Solutions to FIT5047 Tutorial on Probability

Exercise 1: Prior probabilities and conditionalization

You work at the treasury and know for a fact that only 1 coin in 1 million is biased. If it is biased, it is biased so that Heads comes up 2/3 instead of 1/2 the time. You pick a new coin at random from a bin, and toss it 10 times in a row. It comes up heads 10 times in a row. Which is more probable, and why?

(a) The coin is fair: you were just lucky in your tosses.

(b) The coin is biased towards heads.

(c) The first two answers are almost equally probable.

(d) The coin is biased towards tails.

SOLUTION:
The prior probability of bias is 1×10^{-6}. The probability of the evidence e (10 heads in a row) given bias is $(\frac{2}{3})^{10} = 0.01734153$. The probability of the evidence given no bias is $(\frac{1}{2})^{10} = 0.0009765625$.

According to Bayes theorem:

$$\Pr(\text{bias}|e) = \frac{\Pr(e|\text{bias})\Pr(\text{bias})}{\Pr(e|\text{bias})\Pr(\text{bias}) + \Pr(e|\neg\text{bias})\Pr(\neg\text{bias})}$$

This yields:

$$\Pr(\text{bias}|e) = \frac{0.01734153 \times 10^{-6}}{0.01734153 \times 10^{-6} + 0.0009765625 \times (1 - 10^{-6})} = 0.00001775$$

This means that the prior is strong enough against bias, so that a much larger discrepancy between heads and tails is needed to overcome it. So, the best answer is (a), no bias.

Exercise 2: Independence

A roulette wheel in a casino has 32 paying numbers (from 1 to 32) and two non-paying numbers (0 and 00). Each of these numbers has a 1/34 chance of being selected on any one roll. Supposing the roulette is functioning normally, which sequence of numbers is most probable?:

(a) 12, 12, 12, 5

(b) 7, 30, 0, 17

(c) 12, 1, 12, 1

(d) They are all equally probable

SOLUTION:
These are independent draws, so the sequences are all equally probable. All sequences of the same length are equally probable, no matter how odd it might look.

Exercise 3: Joint distribution

You are the ruler of a distant kingdom, and you have just gotten engaged to your beloved. You have set a date for the wedding, but you now need to send a message to your soon-to−be in-laws. Hopefully, they will attend the wedding (A = +a), but they may not (A = −a).

You dispatch your smartest and fastest messenger across the barren wastes. Unfortunately, there are a number of things that could go wrong. Your messenger could be beset by a pack of wild bears (B = +b). Also, your messenger may be captured (C = +c) by a cohort of cave trolls. It is even possible that both ills could have befallen the messenger! Using your knowledge about the dangers of bears, of cave trolls, and of your in-laws, you have tallied out the following joint probability table:

A	**B**	**C**	**Pr(A,B,C)**
+a	+b	+c	0.0000
+a	+b	−c	0.0008
+a	−b	+c	0.0396
+a	−b	−c	0.6336
−a	+b	+c	0.0020
−a	+b	−c	0.0072
−a	−b	+c	0.1584
−a	−b	−c	0.1584

(a) What is the distribution Pr(B,C)? Your answer should be in the form of a table.
SOLUTION:
By marginalizing over A.

B	**C**	**Pr(B,C)**
+b	+c	0.0020
+b	−c	0.0080
−b	+c	0.1980
−b	−c	0.7920

(b) Are B and C independent? Justify your answer using the probabilities computed in part (a).
SOLUTION:
Yes, they are, because the joint distribution in (a) can be obtained from the marginal distributions of B and C.

B	**Pr(B)**
+b	0.01
−b	0.99

C	**Pr(C)**
+c	0.2
−c	0.8

(c) Because you are also a naturalist, you are curious as to what the probability of a bear attack is in your model. What is Pr(B), the marginal distribution over B given no evidence? (i.e., the prior distribution of a bear attack)

SOLUTION:
The prior distribution of a bear attack is given in the table for B in part (b).

(d) If your in-laws do not attend the wedding (A = −a), what is the posterior distribution of a bear attack, $\Pr(B|A = -a)$? Does it make intuitive sense how the answer has shifted from part (c)?

SOLUTION:

- We obtain $\Pr(-a, +b)$ by marginalizing C in the initial table: 0.0092.
- We obtain $\Pr(-a)$ by marginalizing B and C in the initial table:

A	**Pr(A)**
+a	0.6740
−a	0.3260

$$\Pr(+b|-a) = \frac{\Pr(-a, +b)}{\Pr(-a)} = \frac{0.0092}{0.3260} = 0.0282$$

$$\Pr(-b|-a) = \frac{\Pr(-a, -b)}{\Pr(-a)} = \frac{0.3168}{0.3260} = 0.9718$$

This result makes intuitive sense because given that the parents are not attending the wedding, there is a higher chance that something bad happened to the messenger, e.g., a bear attack.

(e) Suppose you further learn that your messenger was captured by cave trolls. What is the new posterior distribution $\Pr(B|A = -a, C = +c)$? Is the conditional probability of bear attack higher or lower than from part (d)? Does this make intuitive sense?

The general phenomenon where the discovery that one possible cause of an event is true decreases belief in other possible causes is called *explaining away*.

SOLUTION:

- We obtain $\Pr(+b, -a, +c)$ directly from the initial table: 0.0020.
- We obtain $\Pr(-a, +c)$ by marginalizing B in the initial table: 0.1604.

$$\Pr(+b|A = -a, C = +c) = \frac{\Pr(+b, -a, +c)}{\Pr(-a, +c)} = \frac{0.0020}{0.1604} = 0.0125$$

$$\Pr(-b|A = -a, C = +c) = \frac{\Pr(-b, -a, +c)}{\Pr(-a, +c)} = \frac{0.1584}{0.1604} = 0.9875$$

This result makes sense because given the parents are not attending and there was a capturing by cave trolls, the probability of a bear attack should decrease (it is called *explaining away*).