FIT1045: Algorithms and Programming Fundamentals in Python

Lecture 12

Decrease and Conquer

Objectives

Objectives of this lecture are to:

1. Know to search efficiently in ordered sequence (Binary Search)
2. Understand design paradigm decrease-and-conquer and recognise situations with logarithmic complexity
3. Demonstrate the time complexity of Euclid's algorithm

This covers learning outcomes:

- 2 – choose and implement appropriate problem solving strategies
- 5 – determine the computational cost and limitations of algorithms

Overview

1. The Ordered Search Problem
2. Binary Search
3. Revisiting Euclid's Algorithm

Search in Ordered Sequence

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…

Problem: find (the position of) a name in a phone book.

Search in Ordered Sequence

ato.gov.au 180.149.195.3

cancer.org.au 52.187.229.23

facebook.com 31.13.71.36

google.com 172.217.12.142

monash.edu 43.245.43.30

newscientist.com 45.60.19.101

news.com.au 23.221.48.198

wikipedia.org 208.80.154.224

…

Problem: find URL in DNS records.

Sequential search solves problem

```
def sequential_search(v, seq):
    """I: value v and sequ
       O: an index of seq
          (if no such inde
    """
    n = len(seq)
    i = 0
    while i < n:
        if seq[i] == v:
            return i
        i += 1
    return None
```

Quiz time (https://flux.qa)
Clayton: AXXULH
Malaysia: LWERDE

`n = len(seq)`	$O(0)$	$O(0)$
`i = 0`	$O(0)$	$O(0)$
`while i < n:`	$O(1)$	$O(n)$
`if seq[i] == v:`	$O(1)$	$O(n)$
`return i`	$O(1)$	$O(0)$
`i += 1`	$O(0)$	$O(n)$
`return None`	$O(0)$	$O(0)$
	$O(1)$	$O(n)$

Example Scenario (e.g. Web search):

- need to find 10.000.000 values from sequence of 2.000.000.000 entries
- cost of 1 elementary step is 1ns (and constant factor in O-notation is 1)

Outcome: need about 40 years

Can we solve problem *more efficiently* for ordered sequences?

```
def sequential_search(v, seq):
    """I: value v and ordered sequence seq
       O: an index of seq with value v or None
          (if no such index exists)
    """
    n = len(seq)
    i = 0
    while i < n:
        #I) v in seq[i:] or v not in seq
        if seq[i] == v:
            return i
        i += 1
    return None
```

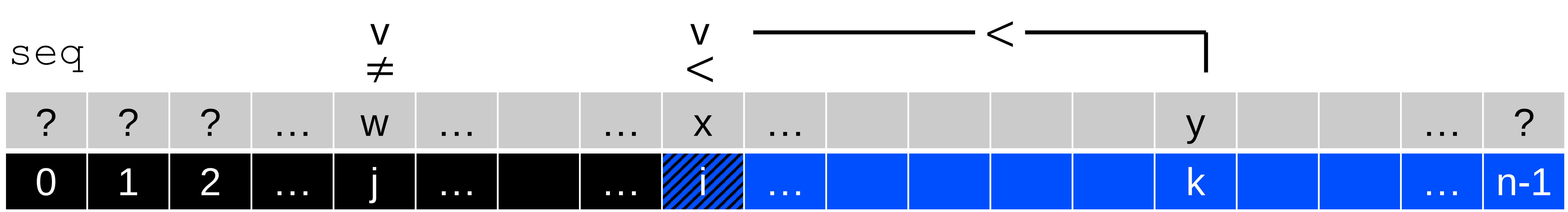
seq

v
≠

?	?	?	…	w	…		…	x	…								…	?
0	1	2	…	j	…		…	i	…								…	n-1

However: order of sequence *implies* many more comparisons

```
def sequential_search(v, seq):
    n = len(seq)
    i = 0
    while i < n:
        #I) v in seq[i:] or v not in seq
        if seq[i] == v:
            return i
        i += 1
    return None
```



However: order of sequence *implies* many more comparisons

```python
def sequential_search2(v, seq):
    n = len(seq)
    i = 0
    while i < n:
        #I) v in seq[i:] or v not in seq
        if seq[i] == v:
            return i
        if seq[i] > v:
            # v not in seq[:i]
            return None
        i += 1
    return None
```

Does that reduce worst-case complexity?

					v ≠				v <				<						
seq	?	?	?	…	w	…		…	x	…					y			…	?
	0	1	2	…	j	…		…	i	…					k			…	n-1

What single comparison rules out most positions (worst case)?

?	?	?	…														…	?
0	1	2	…														…	n-1

What single comparison rules out most positions (worst case)?

Case 1: v = q

v
=

≤q	≤q	≤q	…				…	≤q	q	≥q	…						…	≥q
0	1	2	…					…	n//2	…							…	n-1

What single comparison rules out most positions (worst case)?

Case 1: v = q

v
=

≤q	≤q	≤q	…				…	≤q	q	≥q	…						…	≥q
0	1	2	…					…	n//2	…							…	n-1

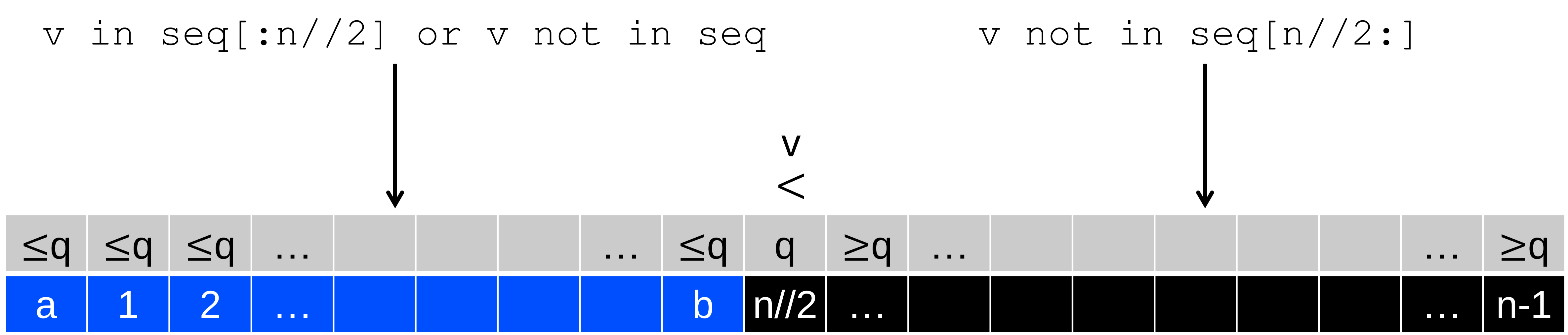
What single comparison rules out most positions (worst case)?

Case 2: v < q

									v									
									<									
≤q	≤q	≤q	…				…	≤q	q	≥q	…						…	≥q
0	1	2	…					…	n//2	…							…	n-1

What single comparison rules out most positions (worst case)?

Case 2: v < q



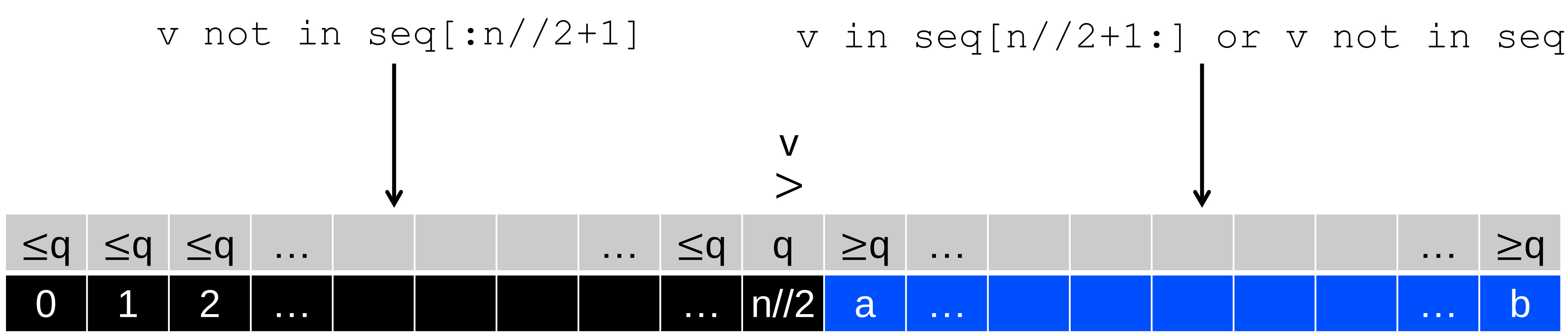
What single comparison rules out most positions (worst case)?

Case 3: $v > q$

									v $>$									
$\le q$	$\le q$	$\le q$	…				…	$\le q$	q	$\ge q$	…						…	$\ge q$
0	1	2	…					…	n//2	…							…	n-1

What single comparison rules out most positions (worst case)?

Case 3: v > q



Overview

1. The Ordered Search Problem
2. Binary Search
3. Revisiting Euclid's Algorithm

Decrease-and-Conquer: reduce problem to smaller subproblem

```python
def probing_search(v, seq):
    a, b = 0, len(seq)-1
    c = b // 2
    if seq[c] == v:
        return c
    elif v < seq[c]:
        b = c - 1
    else:
        a = c + 1

    # search between a and b
```

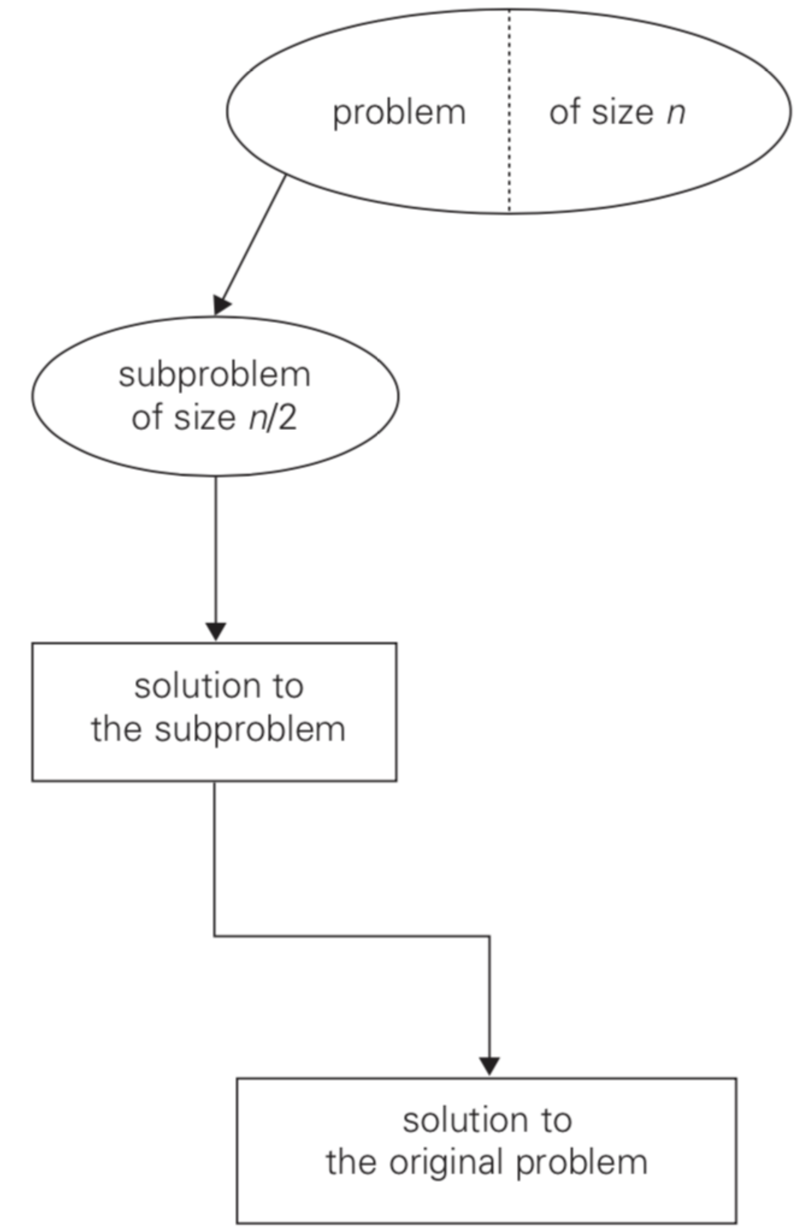


illustration: [Levitin, p. 133]

v
>

≤q	≤q	≤q	…				…	≤q	q	≥q	…						…	≥q
0	1	2	…					…	n//2	a	…						…	b

We could solve remaining sub-problem as before…

```python
def probing_search(v, seq):
    a, b = 0, len(seq)-1
    c = b // 2
    if seq[c] == v:
        return c
    elif v < seq[c]:
        b = c - 1
    else:
        a = c + 1
    i = a
    while a <= i <= b:
        if seq[i] == v:
            return i
        i += 1
    return None
```

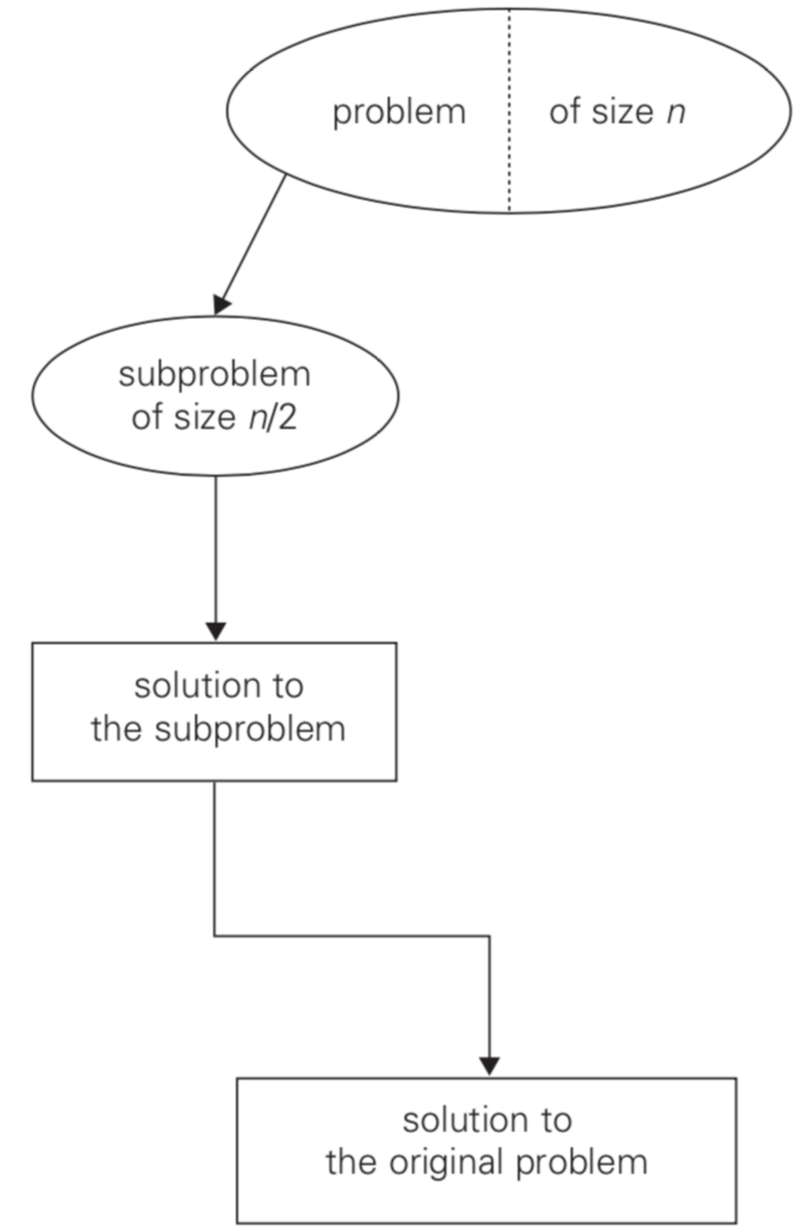


illustration: [Levitin, p. 133]

v
>

≤q	≤q	≤q	…				…	≤q	q	≥q	…						…	≥q
0	1	2	…					…	n//2	a	…						…	b

Instead: let's re-apply same principle

```python
def probing_search(v, seq):
    a, b = 0, len(seq)-1
    c = b // 2
    if seq[c] == v:
        return c
    elif v < seq[c]:
        b = c - 1
    else:
        a = c + 1
    i = a
    while a <= i <= b:
        if seq[i] == v:
            return i
        i += 1
    return None
```

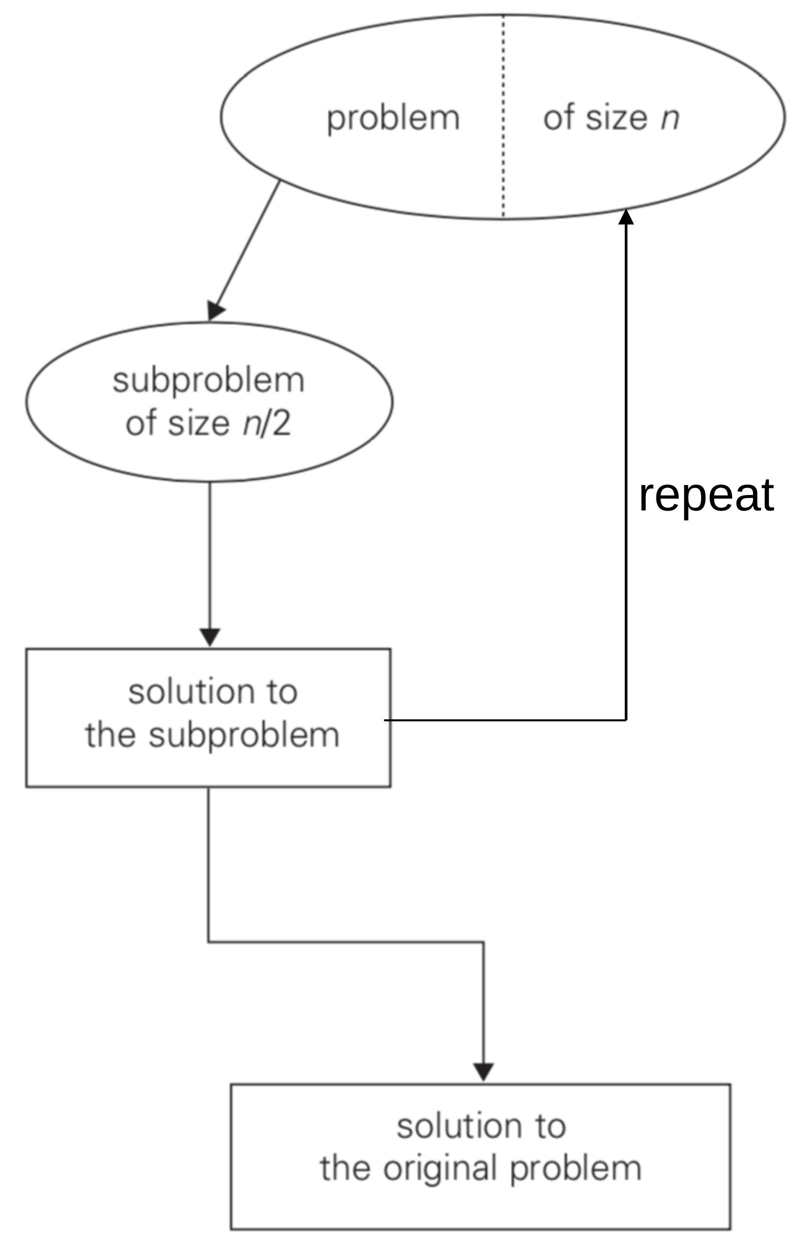
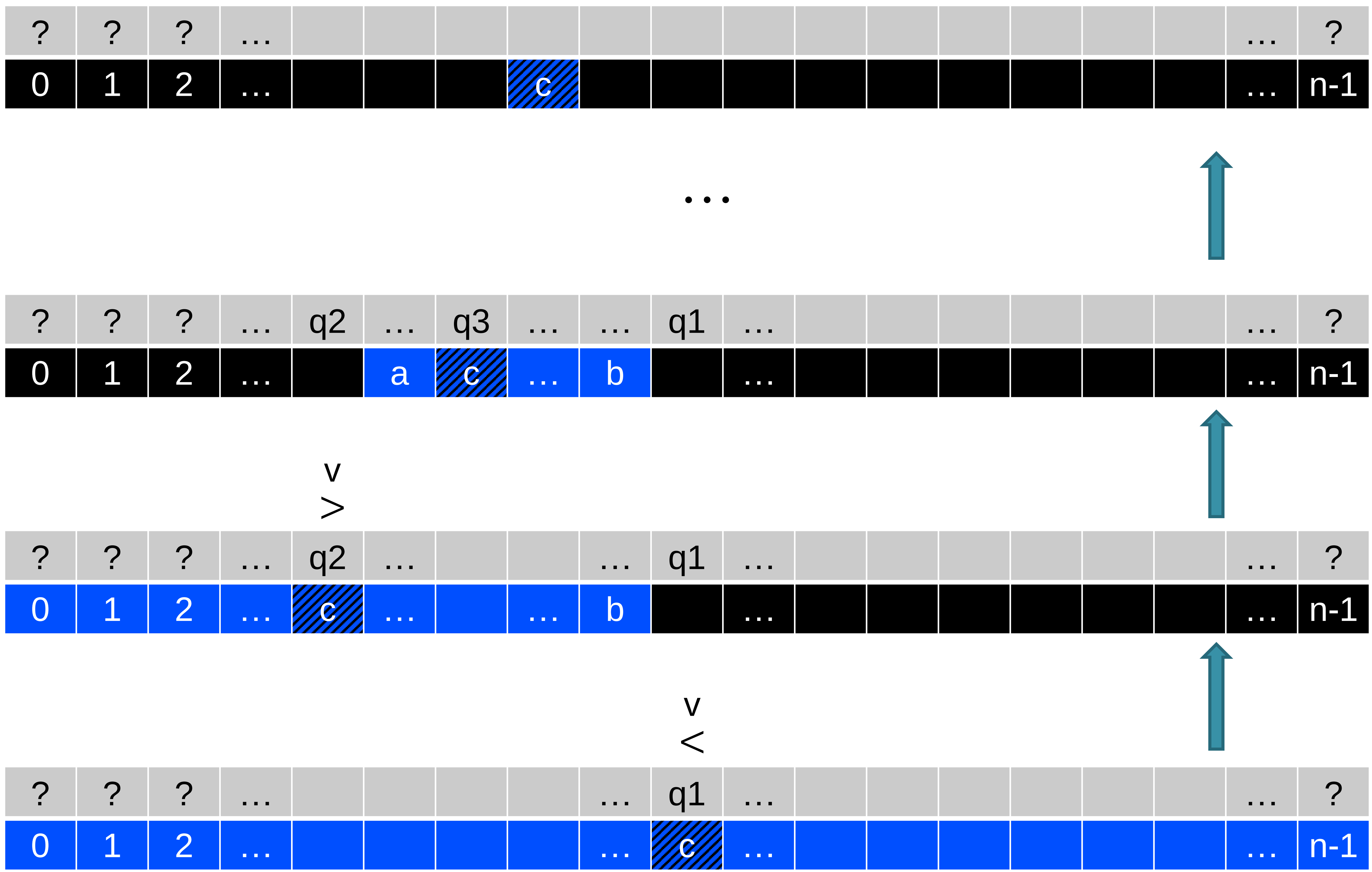


illustration: [Levitin, p. 133]

v
>

≤q	≤q	≤q	…				…	≤q	q	≥q	…						…	≥q
0	1	2	…					…	n//2	a	…						…	b

Instead: let’s re-apply same principle



Binary Search

```
def binary_search(v, seq):
    a = 0
    b = len(seq) - 1
```

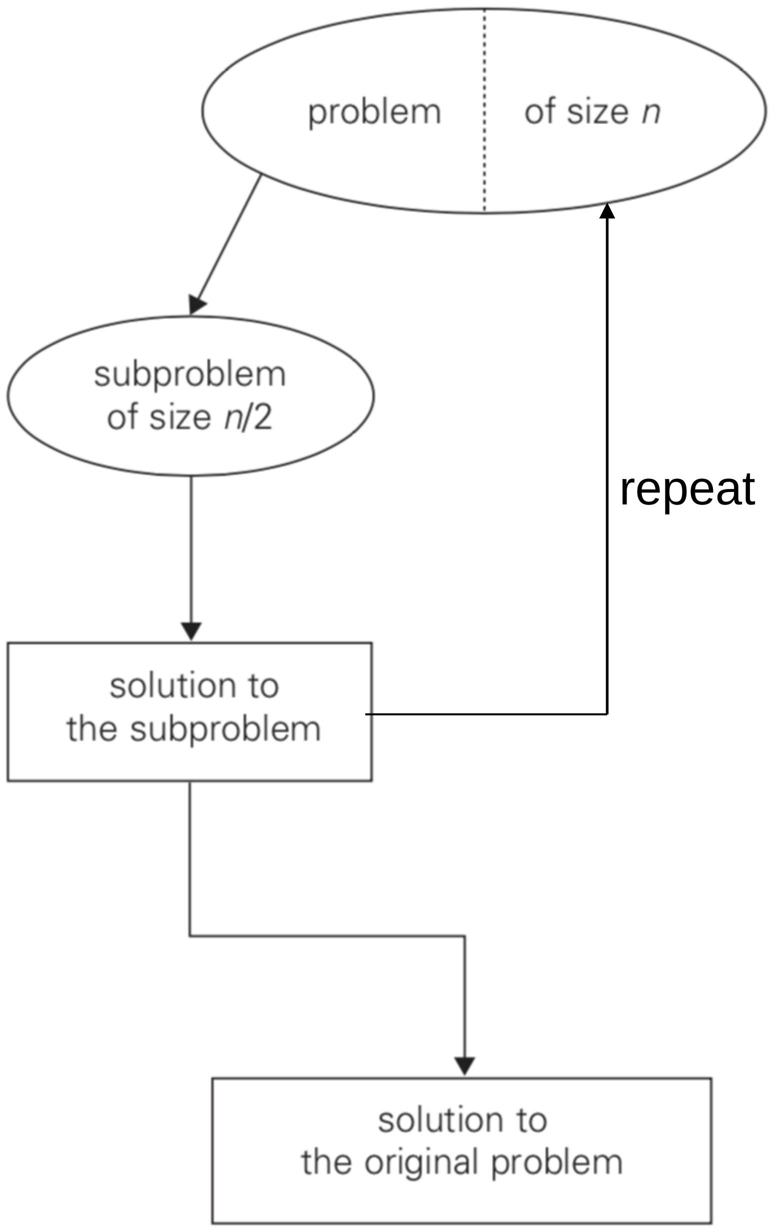


illustration: [Levitin, p. 133]

Binary Search

```
def binary_search(v, seq):
    a = 0
    b = len(seq) - 1
    # in iteration i:
        c = (a + b) // 2
        if seq[c] == v:
            return c
        elif seq[c] > v:
            b = c - 1
        else:
            a = c + 1
```

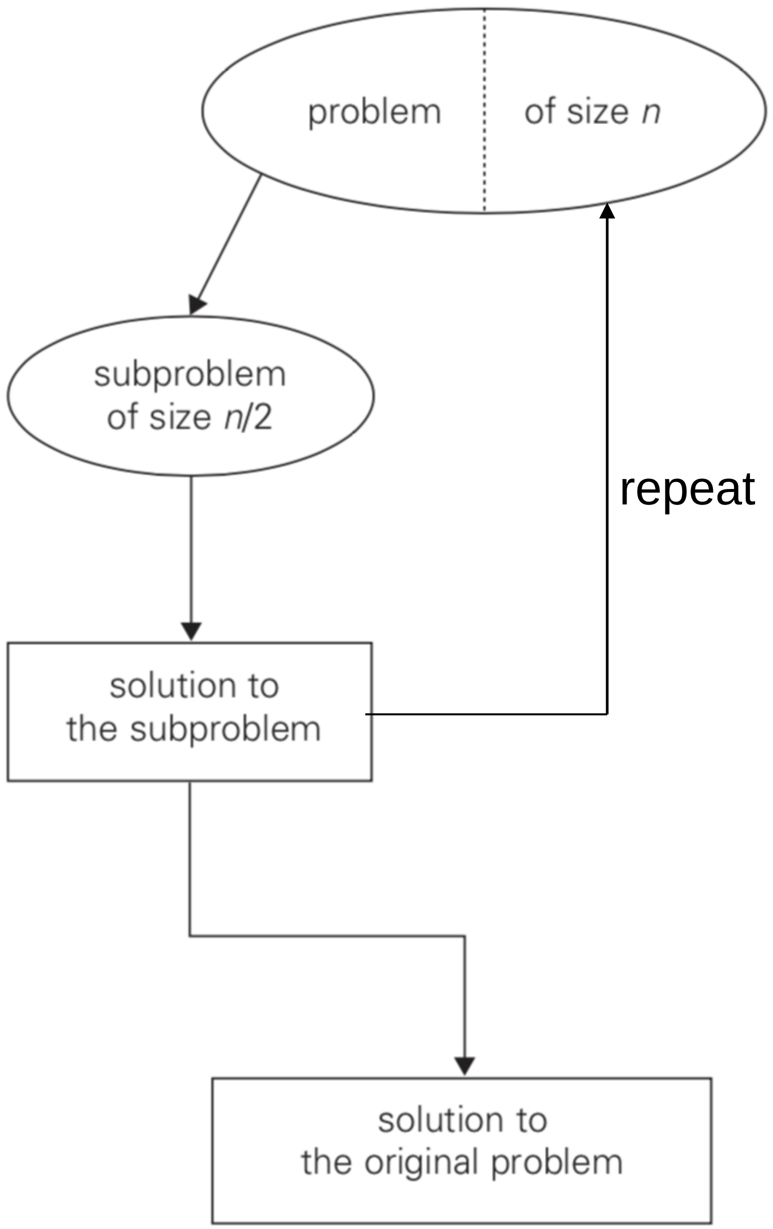


illustration: [Levitin, p. 133]

Binary Search

```
def binary_search(v, seq):
    a = 0
    b = len(seq) - 1
    while a <= b:
        c = (a + b) // 2
        if seq[c] == v:
            return c
        elif seq[c] > v:
            b = c - 1
        else:
            a = c + 1
    return None
```

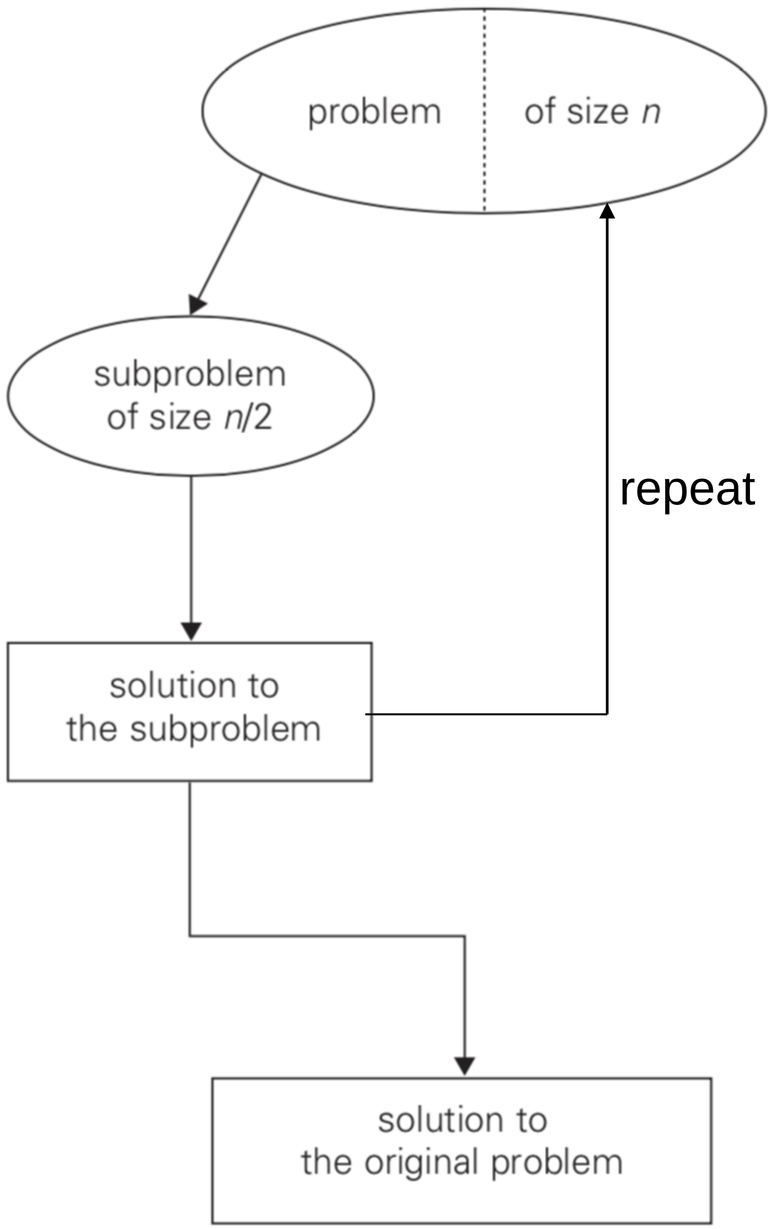
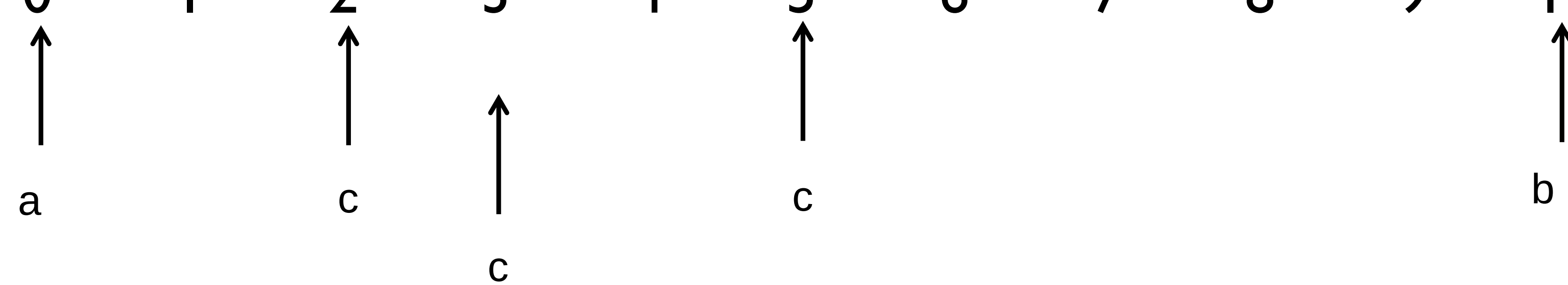


illustration: [Levitin, p. 133]

Binary Search

```python
def binary_search(v, seq):
    a = 0
    b = len(seq) - 1
    while a <= b:
        c = (a + b) // 2
        if seq[c] == v:
            return c
        elif seq[c] > v:
            b = c - 1
        else:
            a = c + 1
    return None
```

seq:	5	10	15	20	25	30	35	40	45	50	55
(v = 20)	0	1	2	3	4	5	6	7	8	9	10



Computational complexity of Binary Search

```python
def binary_search(v, seq):
    a = 0
    b = len(seq) - 1
    while a <= b:
        c = (a + b) // 2
        if seq[c] == v:
            return c
        elif seq[c] > v:
            b = c - 1
        else:
            a = c + 1
    return None
```

`def binary_search(v, seq):`	$n = \text{len}(\text{lst})$
`a = 0`	$O(0)$
`b = len(seq) - 1`	$O(1)$
`while a <= b:`	$O(1)$
`c = (a + b) // 2`	$O(1)$
`if seq[c] == v:` … `a = c + 1`	$O(1)$
`return None`	$O(0)$
	$O(1)$

Quiz time (https://flux.qa)

Clayton: AXXULH

Malaysia: LWERDE

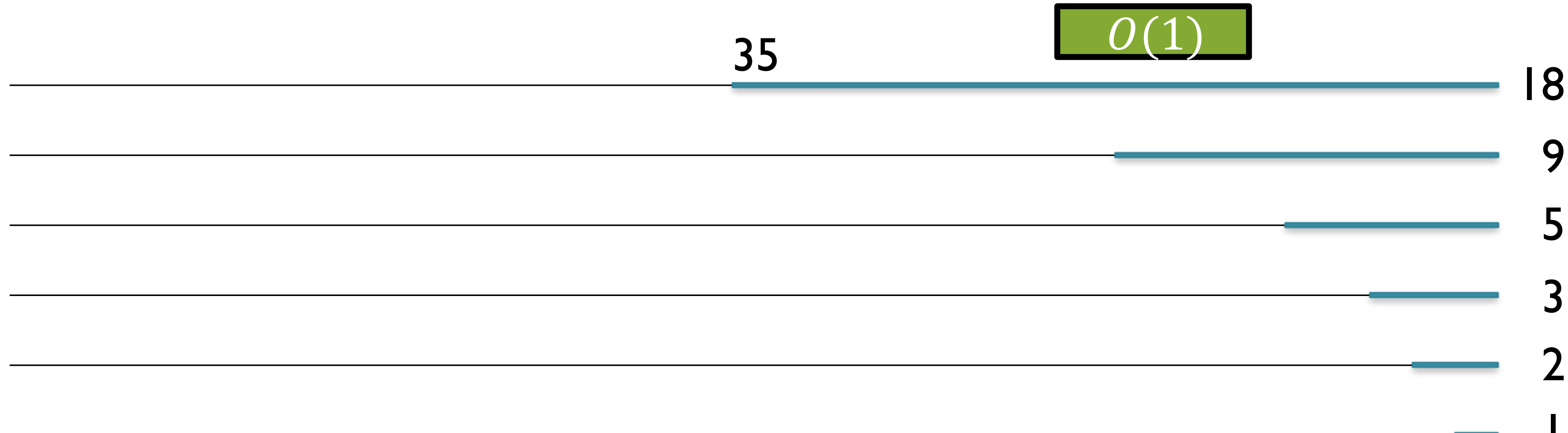
Computational complexity of Binary Search

```python
def binary_search(v, seq):
    a = 0
    b = len(seq) - 1
    while a <= b:
        c = (a + b) // 2
        if seq[c] == v:
            return c
        elif seq[c] > v:
            b = c - 1
        else:
            a = c + 1
    return None
```

$n = \text{len(lst)}$

$O(0)$	$O(0)$
$O(1)$	$O(1)$
$O(1)$	?
$O(1)$	
$O(1)$	
$O(0)$	

$\boxed{O(1)}$



Computational complexity of Binary Search

```
def binary_search(v, seq):
    a = 0
    b = len(seq) - 1
    while a <= b:
        c = (a + b) // 2
        if seq[c] == v:
            return c
        elif seq[c] > v:
            b = c - 1
        else:
            a = c + 1
    return None
```

Line		
`def binary_search(v, seq):`	$n = \text{len(lst)}$	
`a = 0`	$O(0)$	$O(0)$
`b = len(seq) - 1`	$O(1)$	$O(1)$
`while a <= b:`	$O(1)$	$O(\log n)$
`c = (a + b) // 2`	$O(1)$	$O(\log n)$
`if … a = c + 1`	$O(1)$	$O(\log n)$
`return None`	$O(0)$	$O(1)$
	$O(1)$	$O(\log n)$

- Let $n_i = b_i - a_i + 1$ be problem size after i iterations of loop
- In the beginning: $n_0 = n$
- In every iteration size is cut in half: $n_i = \lceil n_{i-1}/2 \rceil$, i.e., $n_i = \lceil n/2^i \rceil$
- After $k = \lceil \log_2 n \rceil$ iterations: $n_k = \lceil n/2^{\log_2 n} \rceil = 1$, i.e., $b_k = a_k$
- So at most $O(\log_2 n)$ loop iterations

Computational complexity of Binary Search

```python
def binary_search(v, seq):
    a = 0
    b = len(seq) - 1
    while a <= b:
        c = (a + b) // 2
        if seq[c] == v:
            return c
        elif seq[c] > v:
            b = c - 1
        else:
            a = c + 1
    return None
```

Code	$n = \mathrm{len(lst)}$	
`a = 0`	$O(0)$	$O(0)$
`b = len(seq) - 1`	$O(1)$	$O(1)$
`while a <= b:`	$O(1)$	$O(\log n)$
`c = (a + b) // 2`	$O(1)$	$O(\log n)$
`if` … `a = c + 1`	$O(1)$	$O(\log n)$
`return None`	$O(0)$	$O(1)$
	$O(1)$	$O(\log n)$

Example Scenario (e.g. Web search):

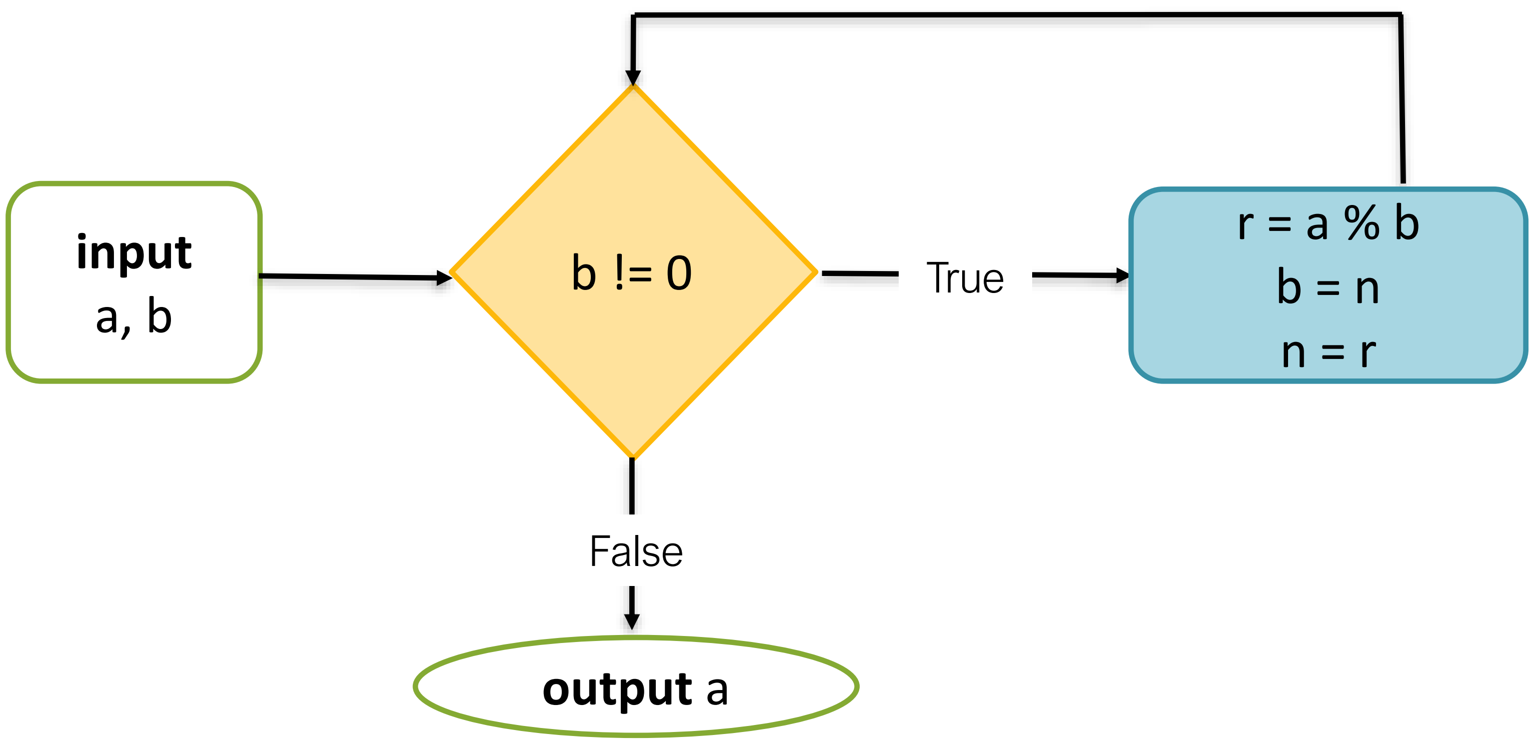
- need to find 10.000.000 values from sequence of 2.000.000.000 entries
- cost of 1 elementary step is 1ns (and constant factor in O-notation is 1)

Outcome: need about 0.3 seconds

Overview

1. The Ordered Search Problem
2. Binary Search
3. Revisiting Euclid's Algorithm

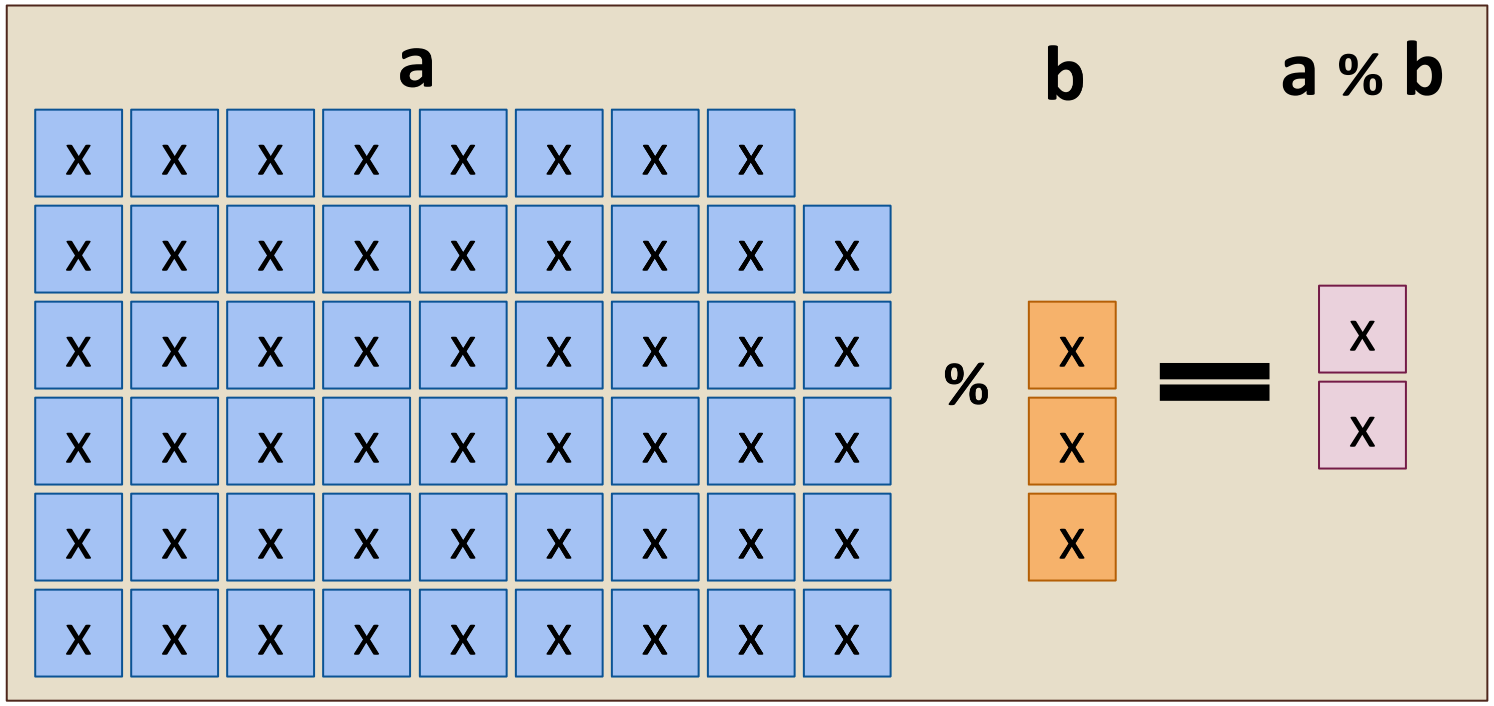
Recall Euclid’s Algorithm

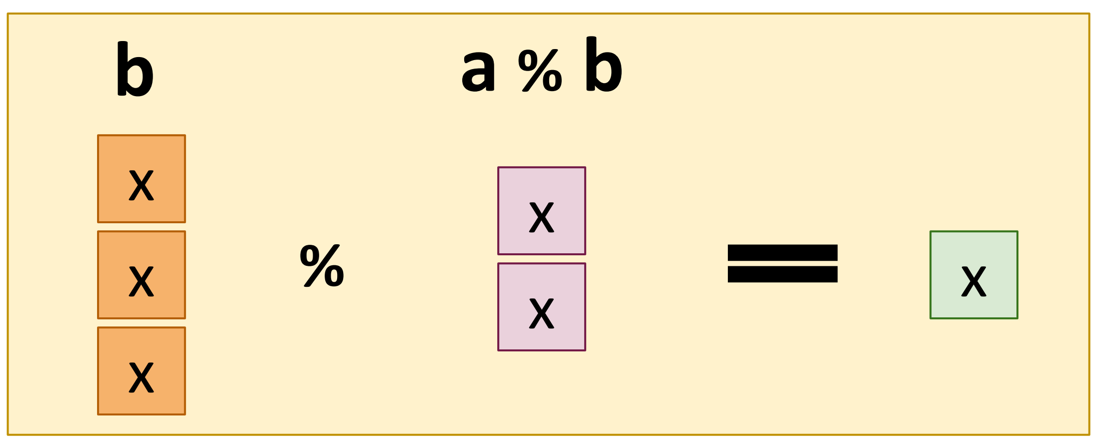


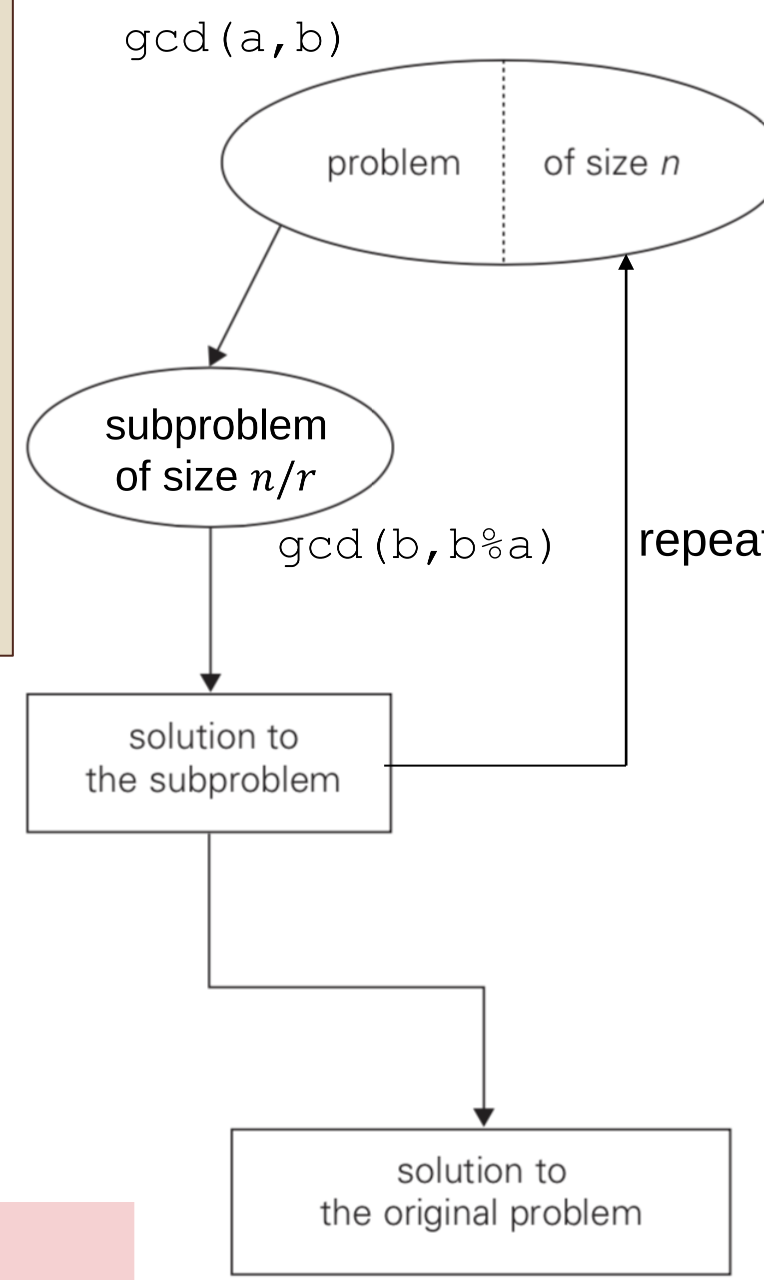
Eukleides of Alexandria
3xx BC – 2xx BC

```
def gcd_euclid(a, b):
    """
    Input : integers a and b such that not a==b==0
    Output: the greatest common divisor of a and b
    """
    while b != 0:
        a, b = b, a % b
    return a
```

Instance of decrease and conquer



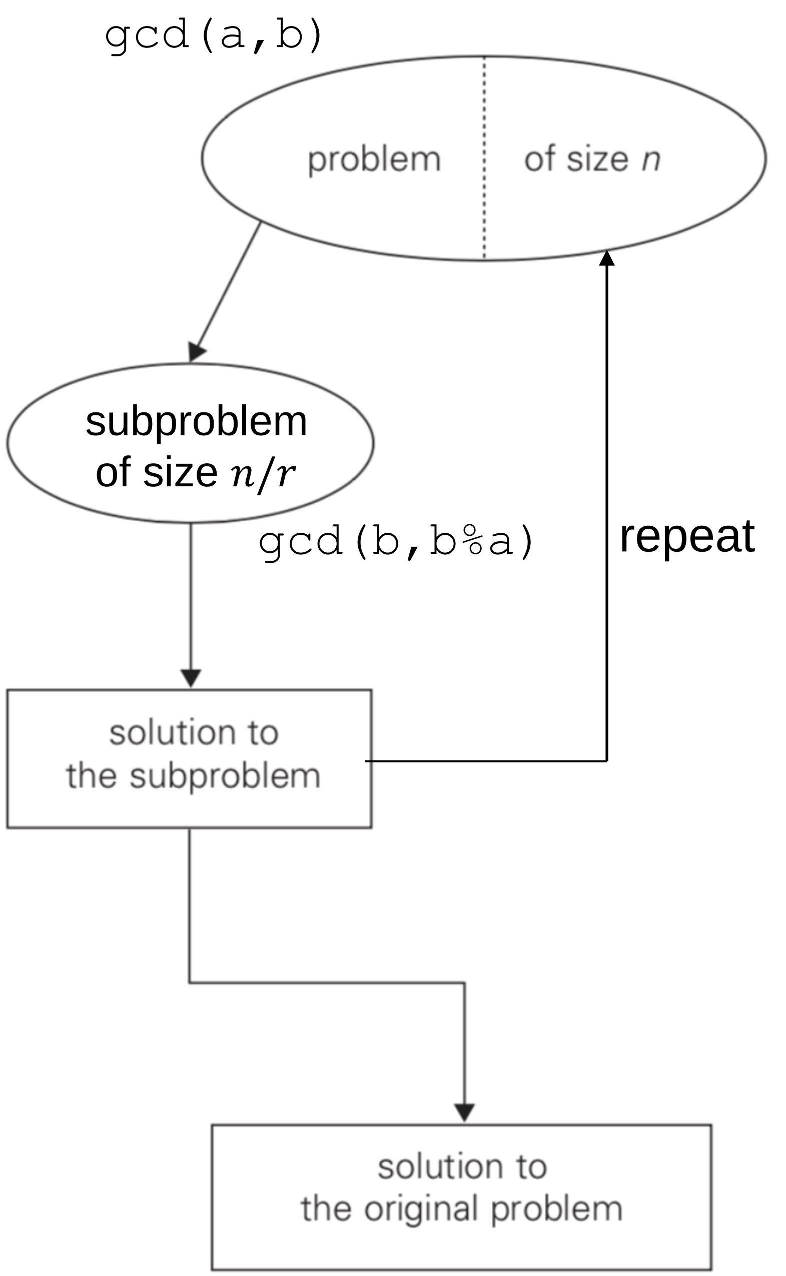





```python
def gcd_euclid(a, b):
    while b != 0:
        a, b = b, a % b
    return a
```

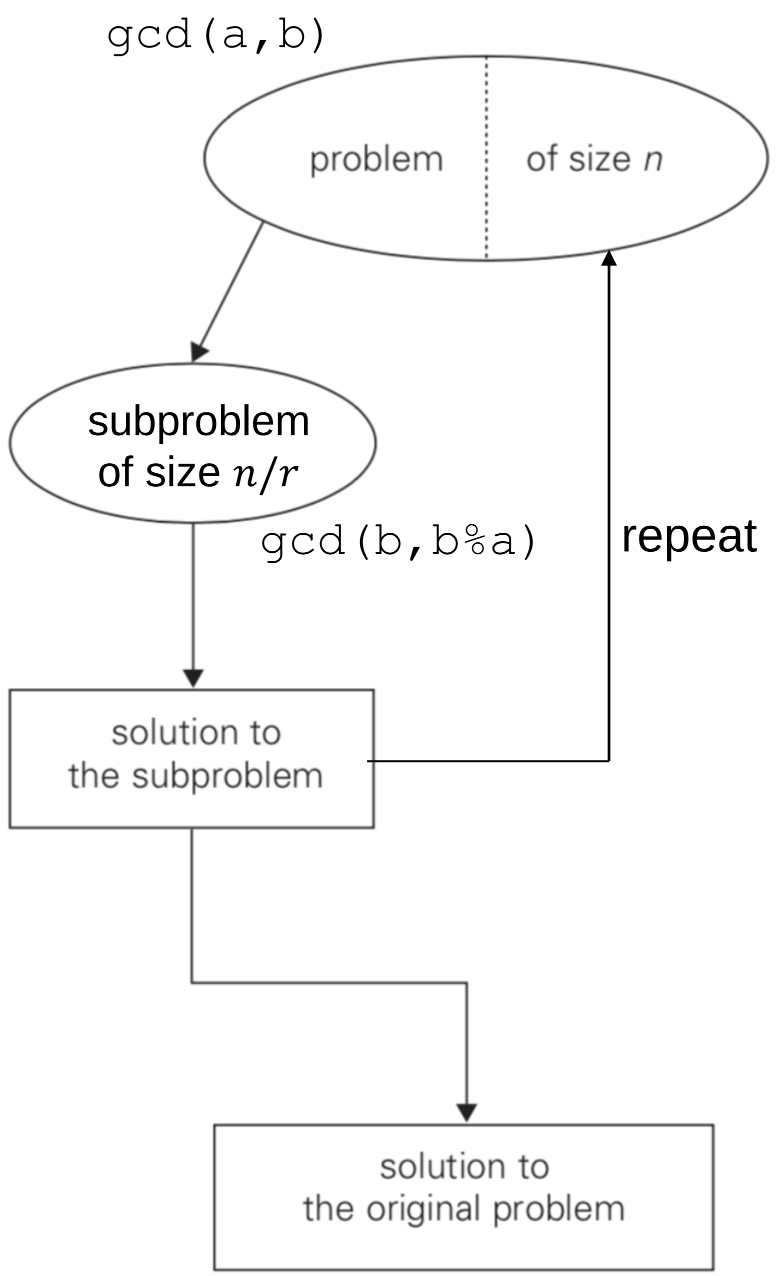
Exercise: correctness via loop invariant

```
def gcd_euclid(a, b):
    """
    I: integers a0 and b0 such
       that not a0==a0==0
    O: gcd(a0,b0)
    """
    while b != 0:
        a, b = b, a % b
    return a
```



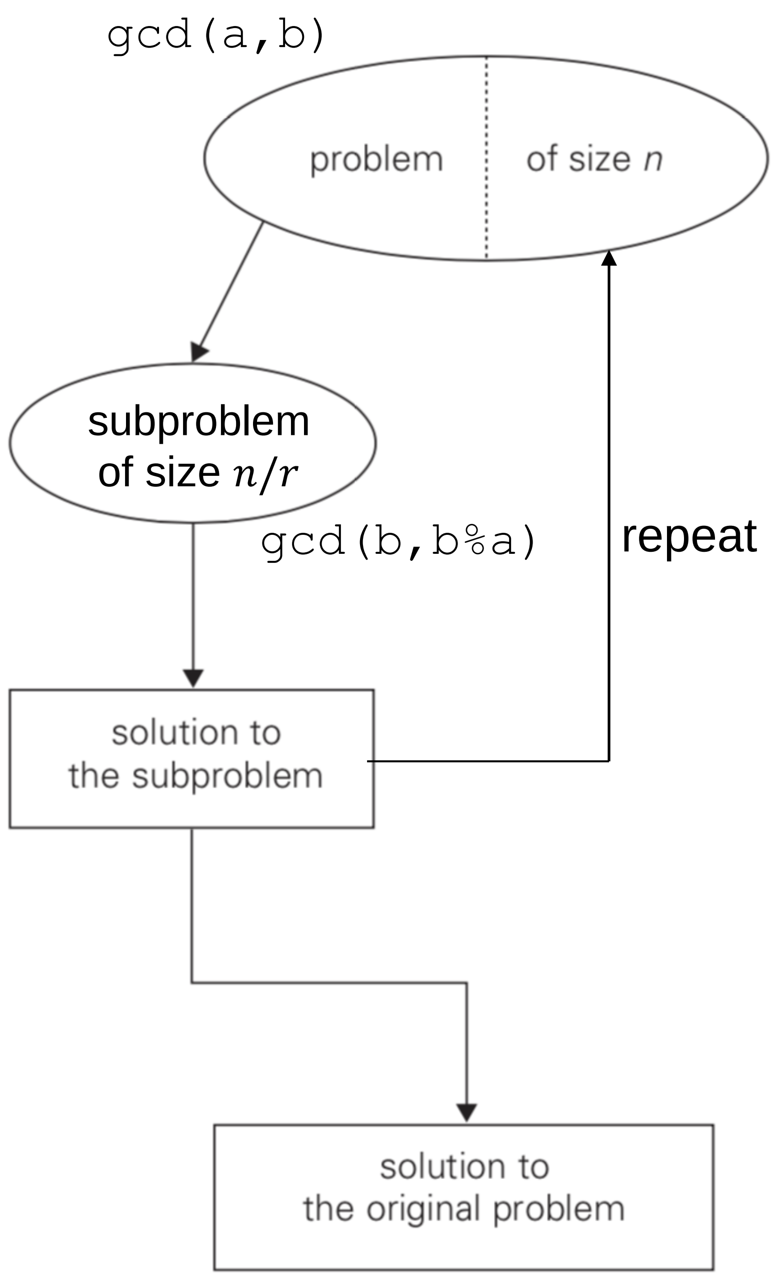
Exercise: correctness via loop invariant

```
def gcd_euclid(a, b):
    """
    I: integers a0 and b0 such
       that not a0==a0==0
    O: gcd(a0,b0)
    """
    #PRC: a,b==a0,b0 (original input)
    while b != 0:
        a, b = b, a % b
    return a
```

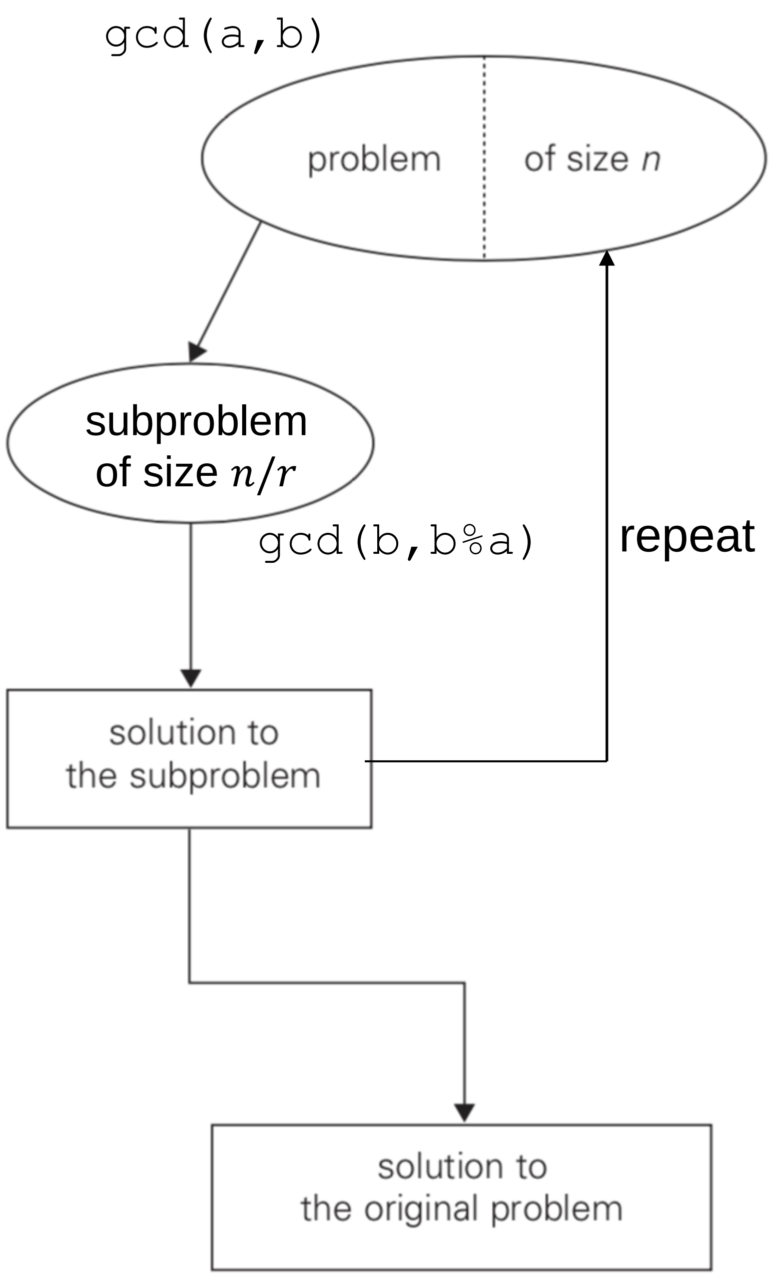


Exercise: correctness via loop invariant

```
def gcd_euclid(a, b):
    """
    I: integers a0 and b0 such
       that not a0==a0==0
    O: gcd(a0,b0)
    """
    #PRC: a,b==a0,b0 (original input)
    while b != 0:
        #I: gcd(a,b)==gcd(a0,b0)
        a, b = b, a % b
        #I: gcd(a,b)==gcd(a0,b0)
    return a
```



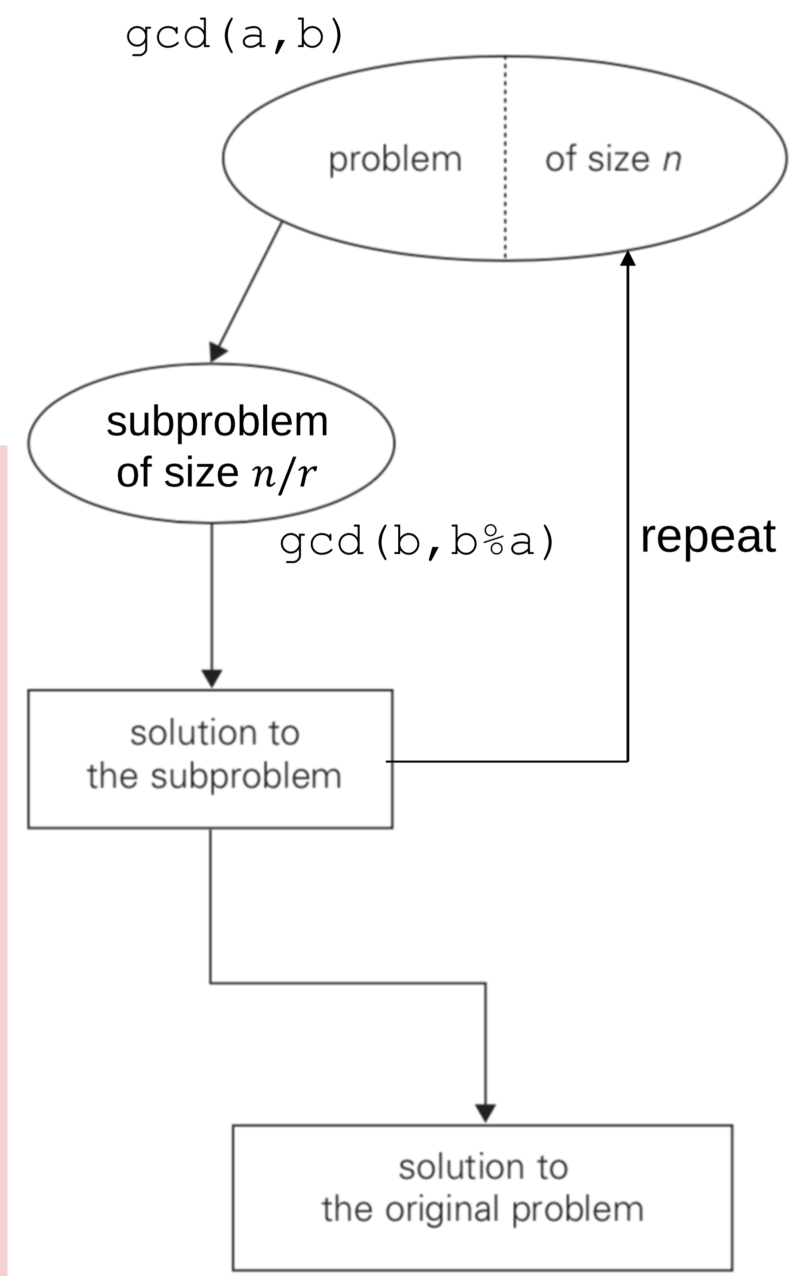
Exercise: correctness via loop invariant




```
def gcd_euclid(a, b):
    """
    I: integers a0 and b0 such
       that not a0==a0==0
    O: gcd(a0,b0)
    """
    #PRC: a,b==a0,b0 (original input)
    while b != 0:
        #I: gcd(a,b)==gcd(a0,b0)
        a, b = b, a % b
        #I: gcd(a,b)==gcd(a0,b0)
    #EXC: b==0
    return a
```

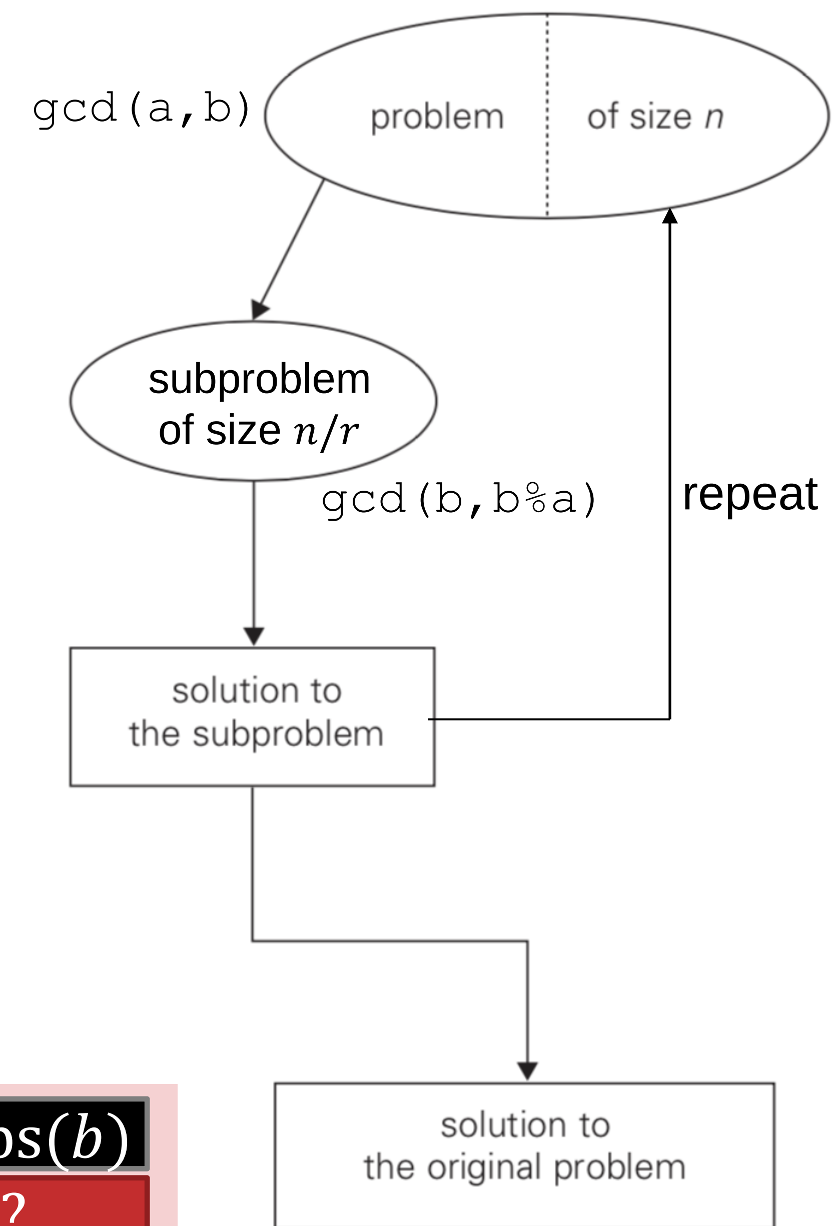
Exercise: correctness via loop invariant

```
def gcd_euclid(a, b):
    """
    I: integers a0 and b0 such
       that not a0==a0==0
    O: gcd(a0,b0)
    """
    #PRC: a,b==a0,b0 (original input)
    while b != 0:
        #I: gcd(a,b)==gcd(a0,b0)
        a, b = b, a % b
        #I: gcd(a,b)==gcd(a0,b0)
    #EXC: b==0
    #POC: a==gcd(a,b)==gcd(a0,b0)
    return a
```



Can we analyse computational complexity as for binary search?

Need to determine how many iterations we can have in worst case!



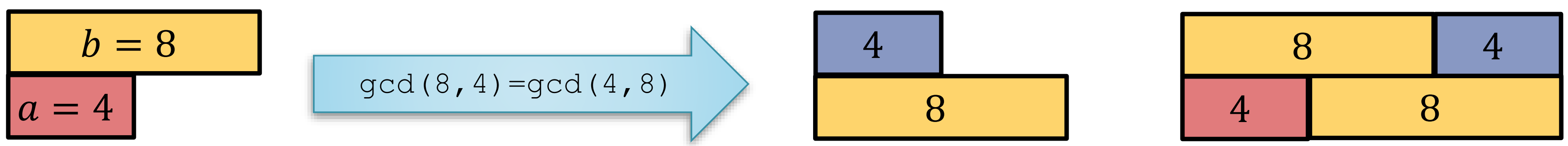

```
def gcd_euclid(a, b):
    while b != 0:
        a, b = b, a % b
    return a
```

Line		
`def gcd_euclid(a, b):`	$n = \text{abs}(a) + \text{abs}(b)$	
`while b != 0:`	$O(1)$	?
`a, b = b, a % b`	0	?
`return a`	$O(1)$	$O(1)$
	$O(1)$	?

By what factor is problem decreased per iteration of Euclid's Algorithm?

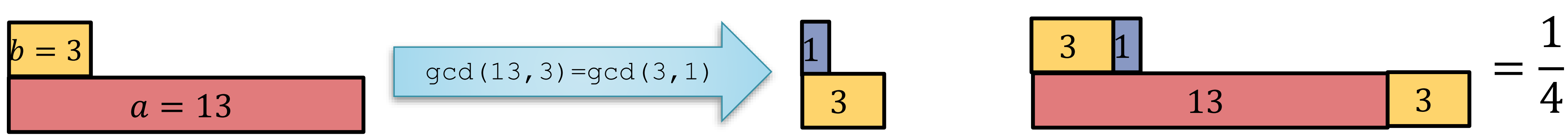
Example 1

No decrease in problem size!

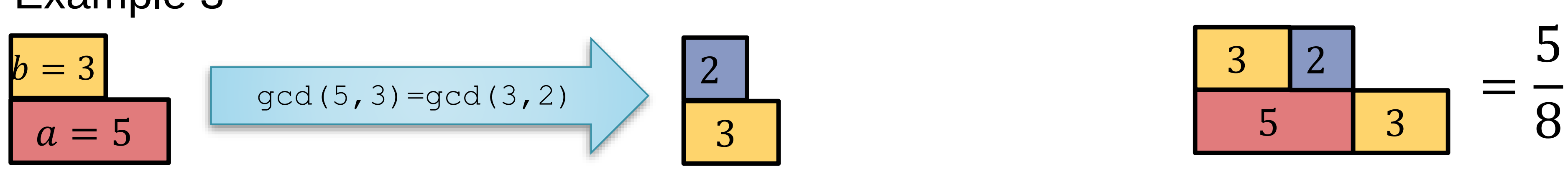


But this can only happen once in the beginning (afterwards $b > a\%b$ guarantees $a > b$)

Example 2



Example 3



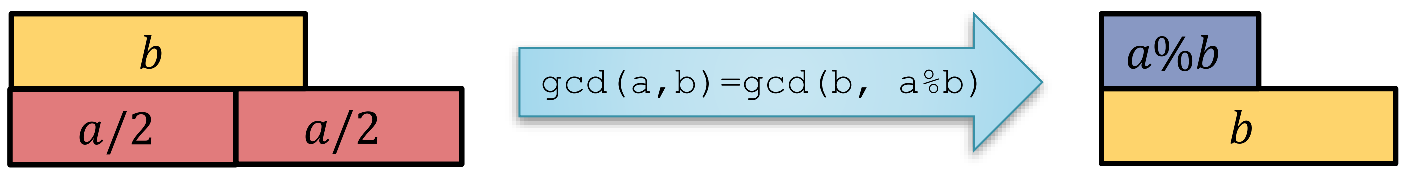
When $a > b$ there is decrease but at varying rate!

Do we need a fixed rate of decrease for logarithmic complexity?

No, just guarantee that reduction factor is always at least some $\alpha > 1$

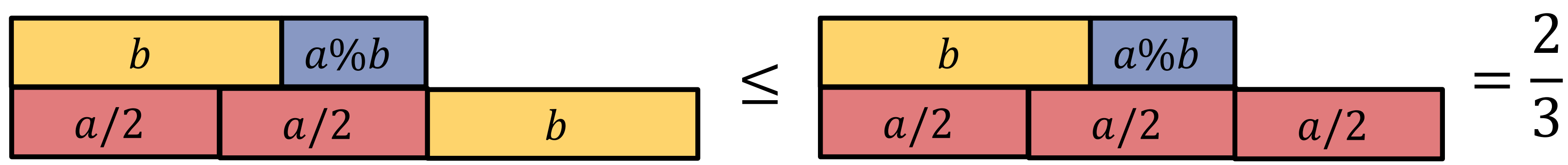
First case: "large b"

Case $b \geq a/2$



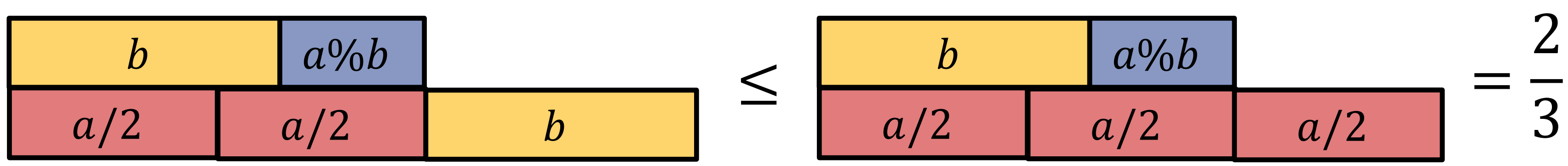
Relative size of decreased problem with large b

Case $b \geq a/2$

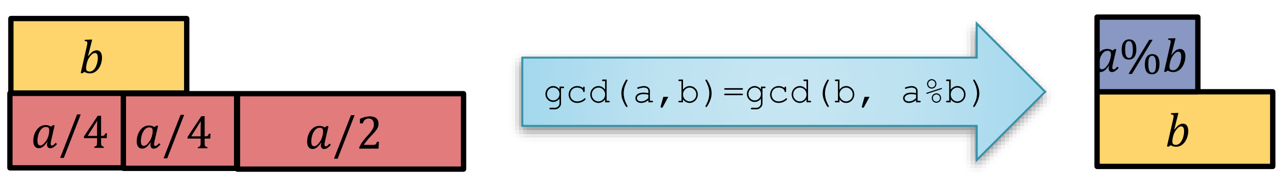


Second case: "small b"

Case $b \geq a/2$

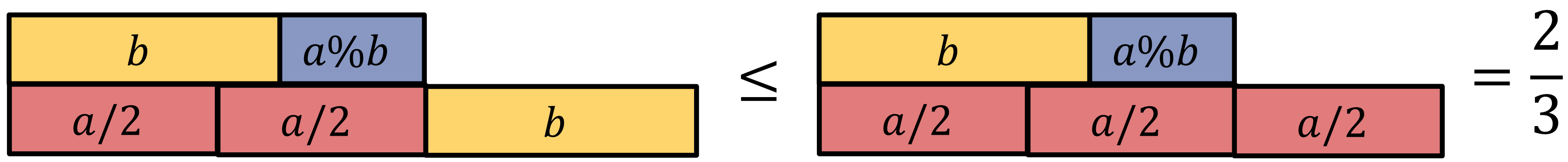


Case $a/2 > b \geq a/4$

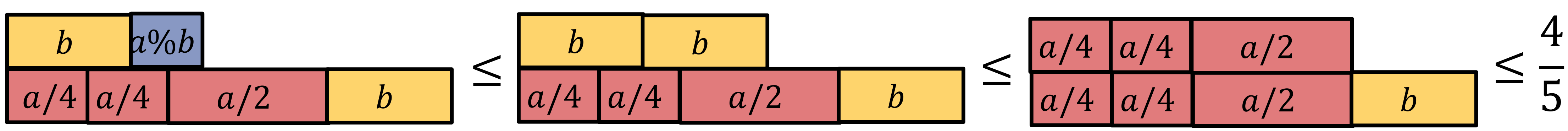


Relative size of decreased problem in second case

Case $b \geq a/2$

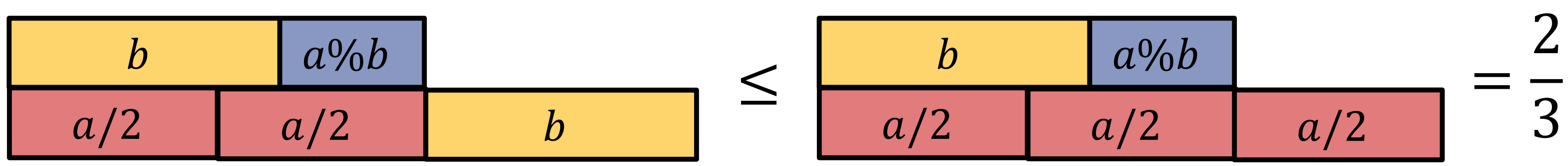


Case $a/2 > b \geq a/4$

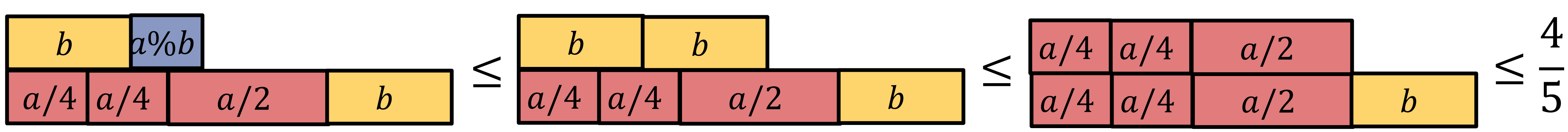


Final case: "tiny b"

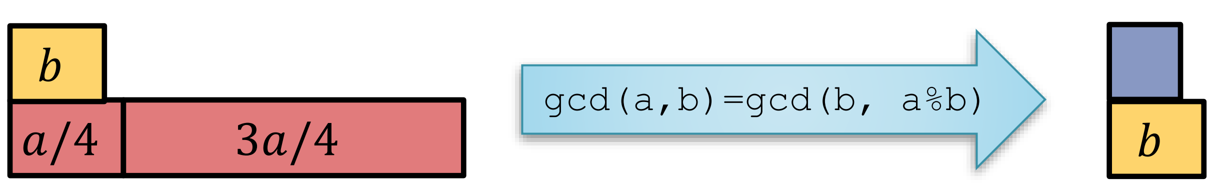
Case $b \geq a/2$



Case $a/2 > b \geq a/4$

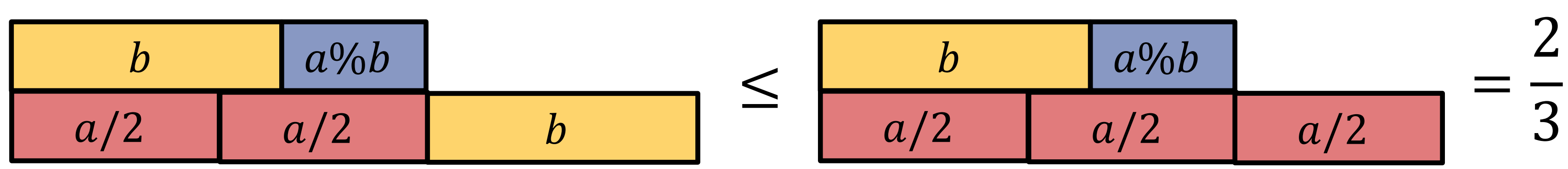


Case $a/4 > b$

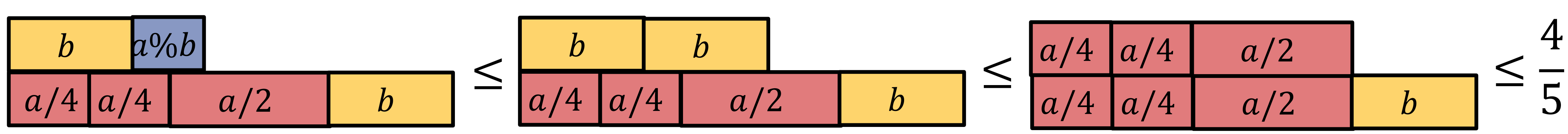


Relative size of decreased problem in final case

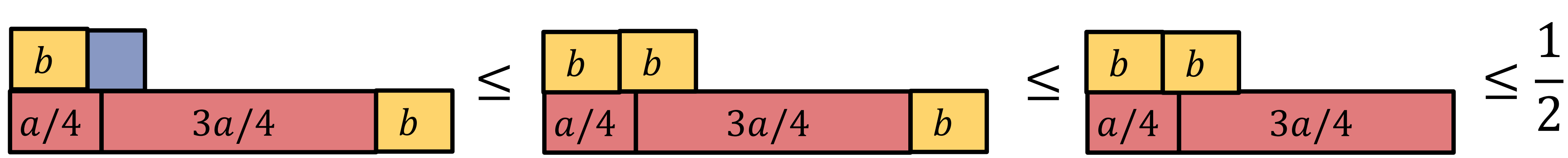
Case $b \geq a/2$



Case $a/2 > b \geq a/4$



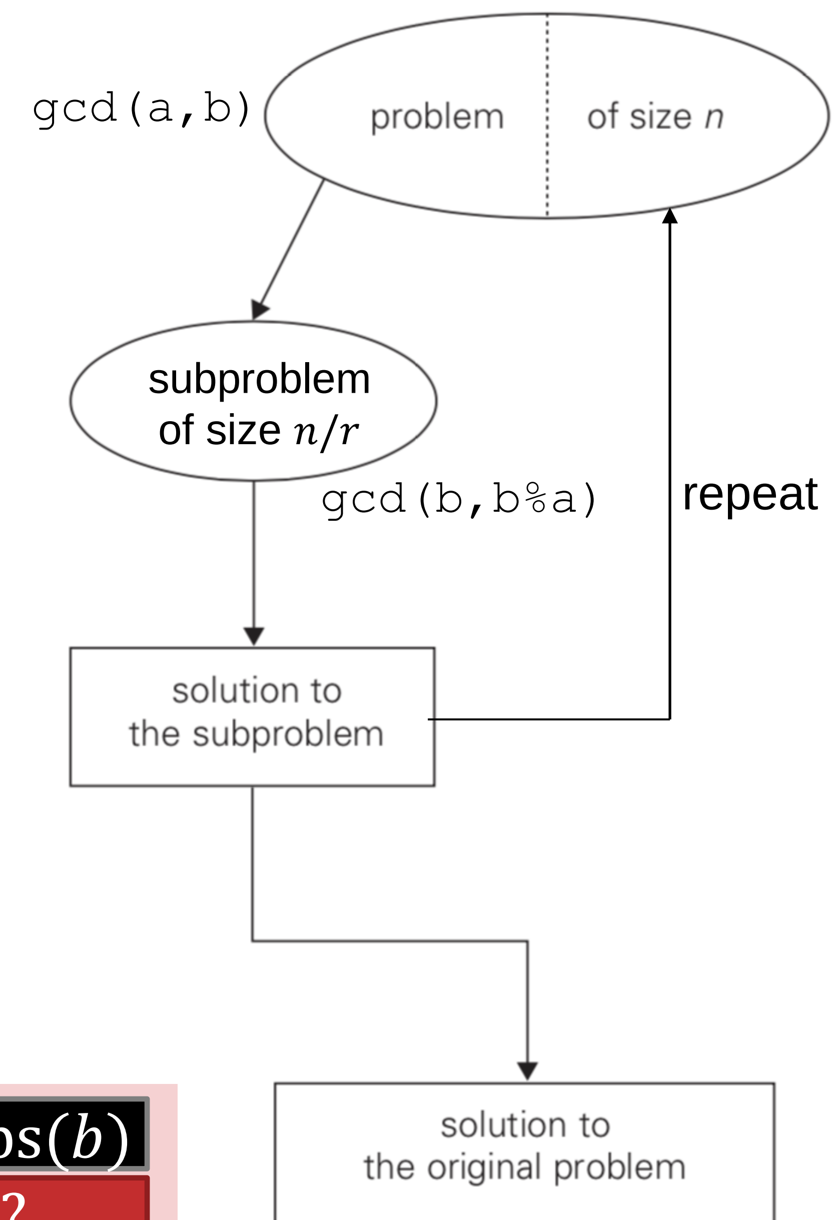
Case $a/4 > b$



In all cases: problem is at least decreased by a rate r of 5/4

Almost identical analysis as for binary search

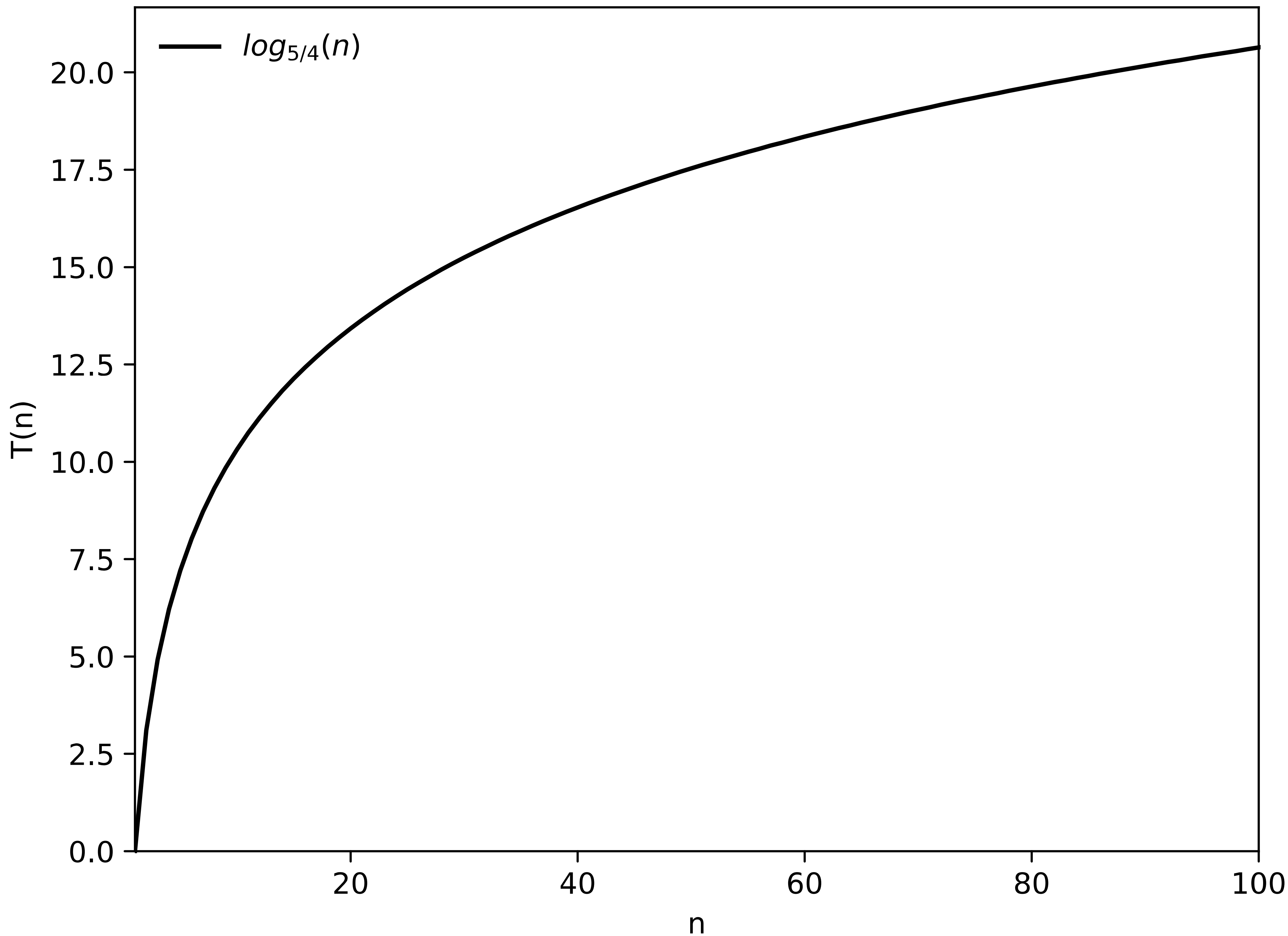
- Let $n_i = a_i + b_i$ be problem size after i iterations of loop
- In the beginning: $n_0 = n$
- Per iteration size is reduced by at least: $n_i = \lceil n_{i-1}/r \rceil$, i.e., $n_i = \lceil n/r^i \rceil$
- After at most $k = \lceil \log_r n \rceil$ iterations: $n_k = \lceil n/r^{\log_r n} \rceil = 1$, i.e., $b_k = 0$ and $a_k = 1$
- So at most $O(\log_r n)$ loop iterations



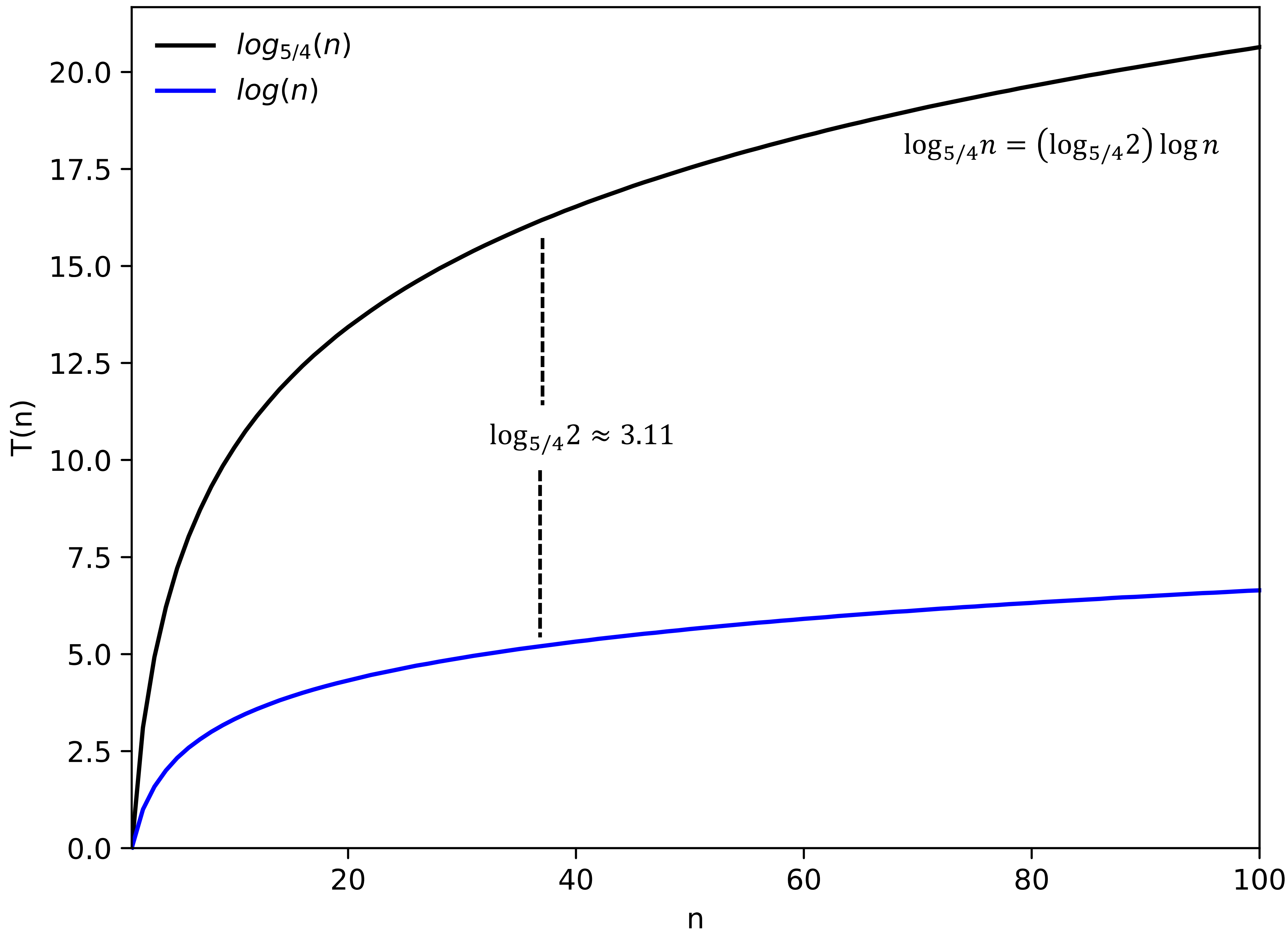

```
def gcd_euclid(a, b):
    while b != 0:
        a, b = b, a % b
    return a
```

`def gcd_euclid(a, b):`	$n = \text{abs}(a) + \text{abs}(b)$	
`while b != 0:`	$O(1)$	?
`a, b = b, a % b`	0	?
`return a`	$O(1)$	$O(1)$
	$O(1)$	?

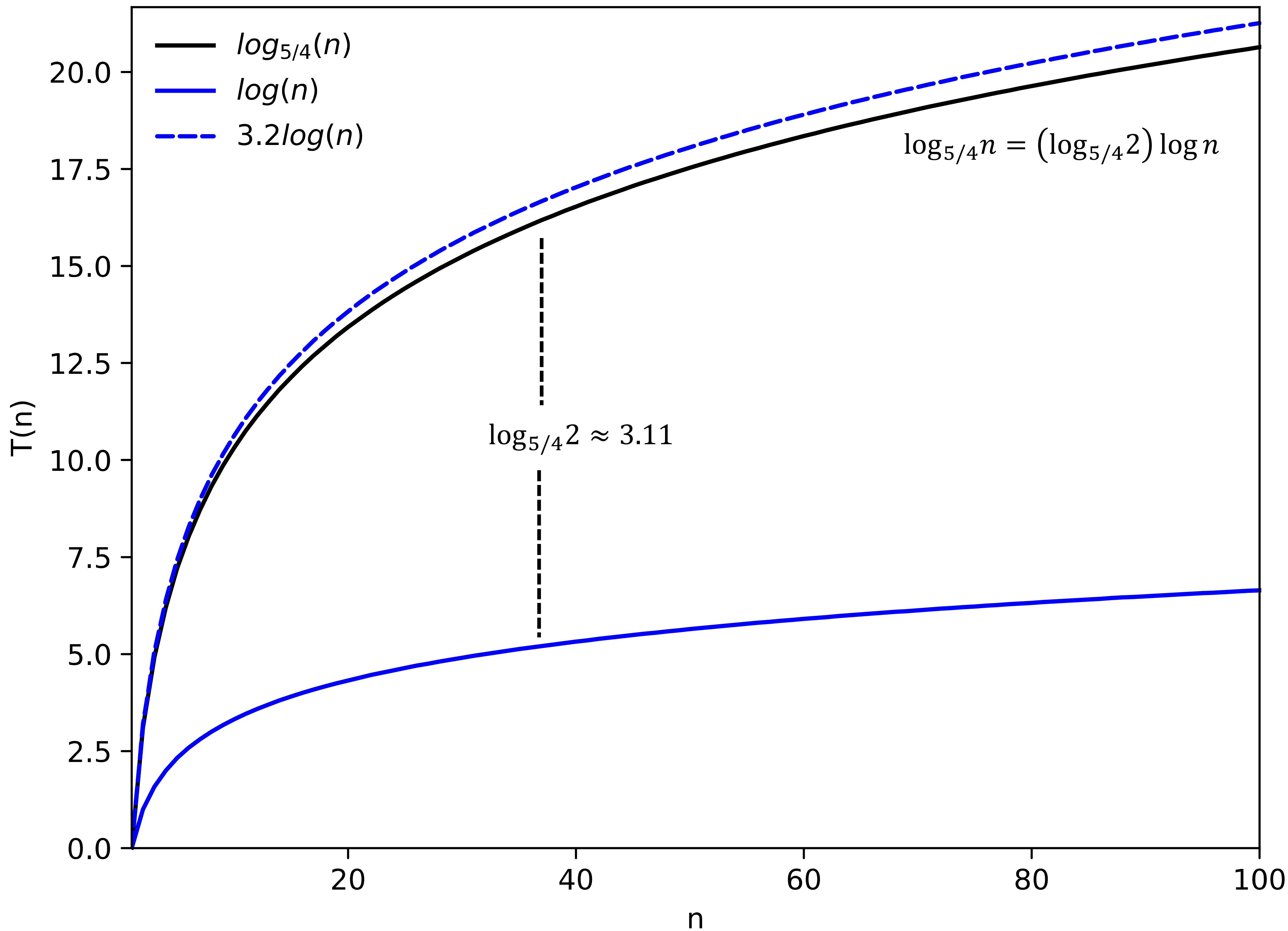
What does this mean in terms of order of growth?



Is order of growth log base 5/4 higher than log base 2?



No: $O(\log n) = O(\log_r n)$



Almost identical analysis as for binary search

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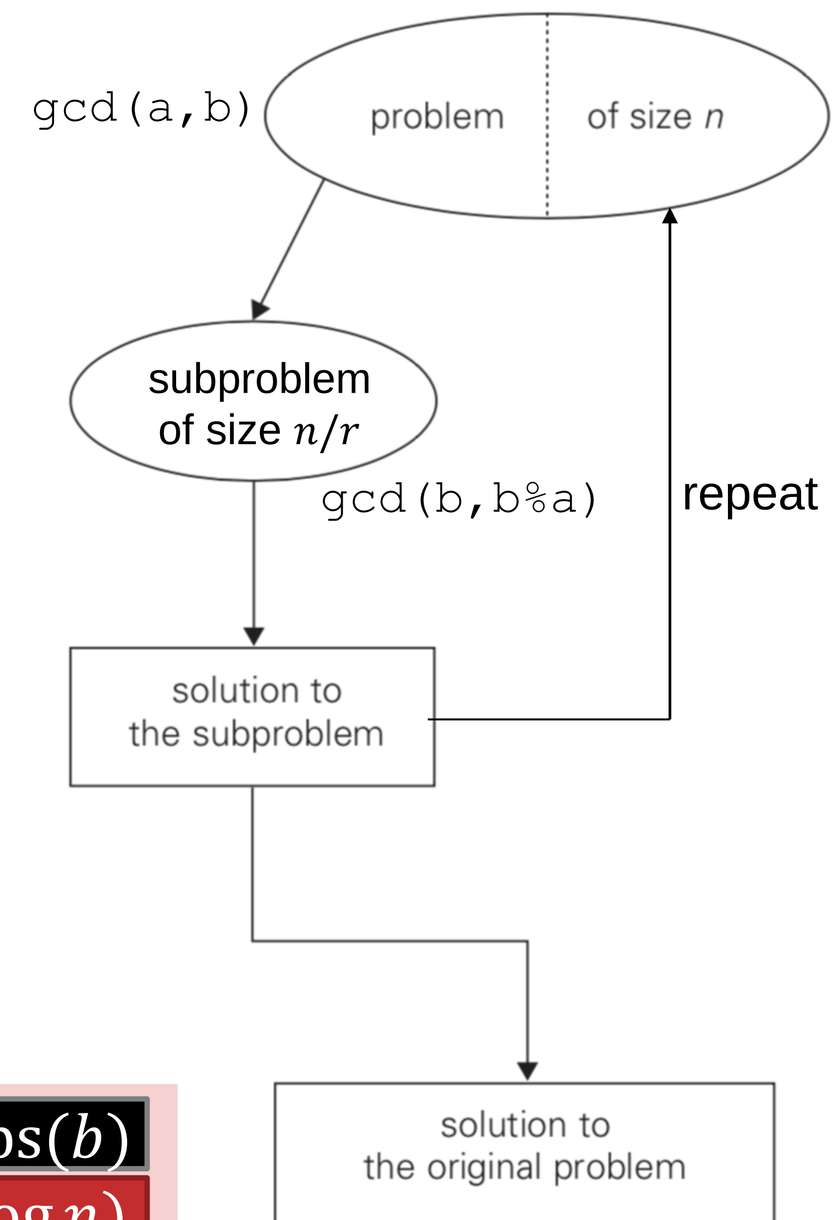


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```

Line		
`def gcd_euclid(a, b):`	$n = \mathrm{abs}(a) + \mathrm{abs}(b)$	
`while b != 0:`	$O(1)$	$O(\log n)$
`a, b = b, a % b`	0	$O(\log n)$
`return a`	$O(1)$	$O(1)$
	$O(1)$	$O(\log n)$

Summary

Algorithmic paradigm: decrease-and-conquer

- decreasing problem size by at least some rate r>1 leads to trivial subproblems after logarithmically many reductions
- if not too much overhead: allows to replace linear complexity term by logarithmic term

Binary Search allows logarithmic time look-up of value in sorted sequence

Euclid's Algorithm finds gcd in time logarithmically in sum of input abs(a) + abs(b)

Coming Up

- More examples for algorithm analysis
- Divide and conquer