FIT5047 Week 3
Knowledge representation

授课老师：Joe

Propositional Logic

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Literal: a proposition or its negation

- E.g., P, $\neg P$

Clause: a disjunction of literals

- E.g., $\neg P \vee Q \vee A$

Negation: If S is a sentence, $\neg S$ is a sentence

Conjunction:
If S_1 and S_2 are sentences, $S_1 \wedge S_2$ is a sentence

Disjunction:
If S_1 and S_2 are sentences, $S_1 \vee S_2$ is a sentence

Implication:
If S_1 and S_2 are sentences, $S_1 \Rightarrow S_2$ is a sentence

Biconditional:
If S_1 and S_2 are sentences, $S_1 \Leftrightarrow S_2$ is a sentence

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$\neg S$	is true iff S is false	
$S_1 \wedge S_2$	is true iff S_1 is true and S_2 is true	
$S_1 \vee S_2$	is true iff S_1 is true or S_2 is true	
$S_1 \Rightarrow S_2 \equiv \neg S_1 \vee S_2$	is true iff S_1 is false or S_2 is true	
$S_1 \Leftrightarrow S_2$	is true iff $S_1 \Rightarrow S_2$ is true and $S_2 \Rightarrow S_1$ is true	

P	Q	$\neg P$	$P \wedge Q$	$P \vee Q$	$P \Rightarrow Q$	$P \Leftrightarrow Q$
false	*false*	*true*	*false*	*false*	*true*	*true*
false	*true*	*true*	*false*	*true*	*true*	*false*
true	*false*	*false*	*false*	*true*	*false*	*false*
true	*true*	*false*	*true*	*true*	*true*	*true*

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- Two sentences are **logically equivalent** iff they are both true in the same models: $\alpha \equiv \beta$ iff $\alpha \models \beta$ and $\beta \models \alpha$

$(\alpha \wedge \beta)$	$\equiv (\beta \wedge \alpha)$	commutativity of $\wedge$
$(\alpha \vee \beta)$	$\equiv (\beta \vee \alpha)$	commutativity of $\vee$
$((\alpha \wedge \beta) \wedge \gamma)$	$\equiv (\alpha \wedge (\beta \wedge \gamma))$	associativity of $\wedge$
$((\alpha \vee \beta) \vee \gamma)$	$\equiv (\alpha \vee (\beta \vee \gamma))$	associativity of $\vee$
$\neg(\neg\alpha)$	$\equiv \alpha$	double-negation elimination
$(\alpha \Rightarrow \beta)$	$\equiv (\neg\beta \Rightarrow \neg\alpha)$	contraposition
$(\alpha \Rightarrow \beta)$	$\equiv (\neg\alpha \vee \beta)$	implication elimination
$(\alpha \Leftrightarrow \beta)$	$\equiv ((\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \alpha))$	biconditional elimination
$\neg(\alpha \wedge \beta)$	$\equiv (\neg\alpha \vee \neg\beta)$	de Morgan
$\neg(\alpha \vee \beta)$	$\equiv (\neg\alpha \wedge \neg\beta)$	de Morgan
$(\alpha \wedge (\beta \vee \gamma))$	$\equiv ((\alpha \wedge \beta) \vee (\alpha \wedge \gamma))$	distributivity of $\wedge$ over $\vee$
$(\alpha \vee (\beta \wedge \gamma))$	$\equiv ((\alpha \vee \beta) \wedge (\alpha \vee \gamma))$	distributivity of $\vee$ over $\wedge$

- **A sentence is valid if it is true in all models**
  - E.g., *True*, $A \lor \neg A$, $A \Rightarrow A$
- **Validity is connected to inference via the Deduction Theorem**
  - $\alpha \models \beta$ iff $\alpha \Rightarrow \beta$ is valid
- **A sentence is satisfiable if it is true in some model**
  - E.g., $A \lor B$, C
- **A sentence is unsatisfiable if it is true in no model**
  - E.g., $A \land \neg A$
- **Satisfiability and validity are connected**
  - α is valid iff $\neg\alpha$ is unsatisfiable
  - α is satisfiable iff $\neg\alpha$ is not valid
  - $\alpha \models \beta$ iff $\alpha \land \neg\beta$ is unsatisfiable

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Prove that $(A \wedge (A \Rightarrow B)) \Rightarrow B$ **is valid**

- $(A \wedge (\neg A \vee B)) \Rightarrow B$
- $((A \wedge \neg A) \vee (A \wedge B)) \Rightarrow B$
- $(A \wedge B) \Rightarrow B$
- $\neg(A \wedge B) \vee B$
- $\neg A \vee \neg B \vee B$
- True

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- **Prove that α is valid iff ¬α is unsatisfiable**
  - If α is valid, it is true in all models ➔ there does not exist a model for which ¬α is true ➔ ¬α is unsatisfiable
  - If ¬α is unsatisfiable ➔ there does not exist a model for which ¬α is true ➔ α is true in all models ➔ α is valid
- **Prove that α is satisfiable iff ¬α is not valid**
  - If α is satisfiable, it is true in some models ➔ ¬α is not true in these models ➔ ¬α is not valid
  - If ¬α is not valid➔ there exist some models for which ¬α is false ➔ α is true in these models ➔ α is satisfiable

Resolution

1. Eat $\Rightarrow$ HaveLessMoney
2. $\neg$Eat $\Rightarrow$ Hungry

Resolvent: HaveLessMoney $\vee$ Hungry

Conversion to conjunctive normal form

1. Eliminate $\Leftrightarrow$, replacing $\alpha \Leftrightarrow \beta$ with $(\alpha \Rightarrow \beta)\wedge(\beta \Rightarrow \alpha)$
2. Eliminate $\Rightarrow$, replacing $\alpha \Rightarrow \beta$ with $\neg\alpha\vee\beta$
3. Move $\neg$ inwards by repeated application of the following equivalences:
   - double-negation: $\neg(\neg\alpha) \equiv \alpha$
   - de Morgan $\neg(\alpha \wedge \beta) \equiv \neg\alpha \vee \neg\beta$
   - de Morgan $\neg(\alpha \vee \beta) \equiv \neg\alpha \wedge \neg\beta$
4. Apply distributivity law ($\wedge$ over $\vee$) and flatten

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$A \Leftrightarrow (B \vee C)$

1. Eliminate $\Leftrightarrow$: $(A \Rightarrow (B \vee C)) \wedge ((B \vee C) \Rightarrow A)$
2. Eliminate $\Rightarrow$: $(\neg A \vee (B \vee C)) \wedge (\neg (B \vee C) \vee A)$
3. Move $\neg$ inwards: $(\neg A \vee (B \vee C)) \wedge ((\neg B \wedge \neg C) \vee A)$
   $(\neg A \vee B \vee C) \wedge ((\neg B \wedge \neg C) \vee A)$
4. Apply distributivity law:
   $(\neg A \vee B \vee C) \wedge (\neg B \vee A) \wedge (\neg C \vee A)$

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- **Proof by refutation**
  1. Negate the goal and add the negation to the set of clauses
  2. Apply resolution to the clauses in the set of clauses until a contradiction is reached
- **Answer extraction**
  1. Build a tautology by appending the goal itself to the negation of the goal
  2. When the negated goal is contradicted, the answer resides in the goal

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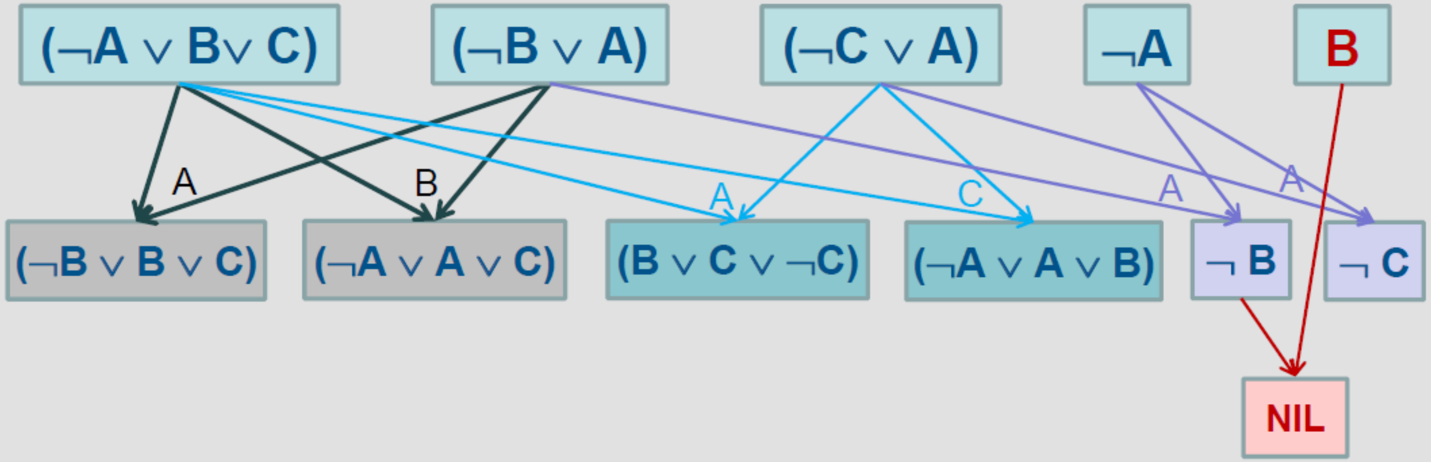
- ***KB*** **= (A ⇔ (B∨ C)) ∧ ¬A**
  (¬A ∨ B∨ C) ∧ (¬B ∨ A) ∧ (¬C ∨ A) ∧ ¬A
- **Prove: α = ¬B**



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- **KB:**
  - R1: $P \Rightarrow Q$
  - R2: $L \wedge M \Rightarrow P$
  - R3: $B \wedge L \Rightarrow M$
  - R4: $A \wedge P \Rightarrow L$
  - R5: $A \wedge B \Rightarrow L$
  - A
  - B
- **Prove Q**

- **KB:**
  - R1: $\neg P \vee Q$
  - R2: $\neg L \vee \neg M \vee P$
  - R3: $\neg B \vee \neg L \vee M$
  - R4: $\neg A \vee \neg P \vee L$
  - R5: $\neg A \vee \neg B \vee L$
  - A
  - B
- **Negate Q: $\neg Q$**

$\underline{\neg \mathbf{Q}}$ R1: $\neg P \vee \underline{Q}$

$\underline{\neg \mathbf{P}}$

R2: $\neg L \vee \neg M \vee \underline{P}$

$\neg \mathbf{L} \vee \underline{\neg \mathbf{M}}$

R3: $\neg B \vee \neg L \vee \underline{M}$

$\neg \mathbf{B} \vee \underline{\neg \mathbf{L}}$

R5: $\neg A \vee \neg B \vee \underline{L}$

$\underline{\neg \mathbf{A}} \vee \neg \mathbf{B}$

$\underline{A}$

$\underline{\neg \mathbf{B}}$

$\underline{B}$

nil

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Definite clause – a disjunction of literals of which <u>exactly one</u> is positive

$\neg A \lor \neg B \lor C$

Horn clause – a disjunction of literals of which <u>at most one</u> is positive

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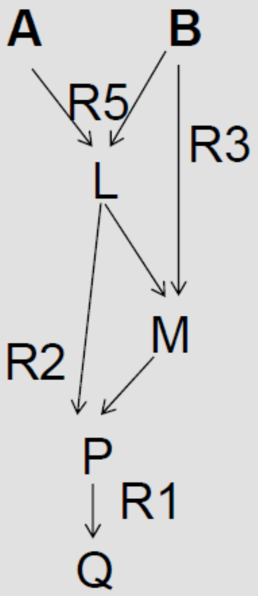
- **KB:**
  - R1: $P \Rightarrow Q$
  - R2: $L \wedge M \Rightarrow P$
  - R3: $B \wedge L \Rightarrow M$
  - R4: $A \wedge P \Rightarrow L$
  - R5: $A \wedge B \Rightarrow L$
  - A
  - B
- **Prove Q**



Agenda	Count					Inferred
	R1	R2	R3	R4	R5	
A B	1	2	2	2	2	
L						
M						

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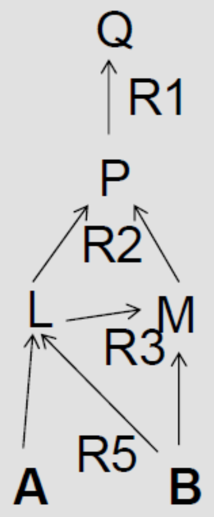
- **KB:**
  - R1: $P \Rightarrow Q$
  - R2: $L \wedge M \Rightarrow P$
  - R3: $B \wedge L \Rightarrow M$
  - R4: $A \wedge P \Rightarrow L$
  - R5: $A \wedge B \Rightarrow L$
  - A
  - B
- **Prove Q**



First order logic

Notation

Predicate symbol – MARRIED
Constant – *John* or *A*
Function – *Mother* or *f*
Variable – *x*

Predicate – a function that only returns Boolean
Function can return any value

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- disjunction – $P(x) \vee Q(y) \vee W(x,y)$
- conjunction – $P(x) \wedge Q(y)$
- implication – $P(x) \Rightarrow W(x,y) \equiv \neg P(x) \vee W(x,y)$

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Quantification

predicate constant

- Universal (∀) –
  $\forall x\ [\text{ELEPHANT}(x) \Rightarrow \text{COLOUR}(x,\text{Gray})]$
- Existential (∃) –
  $\exists x\ \text{WRITE}(x,\text{Computer-Chess})$

variable

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$$\exists x(\forall y[P(x,y)\wedge Q(x,y)]\Rightarrow R(x))$$

$$\neg f(A)$$

$$\neg P(A,g(A,B,A))$$

$$f(P(A))$$

$$\forall P\ P(A)$$

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- Existential inside the scope of universal
  E.g., $\forall s \exists c$ Eats(s,c)

- Universal inside the scope of existential
  E.g., $\exists c \forall s$ Eats(s,c)

- $\neg(\exists x)P(x) \equiv (\forall x)\,[\neg P(x)]$
  There does not exist an x such that P(x) is true ≡
  For all x, P(x) is false
- $\neg(\forall x)P(x) \equiv (\exists x)\,[\neg P(x)]$
  It is not true that for all x P(x) is true ≡
  There exists an x, such that ¬P(x)
- $(\forall x)[P(x) \land Q(x)] \equiv (\forall x)P(x) \land (\forall y)Q(y)$
  For all x, P(x) and Q(x) are true ≡
  For all x, P(x) is true, and for all y Q(y) is true
- $(\exists x)[P(x) \lor Q(x)] \equiv (\exists x)P(x) \lor (\exists y)Q(y)$
  There is an x, such that P(x) is true or Q(x) is true ≡
  There is an x, such that P(x) is true, or
  there is a y, such that Q(y) is true

$(\forall x)[P(x) \lor Q(x)]$ $(\forall x)P(x) \lor (\forall y)Q(y)$

Substitution

Substitution is a set of ordered pairs
$s=\{v_1|t_1, v_2|t_2, \ldots, v_n|t_n\}$
where $v_i|t_i$ means that term t_i substitutes variable v_i throughout

- Example:
  $P(x,y)\ \{x|A, y|B\} \rightarrow P(A,B)$

Semantics: all elements are applied *simultaneously*

- Example:
  $P(w, y, g(z), x)\ \{x|g(y), y|h(z), z|x\} \rightarrow P(w, h(z), g(x), g(y))$

Substitution

$s_i s_j$ – composition of two substitutions

- Apply s_j to the terms of s_i
- Add any pairs of s_j having variables not in s_i

Example: term variable

s1 = {z|g(x,y)}   s2 = {x|A, y|B, w|C, z|D}

s1s2 = {z|g(x,y)}{x|A, y|B, w|C, z|D}={z|g(A,B), x|A, y|B, w|C}

s2s1 = {x|A, y|B, w|C, z|D} {z|g(x,y)} = {x|A, y|B, w|C, z|D}

NOT commutative: s1s2 ≠ s2s1

Unification

s={x|A, y|B} unifies P(x,f(y)) with P(x,f(B))

You can substitute anything into variable, not the other way around

Converting wffs into clauses

1. Eliminate implication symbols
2. Reduce scopes of negation symbols
3. Standardize variables (for each quantifier)
4. Eliminate existential quantifiers (*skolemize*)
5. Move all universal quantifiers to the front
6. Put result in conjunctive normal form (CNF)
7. Eliminate universal quantifiers
8. Eliminate $\wedge$ symbols
9. Rename variables (standardize variables apart for each clause)

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$\forall x$ [CanRead(x) $\Rightarrow$ Intelligent(x)]

1. Eliminate $\Rightarrow$: $\forall x$ [$\neg$CanRead(x) v Intelligent(x)]

7. Eliminate $\forall$: $\neg$CanRead(x) v Intelligent(x)

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$\forall x\ [\neg\ (\forall y)\ [\ P(x,y) \Rightarrow Q(x,y)]\ ]$

1. Eliminate $\Rightarrow$: $\forall x\ [\neg\ (\forall y)\ [\ \neg P(x,y) \vee Q(x,y)]\ ]$
2. Reduce scope of $\neg$: $\forall x\ [\ \exists y\ \neg\ [\ \neg P(x,y) \vee Q(x,y)]\ ]$
   $\forall x\ \ \exists y\ \ [\ P(x,y) \wedge \neg Q(x,y)\ ]$
4. Eliminate $\exists$: $\forall x\ [\ P(x,g(x)) \wedge \neg Q(x,g(x))\ ]$
7. Eliminate $\forall$: $P(x,g(x)) \wedge \neg Q(x,g(x))$
8. Eliminate $\wedge$ symbols: $\{\ P(x,g(x)),\ \neg Q(x,g(x))\ \}$
9. Standardize variables apart: $\{\ P(x_1,g(x_1)),\ \neg Q(x_2,g(x_2))\ \}$

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1. **If a unit is easy, there are some students who are enrolled in it who are happy**

   $\forall u\ [\ Easy(u) \Rightarrow \exists s\ [\ Enrolled(s,u) \land Happy(s)\ ]\ ]$

2. **If a unit has a final exam, no students that are enrolled in it are happy**

   $\forall u\ [\ HasFinal(u) \Rightarrow \neg \exists s\ [\ Enrolled(s,u) \land Happy(s)\ ]\ ]$

3. **Prove that if a unit has a final exam, the unit is not easy**

   $\forall u\ [\ HasFinal(u) \Rightarrow \neg\ Easy(u)\ ]$

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1. $\forall u\ [\ Easy(u) \Rightarrow \exists s\ [\ Enrolled(s,u) \wedge Happy(s)\ ]\ ]$

Eliminate $\Rightarrow$: $\forall u\ [\ \neg\ Easy(u) \vee \exists s\ [\ Enrolled(s,u) \wedge Happy(s)\ ]\ ]$

Eliminate $\exists$: $\forall u\ [\ \neg\ Easy(u) \vee [\ Enrolled(g(u),u) \wedge Happy(g(u))\ ]\ ]$

Convert to CNF: $\forall u\ [\ [\ \neg\ Easy(u) \vee Enrolled(g(u),u)\ ] \wedge$
$[\ \neg\ Easy(u) \vee Happy(g(u))\ ]\ ]$

Eliminate $\forall$: $[\ \neg\ Easy(u) \vee Enrolled(g(u),u)\ ] \wedge$
$[\ \neg\ Easy(u) \vee Happy(g(u))\ ]$

Eliminate $\wedge$ and standardize variables apart:

1.1 $\neg\ Easy(u_1) \vee Enrolled(g(u_1),u_1)$

1.2 $\neg\ Easy(u_2) \vee Happy(g(u_2))$

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2. $\forall u\ [\ HasFinal(u) \Rightarrow \neg\ \exists s\ [\ Enrolled(s,u) \wedge Happy(s)\ ]\ ]$

Eliminate $\Rightarrow$: $\forall u\ [\ \neg\ HasFinal(u) \vee$

$\neg\ \exists s\ [\ Enrolled(s,u) \wedge Happy(s)\ ]\ ]$

Reduce scope of $\neg$:

$\forall u\ [\neg\ HasFinal(u) \vee \forall s\ \neg[\ Enrolled(s,u) \wedge Happy(s)\ ]\ ]$

$\forall u\ [\neg\ HasFinal(u) \vee \forall s[\ \neg Enrolled(s,u) \vee \neg Happy(s)\ ]\ ]$

Move $\forall$ to the front: $\forall u\ \forall s\ [\ \neg\ HasFinal(u) \vee$

$\neg\ Enrolled(s,u) \vee \neg Happy(s)\ ]$

Eliminate $\forall$ and standardize variables apart:

$\neg\ HasFinal(u_3) \vee \neg Enrolled(s_3,u_3) \vee \neg Happy(s_3)$

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Using resolution to prove statement 3

Negate the goal:

3'. $\neg \forall u\, [\, HasFinal(u) \Rightarrow \neg\, Easy(u)\,]$

Eliminate $\Rightarrow$: $\neg \forall u\, [\neg\, HasFinal(u) \vee \neg\, Easy(u)\,]$

Reduce scope of $\neg$: $\exists u\, [\, HasFinal(u) \wedge Easy(u)\,]$

Eliminate $\exists$: $HasFinal(A) \wedge Easy(A)$

Eliminate $\wedge$: 3'.1 $HasFinal(A)$

3'.2 $Easy(A)$

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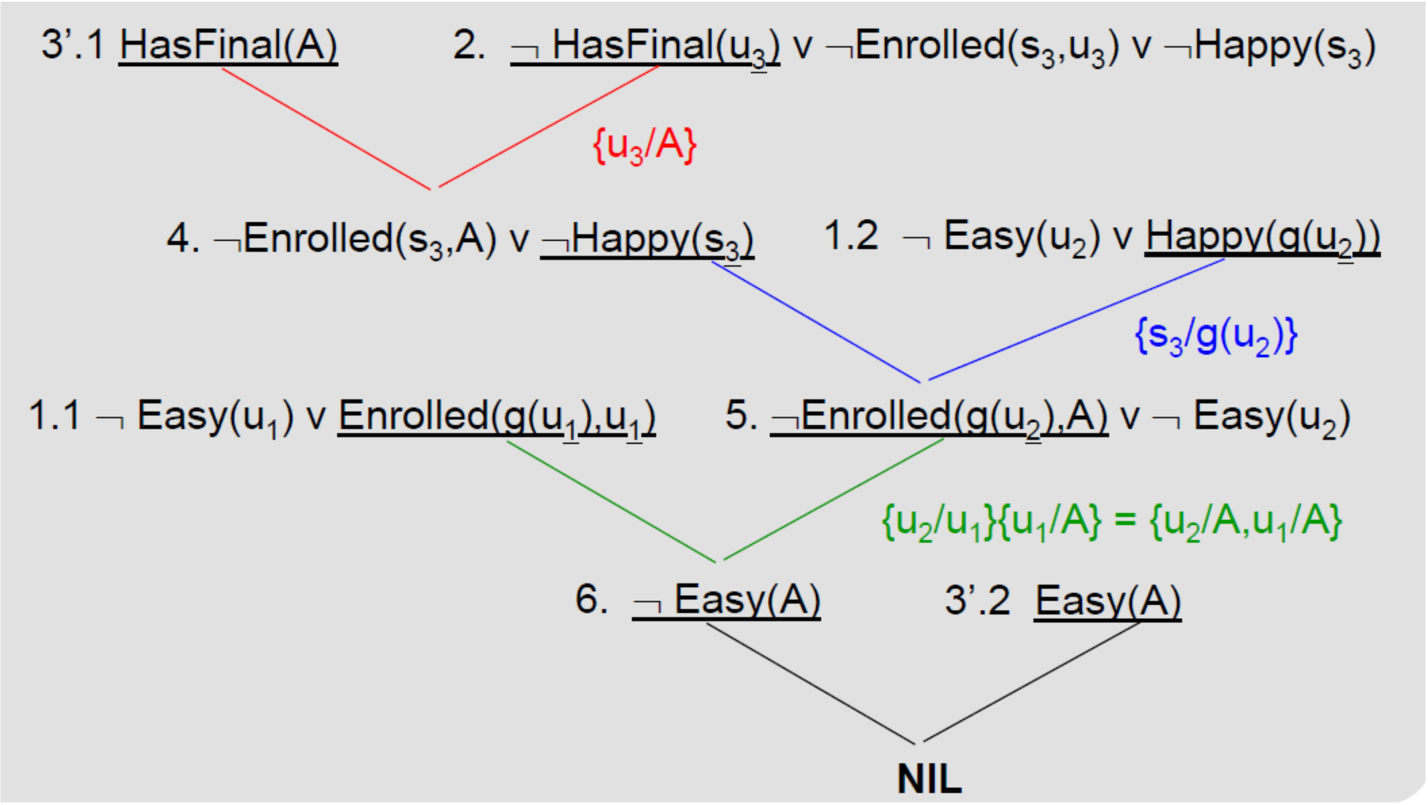

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Question answering

1. MANAGER(Purchasing-dept., John-Jones)
2. WORKSIN(Purchasing-dept., Joe-Smith)
3. $\forall x\ \forall y\ \forall z\ [\text{WORKSIN}(x,y) \land \text{MANAGER}(x,z)] \Rightarrow \text{BOSSOF}(y,z)$
4. **Goal:** Who is the boss of Joe Smith?

   $\exists x\ \text{BOSSOF}(\text{Joe-Smith},x)$

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3. ¬ WORKSIN(x_1,y) v ¬ MANAGER(x_1,z) v BOSSOF(y,z)

4. ¬ BOSSOF(Joe-Smith,x_2) v BOSSOF(Joe-Smith,x_2)

{y|Joe-Smith, x_2|z}

5. ¬ WORKSIN(x_1,Joe-Smith) v ¬ MANAGER(x_1,z) BOSSOF(Joe-Smith,z)

1. MANAGER(Purchasing-dept., John-Jones)

{x_1|Purchasing-dept, z|John-Jones}

6. ¬ WORKSIN(Purchasing-dept,Joe-Smith) BOSSOF(Joe-Smith,John-Jones)

2. WORKSIN(Purchasing-dept,Joe-Smith)

NIL