational polyhedra $P_i \subseteq \mathbb{R}^n$ with the same recession cones and $\{r^j\}_{j=1}^{L} \subseteq \mathbb{Z}^n$ such that

  – $S = \bigcup_{i=1}^{K} P_i + \left\{\sum_{j=1}^{L} \lambda_j r^j \,:\, \lambda \in \mathbb{Z}_+^L\right\}$

- Can always assume the P_i are polytopes

What if Recession Cones are Different?



$$f : [0, \infty) \to \mathbb{R}$$

$$f(x) = \begin{cases} 0 & x = 0 \\ 1 & x > 0 \end{cases}$$

$$\operatorname{epi}(f) = \{(x, z) \,:\, f(x) \leq z\}$$

$$\operatorname{conv}(\operatorname{epi}(f)) = \left\{(x, z) \in \mathbb{R}^2 \,:\, x \geq 0, \quad z > 0\right\} \cup \{(0, 0)\}$$

Not MIP representable: bounded or general

- Useful lemma:
  - If S has a MIP formulation then S is closed and $\operatorname{conv}(S)$ is a rational polyhedron

Rational Polyhedral Convex Hull is Not Sufficient

- $$S = \left\{ x \in \mathbb{Q}^n \; : \; x_n = \max_{i \in [n-1]} x_i, \quad x_j \geq 0 \quad \forall j \in [n] \right\}$$
$$= \bigcup_{i=1}^{n-1} \{ x \in \mathbb{Q}^n \; : \; x_n = x_i, \quad x_n \geq x_j \geq 0 \quad \forall j \in [n] \}$$
- $$\operatorname{conv}(S) = \left\{ x \in \mathbb{Q}^n \; : \; x_n \leq \sum_{i=1}^{n-1} x_i, \quad x_n \geq x_j \geq 0 \quad \forall j \in [n] \right\}$$
- S does not have a MIP formulation

Rationality is Crucial

- $\operatorname{conv}(\{x \in \mathbb{Z}^2 : x_2 - \sqrt{2}\, x_1 \leq 0, x_2 \geq 0\})$ is not even closed

(Convex) MICP Formulations and Representability

- A set $S \subseteq \mathbb{R}^n$ is MICP representable (MICPR) if it has an MICP formulation:
  - A closed convex set $M \subseteq \mathbb{R}^{n+p+d}$
  - auxiliary continuous variables $y \in \mathbb{R}^p$
  - auxiliary integer variables $z \in \mathbb{Z}^d$

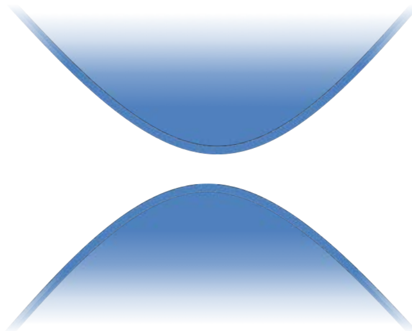
$$x \in S \quad \Leftrightarrow \quad \begin{array}{c} \exists (y, z) \in \mathbb{R}^p \times \mathbb{Z}^d \text{ s.t.} \\ (x, y, z) \in M \end{array}$$

or equivalently

$$S = \operatorname{proj}_x \left(M \cap \left(\mathbb{R}^{n+p} \times \mathbb{Z}^d \right) \right)$$

What Sets are MICP Representable (MICPR) ?

Two sheet hyperbola?



$$\left\{x \in \mathbb{R}^2 \;:\; 1 + x_1^2 \leq x_2^2\right\}$$

Spherical shell?

$$\left\{x \in \mathbb{R}^2 \;:\; 1 \leq \|x\| \leq 2\right\}$$

- Discrete subsets of the real line or natural numbers:
  - Dense discrete set? $\left\{\sqrt{2}x - \left\lfloor \sqrt{2}x \right\rfloor \;:\; x \in \mathbb{N}\right\} \subseteq [0,1]$
  - Set of prime numbers?

"God made the integers, all else is the work of man"

- Leopold Kronecker

A Simple Lemma for non-MICP Representability

- Obstruction for MICP representability of S :

$$\text{infinite } R \subseteq S \quad \text{s.t.} \quad \frac{u+v}{2} \notin S \quad \forall u, v \in R,\ u \neq v$$

Proof: Assume for contradiction there exists M such that:

$$S = \operatorname{proj}_x \left(M \cap \left(\mathbb{R}^{n+p} \times \mathbb{Z}^d \right) \right)$$

$$\begin{array}{l} (u, y_u, z_u) \in M \\ (v, y_v, z_v) \in M \end{array} \quad \Rightarrow \quad \frac{z_u + z_v}{2} \notin \mathbb{Z}^d$$

$$z_u \equiv z_v \pmod 2 \text{ component-wise} \quad \Rightarrow \quad \frac{z_u + z_v}{2} \in \mathbb{Z}^d$$

$$\text{component-wise parity classes} = 2^d < |R| = \infty \qquad \Rightarrow\!\Leftarrow$$

A Simple Lemma for non-MICP Representability

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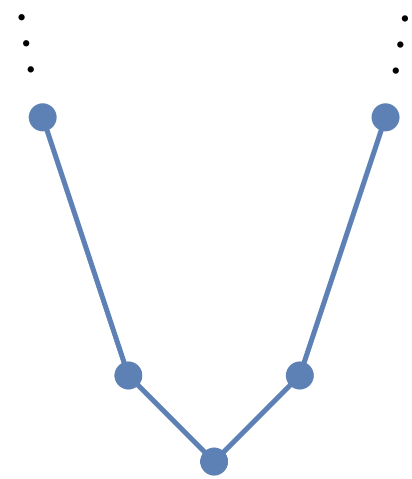
$$\text{infinite } R \subseteq S \quad \text{s.t.} \quad \frac{u+v}{2} \notin S \quad \forall u, v \in R,\, u \neq v$$

✗ **Spherical shell** $\{x \in \mathbb{R}^2 : 1 \leq \|x\| \leq 2\}$

✗ **Set of prime numbers**

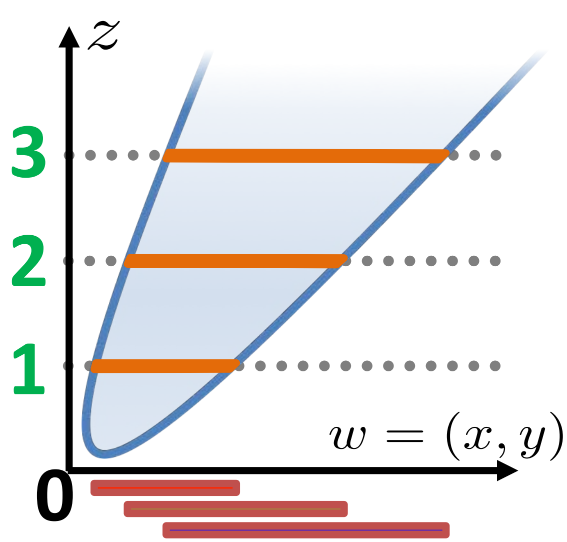
- Does have **non-convex** polynomial MIP



✗ Set of Matrices of rank at most k

✗ Piecewise linear interpolation of x^2 at all integers

MICPR = Convex Sets Indexed by Integers in Convex



$$S = \bigcup_{z \in I \cap \mathbb{Z}^d} \operatorname{proj}_x (B_z)$$

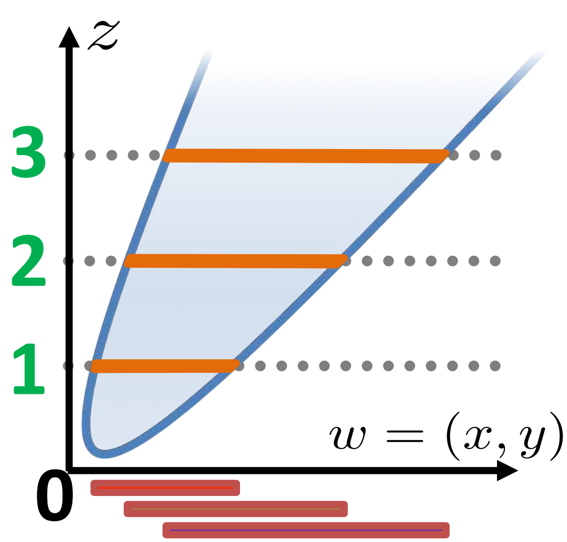
$$I = \operatorname{proj}_z (M) \quad \text{convex}$$

- For rational polyhedral M (Jeroslow and Lowe '84):
  - $S = \bigcup_{i=1}^{k} P_i + \left\{ \sum_{i=1}^{t} \lambda_i r^i \ : \ \lambda \in \mathbb{Z}_+^t \right\}$

1: Rational polyhedra with the same recession cone

2: Finite # of shapes + periodic translations

MICPR = Convex Sets Indexed by Integers in Convex



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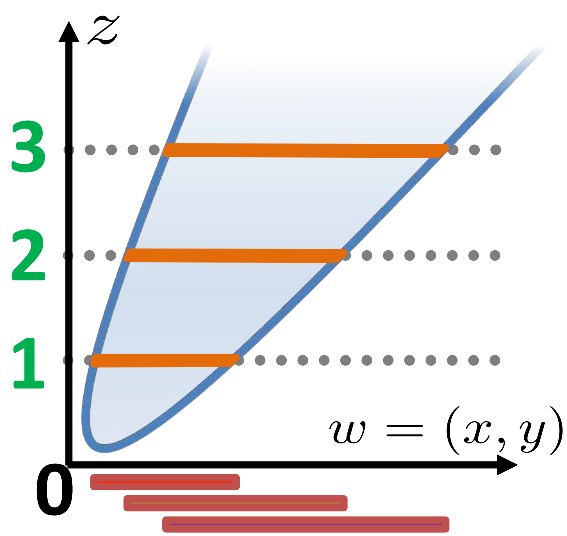
- Extensions $S = \bigcup_{i=1}^{k} P_i + \left\{ \sum_{i=1}^{t} \lambda_i r^i \ : \ \lambda \in \mathbb{Z}_+^t \right\}$
  - $M = \{x \in \mathbb{Z}^2 \ : \ x_1 \cdot x_2 \geq \alpha\} \implies P_i$ = points (Dey & Moran ’13)
  - M = Rational Polyhedron ∩ “Rational” Ellipsoidal Cylinder $\implies$ P_i = Rational Ellipsoid ∩ Polytope (Del Pia & Poskin ‘16)
  - M = Compact Convex + Rational Polyhedron Cone $\implies$ P_i = Compact Convex (Lubin, Zadik & V. 17’)

MICPR = Convex Sets Indexed by Integers in Convex



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1: Rational polyhedra with the same recession cone

2: Finite # of shapes + periodic translations

Extra from MICP 1: Non-Polyhedral Unions

$$S = \bigcup_{z \in I \cap \mathbb{Z}^d} \mathrm{proj}_x \left(B_z \right)$$

1. Unions of **Non-Polyhedral sets**

Plus Projection:

2. Unions of non-closed sets
3. Unions of convex sets with different recession cones

Two sheet hyperbola?

$$\left\{ x \in \mathbb{R}^2 \ : \ 1 + x_1^2 \leq x_2^2 \right\}$$

$$\left\{ \left\{ (x, t) \ : \ x \in S_i, \quad \|x\|_2^2 \leq t \right\} \right\}_{i=1}^{k} \text{ have the same recession cone}$$

Extra from MICP 2: Non-Polyhedral I

- An **infinite set** S is **periodic** if and only if:

$$\exists r \in \mathbb{R}^n \quad \forall \lambda \in \mathbb{Z}_+,\ x \in S \quad x + \lambda r \in S$$

- Non-periodic MICPR sets
  - Dense discrete set $\left\{\sqrt{2}x - \left\lfloor \sqrt{2}x \right\rfloor \,:\, x \in \mathbb{N}\right\} \subseteq [0,1]$

$$\begin{aligned} &\|(z_1, z_1)\|_2 \le z_2 + 1, \quad &&\|(z_2, z_2)\|_2 \le 2z_1, \quad && x_1 = y_1 - z_2, \\ &\|(z_1, z_1)\|_2 \le y_1, &&\|(y_1, y_1)\|_2 \le 2z_1, && z \in \mathbb{Z}^2 \end{aligned}$$

  - Set of naturals $\left\{x \in \mathbb{N} \,:\, \sqrt{2}x - \left\lfloor \sqrt{2}x \right\rfloor \notin \left(\varepsilon, 1 - \sqrt{2}\varepsilon\right)\right\}$

$$\begin{aligned} &\|(x_1, x_1)\|_2 \le x_2 + \varepsilon, \\ &\|(x_2, x_2)\|_2 \le 2x_1 + 2\varepsilon, \quad x \in \mathbb{Z}^2_+ \end{aligned}$$

A Definition Rational MICPR (R-MICPR)

$$S = \bigcup_{z \in I \cap \mathbb{Z}^d} \operatorname{proj}_x (B_z) \qquad S = \operatorname{proj}_x \left(M \cap \left(\mathbb{R}^{n+p} \times \mathbb{Z}^d \right) \right)$$

$$I = \operatorname{proj}_z (M)$$

- Any rational affine mapping of index set I :
  - Is bounded, or
  - Has an integer (rational) recession direction
- Irrational directions can hide!
  - R-MICPR $\not\Leftrightarrow$ $\operatorname{span}(\operatorname{rec}(I))$ and/or $\operatorname{aff}(I)$ = rational space

$$\left(z_1 + \sqrt{2} z_2 \right)^2 \leq z_3 \qquad \operatorname{span}(\operatorname{rec}(I)) = \operatorname{span}(\{\mathbf{e}_3\})$$

$$\left(z_2 - \sqrt{2} z_1 \right)^2 \leq 1 \qquad \operatorname{rec}(\operatorname{proj}_{z_1, z_2}(I)) = \operatorname{span}(\{(1, \sqrt{2})\})$$

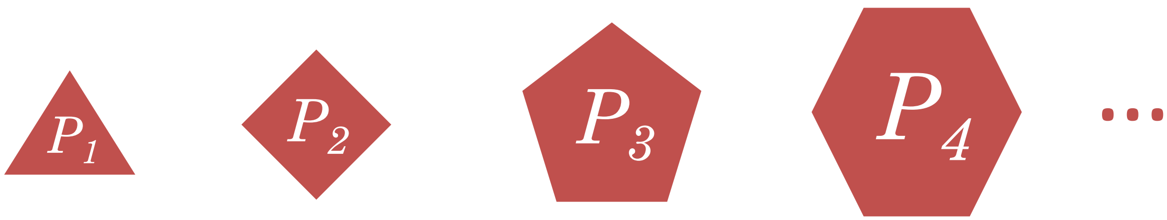
Properties of Rational MICPR (R-MICPR)

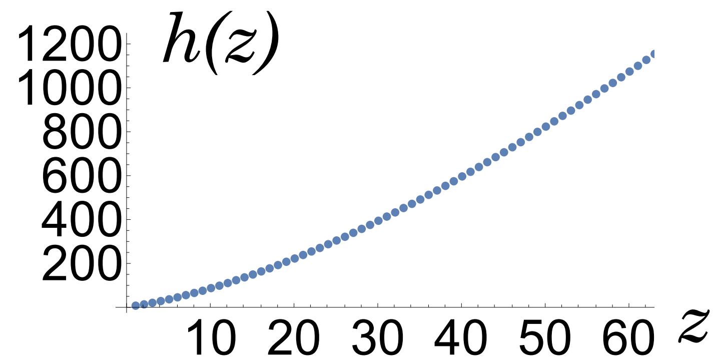
- For **compact** S :
  - **Finite unions of compact** convex sets
- For S **infinite unions of "uniformly bounded" closed convex sets** :
  - **Finite union of periodic**
  - **Dense discrete** and **non-periodic naturals** NOT R-MICPR
- Rational MICP Representability:
  - **Closed** under: **Finite Union, Cartesian Product and Minkowski sum**
  - **NOT Closed** under **intersection**.

R-MICPR: Periodicity for Natural #s

- An **infinite set** of naturals S is **periodic** if and only if:
  - $S = \bigcup_{i=1}^{k} \{s_i\} + \mathrm{intcone}\left(\{r\}\right)$
  - It is **rational MILP representable**
- A subset S of the naturals is **R-MICPR** if and only if:
  - It is the **union** of a **finite** and an **infinite periodic** set

R-MICPR does Not Imply Finite Shapes





- There exists increasing functions h such that:
  - $P_z \subseteq \mathbb{R}^2$ regular $h(z)$-gon centered at $(z, 0)$
  - $P_z \cap P_{z'} = \emptyset, \quad z \neq z'$
  - $S = \bigcup_{z=1}^{\infty} P_z$ is R-MICPR and periodic

- Equal volume ⇒ Finite # of Shapes